Lattice QCD determinations of α_s and results by the ALPHA collaboration

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Outline

- FLAG = FLavour Lattice Averaging Group, summary plots
- Some generalities on α_s extraction
- Effective coupling, Λ-parameter, FLAG criteria
- Problem of large scale differences
- & its solution by the Step Scaling method
- Case study of vacuum polarization
- Summary of results in FLAG 2024 report
- Decoupling across charm and bottom thresholds
- Decoupling as a tool to determine α_s.
- Details on the decoupling method that entered FLAG 2024
- Substantial improvements to decoupling and step-scaling results since FLAG 2024
- Conclusions

FLAG = FLavour Lattice Averaging Group

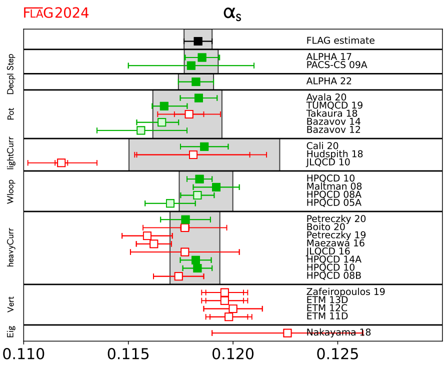
- Effort by the international lattice QCD community to provide the wider high energy physics community with lattice results for quantities of phenomenological interest, satisfying clearly defined quality criteria
- Original focus was on flavour physics, but now FLAG includes also sections on α_s, nucleon matrix elements and scale setting.
- FLAG website: `flag.unibe.ch`
- FLAG has 31 members in 8 WG's, membership by invitation, FLAG aims to have representatoin from the main collaborations.

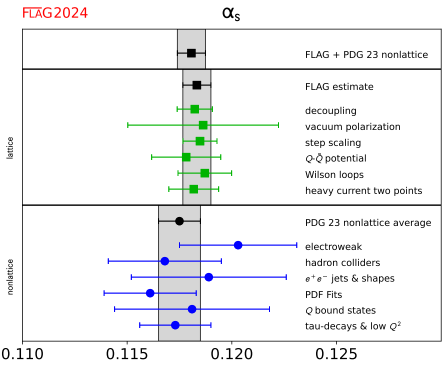
  Working group on α_s: L. Del Debbio, P. Petreczky and S. Sint
- Reports appear every 2-3 years; FLAG requires acceptance by/publication in a peer reviewed journal by a cutoff date; for FLAG 2024 this was 30 April 2024; Possibility of occasional web updates (by section), if there are significant new results.
- FLAG 2024 report: `https://arxiv.org/abs/2411.04268`
- FLAG 2021 report: `https://arxiv.org/abs/2111.09849`,

  published in Eur. Phys. J. C 82 (2022) 10, 869

N.B. Anyone using FLAG results should cite the original sources which enter the relevant average.

FLAG 2024 plots





Some general observations on determinations of $\alpha_s(m_Z)$

- FLAG 24 average: $\alpha_s(m_Z) = 0.1183(7)$, the uncertainty is 0.6%
- All but one determinations: $N_\mathrm{f} = 2+1$, combined with 4-loop matching across charm and bottom thresholds
- A 1% error on α_s requires $\Delta\Lambda^{(N_\mathrm{f}=3)}_{\overline{\mathrm{MS}}} < 5\%$

$\Rightarrow$ isospin breaking due to electromagnetism & $m_u \neq m_d$ is not yet relevant for α_s!

All but 2 categories affected predominantly by systematics, in particular:

- Perturbative truncation errors: requires $\mu \gg \Lambda_{\overline{\mathrm{MS}}}$
- continuum limit: requires $\mu \ll 1/a$

Note: given the very good quantitative perturbative description of decoupling across charm and bottom threshold [cf. Athenodorou et al (ALPHA '18)] the determination of α_s is equivalent to a non-perturbative result for the Λ-parameter with $N_\mathrm{f} = 3, 4$

Reminder: definition of effective coupling, α_{eff}

Starting point for all α_s determinations: Euclidean short distance quantity Q, that

- can be measured in a lattice simulation
- has a perturbative expansion, $Q = c_0 + c_1\alpha + c_2\alpha^2 + \ldots$.

We associate an effective coupling to Q, by normalizing

$$\alpha_{\text{eff}} = (Q - c_0)/c_1$$

- Advantage: no need to refer to a particular scale, α_{eff} is measured, possibly after chiral and continuum extrapolations (exception: couplings at $1/a$, e.g. from small Wilson loops).
- Loop counting: Relate to the MS scheme:

$$\alpha_{\text{eff}} = \alpha_{\overline{\text{MS}}} + d_1\alpha_{\overline{\text{MS}}}^2 + d_2\alpha_{\overline{\text{MS}}}^3 + d_3\alpha_{\overline{\text{MS}}}^4 + \ldots$$

  If d_k are known up to $k = n_l$ the loop order is n_l. Currently best cases have $n_l = 3$ (plus partial information on $n_l = 4$ for static potential/force)

FLAG qualilty criteria for α_s (unchanged since FLAG 2019)

Renormalization scale

- ★ all points in the analysis have $\alpha_{\text{eff}} < 0.2$
- ○ all points have $\alpha_{\text{eff}} < 0.4$ and at least 1 with $\alpha_{\text{eff}} < 0.25$
- ■ otherwise

Continuum limit: at a reference point of $\alpha_{\text{eff}} = 0.3$ (or less) require

- ★ three lattice spacings with $\mu a < 1/2$ and full O(a) improvement, or three lattice spacings with $\mu a \leq 1/4$ and 2-loop O(a) improvement, or $\mu a \leq 1/8$ and 1-loop O(a) improvement
- ○ three lattice spacings with $\mu a < 3/2$ reaching down to $\mu a = 1$ and full O(a) improvement, or three lattice spacings with $\mu a \leq 1/4$ and 1-loop O(a) improvement
- ■ otherwise

plus convention for μ in different quantities (e.g. $\mu = q$ in momentum space observables, or $\mu = 1/L$ for step-scaling)

Perturbative behaviour: assessed in terms of the Λ-parameter, s. below.

The QCD Λ-parameter vs. $\alpha_s(\mu) = \bar{g}^2(\mu)/4\pi$

The coupling $\alpha_s(\mu)$ can be traded for its associated Λ-parameter:

$$\Lambda = \mu\varphi(\bar{g}(\mu)) = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- <u>exact</u> solution of Callan-Symanzik equation: $\left(\mu\frac{\partial}{\partial\mu} + \beta(\bar{g})\frac{\partial}{\partial\bar{g}}\right)\Lambda = 0$
- Number N_f of massless quarks is fixed.
- If the coupling $\bar{g}(\mu)$ non-perturbatively defined so is its β-function!
- $\beta(g)$ has asymptotic expansion $\beta(g) = -b_0 g^3 - b_1 g^5 - b_2 g^7 ..$

  $$b_0 = (11 - \tfrac{2}{3}N_\mathrm{f})/(4\pi)^2, \qquad b_1 = (102 - \tfrac{38}{3}N_\mathrm{f})/(4\pi)^4, \quad \ldots$$

  $b_{0,1}$ are universal, scheme-dependence starts with 3-loop coefficient b_2.
- Scheme dependence of Λ <u>almost</u> trivial:

  $$g^2_\mathrm{X}(\mu) = g^2_\mathrm{Y}(\mu) + c_\mathrm{XY} g^4_\mathrm{Y}(\mu) + \ldots \quad \Rightarrow \quad \frac{\Lambda_\mathrm{X}}{\Lambda_\mathrm{Y}} = \mathrm{e}^{c_\mathrm{XY}/2b_0}$$

$\Rightarrow$ can use $\Lambda_{\overline{\mathrm{MS}}}$ as reference (even though the $\overline{\mathrm{MS}}$-scheme is purely perturbative!)

The QCD Λ-parameter and $\alpha_s(\mu) = \bar{g}^2(\mu)/4\pi$

$$\Lambda = \mu\varphi(\bar{g}(\mu)) = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- Continuum relation, exact at any scale μ:
  - require large μ to evaluate integral perturbatively
  - require small μ to match hadronic scale

$\Rightarrow$ problem of large scale differences:

  - The scale μ must reach the perturbative regime: $\mu \gg \Lambda$
  - lattice cutoff must still be larger: $\mu \ll a^{-1}$
  - spatial volume must be large enough to contain pions: $L \gg 1/m_\pi$
  - Taken together a naive estimate gives

$$L/a \gg \mu L \gg m_\pi L \gg 1 \quad \Rightarrow L/a \simeq \mathrm{O}(10^3)$$

$\Rightarrow$ widely different scales cannot be resolved simultaneously on a single lattice!

FLAG criterion on perturbative behaviour

Λ-parameter in mass-independent renormalization scheme:

$$\Lambda_{\overline{\mathrm{MS}}} = \mu\varphi\left(\bar{g}(\mu)\right)$$

$$\varphi\left(\bar{g}\right) = \left[b_0\bar{g}^2\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2}} \exp\Big\{\underbrace{-\int_0^{\bar{g}} dg\left[\frac{1}{\color{red}{\beta(g)}} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]}_{=I[\bar{g};\beta]}\Big\}$$

At large μ, use β-function truncated to n_l loops ($\beta^{(n_l)}$, with known b_{n_l-1})

$$I[g;\beta] \overset{g\to 0}{\simeq} I[g,\beta^{(n_l)}] + O\left(g^{2n_l}\right) \quad \Rightarrow \quad \Lambda^{\text{estimated}}_{\overline{\mathrm{MS}}}/\Lambda_{\overline{\mathrm{MS}}} = 1 + O\left(\alpha^{n_l}(\mu)\right)$$

<u>Perturbative behaviour</u>

- ★ verified over a range of a factor 4 change in $\alpha_{\text{eff}}^{n_l}$ ($=$ parametric uncertainty in Λ) without power corrections or alternatively $\alpha_{\text{eff}}^{n_l} < \frac{1}{2}\Delta\alpha_{\text{eff}}/(8\pi b_0\alpha_{\text{eff}}^2)$ is reached.

$$\Delta\Lambda|_{\Delta\alpha} = \Delta\alpha\frac{\partial\Lambda}{\partial\alpha} = \frac{2\pi\Delta\alpha}{-g\beta(g)}\Lambda \approx \frac{\Delta\alpha}{8\pi b_0\alpha^2}\Lambda$$

- ○ verified over a range of a factor $(3/2)^2$ change in $\alpha_{\text{eff}}^{n_l}$ possibly fitting with power corrections or alternatively $\alpha_{\text{eff}}^{n_l} < \Delta\alpha_{\text{eff}}/(8\pi b_0\alpha_{\text{eff}}^2)$ is reached.

- ■ otherwise

Collaboration	Ref.	N_f	publication status	renormalization scale	perturbative behaviour	continuum extrapolation	$\alpha_{\overline{\rm MS}}(M_{\rm Z})$	Remark	Tab.
ALPHA 17	[85]	2+1	A	★	★	★	0.11852(84)	step scaling	57
PACS-CS 09A	[86]	2+1	A	★	★	○	0.11800(300)	step scaling	57
pre-range (average)							0.11848(81)		
AlPHA 22	[80]	2+1	A	★	★	★	0.11823(84)	decoupling $N_f = 3$ to $N_f = 0$ & step scaling	58
pre-range (average)							0.11823(84)		
Ayala 20	[82]	2+1	A	○	★	○	0.11836(88)	Q-$\bar{Q}$ potential	59
TUMQCD 19	[83]	2+1	A	○	★	○	$0.11671(^{+110}_{-57})$	Q-$\bar{Q}$ potential (and free energy)	59
Takaura 18	[780, 781]	2+1	A	■	○	○	$0.11790(70)(^{+130}_{-120})$	Q-$\bar{Q}$ potential	59
Bazavov 14	[782]	2+1	A	○	★	○	0.11660(100)	Q-$\bar{Q}$ potential	59
Bazavov 12	[783]	2+1	A	○	○	○	$0.11560(^{+210}_{-220})$	Q-$\bar{Q}$ potential	59
pre-range with estimated pert. error							0.11782(165)		
Cali 20	[84]	2+1	A	○	★	★	0.11863(114)	vacuum pol. (position space)	60
Hudspith 18	[727]	2+1	P	○	★	■	$0.11810(270)(^{+80}_{-220})$	vacuum polarization	60
JLQCD 10	[726]	2+1	A	■	○	■	$0.11180(30)(^{+160}_{-170})$	vacuum polarization	60
pre-range with estimated pert. error							0.11863(360)		
HPQCD 10	[15]	2+1	A	○	★	★	0.11840(60)	Wilson loops	61
Maltman 08	[87]	2+1	A	○	○	★	0.11920(110)	Wilson loops	61
pre-range with estimated pert. error							0.11871(128)		
Petreczky 20	[81]	2+1	A	○	○	★	0.11773(119)	heavy current two points	62
Boito 20	[830, 835]	2+1	A	■	■	○	0.1177(20)	use published lattice data	62
Petreczky 19	[31]	2+1	A	■	■	★	0.1159(12)	heavy current two points	62
JLQCD 16	[30]	2+1	A	■	○	○	0.11770(260)	heavy current two points	62
Maezawa 16	[230]	2+1	A	■	■	○	0.11622(84)	heavy current two points	62
HPQCD 14A	[18]	2+1+1	A	○	★	○	0.11822(74)	heavy current two points	62
HPQCD 10	[15]	2+1	A	○	★	○	0.11830(70)	heavy current two points	62
HPQCD 08B	[244]	2+1	A	■	■	■	0.11740(120)	heavy current two points	62
pre-range with estimated pert. error							0.11818(119)		
Zafeiropoulos 19	[836]	2+1	A	■	■	■	0.1172(11)	gluon-ghost vertex	66 in [5]
ETM 13D	[837]	2+1+1	A	○	○	■	0.11960(40)(80)(60)	gluon-ghost vertex	66 in [5]
ETM 12C	[838]	2+1+1	A	○	○	■	0.12000(140)	gluon-ghost vertex	66 in [5]
ETM 11D	[839]	2+1+1	A	○	○	■	$0.11980(90)(50)(^{+0}_{-50})$	gluon-ghost vertex	66 in [5]
Nakayama 18	[840]	2+1	A	★	○	■	0.12260(360)	Dirac eigenvalues	67 in [5]

The step scaling solution

- Widely different scales cannot be resolved simultaneously on a *single* lattice

$\Rightarrow$ break calculation up in steps [Lüscher, Weisz, Wolff '91; Jansen et al. '95]:

1. define $\bar{g}^2(L)$ that runs with the space-time volume, i.e. $\mu = 1/L$
2. construct the step-scaling function

$$\sigma(u) = \bar{g}^2(2L)\Big|_{u=\bar{g}^2(L)}$$

   for a range of values $u \in [u_{\mathrm{min}}, u_{\mathrm{max}}]$

3. iteratively step up/down in scale by factors of 2:

$$\bar{g}^2(L_{\mathrm{max}}) = u_{\mathrm{max}} \equiv u_0, \quad u_k = \sigma(u_{k+1}) = \bar{g}^2(2^{-k}L_{\mathrm{max}}), \quad k = 0, 1, \ldots$$

4. match to hadronic input at a hadronic scale L_{max}, i.e. $F_K L_{\max} = \mathrm{O}(1)$
5. once arrived in the perturbative regime $L_{\mathrm{pert}} = 2^{-n}L_{\mathrm{max}}$ one now knows $u_n = \bar{g}^2(L_{\mathrm{pert}})$; determine $L_{\mathrm{pert}}\Lambda$ and combine to obtain Λ/F_K.

Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

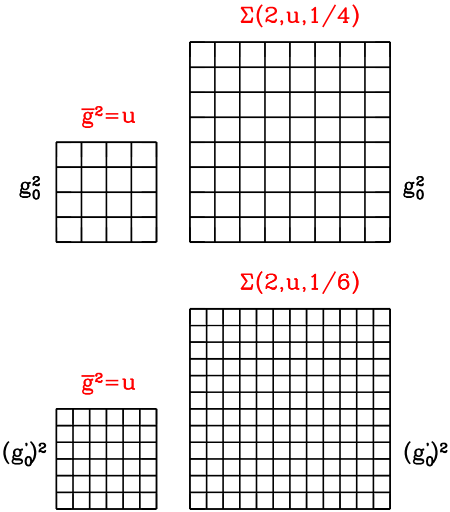
- choose g_0 and $L/a = 4$, measure $\bar{g}^2(L) = u$ (this sets the value of u)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

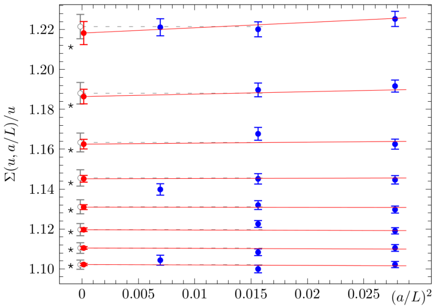
- now choose $L/a = 6$ and tune g_0' such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

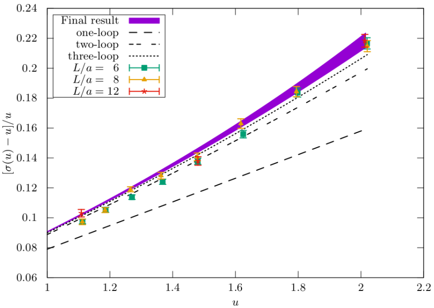
$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- $\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$.
- change u and repeat...

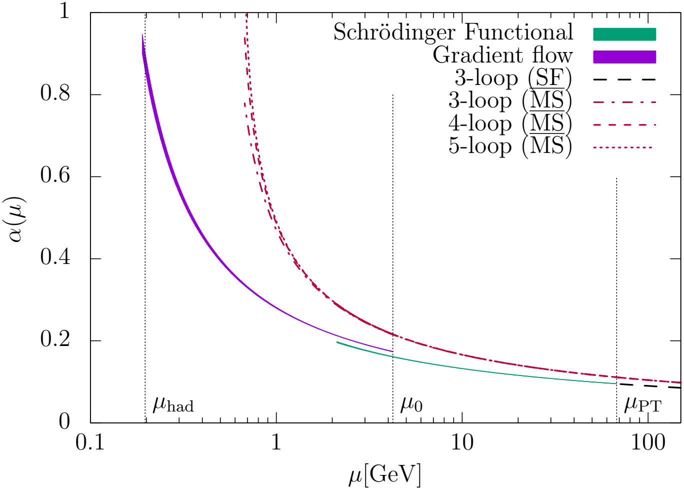


Continuum limit $\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$



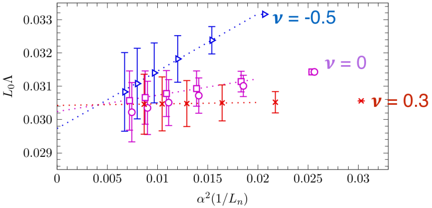


Non-perturbative running of α_s in $N_\mathrm{f} = 3$ QCD



Result for Λ

- The Λ-parameter can now be evaluated at a very high perturbative scale μ_{PT} with $\mu_{\mathrm{had}}/\mu_{\mathrm{PT}}$ known (a power of 2 if related by step-scaling).
- The remaining uncertainty is parametrically $\propto \alpha^2(\mu_{\mathrm{PT}})$ if the β-function is



  known to 3-loop order

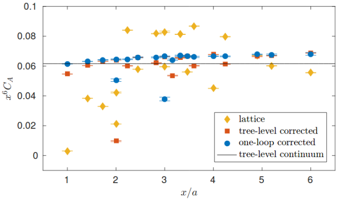
- Note: observation of this dependence requires data over a large range of scales, so that α^2 varies significantly!
- The result

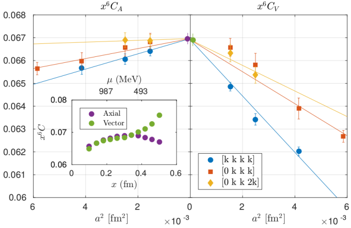
$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 341(12)\mathrm{MeV}$$

  translates to $\alpha_s(m_Z) = 0.11852(84)$ (with 4-loop matching across charm and bottom thresholds)

$\Lambda_{\overline{\rm MS}}$ from vacuum polarization, Cali 20

$$C_{\mathsf{A,V}}(x) = \sum_\mu \left\langle J^\mu_{\mathsf{A,V}}(x) J^\mu_{\mathsf{A,V}}(0) \right\rangle = \frac{6}{\pi^4 (x^2)^3} \left(1 + \frac{\alpha}{\pi} + O(\alpha^2)\right)$$



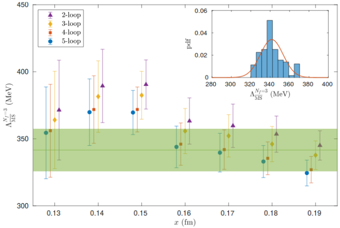


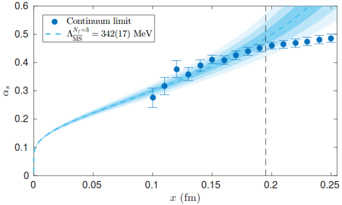
- $\alpha_{\mathsf{eff}}(\mu = 1/|x|) = \pi[(x^2)^3 (\pi^4/6) C_{\mathsf{A,V}}(x) - 1]$
- use $|x| = 0.13 - 0.19\,\mathrm{fm}$, CLS, lattice spacings $a = 0.039 - 0.076\,\mathrm{fm}$, $\alpha_{\mathsf{eff}} = 0.235 - 0.308$, extrapolated to chiral limit.
- Non-perturbatively O(a) improved with 3 lattice spacings at $\mu^{-1} = |x| = 0.13\mathrm{fm}$ with $a\mu < 1/2$ and $\alpha_{\mathsf{eff}} \approx 0.3$

$\Rightarrow$ ★ for continuum extrapolation

- HOWEVER: 1-loop subtraction (using NSPT) of hypercubic lattice artifacts crucial!

Result $\Lambda_{\overline{\rm MS}}^{N_f=3} = 342(17)\,\mathrm{MeV}$ from weighted average over $|x| = 0.13 - 0.19\,\mathrm{fm}$:





Apply FLAG criteria:

- Perturbative behaviour: $\alpha^3_{\rm eff}$ covers a range of 2.2 close to $(3/2)^2$. Also the error $\Delta\alpha_{\rm eff} \approx 4-6\%$ are comfortably larger than the relative parametric uncertainty of Λ, with the stronger criterion still ok, so ★
- Renormalization scale: $\alpha_{\rm eff}$ reaches $0.235 < 0.25 \Rightarrow$ ○

Conclusion: ★ in continuum limit and perturbative behaviour and ○ for the renormalization scale.
$\Rightarrow$ passes all FLAG criteria!

Discussion of Cali 20 and vacuum polarization category

Nevertheless the error estimate seems rather optimistic:

- Cali 20 show that vector and axial vector correlators yield compatible results within errors

$\Rightarrow$ absence of chirality breaking effects, however, one may expect other non-perturbative effects at these low energies.

- Cali 20 convert to the $\alpha_{\overline{\mathrm{MS}}}$ by solving numerically the truncated expansion

$$\alpha_{\mathrm{eff}}(\mu = 1/|x|) = \alpha_{\overline{\mathrm{MS}}}(\mu) + c_1\alpha^2_{\overline{\mathrm{MS}}}(\mu) + c_2\alpha^3_{\overline{\mathrm{MS}}}(\mu) + c_3\alpha^4_{\overline{\mathrm{MS}}}(\mu)$$

- If one instead inverts perturbatively, one obtains $\Lambda_{\overline{\mathrm{MS}}}$ in the range $409 - 468$ MeV, i.e. $15 - 30\%$ higher!
- The difference decreases roughly proportionally to the expected α^3_{eff}.

FLAG decision for pre-range of vacuum polarization category:

- Systematic error as the difference 54 MeV between $\Lambda_{\overline{\mathrm{MS}}}$ estimates at $\mu = 1.5$ GeV

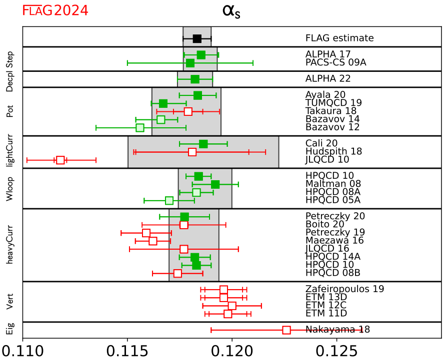
$\Rightarrow$ take $\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 342(54)$ MeV as range for vacuum polarization. ,

FLAG estimate for α_s

- Form pre-ranges for each of the 6 categories
- Decoupling and step-scaling results statistics dominated

$\Rightarrow$ average taking into account known correlation (28 percent of variance due to common scale setting)

- Use weighted average of all results to obtain central value
- Use statistics dominated error as proxy for total error



$$\Lambda_{\overline{\mathrm{MS}}} = 338(10)\,\mathrm{MeV} \quad \Rightarrow \quad \alpha_s(m_Z) = 0.1183(7)$$

Summary FLAG 2024

- From FLAG 2021 to FLAG 2024 estimate moves slightly from $\alpha_s = 0.1184(8)$ to $\alpha_s = 0.1183(7)$, with error reduction due to the decoupling method.
- The current FLAG criteria (unchanged since FLAG 2019) are not very stringent, as the example of Cali 20 illustrates.
- In addition to the FLAG criteria, scale variations were performed, as a further way to assess perturbative truncation errors (cf. review by Del Debbio and Ramos '20).
- Systematic errors dominate most result, mainly due to using perturbation theory at low scales.

$\Rightarrow$ some improvement might come from using elements of the step-scaling solution or similar ideas.

- Decoupling method works very well: non-perturbative decoupling of $N_\mathrm{f} = 3$ quarks in combination with step-scaling in the $N_\mathrm{f} = 0$ theory (s. below).

$\Rightarrow$ Side effect of the decoupling method: need for reliable results for $N_\mathrm{f} = 0$ (pure SU(3) gauge theory) which acquire physical significance (cf. FLAG 2024 report).

ALPHA's new results since FLAG 2024: arXiv:2501.06633 [hep-ph]

Work in collaboration with M. Dalla Brida, R. Höllwieser, F. Knechtli, T. Korzec, A. Ramos, R. Sommer:

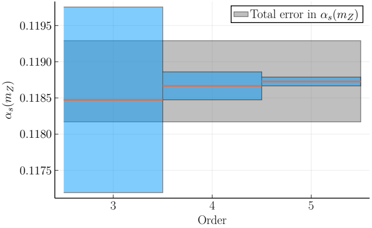
- Significant update and improvement of ALPHA 17 (step-scaling with $N_\mathrm{f} = 3$)
  - Step scaling at low energy: larger lattice sizes at selected values

  $\Rightarrow$ improved control over continuum limit
  - Improved treatement of O(a) boundary effects in SF coupling (high energy running)
  - hadronic scale setting much improved.
- Significant update and improvement of ALPHA 22 (decoupling $N_\mathrm{f} = 3 \to 0$ & step scaling with $N_\mathrm{f} = 0$)
  - Complete elimination of a major O(am) uncertainty (b_g-counterterm determination).
  - Common scale setting with $N_\mathrm{f} = 3$ step-scaling
- Overlap due to common scale setting is small

$\Rightarrow$ Combine results!

Decoupling across charm and bottom thresholds

- Step scaling method for $\Lambda^{(3)}_{\overline{\mathrm{MS}}}$ completely avoids the use of perturbation theory at low energy. PT is used only at $O(100)$ GeV & can be tested over a wide range.
- The result is $\Lambda^{(3)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{had}}$ in terms of a hadronic scale μ_{had}.
- But $\alpha_s(m_Z)$ requires to get from $N_\mathrm{f} = 3$ to $N_\mathrm{f} = 5$ across charm and bottom thresholds! $\Rightarrow$ How good is PT for this step?
- Standard matching procedure known to 4 loops and converges very well: use 5-loop β-function and 4-loop decoupling relations at $\mu = m^\star = \bar{m}(m^\star)$ for $m_c^\star = 1.275\,\mathrm{GeV}$ and $m_b^\star = 4.171\,\mathrm{GeV}$; reduce loop order one by one.
- scale variations by a factor 2 give similar uncertainties.



References: [Bernreuther and Wetzel '82; Grozin et al. '11; Chetyrkin et al.'05; Schröder and Steinhauser '05; Kniehl et al. '06; Gerlach, Herren and Steinhauser '18]

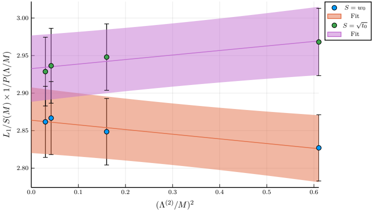
Non-perturbative effects of decoupling of charm

[Athenodorou et al.(ALPHA) '18]

- PT assumes that power corrections $\propto \Lambda^2_{\overline{\mathrm{MS}}}/M^2$ are small.
- These terms are observable-dependent, not universal!
- Non-perturbative test in the charm region, decoupling from $N_\mathrm{f} = 2$ to $N_\mathrm{f} = 0$, with 2 different quantities.

$\Rightarrow$ Slopes are very small, effect on $\Lambda_{\overline{\mathrm{MS}}}$ of order of 0.05% per quark of charm mass;

- Allow for 10 times larger non-perturbative charm decoupling effects; still negligible compared to total error!



Decoupling of heavy quarks

- Consider QCD with $N_\mathrm{f} = 3$ heavy quarks of RGI mass M

$$M = \overline{m}_s(\mu) \left[2b_0 \bar{g}_s^2(\mu)\right]^{-\frac{d_0}{2b_0}} \exp\left\{ -\int_0^{\bar{g}_s(\mu)} \left[\frac{\tau_s(x)}{\beta_s(x)} - \frac{d_0}{b_0 x}\right] dx \right\},$$

  with $\overline{m}_s(\mu)$ the running mass in scheme s.

- At scales $\mu \ll M$, the fundamental theory ($N_\mathrm{f} = 3$-flavour QCD) can be described by an effective theory, $N_\mathrm{f} = 0$ QCD (i.e. pure Yang-Mills theory):

$$\bar{g}_s^{(3)}(\mu/\Lambda_s^{(3)}, M) = \bar{g}_s^{(0)}(\mu/\Lambda^{(0)}) + \mathrm{O}(\mu^2/M^2)\,, \qquad (1)$$

- in PT this leads to

$$[\bar{g}_{\overline{\mathrm{MS}}}^{(0)}(\mu)]^2 = C\left(\bar{g}_{\overline{\mathrm{MS}}}^{(3)}(m_\star)\right)[\bar{g}_{\overline{\mathrm{MS}}}^{(3)}(m_\star)]^2, \qquad m_\star = \overline{m}_{\overline{\mathrm{MS}}}(m_\star),$$

  and for $\mu = m_\star$ one finds $C(x) = 1 + c_2 x^4 + c_3 x^6 + c_4 x^8 + \ldots$.

- Reformulation with $P = \varphi_{\overline{\mathrm{MS}}}^{(0)}\left(g_\star\sqrt{C(g_\star)}\right)/\varphi_{\overline{\mathrm{MS}}}^{(3)}(g_\star)$, $g_\star = g_{\overline{\mathrm{MS}}}^{(3)}(m_\star)$:

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_\mathrm{dec}} = \frac{\Lambda_{\overline{\mathrm{MS}}}^{(0)}}{\Lambda_s^{(0)}} \times \lim_{M/\mu_\mathrm{dec}\to\infty}\left[\frac{\varphi_s^{(0)}\left(\bar{g}_s^{(3)}(\mu_\mathrm{dec}, M)\right)}{P\left(\frac{M}{\mu_\mathrm{dec}} \Big/ \frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_\mathrm{dec}}\right)}\right]$$

Coupling definitions

Set-up such that it benefits from various previous projects: running quark mass [Campos et al '18], $N_\mathrm{f} = 3$ coupling [ALPHA'17]

- Definition of massless renormalized couplings: use gradient flow GF scheme in finite volume

$$\bar{g}^2_{\mathsf{GF}}(\mu) = \mathcal{N}^{-1} \sum_{k,l=1}^{3} \frac{t^2 \langle \mathrm{tr}\,\{G_{kl}(t,x)G_{kl}(t,x)\}\,\delta_{Q,0}\rangle}{\langle \delta_{Q,0}\rangle}\Bigg|^{x_0=T/2,\; c=\sqrt{8t}/L}_{\mu=1/L, T=L,\; M=0}$$

- use both $T = L$ (GF scheme) and $T = 2L$ (GFT scheme) with projection to topological charge $Q = 0$ sector (part of scheme definition)
- 1-parameter families of schemes, parameter $c = \sqrt{8t}/L$
- $T = 2L$ chosen to suppress both cutoff effects linear in a and large mass effects linear in $1/M$ from Euclidean time boundaries.

Numerical set-up

Lines of constant physics:

- $\bar{g}^2_{\mathsf{GF}}(\mu_{\mathsf{dec}}) = 3.949 \quad \Rightarrow \mu_{\mathsf{dec}} = 789(15)\mathrm{MeV}$
  Varying $L/a = 1/(a\mu_{\mathsf{dec}})$ between $L/a = 12 - 48$ defines a sequence of values $\beta = 6/g_0^2 \in [4.3, 5.2]$
- Define range of values $z = M/\mu_{\mathsf{dec}} \in \{1.972, 4, 6, 8, 10, 12\}$ up to $\mathrm{O}(a^2)$ effects (non-trivial!) and find corresponding bare mass values.
- At these values of the bare parameters choose $T = 2L$ and compute the couplings in a massive scheme

$$\bar{g}^{(3)}_{\mathsf{GFT},c}(\mu_{\mathsf{dec}}, z)$$

- require aM to be small and $z = ML = M/\mu_{\mathsf{dec}} \gg 1$

$\Rightarrow$ potentially a difficult multiscale problem; using $\mu = 1/L$ alleviates part of it.

O(a) improvement, rôle of b_{g}

- Lattice QCD with Wilson quarks is affected by lattice artefacts of O(a), due to explicit chiral symmetry breaking.

$\Rightarrow$ can be restored by tuning the corresponding counterterms in the action and composite fields

- In particular $am_{\mathrm{q}}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$ rescales the bare gauge coupling g_0^2 With $m_{\mathrm{q}} = m_0 - m_{\mathrm{cr}}(g_0^2)$, the O($a$) improved bare coupling is

$$\tilde{g}_0^2 = g_0^2(1 + b_{\mathrm{g}}(g_0^2)am_{\mathrm{q}}),$$

and the O(a) improved RGI mass can be written as

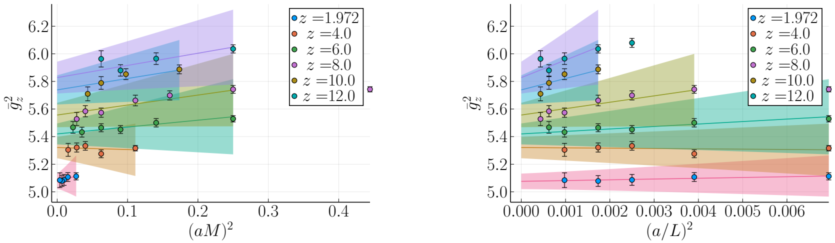
$$M = Z_M(\tilde{g}_0^2)m_{\mathrm{q}}(1 + b_{\mathrm{m}}(g_0^2)am_{\mathrm{q}}), \qquad Z_M = \underbrace{\frac{M}{\overline{m}(\mu)}}_{\text{RG running}} \times Z_m(\tilde{g}_0^2, a\mu)$$

- Variation of the quark mass at fixed lattice spacing at fixed $\tilde{g}_0^2$. This requires b_{g}, given to 1-loop order by,

$$b_{\mathrm{g}}(g_0^2) = 0.01200 \times N_{\mathrm{f}} g_0^2 + \mathrm{O}(g_0^4)$$

$\Rightarrow$ in ALPHA '22 we assumed an uncertainty of b_{g} of the same size as the one-loop term.

Continuum extrapolations



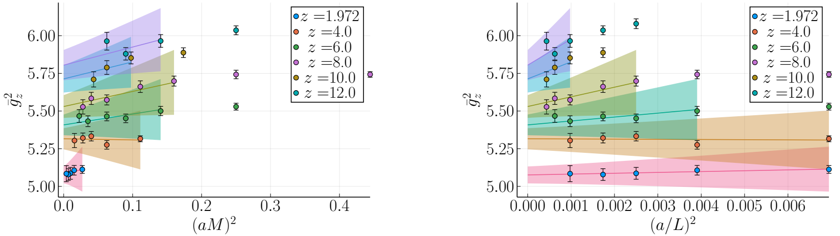
Data for $z = M/\mu_{\rm dec} \in \{1.972, 4, 6, 8, 10, 12\}$ (here with $c = 0.36$), extrapolated to $a = 0$ using

- individual fits for each z-value ($b_{\rm g}$-uncertainty not included in error bars):

$$\bar{g}^2(z_i, a) = C_i + p_i \left[\alpha_{\overline{\rm MS}}(a^{-1})\right]^{\hat{\Gamma}} (a\mu_{\rm dec})^2$$

- fit form motivated by Symanzik expansion with RG improvement [Balog et al. '09; Husung et al.'19]
- 2 cuts in $(aM)^2 < 0.16, 0.25$, fits are carried out for various $\hat{\Gamma} \in [-1, 1]$ (lines in plot for $\hat{\Gamma} = 0$.

Continuum extrapolations



Data for $z = M/\mu_{\rm dec} \in \{1.972, 4, 6, 8, 10, 12\}$ (here with $c = 0.36$), extrapolated to $a = 0$ using

- global fits (bands in plot, contain $b_{\rm g}$-uncertainty);

$$\bar{g}^2(z_i, a) = C_i + p_1[\alpha_{\overline{\rm MS}}(a^{-1})]^{\hat{\Gamma}}(a\mu_{\rm dec})^2 + p_2[\alpha_{\overline{\rm MS}}(a^{-1})]^{\hat{\Gamma}'}(aM_i)^2 \, .$$

- fit form motivated by Symanzik and large mass expansions.
- 2 cuts in $(aM)^2 < 0.16, 0.25$, with fixed $\hat{\Gamma} \in [-1, 1]$ and $\hat{\Gamma}' \in [-1/9, 1]$
- $z = 1.972$ seems to be at the edge of large mass regime; precautioniary measure: cut $z > 2$ and include $z = 1.972$ with different slope parameter

Combining with pure gauge theory results [Dalla Brida & Ramos '19]

- The continuum values of the massive couplings $g^{(3)}_{\mathrm{GFT}}(\mu_{\mathrm{dec}}, M)$ can now be matched with the corresponding $g^{(0)}_{\mathrm{GFT}}(\mu_{\mathrm{dec}})$, up to power corrections $1/M$.
- The step-scaling procedure in pure gauge theory gives us the function $\varphi^{(0)}_{\mathrm{GF}}(g)$

$$\frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} = \frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\Lambda^{(0)}_{\mathrm{GF}}} \times \varphi^{(0)}_{\mathrm{GF}}\left(\bar{g}^{(0)}_{\mathrm{GF}}(\mu_{\mathrm{dec}})\right) .$$

$\Rightarrow$ requires matching GFT,c with the $T = L, c = 0.3$ scheme GF

$$\bar{g}^{(0)}_{\mathrm{GF}}(\mu) = \chi_{\mathrm{c}}\left(\bar{g}^{(0)}_{\mathrm{GFT,c}}(\mu)\right)$$

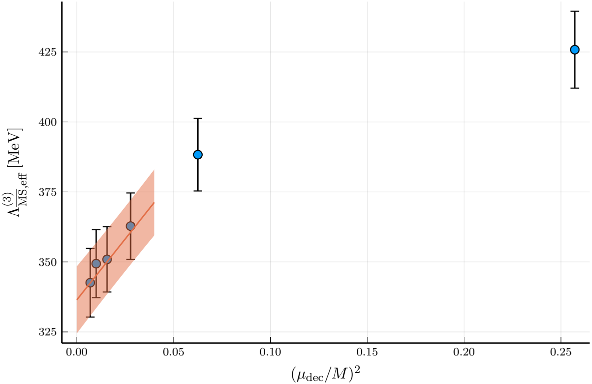
$\Rightarrow$ define $g = \chi_{\mathrm{c}}\left(\bar{g}^{(3)}_{\mathrm{GFT,c}}(\mu, M)\right)$ as input to $\varphi^{(0)}_{\mathrm{GF}}(g)$

- Combining all this at $\mu = \mu_{\mathrm{dec}}$, solve equation for target ρ,

$$\rho \times \underbrace{P\left(z/\rho\right)}_{\mathrm{PT} + \mathrm{O}\left(\alpha^4_{\overline{\mathrm{MS}}}(m_\star)\right)} = \frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}}, \qquad \rho = \frac{\Lambda^{(3)}_{\overline{\mathrm{MS}},\mathrm{eff}}}{\mu_{\mathrm{dec}}} = \frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} + \mathrm{O}(1/z^2)$$

- Corrections $\mathrm{O}(1/z)$ from boundaries strongly suppressed due to $T = 2L$.

Decoupling limit extrapolation



Extrapolate continuum results for $z \to \infty$:

$$\Lambda^{(3)}_{\overline{\mathrm{MS}},\ \mathrm{eff}} = \Lambda^{(3)}_{\overline{\mathrm{MS}}} + \frac{B}{z^2}\left[\alpha_{\overline{\mathrm{MS}}}(m_\star)\right]^{\hat{\Gamma}_m}$$

Extrapolation at

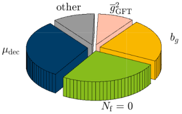
- fixed $c \in [0.3, 0.42]$ (here $c = 0.36$)
- fixed $\hat{\Gamma}_m \in [0, 1]$ (here $\hat{\Gamma}_m = 0$, variation with $\hat{\Gamma}_m$ is used as error estimate)

Result from decoupling, ALPHA '22

Our best estimate:

$$\Lambda_{\overline{\mathrm{MS}}}^{(3)} = 336(10)(6)_{b_\mathrm{g}}(3)_{\hat{\Gamma}_m}\,\mathrm{MeV} = 336(12)\mathrm{MeV} \quad \Rightarrow \quad \alpha_s(m_Z) = 0.11823(84)$$

- Total error is of the same size as in ALPHA '17 $(341(12)\mathrm{MeV})$



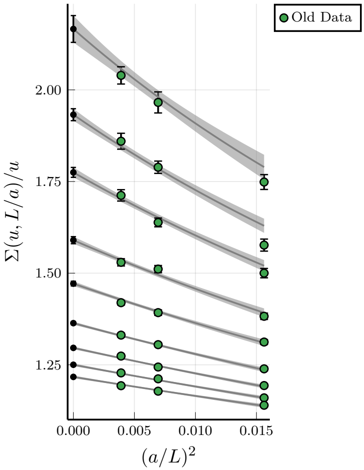
contributions to error^2

- only 28% common (squared) error with ALPHA '17 $\Rightarrow$ combine:
  $\Lambda_{\overline{\mathrm{MS}}}^{(3)} = 339.5(9.6) \qquad \Rightarrow \qquad \alpha_s(m_Z) = 0.1184(7)$ (statistics dominated!)
- Clear path to further error reduction:
  - Improve determination of $\Lambda_{\overline{\mathrm{MS}}}^{(0)}/\mu_{\mathrm{dec}}$
  - Improve physical scale setting from CLS ensembles
  - Improve continuum extrapolation of SSF at low energies
  - Non-perturbative determination of b_g

$\Rightarrow$ all these improvements are done or underway.

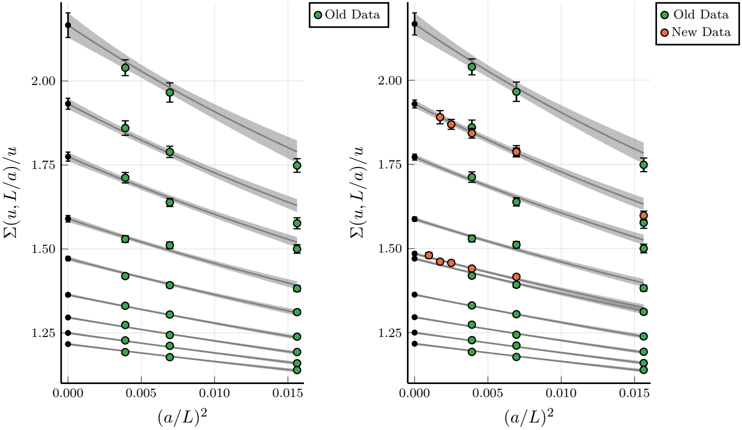
Improvement 1: ALPHA 17 SSF for GF coupling (low energy running)

old data: significant cutoff effects, small lattices ($L/a = 8$) given smaller weight:

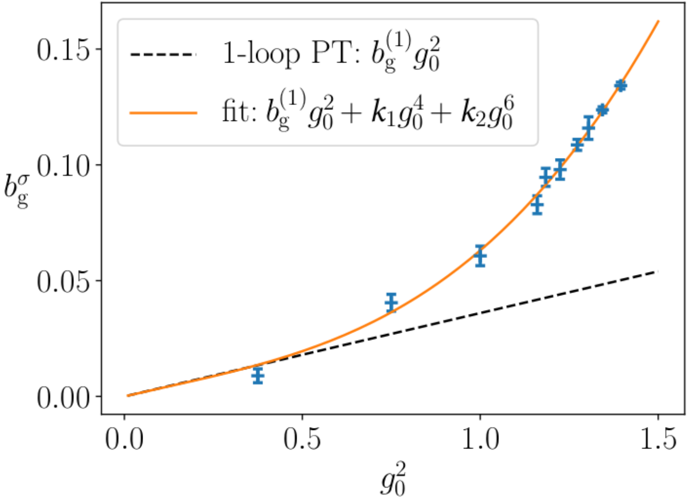


Improvement 1: ALPHA 17 SSF for GF coupling (low energy running)

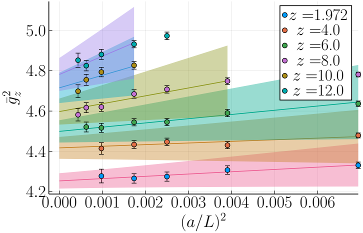
with new data from ALPHA coll. HQET project (courtesy Fritzsch, Kuberski, Heitger): very nice confirmation & improvement of old continuum extrapolations!



Nonperturbative result for b_g, ALPHA '24

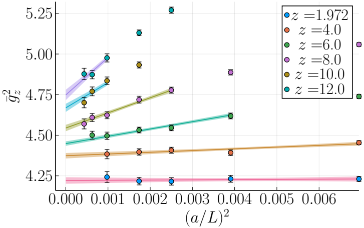


Improvement 2. The continuum extrapolation of massive couplings



Previous determination, ALPHA '22

- Most of error from estimate of $b_g - b_g^{1-\text{loop}}$
- This is a systematic!
- Even so, error in $\bar{g}^2(\mu, M)$ was subdominant (assuming 100 percent error on 1-loop b_g).



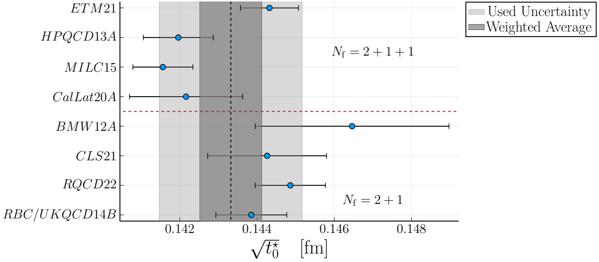
NP determination of b_g (ALPHA '24)

- Much more precise continuum values
- Completely removes largest systematic effect in α_s from the decoupling method.

Scale setting (cf. 2501.06633 [hep-ph] for references)

Match to the gradient flow scale $\sqrt{t_0}$, as determined by all major collaborations in terms of m_Ω, F_π, $F_K, \ldots$: Define: $\bar{g}^2(\mu_{\text{had}}) = 11.31$ and determine (with $\sqrt{t_0/t_0^\star} = 1.0003(30)$ and $\sqrt{t_0} = 0.1434(18)$fm)

$$t_0^\star \times \mu_{\text{had}} = 0.146(11) \Rightarrow \mu_{\text{had}} = 200.5(3.0)\,\text{MeV},$$



Conclusions

- Re-analysis with major improvements by the ALPHA collaboration
- Updated/improved $N_\mathrm{f} = 3$ step-scaling result: $\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 347(11)\,\mathrm{MeV}$

$\Rightarrow$ corroborated earlier continuum extrapolations, new scale determination

- Decoupling result with major improvement: $\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 341.9(9.6)\,\mathrm{MeV}$ Eliminated main source of systematic errors (b_g), better scale determination
- Combined result, taking into account correlation from common scale setting

$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 343.9(8.4)\,\mathrm{MeV} \quad \Rightarrow \quad \alpha_s(m_Z) = 0.11872(56) \qquad [0.47\%]$$

$\Rightarrow$ error is 0.47% while still being statistics dominated!

- Some further error reduction still feasible:
  - Reduce error in $N_\mathrm{f} = 0$ step-scaling; straightforward, just needs to be done.
  - Some error reduction in $N_\mathrm{f} = 3$ step-scaling at high energies (significant computational costs!)
  - The scale setting error is chosen generously to cover all values from the literature, which show some inconsistencies at the level of the quoted errors. This situation should be resolved by the wider lattice community!
- Substantial error reduction requires a thorough re-evaluation; only sensible if the total error remains statistics dominated!

Question to the perturbative QCD and experimental communities:

- Determination of a fundamental parameter such as $\alpha_s(m_Z)$ in many different ways is important;
- Question: Can we learn more from the data, by changing perspective?
- What if the value $\alpha_s(m_Z) = 0.11872(56)$ by ALPHA coll. were used as input parameter?
- Advantages:
    - Small error, statistics dominated, easy to propagate.
    - Error is completely uncorrelated to other collider data entering e.g. the pdf's; full data sets can be used.
- One could study non-perturbative effects, test factorization assumption and constrain pdf's.