FROM PRECISION TO AMBIGUITY

OR, INSTEAD

(IMPROVED) LOWER BOUND ON THE CHARGE RADIUS

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“

IT IS THE THEORY THAT DECIDES WHAT WE CAN OBSERVE.

— *Albert Einstein*

”

Can we can observe the charge radius in electron scattering?

Extrapolation is a major (theor.) systematics:

Experimental data for $d\sigma/dQ^2, G_E(Q^2)$, *etc.*, are at finite Q^2,

whereas the charge radius is the slope at 0:

$$R_E := \left(-6 \frac{dG_E(Q^2)}{dQ^2} \bigg|_{Q^2=0} \right)^{1/2} \equiv \sqrt{-6G'_E(0)}$$

Presently circumvented by *assuming* a functional form

for the form factor $G_E(Q^2)$.

—> Bias, non-improvable, ambiguity, …, lots of publications!

The **extrapolation problem** in atomic spectroscopy solved via a Taylor expansion

$$G_E(Q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Q^{2n}\, G_E^{(n)}(0)$$

with $Q \sim Z\alpha m_r$, about 1 MeV for light muonic atoms.

Rapidly convergent, systematically improvable expansion!

Strictly speaking,

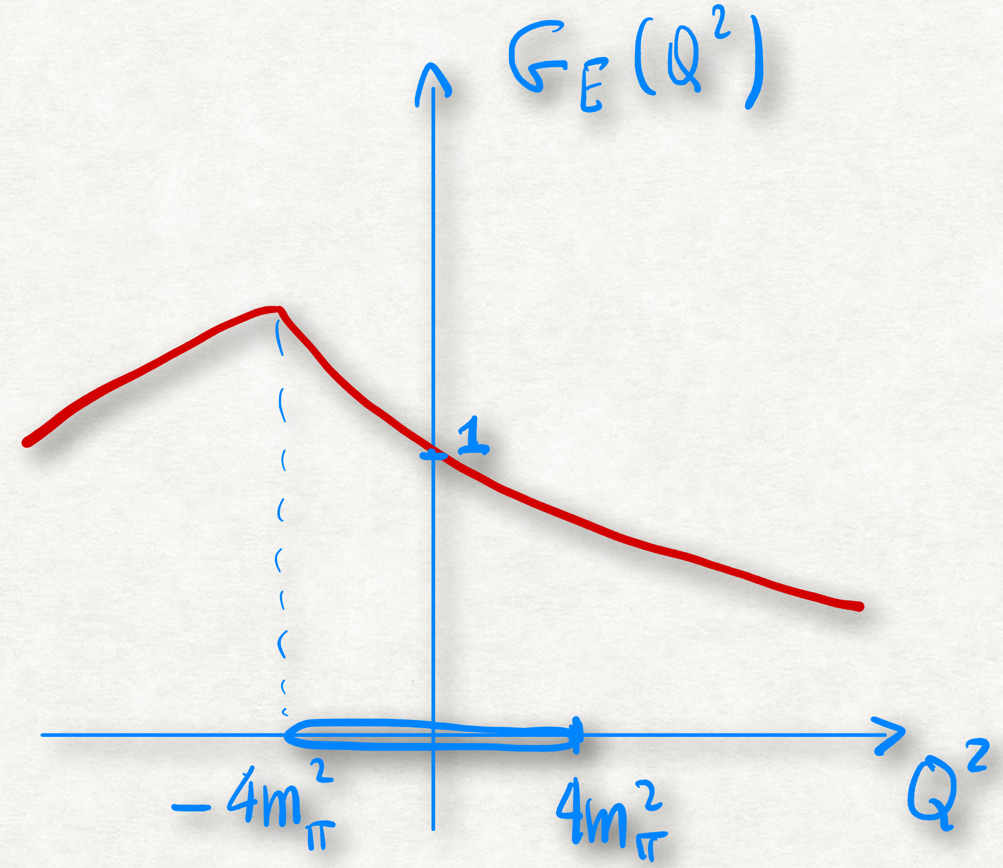
Convoluted wave-functions

$$E_{\mathrm{LS}} = -\frac{2Z\alpha}{\pi}\int_0^\infty dQ\, w_{2P-2S}(Q)\, G_E(Q^2),$$

with $w_{2P-2S}(Q) = 2(Z\alpha m_r)^4 Q^2 \dfrac{(Z\alpha m_r)^2 - Q^2}{\left[(Z\alpha m_r)^2 + Q^2\right]^4}$.

Hagelstein, Miskimen & VP, *PPNP (2016)*

Taylor expansion is unsuitable for electron scattering where $Q \sim 2m_\pi \approx 300$ MeV, $\quad Q^2 \sim 0.1$ GeV^2



Improvable by ChPT and DRs, but still model-dependent!

ChPT cannot predict radii, form factors

LQCD can, in principle, but then it's LQCD+expt ?

WHAT TO DO?

> “
> ADMITTING TO A PROBLEM IS THE FIRST STEP TOWARD FINDING A SOLUTION
>
> — *John Perkins*
> ”

FROM EXTRAPOLATION TO BOUNDS

My main assumption (not valid for A1 dataset!):

We obtained precise $G_E(Q^2)$ data, which are (theory) unbiased by a choice of the functional form.

We need to extrapolate the slope from Q^2 to 0:

$$G'_E(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Q^{2n}\, G_E^{(n+1)}(Q^2)$$

$$= G'_E(Q^2) - Q^2 G''_E(Q^2) + \frac{1}{2} Q^4 G'''_E(Q^2) + \ldots$$

Any truncation provides a bound (upper for the slope, lower for R_E)

provided $\quad 0 < G_E(Q^2) \leq 1, \quad$ with $G_E(0) = 1.$

AN OPTIMAL BOUND

Some bounds are better than others, e.g.,

$$G_E'(Q^2) \;\geq\; \frac{G_E'(Q^2)}{G_E(Q^2)} \;\geq\; \frac{1}{Q^2}\log G_E(Q^2) \;\geq G_E'(0)$$

An optimal *lower bound* on the radius:

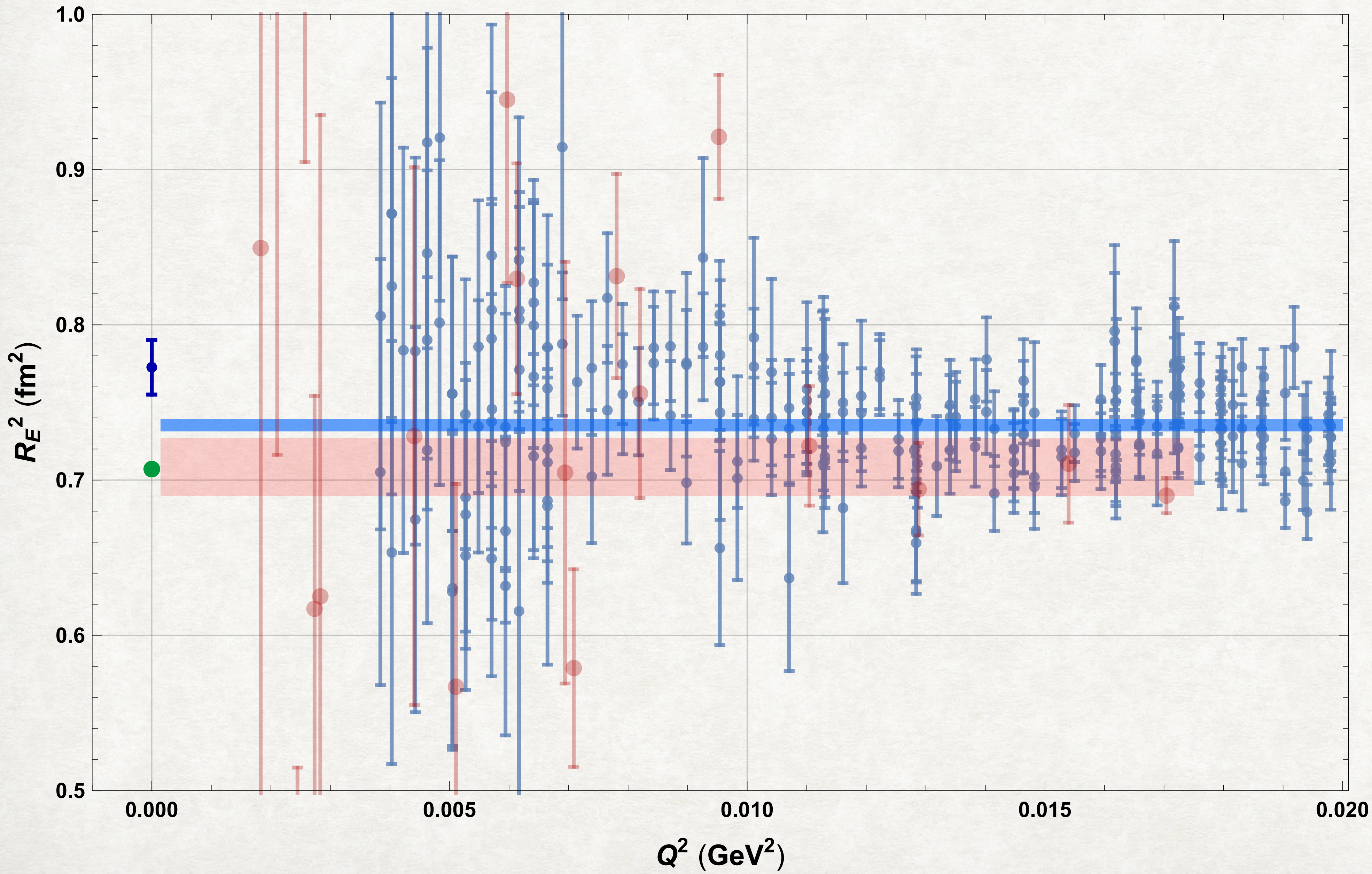
Hagelstein & VP, *PLB 797 (2019)*

$$R_E^2(Q^2) = -\frac{6}{Q^2}\log G_E(Q^2)\,.$$

$$R_E^2(Q^2) \leq R_E^2(0) = R_E^2$$

IN PRACTICE (A1 EXP.)

$$R_E^2(Q^2) = -\frac{6}{Q^2}\log G_E(Q^2) \underset{Q^2=0}{\longrightarrow} R_E^2$$



Data points from A1 Coll.:
Bernauer et al (2010)
Mihovilovic et al (2017)

Data at $Q^2 = 0$:
Bernauer et al (2010)
CREMA (2013)

PRAD EXP.

M. Horbatsch / Physics Letters B 804 (2020) 135373

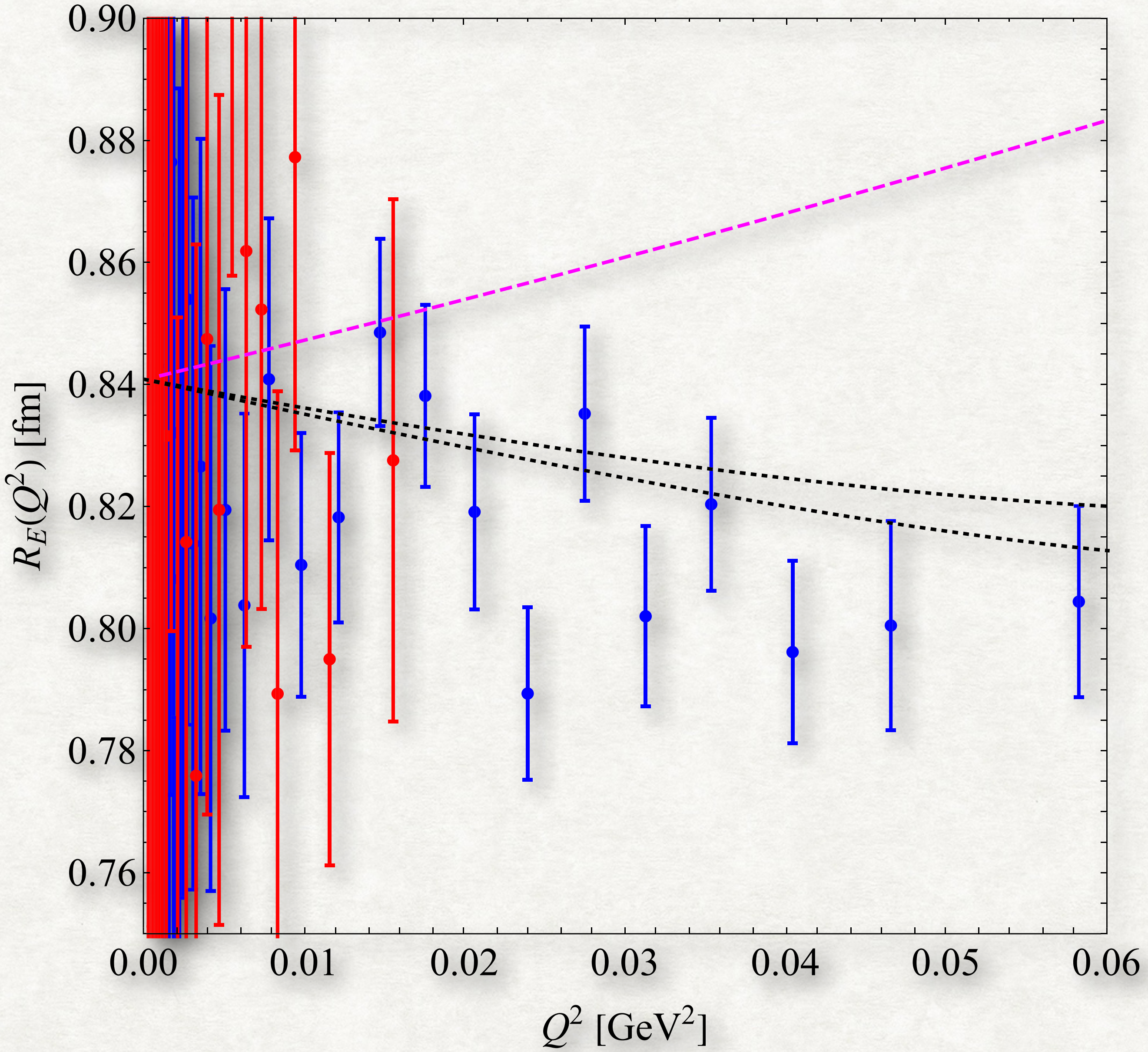


Fig. 3. The proton electric charge radius function $R_E(Q^2)$ in fm obtained from eq. (6). The data points from the PRad experiment are shown in red and blue for the 1.1 and 2.2 GeV data runs respectively, while the dotted curves correspond to the equivalent result in Fig. 1 which is a prediction based on the spectroscopic value of the charge radius and a theoretical prediction for the higher moments. The dashed magenta curve corresponds to the straight-line result in Fig. 1.

IMPROVED BOUNDS

Improve any bound by extrapolating it to zero!

$$R_E^2(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Q^{2n}\, R_E^{2(n)}(Q^2) = R_E^2(Q^2) - Q^2 R_E^{2\,\prime}(Q^2) + \ldots$$

Taking only the first two terms, the improved boundary function:

$$\boxed{\tilde{R}_E^2(Q^2) = -\frac{12}{Q^2} \log G_E(Q^2) + 6\frac{G_E'(Q^2)}{G_E(Q^2)}}$$

$$R_E^2(Q^2) \leq \tilde{R}_E^2(Q^2) \leq R_E^2$$

Further improvements require higher derivatives of $G_E(Q^2)$.

DIPOLE ILLUSTRATION

$$G_{\text{dip.}}(Q^2) = \frac{1}{(1+Q^2/\Lambda^2)^2} \quad \Rightarrow \quad R^2_{\text{dip.}} \equiv -6G'_{\text{dip.}}(0) = \frac{12}{\Lambda^2}$$

1. $$-6G'_{\text{dip.}}(Q^2) = \frac{12}{\Lambda^2}\left(1 - 3\frac{Q^2}{\Lambda^2} + 6\frac{Q^4}{\Lambda^4} + \ldots\right)$$

2. $$-6\frac{G'_{\text{dip.}}(Q^2)}{G_{\text{dip.}}(Q^2)} = \frac{12}{\Lambda^2}\left(1 - \frac{Q^2}{\Lambda^2} + \frac{Q^4}{\Lambda^4} + \ldots\right)$$

3. $$-\frac{6}{Q^2}\log G_{\text{dip.}}(Q^2) = \frac{12}{\Lambda^2}\left(1 - \frac{Q^2}{2\Lambda^2} + \frac{Q^4}{3\Lambda^4} + \ldots\right)$$

4. $$\tilde{R}^2_{\text{dip.}}(Q^2) \equiv 2\times(3.) - (2.) = \frac{12}{\Lambda^2}\left(1 - 4\frac{Q^4}{\Lambda^4} + \ldots\right)$$

The improved bound is within 0.3% of the true value,

for $Q^2 < 0.1\text{GeV}^2$ (with $\Lambda^2 = 0.71\text{GeV}^2$)

SUMMARY

1. Extrapolation from finite Q^2 to 0 is required for the radius extraction

2. Currently solved by an "*a priori*" functional form of $G_E(Q^2)$, bringing bias, ambiguity.

3. Lower bounds avoid this problem, put it on stricter footing, systematically improvable

4. Looks like it does not (yet) work well with present data

“

A CLEVER MAN SOLVES THE PROBLEM,
A WISE MAN AVOIDS IT

— Albert Einstein(?)

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