Topological Phases and Quantum Simulations of Wilson Fermions Coupled to U(1) Gauge Fields

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Outline of talk

- Motivation
- Setup: $(2+1)d$ Wilson fermions coupled to $\mathsf{U}(1)$ background gauge fields
- Topological phases of the $N_f = 1$ theory at zero density
- Topological phases of the $N_f = 2$ theory at finite density
- Agreement with lattice simulations
- Outlook: ongoing work and future directions

Motivation

Motivation

- Past work: Hamiltonian formulation of Chern-Simons-Maxwell theory

Hamiltonian Lattice Formulation of Compact Maxwell-Chern-Simons Theory

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Figure: arXiv 2407.20225

- Today: start w/o CS term in UV and see how it emerges in the IR using the Hamiltonian

Motivation

- In continuum QFT, coupling a $(2+1)d$ Dirac fermion ψ to a background $\mathsf{U}(1)$ gauge field A_μ leads to a low-energy Chern-Simons TQFT, whose level is fixed by $\mathsf{sign}(m_\psi)$.
- Consistently: integrating out a massive Wilson fermion in the lattice Lagrangian formulation leads to CS terms in the IR.
- Our motivation is two-fold
  1. How can we see the existence of such topological phases within the Hamiltonian formulation?
  2. Can such topological phases be simulated on a quantum computer?
- Key observation: staggered fermions do not lead to phases characterized by non-trivial Chern numbers.
- Simulations and Hamiltonian models based on staggered fermions cannot capture such topological phases!

Setup

Setup: Wilson fermions coupled to a U(1) background gauge field

- Hamiltonian ($a, b = 1, \ldots, N_f$ flavor indices, $k = x, y$, $r \in L \times L$ spatial lattice with PBC)

$$H_1 = H_{\text{hop}} + H_{\text{Wilson},1} = \frac{1}{2}\sum_r \left[\psi_a(r)^\dagger \gamma^0 \left(i\gamma^k + R\right) \psi_a(r+\hat{k})\, U_k(r) + \text{H.C.}\right]$$

$$H_2 = H_{\text{mass}} + H_{\text{Wilson},2} = \sum_r \psi_a(r)^\dagger M_{ab} \gamma^0 \psi_b(r)$$

- $M_{ab} = m_{ab} + 2R\delta_{ab}$: m_{ab} is the mass matrix and $R = 1$ is the Wilson coupling.
- $U_k(r)$ denotes the gauge fields that live on links
- $\psi_a(r)$ denotes a two-component spinor field living at the sites
- Focus on the cases $N_f = 1, 2$.
- We may dynamically gauge this theory by introducing:

$$H_B = -\frac{1}{e^2}\sum_r \cos B(r) \qquad\qquad H_E = \frac{e^2}{2}\sum_{k,r} E_k(r)^2$$

  and imposing Gauss' law at each vertex.
- Today: restrict attention to background gauge fields!

Setup: Wilson fermions coupled to a U(1) background gauge field

- Goal: find effective low-energy Hamiltonian
- Requires choice of background gauge fields such that
  1. Translational invariance is unbroken in the vacuum
  2. Gauss' law is satisfied
  3. Energy is minimized
- These conditions are satisfied if (ED + analytic evidence)

$$U_x(r) = e^{iX} \text{ and } U_y(r) = e^{iY}, \text{ where } X, Y \in \mathrm{U}(1) \text{ for all } r$$

- Supressing flavor indices, the Hamiltonian is (with $M_k = \gamma^0 \left(i\gamma^k + R\right)$)

$$H = \frac{1}{2}\sum_r \left[\psi^\dagger_{r+\hat{x}} M_x e^{iX}\psi_r + \psi^\dagger_{r+\hat{y}} e^{iY} M_y \psi_r\right] + \text{ H.C.} + \sum_r \psi(r)^\dagger \gamma^0 M \psi(r)$$

- In momentum space, we obtain a $(2N_f)$-band model: $H = \sum_k \psi^\dagger_k \mathcal{H}(k_x - X, k_y - Y)\psi_k$
- For the observables we study: sum over all $k \Rightarrow X = Y = 0$ by a shift of k_x, k_y!

Topological Phases of the $N_f = 1$ theory at zero density

Spectrum of the two-band model

- Hamiltonian in momentum space

$$\mathcal{H}(k_x, k_y) = \mathcal{H}_x \sigma^x + \mathcal{H}_y \sigma^y + \mathcal{H}_z \sigma^z \tag{1}$$

  where $(\mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_z) = (\sin k_x, \sin k_y, M + \cos k_x + \cos k_y)$
- This is just the Hamiltonian for a Chern insulator!
- Energy levels: $E_\pm(k_x, k_y) = \pm\sqrt{\mathcal{H}_x^2 + \mathcal{H}_y^2 + \mathcal{H}_z^2} \equiv \pm|\mathcal{H}|$
- Band gap: $\Delta(k_x, k_y) = E_+(k_x, k_y) - E_-(k_x, k_y) = 2\sqrt{\mathcal{H}_x^2 + \mathcal{H}_y^2 + \mathcal{H}_z^2}$
- GS energy: $E_0 = \sum_{k,\ell=0}^{L-1} E_-\left(\frac{2\pi k}{L}, \frac{2\pi \ell}{L}\right)$ with $(k_x, k_y) = \left(\frac{2\pi k}{L}, \frac{2\pi \ell}{L}\right)$

Eigenvectors and singularities

- An eigenstate for the lower band is

$$\left|u_-^{(1)}\right\rangle = \frac{1}{\mathcal{N}_1}\begin{pmatrix}\mathcal{H}_z - |\mathcal{H}| \\ \mathcal{H}_x + i\,\mathcal{H}_y\end{pmatrix} \tag{2}$$

- Singular when $\mathcal{H}_x = \mathcal{H}_y = 0$ and $\mathcal{H}_z > 0$ ($\mathcal{N}_1$ is a normalization constant)
- Gauge-equivalent alternative ($\mathcal{N}_2$ is a normalization constant):

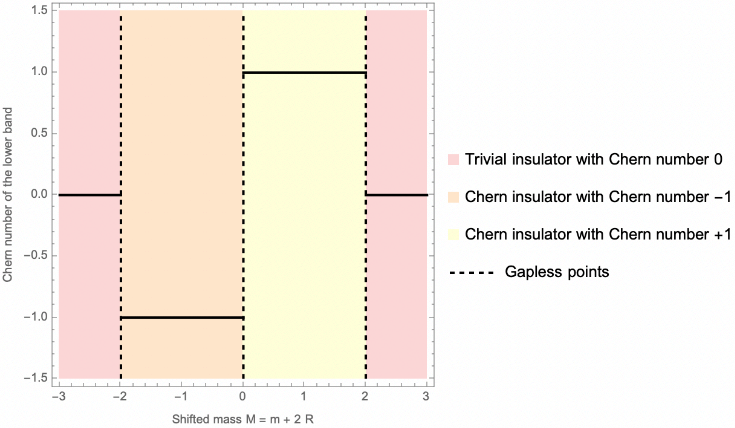
$$\left|u_-^{(2)}\right\rangle = \frac{1}{\mathcal{N}_2}\begin{pmatrix}\mathcal{H}_x + i\mathcal{H}_y \\ \mathcal{H}_z + |\mathcal{H}|\end{pmatrix} = e^{i\lambda(k_x,k_y)}\left|u_-^{(1)}\right\rangle, \quad e^{i\lambda(k_x,k_y)} = \left(\frac{\mathcal{H}_z + |\mathcal{H}|}{\mathcal{H}_x + i\mathcal{H}_y}\right)\left|\frac{\mathcal{H}_z + |\mathcal{H}|}{\mathcal{H}_x + i\mathcal{H}_y}\right|^{-1} \tag{3}$$

- Complementary: singular when $\mathcal{H}_x = \mathcal{H}_y = 0$ and $\mathcal{H}_z < 0$
- Upshot: no globally well-defined wave-function, leads non-trivial topological invariants!

Chern number and phase diagram

- U(1) connection, curvature and first Chern number for the lower band

$$a_i^{(\ell)} = -i \left\langle u_-^{(\ell)} \middle| \partial_{k_i} \middle| u_-^{(\ell)} \right\rangle, \qquad b_i = (\nabla \times a^{(\ell)})_i, \qquad c_1[b] = \int_{\mathcal{B}} \frac{b}{2\pi} \tag{4}$$



Agreement with the lattice Lagrangian formulation

- CS level from lattice Lagrangian (time lattice spacing a_0, spatial lattice spacing $a = 1$) [Sen 2008.01743]

$$c(m, a, a_0) = \frac{1}{2}\Bigg[(-1)^0 \frac{m}{|m|} + (-1)^1 \left(2\frac{m+2}{|m+2|} + \frac{m+\frac{2}{a_0^2}}{\left|m+\frac{2}{a_0^2}\right|}\right) + (-1)^2 \left(\frac{m+4}{|m+4|} + 2\frac{m+2+\frac{2}{a_0^2}}{\left|m+2+\frac{2}{a_0^2}\right|}\right) + (-1)^3 \left(\frac{m+2+\frac{4}{a_0^2}}{\left|m+2+\frac{4}{a_0^2}\right|}\right)\Bigg]$$

- Agreement with Hamiltonian calculation as we take $a_0 \to 0$ with $M = m + 2$

$$c_1[b] = -\lim_{a_0\to 0} c(M, a, a_0) = -\frac{1}{2}\left[\frac{M-2}{|M-2|} - 2\frac{M}{|M|} + \frac{M+2}{|M+2|}\right] = \begin{cases} 0, & M < -2 \\ -1, & M \in (-2, 0) \\ +1, & M \in (0, 2) \\ 0, & M > 2 \end{cases}$$

Why does this not work for staggered fermions?

- Hamiltonian in momentum space: $\mathcal{H}_{\text{staggered}} = \mathcal{H}^s_x\sigma^x + \mathcal{H}^s_y\sigma^y + \mathcal{H}^s_z\sigma^z$ with $\mathcal{H}^s_z = m$.
- $\mathcal{H}^s_z$ has a definite sign $\Rightarrow$ there exists a globally well-defined wavefunction
- Hence, the Chern number is zero, and we are always in a trivial insulator phase
- Consistently: staggered fermion theory is time-reversal invariant, while Wilson fermion theory is not!

Topological Phases of the $N_f = 2$ theory at finite density

Choices of mass matrices and chemical potentials

- Focus on two mass configurations
  1. Equal masses: $M_1 = M_2 = M$
  2. Equal but opposite masses: $M_1 = -M_2 = M$
- We now have the possibility of including a chemical potential

$$H_\mu = \sum_r [\mu_0(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) + \mu(\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2)] \tag{5}$$

- WLOG take $\mu_0 \to 0$: multiple of identity, and focus on the theory at half-filling

$$\frac{1}{4L^2} \sum_r \psi_a(r)^\dagger \psi_a(r) = \frac{1}{2}$$

  where the theory is on an $L \times L$ lattice
- Define the occupation fraction

$$f_a(\mu, M) = \frac{1}{2L^2} \sum_{k \in \mathcal{B}} \left\langle \psi_a(k)^\dagger \psi_a(k) \right\rangle_{\mu, M} \qquad \sum_a f_a(\mu, M) = 1$$

Spectrum for case 1 with $M_1 = M_2 = M$

- Hamiltonian in Fourier space

$$H = \sum_r \begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix}^\dagger \left[\mathcal{H}_x\Gamma^x + \mathcal{H}_y\Gamma^y + \mathcal{H}_z\Gamma^z + \mu\,\sigma^z \otimes \mathbb{I}_2\right] \begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix}$$

  where $\Gamma^k = \mathbb{I}_2 \otimes \sigma^k$ and the functions $\mathcal{H}_i$ are the same as for $N_f = 1$.
- Eigenstates of H_μ are those of σ^z

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  corresponding to the energies $\pm\mu$.
- Eigenstates of the full Hamiltonian are tensor-product states

$$\{|\uparrow\rangle \otimes |u_+(k)\rangle\,, |\downarrow\rangle \otimes |u_+(k)\rangle\,, |\uparrow\rangle \otimes |u_-(k)\rangle\,, |\downarrow\rangle \otimes |u_-(k)\rangle\}$$

- Still have a gauge-ambiguity in the global definitions of $u_\pm$.

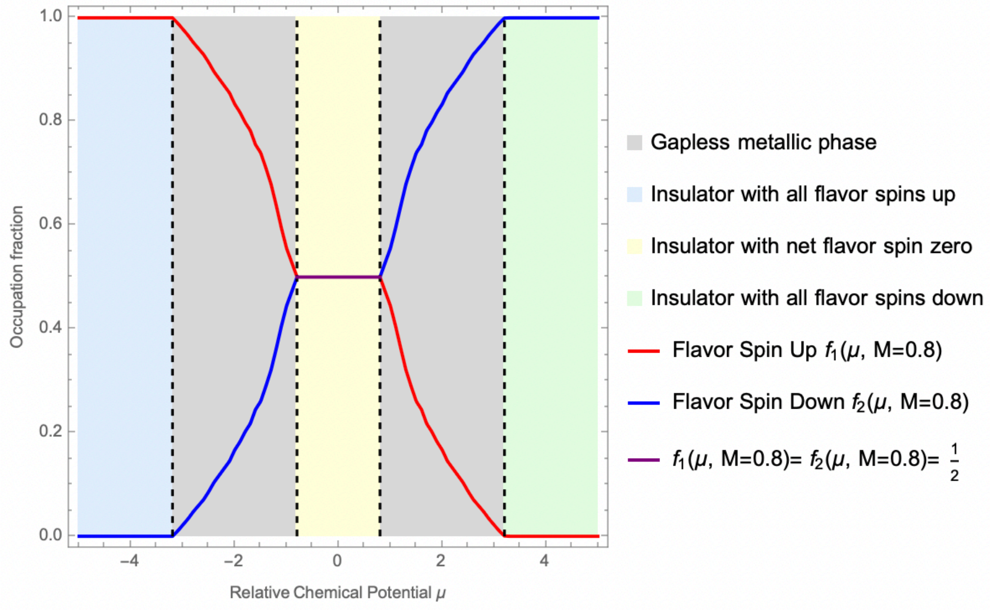
- The corresponding energies of the four bands are given by

$$E = \pm\mu \pm |\mathcal{H}|$$

- When $|\mu| \gg |\mathcal{H}|$, the mass gap $|\Delta| \sim 2|\mu| + \dots$
- When $|\mu| \ll |\mathcal{H}|$, the mass gap $|\Delta| \sim 2|\mathcal{H}| + \dots$
- Heuristically, we see the existence of three distinct gapped phases with vacua:

$$|0\rangle = \begin{cases} \bigotimes_{k\in\mathcal{B}} |\downarrow, u_-(k)\rangle \otimes |\downarrow, u_+(k)\rangle\,, & \mu \gg \sup|\mathcal{H}| \\ \bigotimes_{k\in\mathcal{B}} |\downarrow, u_-(k)\rangle \otimes |\uparrow, u_-(k)\rangle\,, & |\mu| \ll \inf|\mathcal{H}| \\ \bigotimes_{k\in\mathcal{B}} |\uparrow, u_+(k)\rangle \otimes |\uparrow, u_-(k)\rangle\,, & \mu \ll -\sup|\mathcal{H}| \end{cases}$$

Occupation fraction versus chemical potential for fixed $M_1 = M_2 = M$



Full phase diagram in (M, μ) plane for case 1 with $M_1 = M_2 = M$

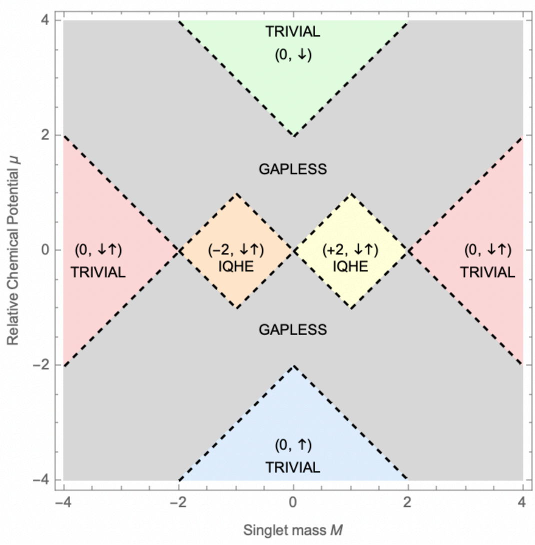


Figure: The pairs denote $(c_1[b], \langle s\rangle)$ with $\langle s\rangle = \uparrow, \downarrow, \uparrow\downarrow$ ($\uparrow\downarrow$ denotes zero spin.)

Full phase diagram in (M, μ) plane for case 2 with $M_1 = -M_2 = M$

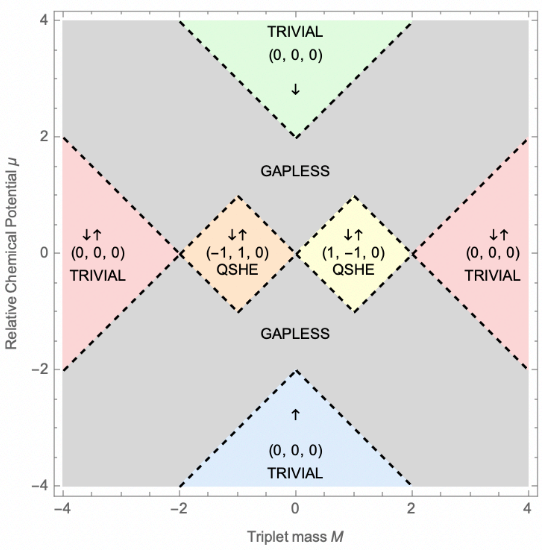
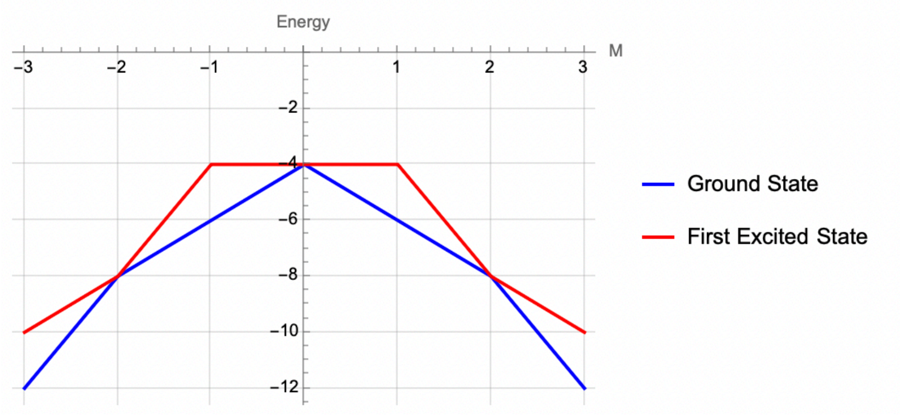


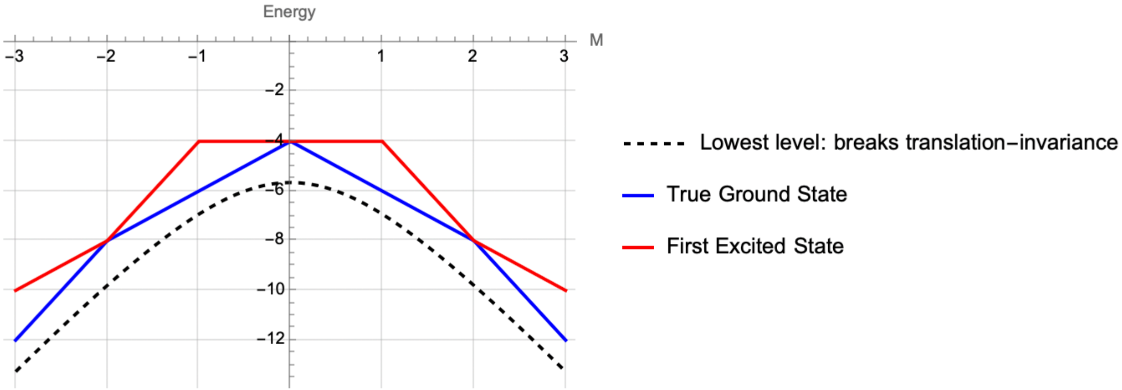
Figure: The pairs denote $(c_\uparrow, c_\downarrow, c_{\text{tot}})$ with $\langle s \rangle = \uparrow, \downarrow, \uparrow\downarrow$ ($\uparrow\downarrow$ denotes zero spin.)

Agreement with lattice simulations

Level crossing and topological transitions for the $N_f = 1$ theory on a 2×2 lattice



The $N_f = 1$ simulation with $\mathbb{Z}_2$ gauge fields on a 2×2 lattice



Outlook: ongoing work and future directions

Ongoing research and future directions

- Purely on the lattice: there exist weaker notions of translational invariance than what was discussed today (i.e. controlled non-invariances that go away as $L \to \infty$ and $a \to 0$). Does the topological phase diagram change for this case?
- Can such topological phases be realized on a quantum computer? Can we prove the existence of such phases with $\mathbb{Z}_N$ truncations or spin-S quantum link models?
- What is the fate of the topological phases/transitions once we dynamically gauge the theory?
- Structure of the phase diagram as a function of the QED coupling?

Thank you!

Spectrum for case 2

- Repeat the same analysis as before, but with a slightly modified Hamiltonian

$$H = \sum_k \begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix}^\dagger \left[\mathcal{H}_x \Gamma^x + \mathcal{H}_y \Gamma^y + \mathcal{H}_z^{(1)} \Gamma^z + M\sigma^z \otimes \sigma^z + \mu\, \sigma^z \otimes \mathbb{I}_2 \right] \begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix}$$

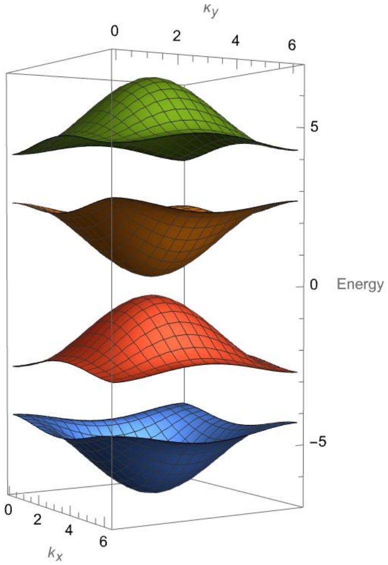
- For convenience, we write $u_\pm^{(s)}(k) = u_\pm(k, sM)$ with $s = \pm$. The eigenvectors that diagonalize the Hamiltonian are

$$\left\{ \left|\downarrow, u_\pm^{(-)}(k)\right\rangle, \left|\uparrow, u_\pm^{(+)}(k)\right\rangle \right\}$$

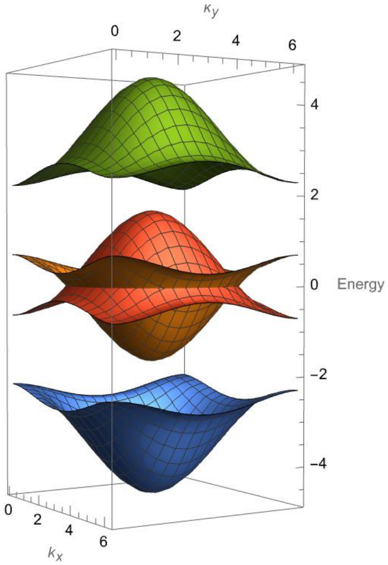
- This is also a four-band model, with three distinct gapped regimes

$$|0\rangle = \begin{cases} \bigotimes_{k\in\mathcal{B}} \left|\downarrow, u_-^{(-)}(k)\right\rangle \otimes \left|\downarrow, u_+^{(-)}(k)\right\rangle, & \mu \gg 1 \\ \bigotimes_{k\in\mathcal{B}} \left|\uparrow, u_-^{(+)}(k)\right\rangle \otimes \left|\downarrow, u_-^{(-)}(k)\right\rangle, & \mu \sim 0 \\ \bigotimes_{k\in\mathcal{B}} \left|\uparrow, u_+^{(+)}(k)\right\rangle \otimes \left|\uparrow, u_-^{(+)}(k)\right\rangle, & \mu \ll -1 \end{cases}$$

The vacuum structure I – fixed mass $M \neq 0, \pm 2$

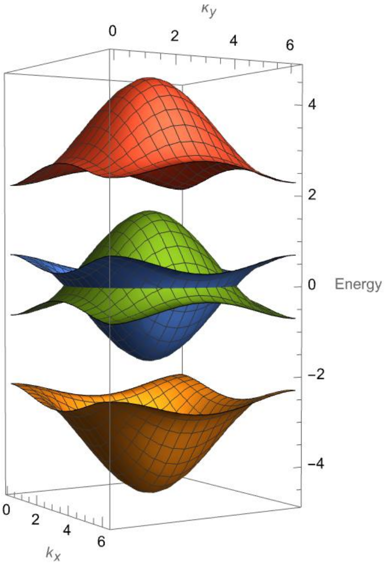


(a) $\mu = -3.5$

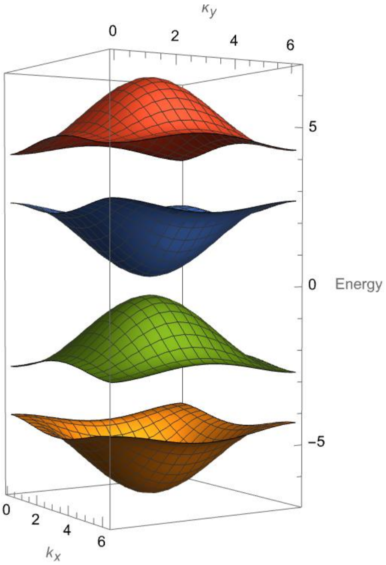


(b) $\mu = -1.5$

The vacuum structure II – fixed mass $M \neq 0, \pm 2$

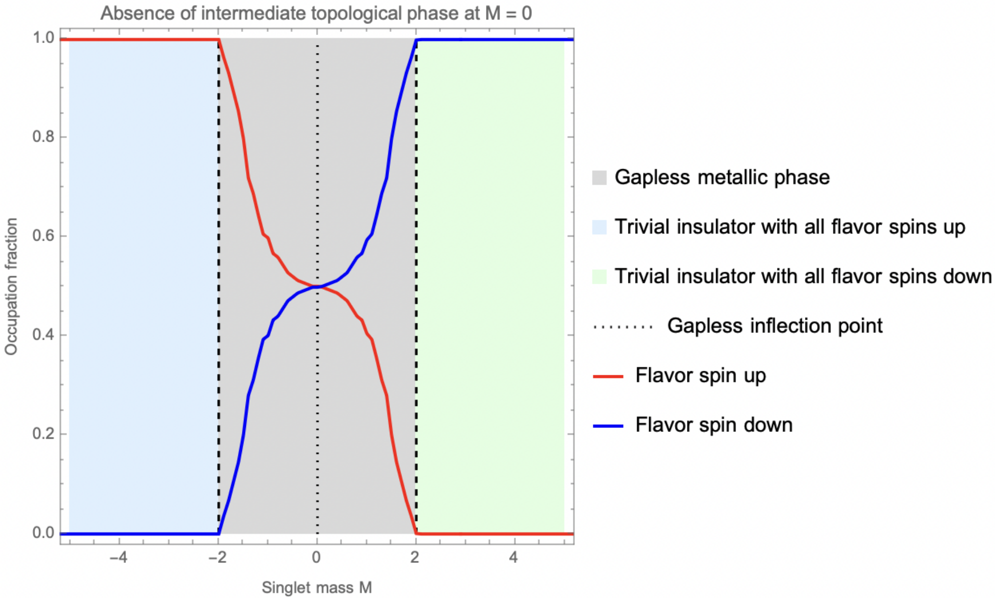


(a) $\mu = 1.5$

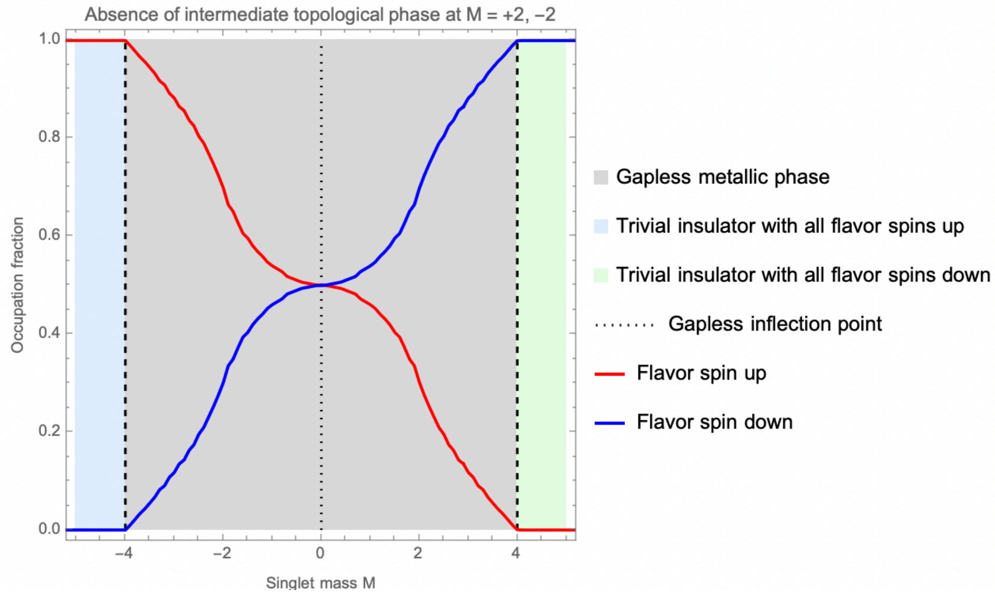


(b) $\mu = 3.5$

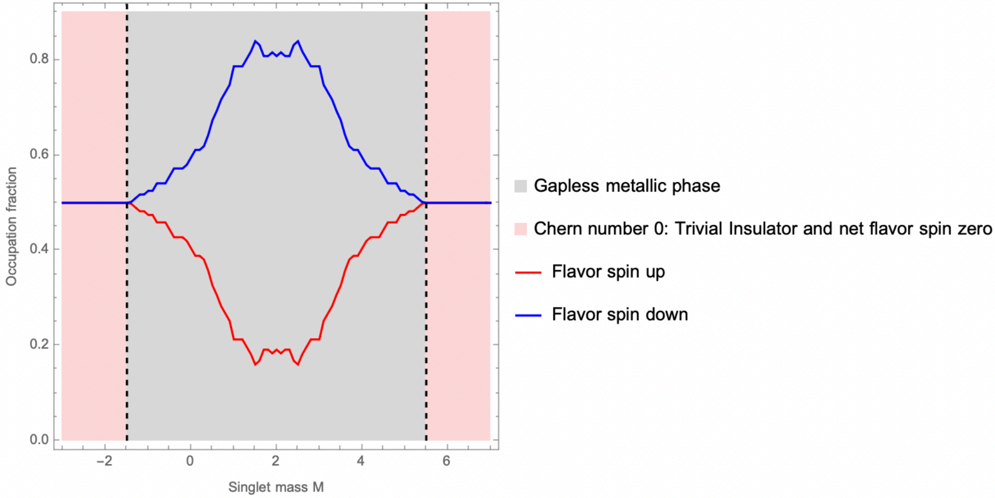
Consistency check absence of intermediate topological phase at $M = 0$



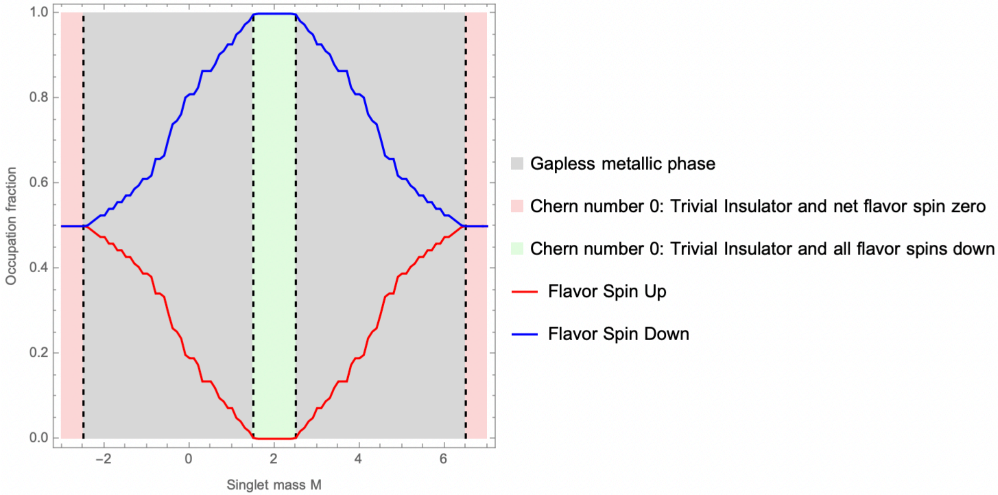
Consistency check: absence of intermediate topological phase at $|M| = 2$



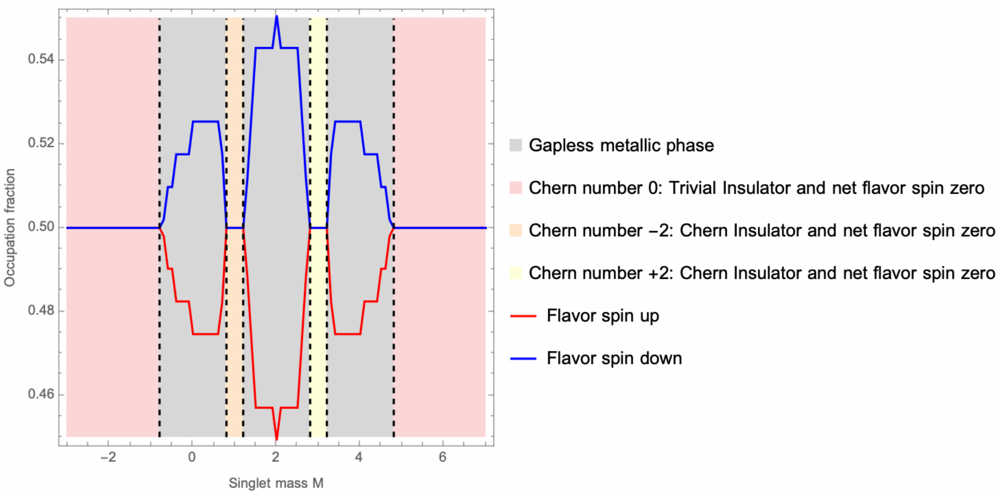
Occupation fraction versus singlet mass M for fixed $\mu = 1.5$: two gapped phases



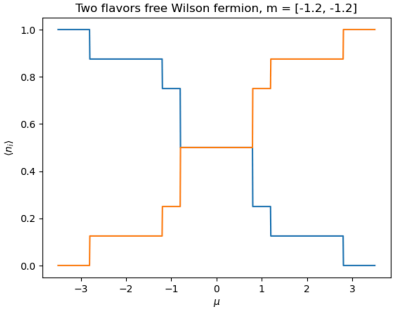
Occupation fraction versus singlet mass M for fixed $\mu = 2.5$: three gapped phases



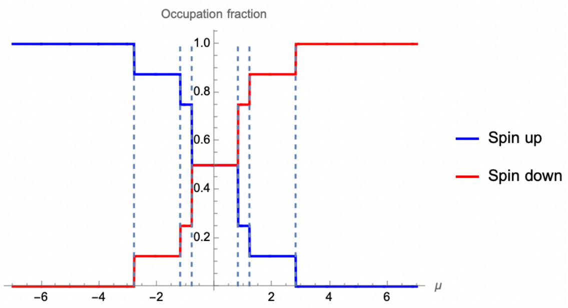
Occupation fraction versus singlet mass M for fixed $\mu = 0.8$: four gapped phases



Occupation fraction for fixed M from 2×2 lattice simulation for the $N_f = 2$ theory



(a) Simulation on 2×2 lattice



(b) "Analytic" result on 2×2 lattice

Color plot of the mass gap in parameter space for the $N_f = 2$ theory

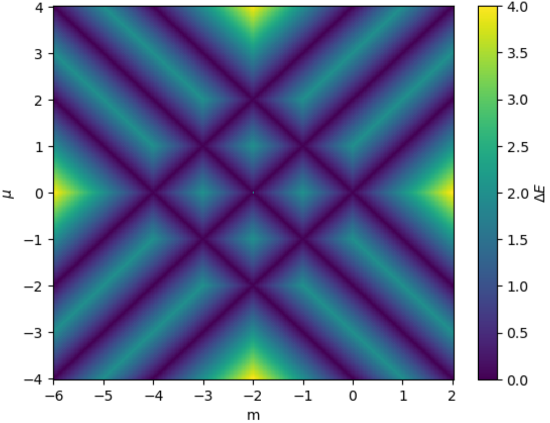


Figure: The plot shows that theory becomes gapless at precisely those points where the analytical results indicate for the 2×2 case