Nucleon spin, form factor and GPD

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About myself:

- PhD from Kyoto university (2004)
- Postdocs at RIKEN BNL and Saclay (France)
- Faculty positions at University of Tsukuba (2008~2013), Kyoto university (2013~2018)
- Moved to Brookhaven National Laboratory in 2018
- Research interest: Hadron structure from QCD: spin, mass, 3D&5D tomography, small-x.

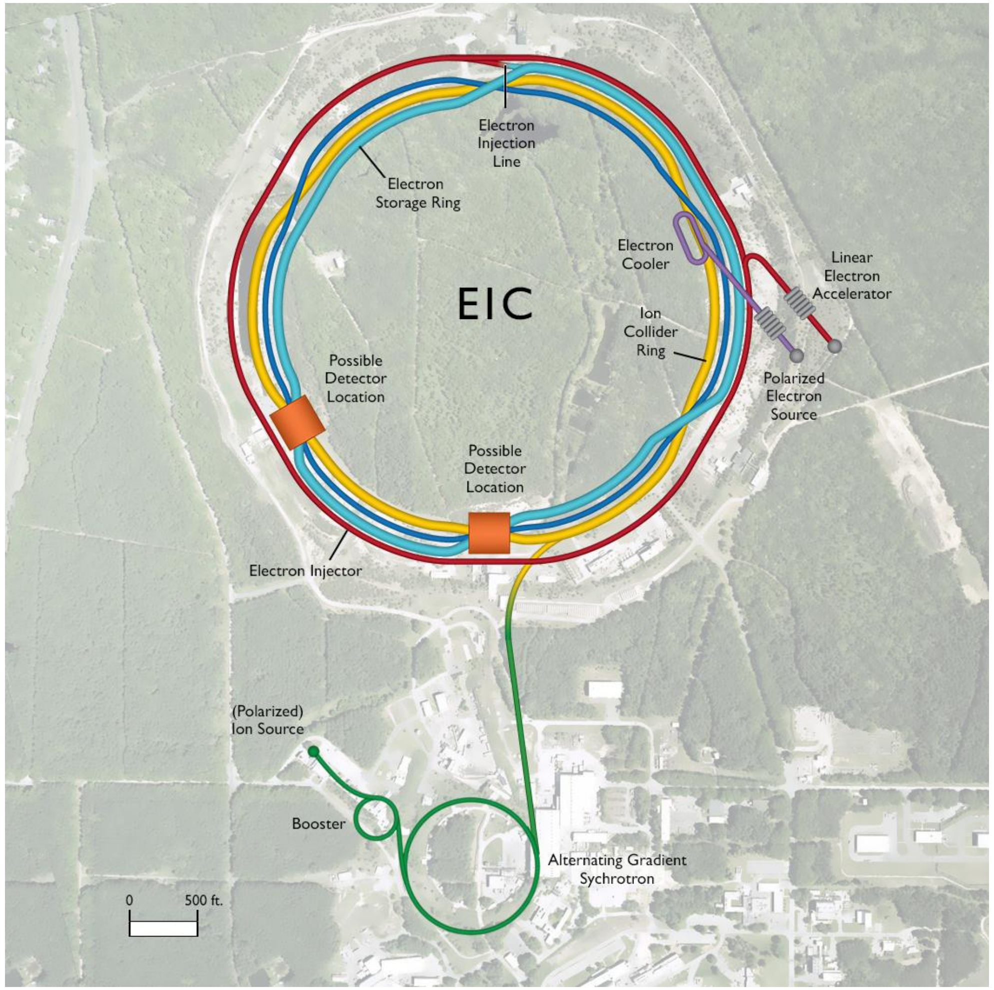
The Electron-Ion Collider (EIC) project

Next-generation (your generation) nuclear physics facility
to be built at Brookhaven National Laboratory, New York

As of now, the only new high energy collider in the world officially approved for construction.

QCD first!

Unravel the structure of proton and nuclei

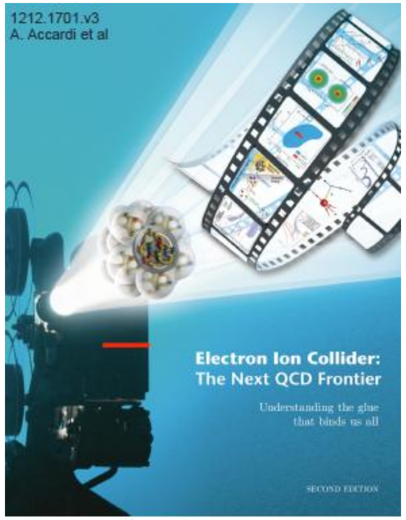


Scientific goals of EIC

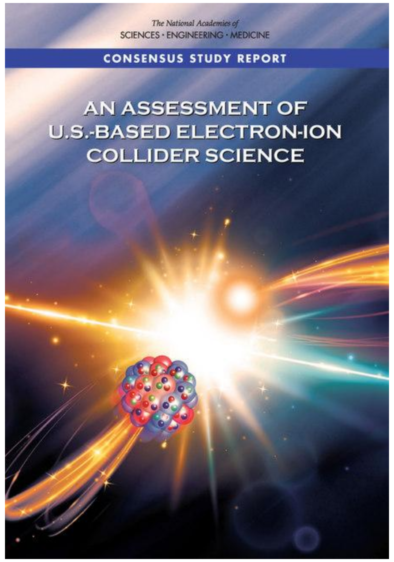
Finding 1: An EIC can uniquely address three profound questions about nucleons-protons—and how they are assembled to form the nuclei of atoms:

- How does the mass of the nucleon arise?
- How does the spin of the nucleon arise?
- What are the emergent properties of dense systems of gluons?

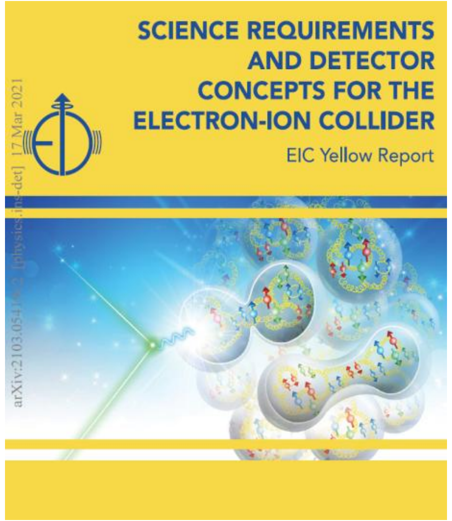
White paper (2012)



NAS report (2018)



Yellow report (2022)



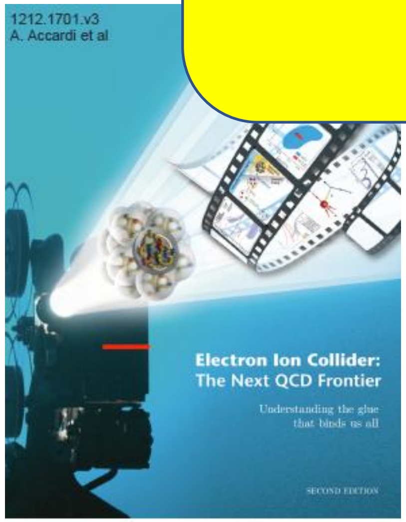
Scientific goals of EIC

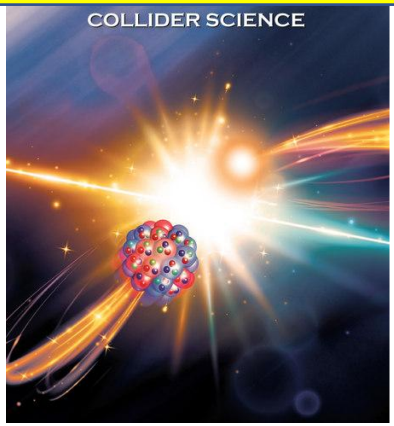
Finding 1: An EIC can uniquely address three profound questions about nucleons-protons—and how they are assembled to form the nuclei of atoms:

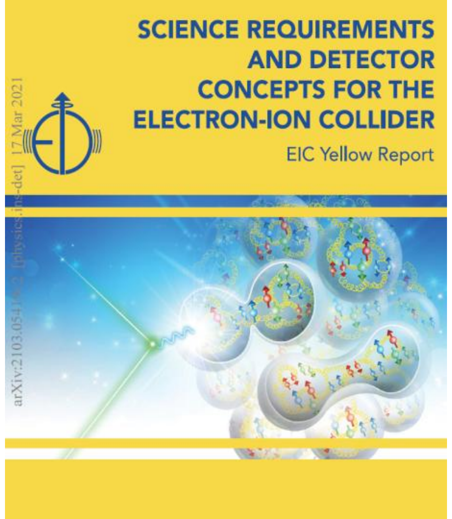
- How does the mass of the nucleon arise?
- How
- What ons?

The era of precision QCD study of nucleon and nuclear structures in the next 20-30 years!

White paper (2012)







Plan

- Proton spin decomposition
- Polarized Deep Inelastic scattering
- Orbital angular momentum
- Generalized parton distribution
- Deeply Virtual Compton Scattering
- Electromagnetic form factor
- Gravitational form factor

Lecture 1

Lecture 2

Lecture 3

Notations

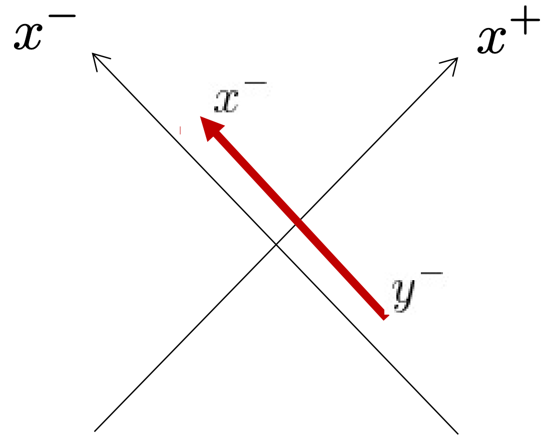
Metric $g^{\mu\nu} = (+1, -1, -1, -1)$ $\mu, \nu = 0, 1, 2, 3$ $i, j = 1, 2$ (transverse)

Light-cone coordinates $P^{\pm} = \frac{1}{\sqrt{2}}(P^0 \pm P^3)$ $g^{+-} = 1$ $P^+ = P_-$

$$P \cdot x = P^+ x^- + P^- x^+ - P_\perp^i x_\perp^i$$

γ_5, antisymmetric tensor $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ $\epsilon^{0123} = +1$ $\epsilon^{12} = \epsilon_{12} = +1$

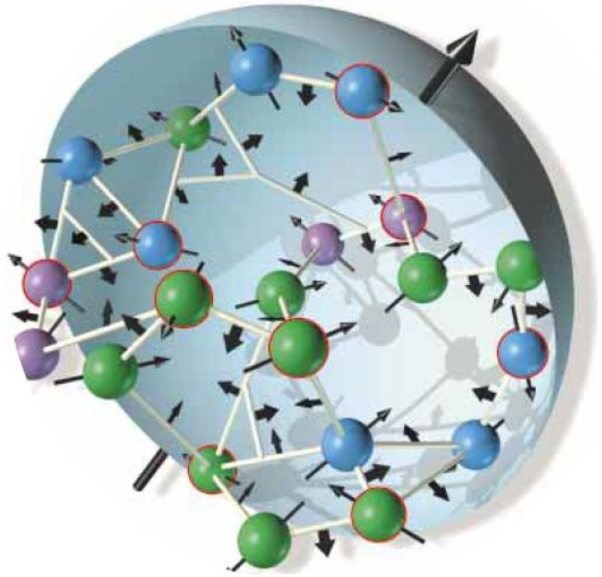
Coupling constant $D^\mu = \partial^\mu + igA^\mu$



Wilson line

$$W[x^-, y^-, x_\perp] = P\exp\left(-ig\int_{y^-}^{x^-} dz^- A^+(z^-, x_\perp)\right)$$

The proton spin problem



The proton has spin ½.

The proton is not an elementary particle.

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g$$

Quarks' helicity

Gluons' helicity

Orbital angular Momentum (OAM)

QCD angular momentum tensor

QCD Lagrangian → Lorentz invariant $x^\mu \to x^\mu + \omega^{\mu\nu} x_\nu$

→ Noether current $\partial_\mu M^{\mu\nu\lambda}_{can} = 0$

QCD angular momentum tensor

$$M^{\mu\nu\lambda}_{can} = x^\nu T^{\mu\lambda}_{can} - x^\lambda T^{\mu\nu}_{can} - \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\bar{\psi}\gamma_\rho\gamma_5\psi + F^{\mu\lambda}A^\nu - F^{\mu\nu}A^\lambda$$

quark helicity gluon helicity

canonical energy momentum tensor

$$T^{\mu\nu}_{can} = \bar{\psi} i\gamma^\mu \overleftrightarrow{\partial}^\nu \psi - F^{\mu\alpha}\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L}$$

→ Quark OAM → Gluon OAM

Exercise: Derive the canonical angular momentum tensor $M^{\mu\nu\lambda}_{can}$

Hint: Under an infinitesimal Lorentz transformation

$$\delta\psi = -\omega^{\mu\nu}\left(\frac{1}{2}(x_\nu\partial_\mu - x_\mu\partial_\nu)\psi - \frac{1}{8}[\gamma_\mu,\gamma_\nu]\psi\right)$$

$$\delta A^\alpha = -w^{\mu\nu}\left(x_\nu\partial_\mu A^\alpha - \frac{1}{2}(\delta^\alpha_\mu g_{\nu\beta} - g_{\mu\beta}\delta^\alpha_\nu)A^\beta\right)$$

$$\delta\mathcal{L} = -w^{\mu\nu}x_\nu\partial_\mu\mathcal{L}$$

Problems

$T^{\mu\nu}_{can}$ is not symmetric, not gauge invariant

$T^{\mu\nu}_{can}$ is conserved wrt the first index $\partial_\mu T^{\mu\nu}_{can} = 0$ but not the second $\partial_\nu T^{\mu\nu}_{can} \neq 0$

Jaffe-Manohar decomposition

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

$\mu\nu\lambda = +12$ component of the canonical angular momentum tensor $M^{\mu\nu\lambda}_{can}$

Operators NOT gauge invariant except the quark helicity $\Delta\Sigma \sim \bar{\psi}\gamma^+\gamma_5\psi$

$$\Delta G \sim \epsilon^{ij} F^{+i} A^j \qquad L^q_{can} \sim \bar{\psi} x \times i\partial\psi \qquad L^g_{can} \sim F x \times \partial A$$

To be understood in the light-cone gauge $A^+ = 0$

Naïve replacement $\partial^\mu \to D^\mu$ does not solve the problem.

proton single-particle state,

Quark helicity: definition

$$2\Delta\Sigma S^\mu = \sum_f \langle PS|\bar{\psi}_f \gamma^\mu \gamma_5 \psi_f|PS\rangle$$

spin 4-vector $$2S^\mu = \bar{u}(PS)\gamma^\mu\gamma_5 u(PS)$$

proton Dirac spinor

Exercise : show that $$P^\mu S_\mu = 0$$

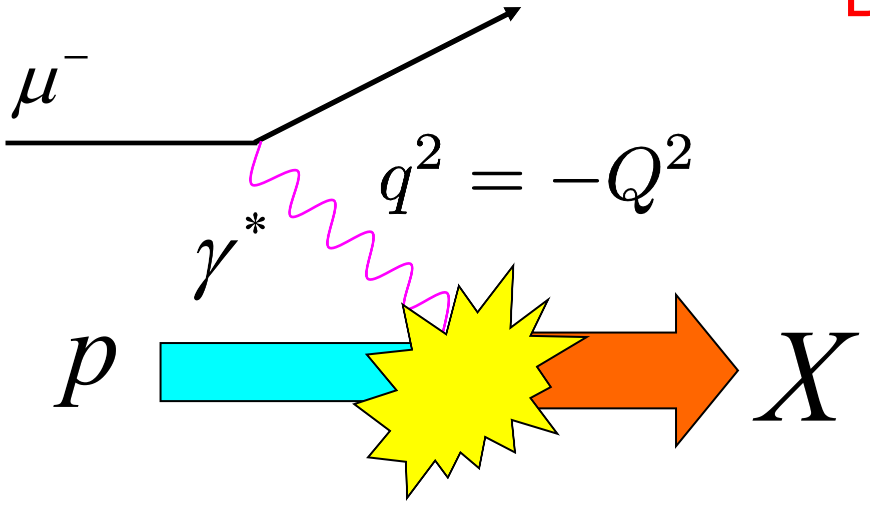
$$u(PS)\bar{u}(PS) = \frac{\not{P}+M}{2}\left(1+\frac{\gamma_5 \not{S}}{M}\right)$$

$$S^2 = -M^2$$

In the quark model, $$\frac{1}{2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \quad \Rightarrow \quad \Delta\Sigma = 1$$

With relativistic effects, $\Delta\Sigma \approx 0.7$

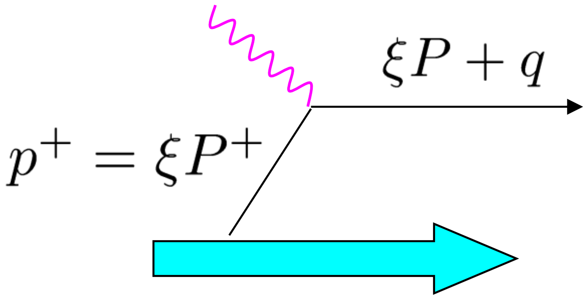
Deep inelastic scattering



Bjorken variable

$$x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{(P+q)^2 + Q^2 - m_p^2}$$
$$= \frac{Q^2}{Q^2 + m_X^2 - m_p^2}$$
$$\sim \frac{Q^2}{s} \quad (x \ll 1)$$

Physical meaning of x : momentum fraction carried by the struck parton



$$(\xi P + q)^2 = \xi^2 m_p^2 + 2\xi P\cdot q - Q^2 = 0$$

$$\xi \approx \frac{Q^2}{2Pq} = x$$

DIS structure functions

Unpolarized

$$\mathrm{Im}\frac{i}{2\pi}\int d^4y e^{iqy}\langle PS|T\{J^\mu(y)J^\nu(0)\}|PS\rangle\Big|_{sym}$$

$$= \left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x,Q^2) + \left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)\frac{F_2(x,Q^2)}{P\cdot q}$$

Polarized

$$\mathrm{Im}\frac{1}{2\pi}\int d^4y e^{iqy}\langle PS|T\{J^\mu(y)J^\nu(0)\}|PS\rangle\Big|_{asym}$$

$$= -\epsilon^{\mu\nu\alpha\beta}\frac{q_\alpha}{P\cdot q}\left[S_\beta g_1 + \left(S_\beta - \frac{S\cdot q}{P\cdot q}P_\beta\right)g_2\right]$$

Exercise

Forward Compton amplitude $q^0 > 0$

$$T^{\mu\nu} = \frac{i}{2\pi}\int d^4x e^{iq\cdot x}\langle PS|\mathrm{T}\{J^\mu(x)J^\nu(0)\}|PS\rangle = T_S^{\mu\nu} + iT_A^{\mu\nu}$$

Not to be confused with real/imaginary decomposition

symmetric antisymmetric

Hadronic tensor

$$W^{\mu\nu} = \frac{1}{2\pi}\int d^4x e^{iq\cdot x}\langle PS|[J^\mu(x), J^\nu(0)]|PS\rangle = W_S^{\mu\nu} + iW_A^{\mu\nu}$$

Show that

$$2\mathrm{Im}T_S^{\mu\nu} = W_S^{\mu\nu}$$

$$2\mathrm{Im}T_A^{\mu\nu} = W_A^{\mu\nu}$$

Light-cone dominance

Want to study the correlator $\int d^4y e^{iqy} \langle P | T\{ J^\mu(y) J^\nu(0) | P \rangle$

in the Bjorken limit $Q^2 \to \infty, \quad P\cdot q \to \infty, \quad x = \frac{Q^2}{2P\cdot q} = const.$

Naively the integral is dominated by $|y^\mu| \sim \frac{1}{|q^\mu|} \to 0 \quad ?$

Proton rest frame (photon in the minus direction) $x = \frac{(q^3 - q^0)(q^3 + q^0)}{2m_p q^0} \simeq \frac{q^3 + q^0}{m_p}$

$$y^+ \sim \frac{1}{q^-} \sim \frac{m_p x}{Q^2} \to 0, \qquad y^- \sim \frac{1}{q^+} \sim \frac{1}{m_p x} \qquad y^2 \sim \frac{1}{Q^2} \to 0$$

finite !

Operator product expansion

$$\int d^4y e^{iqy} \bar{\psi}\gamma^\mu \psi(y)\bar{\psi}\gamma^\nu\psi(0) = \bar{\psi}i(i\partial_\alpha + q_\alpha)\gamma^\mu\gamma^\alpha\gamma^\nu \frac{-1}{Q^2}\sum_n \left(\frac{2iq\cdot\partial}{Q^2}\right)^n \psi(0) + \cdots$$
$$+(\mu \to \nu, q \to -q)$$

c.f., Peskin (18.125)

Pick up the antisymmetric part

$$\gamma^\mu\gamma^\alpha\gamma^\nu = g^{\mu\alpha}\gamma^\nu - g^{\mu\nu}\gamma^\alpha + g^{\alpha\nu}\gamma^\mu + i\epsilon^{\mu\alpha\nu\rho}\gamma_\rho\gamma_5$$

$$\int d^4y e^{iqy}\bar{\psi}\gamma^\mu\psi(y)\bar{\psi}\gamma^\nu\psi(0) = 2\epsilon^{\mu\nu\lambda\alpha}q_\alpha \sum_n^{even} \frac{2q_{\mu_1}\cdots 2q_{\mu_n}}{Q^{2(n+1)}} \underline{\bar{\psi}\gamma_\lambda\gamma_5 i\partial^{\mu_1}\cdots i\partial^{\mu_n}\psi(0)}$$

When $Q^2 \to \infty$, naively, the most important operators are those with smallest dimensions (smallest n)

Twist expansion

However, in the proton matrix element, $i\partial^\mu \to P^\mu$, and $\dfrac{2P\cdot q}{Q^2} = \dfrac{1}{x}$
is not small in the Bjorken limit $Q^2 \to \infty,\ x = const.$

The most important operators are those with lowest twist

(twist) = (dimension) – (spin)

In practice, the number of plus Lorentz indices

$\bar{\psi}\gamma^+\psi$ twist-2 $\bar{\psi}\gamma_\perp\psi$ twist-3

Twist-2 polarized quark operators
(symmetrized in all Lorentz indices and trace subtracted)

$$\bar{\psi}\gamma_5\gamma^{(\lambda}iD^{\mu_1}iD^{\mu_2}\dots iD^{\mu_n)}\psi\ -(\text{traces})$$

Totally symmetric in all indices → twist-2

$$\langle PS|\frac{1}{n+1}\bar{\psi}\Big(\gamma_\lambda\gamma_5 i\partial_{(\mu_1}\cdots i\partial_{\mu_n)}+\sum_{i=1}^{n}\gamma_{\mu_i}\gamma_5 i\partial_{(\mu_1}\cdots i\partial_\lambda\cdots i\partial_{\mu_n)}\Big)\psi|PS\rangle\frac{2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}}$$

$$\equiv\frac{a_n}{n+1}\Big(S_\lambda P_{\mu_1}\cdots P_{\mu_n}+\sum_{i=1}^{n}S_{\mu_i}P_{\mu_1}\cdots P_\lambda\cdots P_{\mu_n}\Big)\frac{2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}}$$

$$=\frac{a_n}{n+1}\frac{1}{x^n}\left(S_\lambda+n\frac{S\cdot q}{P\cdot q}P_\lambda\right)$$

→ longitudinal/transverse polarization $g_1(x), g_2(x)$

Anti-symmetric in λ and $\mu_1,\mu_2,\cdots$ → One twist higher (twist-3)

$$\langle PS|\frac{1}{n+1}\sum_{i=1}^{n}\bar{\psi}\Big(\gamma_\lambda\gamma_5 i\partial_{(\mu_1}\cdots i\partial_{\mu_n)}-\gamma_{\mu_i}\gamma_5 i\partial_{(\mu_1}\cdots i\partial_\lambda\cdots i\partial_{\mu_n)}\Big)\psi|PS\rangle\frac{2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}}$$

$$\equiv\frac{d_n}{n+1}\sum_{i=1}^{n}\Big(S_\lambda P_{\mu_1}\cdots P_{\mu_n}-S_{\mu_i}P_{\mu_1}\cdots P_\lambda\cdots P_{\mu_n}\Big)\frac{2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}}$$

$$=\frac{nd_n}{n+1}\frac{1}{x^n}\left(S_\lambda-\frac{S\cdot q}{P\cdot q}P_\lambda\right)$$

→ transverse polarization $g_2(x)$

g_1 structure function

$$g_1(x) = \frac{1}{2\pi}\mathrm{Im}\sum_{n=0}^{even}\frac{a_n}{x^{n+1}} = \frac{1}{2\pi S^+}\mathrm{Im}\sum_{n=0}^{even}\frac{1}{(P^+)^n x^{n+1}}\langle PS|\bar{\psi}\gamma_5\gamma^+(iD^+)^n\psi|PS\rangle + \cdots$$

Convergent only when $|x| > 1$!
All terms are real!

$$= \frac{1}{2\pi S^+}\mathrm{Im}\sum_{n=0}^{even}\frac{1}{x^{n+1}}\int\frac{dk^+}{2\pi}\left(\frac{k^+}{P^+}\right)^n\int dx^- e^{ik^+x^-}\langle PS\,\bar{\psi}(0)\gamma^+\gamma_5 W[0,x^-]\psi(x^-)|PS\rangle$$

Wilson line

Exercise: show this

$$= \frac{P^+}{4\pi S^+}\mathrm{Im}\int\frac{dk^+}{2\pi}\left(\frac{1}{xP^+ + k^+} + \frac{1}{xP^+ - k^+}\right)\int dx^- e^{ik^+x^-}\langle PS|\,\bar{\psi}(0)\gamma^+\gamma_5 W[0,x^-]\psi(x^-)|PS\rangle$$

Analytic continuation from

$|x| > 1$ to $1 > x > 0$ $\quad x \to x - i\epsilon$

(cf. $s \to s + i\epsilon$)

$$= \frac{P^+}{8\pi S^+} \int dx^- e^{ixP^+x^-} \langle PS|\bar{\psi}(0)\gamma_5\gamma^+ W[0,x^-]\psi(x^-)|PS\rangle + (x \to -x)$$

$$= \frac{1}{2}(\Delta q(x) + \Delta \bar{q}(x))$$

Polarized quark and antiquark distributions

Note the sign difference

$q(-x) = -\bar{q}(x)$ unpolarized quark PDF

$\Delta q(-x) = \Delta \bar{q}(x)$ polarized quark PDF

Exercise: Show that for n even, $\int_0^1 dx x^n g_1(x) = \frac{a_n}{4}$

$g_2(x)$ structure function

Similarly,

$$g_2(x) = \frac{1}{2\pi}\mathrm{Im}\sum_n \frac{n(d_n - a_n)}{n+1}\frac{1}{x^{n+1}} \qquad \int_0^1 dx x^n g_2(x) = \frac{n(d_n - a_n)}{4(n+1)}$$

Invert these relations and get

$$g_2(x) = \underline{-g_1(x) + \int_x^1 \frac{dz}{x} g_1(z)} + \underline{\bar{g}_2(x)}$$

Wandzura, Wilczek (1977)

Wandzura-Wilczek part
related to twist-2 PDF

`genuine twist-3' part
$q\bar{q}g$ correlation functions

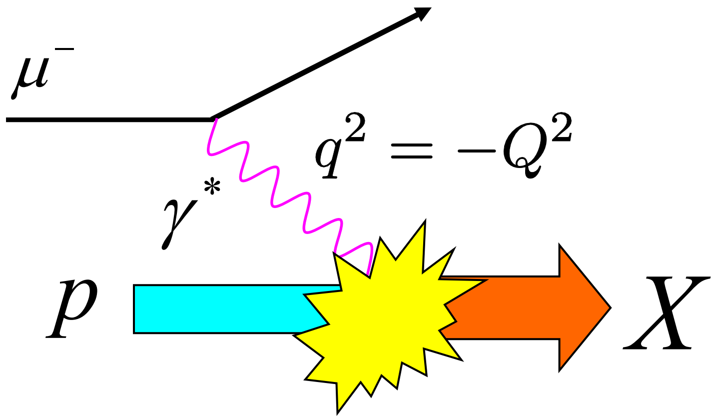
$$\int_0^1 dx x^2 \bar{g}_2(x) = \frac{d_2}{6} \qquad \langle PS|\bar{\psi}\gamma^+ gF^{+i}\psi|PS\rangle = 2d_2(P^+)^2\epsilon^{ij}S_j$$

Shuryak, Vainshtein (1982)

$\Delta\Sigma$ from polarized DIS

Longitudinal double spin asymmetry in polarized DIS

$$A_{LL} = \frac{\mu^\uparrow p^\downarrow - \mu^\uparrow p^\uparrow}{\mu^\uparrow p^\uparrow + \mu^\uparrow p^\downarrow} \sim \left(1 + \frac{\sigma_L}{\sigma_T}\right)\frac{2x g_1}{F_2}$$



$$\int_0^1 dx g_1(x,Q^2) = \frac{1}{2}\sum_f e_f^2 \int_0^1 dx(\Delta q_f(x,Q^2) + \Delta\bar{q}_f(x,Q^2)) + \cdots$$

Flavor SU(3) decomposition

$$\sum_f e_f^2 = \begin{pmatrix} \frac{4}{9} & & \\ & \frac{1}{9} & \\ & & \frac{1}{9} \end{pmatrix} = \frac{2}{9} + \frac{1}{6}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} + \frac{1}{18}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$\int_0^1 dx g_1(x) = \frac{1}{9}\overbrace{(\Delta u + \Delta d + \Delta s)}^{\Delta\Sigma}$$
$$+\frac{1}{12}(\Delta u - \Delta d)$$
$$+\frac{1}{36}(\Delta u + \Delta d - 2\Delta s) + \mathcal{O}(\alpha_s)$$

nucleon isovector axial charge

$$\langle p|\bar{q}\gamma^\mu\gamma_5 t^3 q|p\rangle \sim g_A^{(3)}$$

→ from neutron beta decay

$$n(udd) \to p(uud) + e^- + \bar{\nu}_e$$

$$\langle p|\bar{u}\gamma^\mu\gamma_5 d|n\rangle \sim g_A^{(3)}$$

octet axial charge $\langle p|\bar{q}\gamma^\mu\gamma_5 t^8 q|p\rangle \sim g_A^{(8)}$

→ from hyperon semileptonic decay

$$\Xi^-(dss) \to \Lambda(uds) + e^- + \bar{\nu}_e$$

$$\langle \Lambda|\bar{u}\gamma^\mu\gamma_5 s|\Xi^-\rangle \sim 3F - D = g_A^{(8)}$$

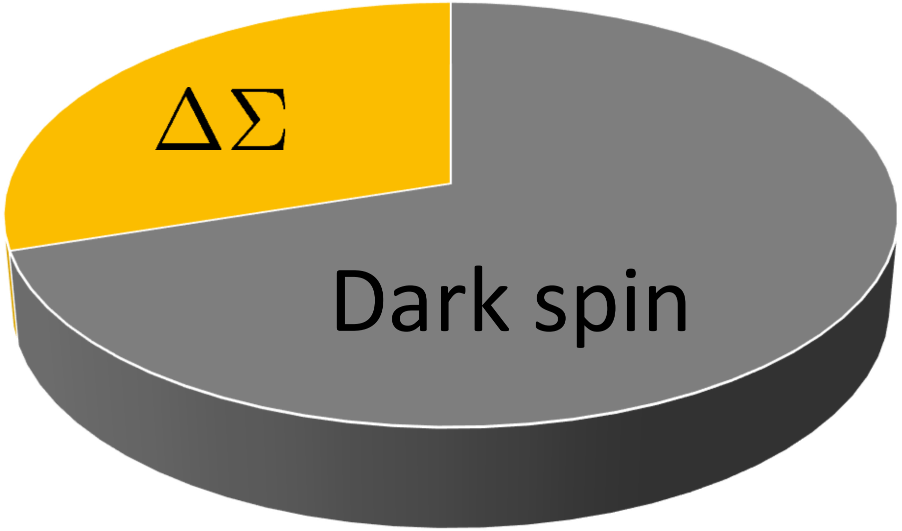
`Spin crisis'

In 1987, EMC (European Muon Collaboration) announced a very small value of the quark helicity contribution

$$\Delta\Sigma = 0.12 \pm 0.09 \pm 0.14 \ \text{!?}$$

Recent value from NLO QCD global analysis

$$\Delta\Sigma = 0.25 \sim 0.3$$



Gluon polarization $\Delta G = \int_0^1 dx \Delta G(x)$

Polarized gluon distribution

$$\Delta G(x) = \frac{i}{xS^+} \int \frac{dy^-}{2\pi} e^{ixP^+y^-} \langle PS | F^{+\alpha}(0) \tilde{F}^+_{\ \alpha}(y^-) | PS \rangle$$

$$\epsilon^\mu_{R/L} = \frac{-1}{\sqrt{2}}(0, \pm 1, i, 0)$$

$$iF^{+i}\tilde{F}^+_{\ i} = (F^{+R})^\dagger F^{+R} - (F^{+L})^\dagger F^{+L}$$

Non-local, even after taking a moment.

$$\int_0^1 dx \Delta G(x) \quad = \quad -\frac{1}{2S^+} \int dy^- \theta(y^-) \langle PS | F^{+\alpha}(0) \tilde{F}^+_{\ \alpha}(y^-) | PS \rangle$$

Depends on the prescription of the pole $1/x$.
The value of ΔG independent of the prescription.

In the light-cone gauge $A^+ = 0$, it reduces to the local operator in the Jaffe-Manohar decomposition.

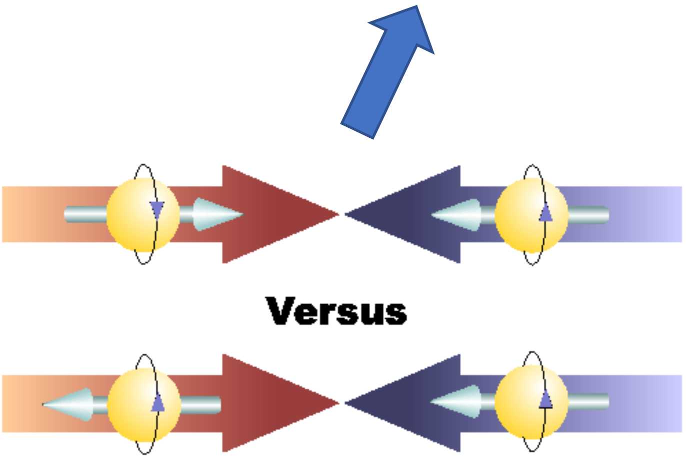
Determination of ΔG at RHIC

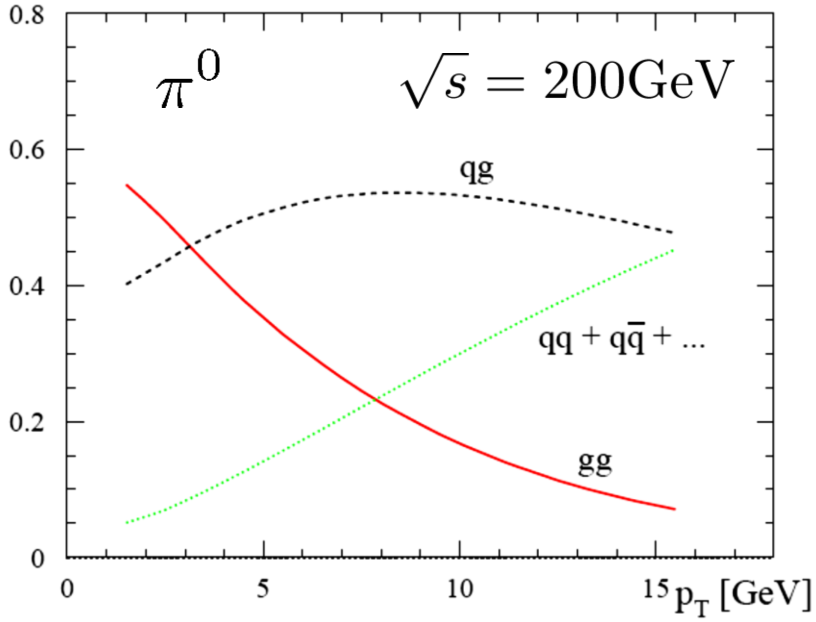
Double spin asymmetry of pions and photons in polarized pp.

$$A_{LL} = \frac{d\sigma^{++} - d\sigma^{+-}}{d\sigma^{++} + d\sigma^{+-}}$$

$$\propto \sum_{a,b} \Delta f_a \otimes \Delta f_b(x) \otimes \Delta\sigma_{ab}$$

pion, photon





Direct Photons Point to Positive Gluon Polarization

Results from 'golden measurement' at RHIC's PHENIX experiment show the spins of gluons align with the spin of the proton they're in

June 21, 2023

ınalysis of data from the PHENIX detector at the Relativistic Heavy Ion
· (RHIC) gives fresh insight into how gluons contribute to proton spin.

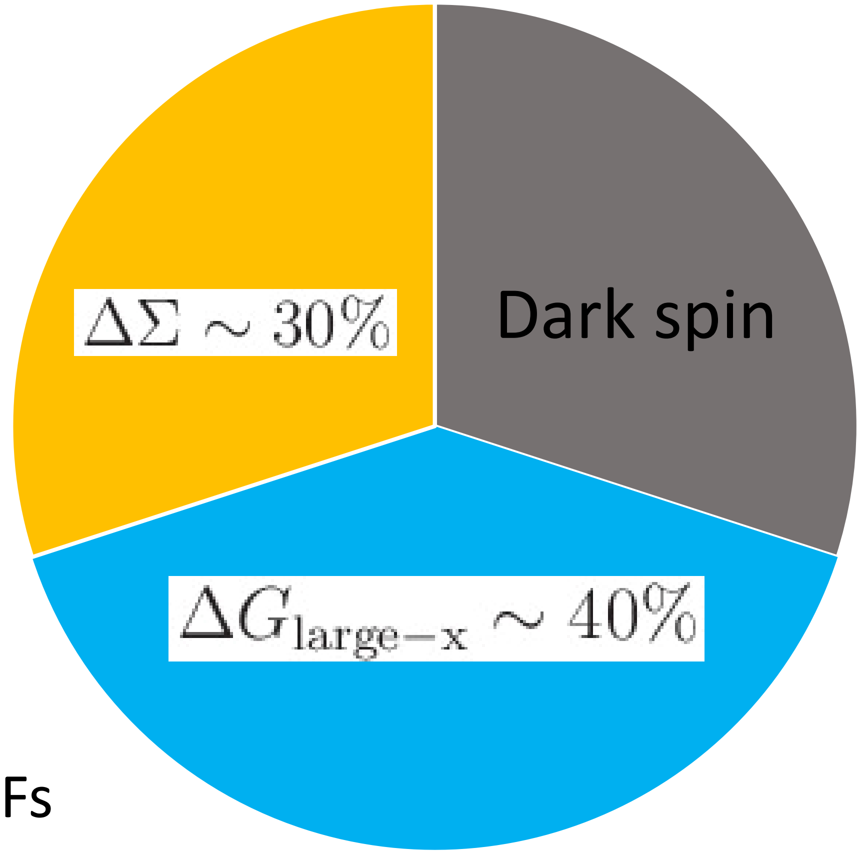
Evidence of nonzero gluon helicity $\Delta G = \int_0^1 dx \Delta G(x)$

A major achievement of the RHIC spin program!

$$\int_{0.05}^{1} dx \Delta G(x, Q^2 = 10\mathrm{GeV}^2) = 0.20^{+.06}_{-.07}$$ DSSV

$$\int_{0.05}^{0.2} dx \Delta G(x, Q^2 = 10\mathrm{GeV}^2) = 0.17 \pm 0.06$$ NNPDF

$$\int_{0.05}^{1} dx \Delta G(x, Q^2 = 10\mathrm{GeV}^2) = 0.23 \pm 0.03$$ JAM



Huge uncertainty from the small-x region → EIC

Renewed interest in the small-x resummation of helicity PDFs

Kovchegov et al. 2015~

NNLO global analysis became available last year. Borsa et al. 2407.11635

Necessity for orbital angular momentum

Polarized DGLAP evolution

$$\frac{d}{d\ln Q^2}\begin{pmatrix}\Delta\Sigma(x)\\ \Delta G(x)\end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dz}{z}\begin{pmatrix}\Delta P_{qq}(z) & \Delta P_{qg}(z)\\ \Delta P_{gq}(z) & \Delta P_{gg}(z)\end{pmatrix}\begin{pmatrix}\Delta\Sigma(x/z)\\ \Delta G(x/z)\end{pmatrix}$$

$$\Delta P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right),$$

$$\Delta P_{qg}(z) = n_f(2z-1),$$

$$\Delta P_{gq}(z) = C_F(2-z),$$

$$\Delta P_{gg}(z) = 6\left(\frac{1}{(1-z)_+} - 2z + 1\right) + \frac{\beta_0}{2}\delta(z-1)$$

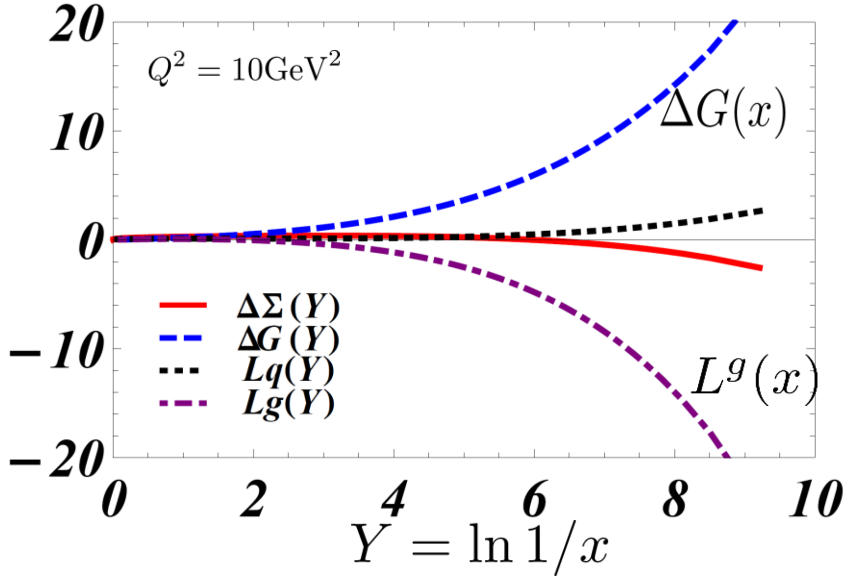
Integrate over x

$$\frac{d}{d\ln Q^2}\left(\frac{1}{2}\Delta\Sigma(Q^2) + \Delta G(Q^2)\right) \neq 0$$

Helicity is not a conserved quantity!

Angular momentum conservation

YH, Yang 1802.02716



Only the sum of helicity and OAM is conserved.

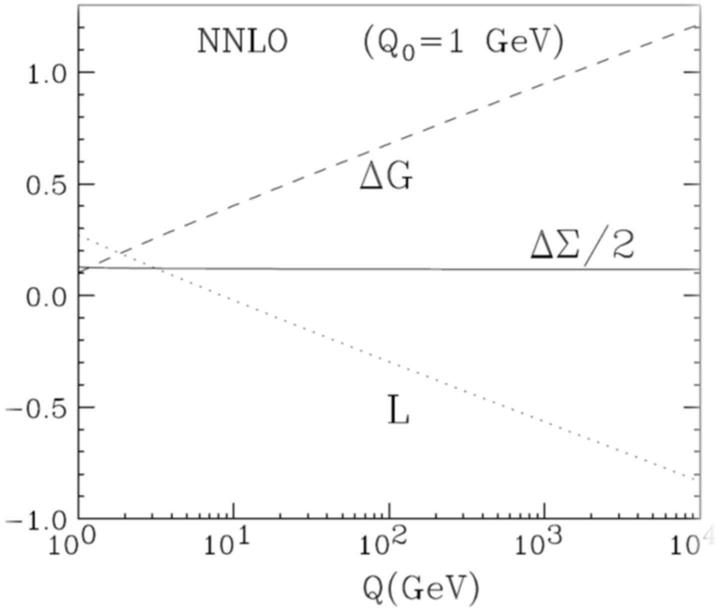
$$\frac{d}{d\ln Q^2}\left(\frac{1}{2}\Delta\Sigma(Q^2) + \Delta G(Q^2) + L_q(Q^2) + L_g(Q^2)\right) = 0$$

In practice, the scale dependence of $\Delta\Sigma$ very weak (starts at 2-loop in perturbation theory)

Kodaira (1980)

$$\frac{d}{d\ln Q^2}\Delta\Sigma = -12C_F T_F n_f \left(\frac{\alpha_s}{4\pi}\right)^2 \Delta\Sigma$$

Can we directly measure quark/gluon OAM at the EIC?
Challenging, but several observables have been proposed.



de Florian, Vogelsang 1902.04636

Gauge invariant completion of JM decomposition

YH (2011) see also Chen et al. 0806.3166

$$\langle PS|\epsilon^{ij}F^{i+}A^{j}_{phys}|PS\rangle = 2S^{+}\Delta G$$

$$\lim_{\Delta\to 0}\langle P'S|\bar{\psi}\gamma^{+}i\overleftrightarrow{D}^{i}_{pure}\psi|PS\rangle = iS^{+}\epsilon^{ij}\Delta_{\perp j}L^{q}_{can}$$

$$\lim_{\Delta\to 0}\langle P'S|F^{+\alpha}\overleftrightarrow{D}^{i}_{pure}A^{phys}_{\alpha}|PS\rangle = -i\epsilon^{ij}\Delta_{\perp j}S^{+}L^{g}_{can}$$

where

$$A^{\mu}_{phys} = -\int_{x^-}^{\infty} dz^- W[x^-,z^-]F^{+\mu}(z^-,x_\perp)$$

$$D^{\mu}_{pure} = D^{\mu} - igA^{\mu}_{phys} \ (= \partial^{\mu} \text{ in the light cone gauge})$$

Wigner distribution in quantum mechanics

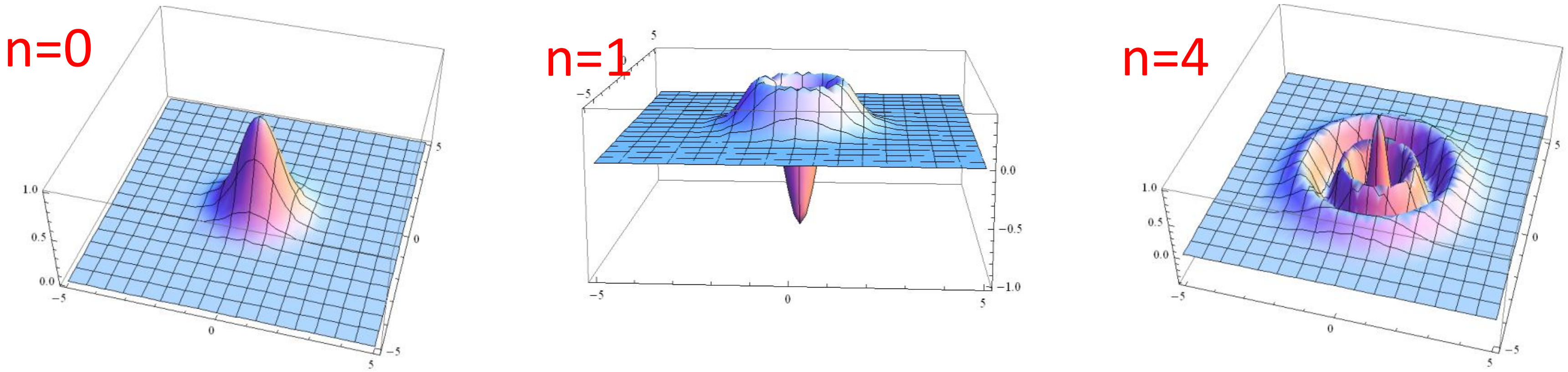
Phase space distribution in QM

$$f_W(q,p) = \int dx e^{-ipx/\hbar} \langle \psi | q - x/2 \rangle \langle q + x/2 | \psi \rangle$$

Reduces to q and p distributions upon integration

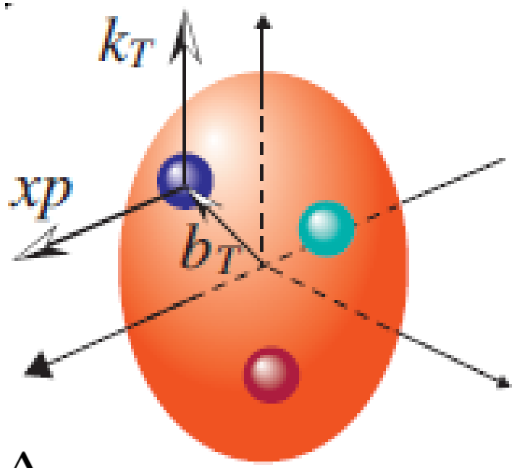
$$\int \frac{dq}{2\pi\hbar} f_W(q,p) = |\langle \psi | p \rangle|^2 , \qquad \int \frac{dp}{2\pi\hbar} f_W(q,p) = |\langle \psi | q \rangle|^2 .$$

Not positive definite, no probabilistic interpretation



n-th excited state of 1D harmonic oscillator

QCD Wigner distribution



Phase space distribution of partons in QCD—the `mother distribution'

Belitsky, Ji, Yuan (2004)

$$W(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$= \int \frac{d^2\Delta_\perp}{(2\pi)^2}\frac{dz^- d^2 z_\perp}{16\pi^3} e^{ixP^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp}\langle P-\frac{\Delta}{2}|\bar{q}(b-z/2)\gamma^+ q(b+z/2)|P+\frac{\Delta}{2}\rangle$$

$\int d\vec{b}_\perp$ $\int d\vec{k}_\perp$

TMD $f(x,\vec{k}_\perp)$ GPD $f(x,\vec{b}_\perp)$

$\int d\vec{k}_\perp$ $\int d\vec{b}_\perp$ $\int dx$

$f(x)$

PDF

$F(\vec{b}_\perp)$ Form factor

$\int dx$ $\int d\vec{b}_\perp$

Q

charge

OAM from the Wigner distribution

Lorce, Pasquini (2011);
YH (2011);

Define $$L^q = \int dx \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^q(x, \vec{b}_\perp, \vec{k}_\perp)$$

Go to momentum space $b_\perp \to \Delta_\perp$ and look for the component

$$W^{q,g} = i\frac{S^+}{P^+}\epsilon^{ij}k_\perp^i \Delta_\perp^j f^{q,g}(x, k_\perp) + \cdots$$

Then $$L^{q,g} = \int dx \int d^2k_\perp k_\perp^2 f^{q,g}(x, k_\perp)$$

Canonical OAM from the light-cone staple Wilson line

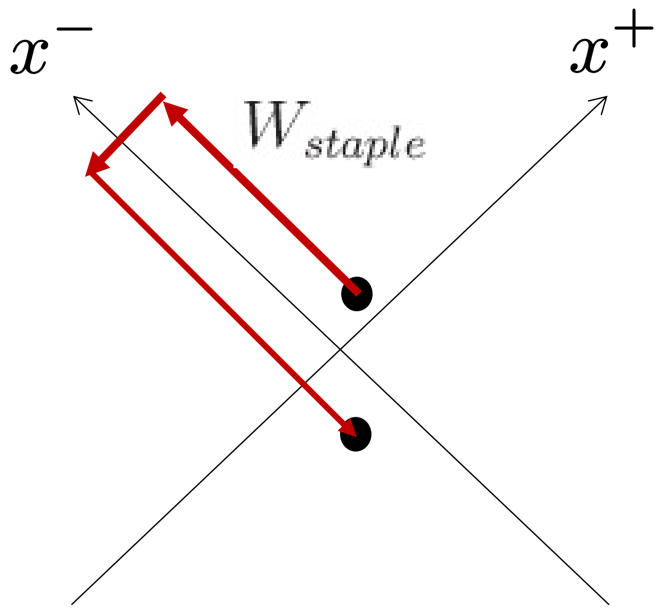
Make the Wigner distribution gauge invariant by attaching a staple-shaped Wilson line

$$W^q \sim \int dz^- d^2 z_\perp e^{ixP^+ z^- - ik_\perp \cdot z_\perp} \langle \bar{\psi}(b - \frac{z}{2})\gamma^+ W_{staple} \psi(b + \frac{z}{2})\rangle$$

The resulting OAM is the gauge invariant canonical OAM

YH (2011)

$$\int d^2 k_\perp (b_\perp \times k_\perp) W^q(b_\perp, k_\perp) = \langle \bar{\psi} b_\perp \times i D_\perp^{pure} \psi \rangle$$



Proof: replace $k_\perp^i \to -i\frac{\partial}{\partial z_\perp^i}$

Wilson line derivative

$$\frac{\partial}{\partial z_\perp^i} W[\infty, z^-, z_\perp]\psi(z^-, z_\perp) = W\partial_{\perp i}\psi(z) - ig\int_{z^-}^{\infty} dx^- W[\infty, x^-]\underbrace{\frac{\partial A^+(x^-, z_\perp)}{\partial z_\perp^i}}_{D^+A_i - F^+{}_i} W[x^-, z^-]\psi(z)$$

Use the trick

$$\int_{z^-}^{\infty} dx^- W[\infty, x^-]D^+A_i(x^-, z_\perp)W[x^-, z^-] = \int_{z^-}^{\infty} dx^- \frac{d}{dx^-}\left(W[\infty, x^-]A_i(x^-)W[x^-, z^-]\right)$$

$$= A_i(\infty)W[\infty, x^-] - W[\infty, x^-]A_i(z^-)$$

Therefore,

$$\frac{\partial}{\partial z_\perp^i} W[\infty, z^-]\psi(z) = W[\infty, z^-]\left(D_{\perp i}\psi + ig\int_{z^-}^{\infty} dx^- W[z^-, x^-]F^+{}_i(x^-)W[x^-, z^-]\psi\right)$$

$$= W[\infty, z^-]D_{\perp i}^{pure}\psi(z)$$

Improved (Belinfante) energy momentum tensor

Return to the canonical angular momentum tensor and write

$$M^{\mu\nu\lambda}_{can} = x^\nu T^{\mu\lambda}_{can} - x^\lambda T^{\mu\nu}_{can} + H^{\mu\nu\lambda}$$

Define $$\tilde{T}^{\mu\nu} = T^{\mu\nu}_{can} + \partial_\rho G^{\rho\mu\nu}$$ ← One can add a total derivative.

where $$G^{\rho\mu\nu} = \frac{1}{2}(H^{\rho\mu\nu} - H^{\mu\rho\nu} - H^{\nu\rho\mu})$$

Exercise: Show that $\tilde{T}^{\mu\nu}$ is symmetric and conserved. Hint: use $\partial_\mu M^{\mu\nu\lambda}_{can} = 0$

Exercise: Show that in QCD, $$\tilde{T}^{\mu\nu} = \bar{\psi} i\gamma^{(\mu} \overleftrightarrow{D}^{\nu)} \psi - F^{\mu\rho}F^{\nu}{}_{\rho} - g^{\mu\nu}\mathcal{L} = \tilde{T}^{\mu\nu}_q + \tilde{T}^{\mu\nu}_g$$

$$A^{(\mu}B^{\nu)} \equiv \frac{A^\mu B^\nu + A^\nu B^\mu}{2}$$

$$\tilde{M}^{\mu\nu\lambda} = x^\nu \tilde{T}^{\mu\lambda} - x^\lambda \tilde{T}^{\mu\nu}$$

Hint: A useful identity

$$\overleftrightarrow{D}^{\mu} = \frac{D^{\mu} - \overleftarrow{D}^{\mu}}{2}$$

$$\overleftarrow{D}^{\mu} = \overleftarrow{\partial}^{\mu} - igA^{\mu}$$

From the Dirac equation $(\not{D} + iM)\psi = \bar{\psi}(\overleftarrow{\not{D}} - iM) = 0$

$$\begin{aligned}
0 &= \bar{\psi}\gamma^{\mu}\gamma^{\nu}(\not{D} + iM)\psi - \bar{\psi}(\overleftarrow{\not{D}} - iM)\gamma^{\nu}\gamma^{\mu}\psi \\
&= \bar{\psi}(g^{\mu\nu}\gamma^{\rho} + g^{\nu\rho}\gamma^{\mu} - g^{\mu\rho}\gamma^{\nu} + i\epsilon^{\mu\nu\rho\sigma}\gamma_{\sigma}\gamma_5)D_{\rho}\psi \\
&\quad -\bar{\psi}\overleftarrow{D}_{\rho}(g^{\rho\nu}\gamma^{\mu} + g^{\nu\mu}\gamma^{\rho} - g^{\rho\mu}\gamma^{\nu} + i\epsilon^{\rho\nu\mu\sigma}\gamma_{\sigma}\gamma_5)\psi + 2iMg^{\mu\nu}\bar{\psi}\psi \\
&= 2\bar{\psi}(\gamma^{\mu}\overleftrightarrow{D}^{\nu} - \gamma^{\nu}\overleftrightarrow{D}^{\mu})\psi + i\epsilon^{\rho\mu\nu\sigma}\partial_{\rho}(\bar{\psi}\gamma_{\sigma}\gamma_5\psi)
\end{aligned}$$

Derivation from general relativity

If you are only interested in the symmetric form, there is a much quicker derivation.
Write down the action in curved space

$$S = \int d^4x \mathcal{L}[\psi, A] \to \int d^4x \sqrt{-g}\mathcal{L}[g^{\mu\nu}, \psi, A]$$

In my convention, $g^{\mu\nu} = (+1, -1, -1, -1)$ in the flat limit.

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g}\mathcal{L} \qquad T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\mathcal{L}$$

beware the sign difference

Don't do this:

$$\bar{\psi}\gamma^\mu \partial_\mu \psi \to \bar{\psi} g^{\mu\nu} \gamma_\mu \partial_\nu \psi \qquad \frac{\delta}{\delta g^{\mu\nu}} \bar{\psi} g^{\mu\nu} \gamma_\mu \partial_\nu \psi = \bar{\psi}\gamma_\mu \partial_\nu \psi$$

Why is this incorrect?

Ji decomposition

Belinfante energy momentum tensor

$$\langle P|J^z_{q,g}|P\rangle = \frac{1}{V}\langle P|\epsilon^{ij}\int d^3x x^i T^{0j}_{q,g}(x)|P\rangle$$
$$= \frac{1}{V}\lim_{P'\to P}\langle P'|\epsilon^{ij}\int d^3x x^i T^{0j}_{q,g}(x)|P\rangle$$
$$= -i\lim_{\Delta\to 0}\epsilon^{ij}\frac{\partial}{\partial\Delta^i}\langle P'|T^{0j}_{q,g}(0)|P\rangle$$

$$\hat{O}(x) = e^{i\hat{P}x}\hat{O}(0)e^{-i\hat{P}x}$$

$$\Delta = P' - P$$
$$\bar{P} = \frac{P+P'}{2}$$

Parametrization (gravitational form factors)

$$\langle P'|T^{\alpha\beta}_q|P\rangle = \bar{u}(P')\left[A_q(t)\gamma^{(\alpha}\bar{P}^{\beta)} + B_q(t)\frac{\bar{P}^{(\alpha}i\sigma^{\beta)\lambda}\Delta_\lambda}{2m_N} + D_q(t)\frac{\Delta^\alpha\Delta^\beta - g^{\alpha\beta}\Delta^2}{4m_N} + \bar{C}_q(t)m_N g^{\alpha\beta}\right]u(P)$$

Use $\bar{u}(P+\Delta)\gamma^i u(P) \approx -i\epsilon^{ijk}\Delta^j\xi'\sigma^k\xi$

$$\Rightarrow \quad \frac{1}{2} = \sum_q J_q + J_g \qquad\qquad J_{q,g} = \frac{1}{2}(A_{q,g}(0) + B_{q,g}(0))$$

$A_q(0), A_g(0)$ Momentum fraction of proton carried by quarks/gluons

$$\sum_q A_q(0) + A_g(0) = 1 \quad \Rightarrow \quad \sum_q B_q(0) + B_g(0) = 0$$

Further decomposition in the quark part (but not in the gluon part)

$$\bar{\psi} i \gamma^{(\mu} \overleftrightarrow{D}^{\nu)} \psi = \bar{\psi} i \gamma^{\mu} \overleftrightarrow{D}^{\nu} \psi - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \partial_\rho (\bar{\psi} \gamma_\sigma \gamma_5 \psi)$$

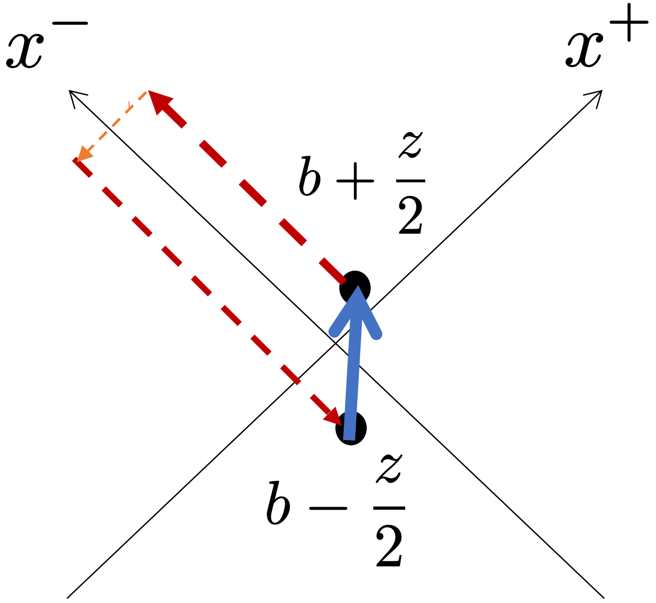
$$J_q = \frac{1}{2} \Delta\Sigma + L^q_{kin}$$ kinetic OAM (features covariant derivative)

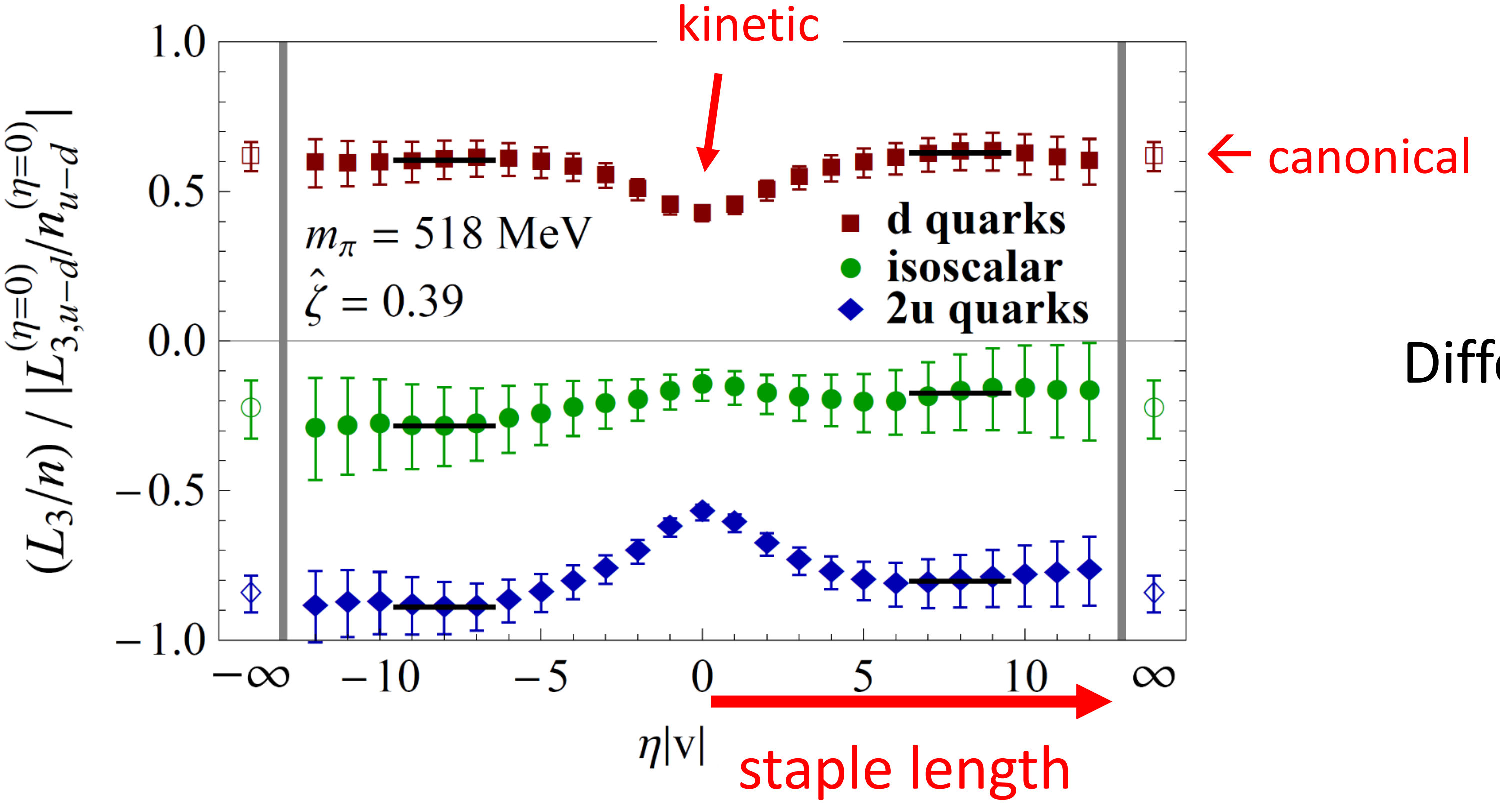
All the operators involved are local and gauge invariant → calculable on a lattice

Kinetic (Ji's) OAM from the straight Wilson line

Ji, Xiong, Yuan (2012)

$$\int d^2k_\perp (b_\perp \times k_\perp) W_{straight}(b_\perp, k_\perp) = \langle \bar{\psi} b_\perp \times iD_\perp \psi \rangle$$





Difference seen in lattice QCD

Engelhardt (2017)

Generalized Parton Distribution

Diehl, hep-ph/0307382
Belitsky, Radyushkin, hep-ph/0504030

Unpolarized

non-forward matrix element $\Delta^\mu = P'^\mu - P^\mu$

$$\bar{P}^+ \int \frac{dz^-}{2\pi} e^{ix\bar{P}^+ z^-} \langle P' | \bar{q}(-z/2)\gamma^+ q(z/2) | P \rangle = \bar{u}(P') \left[\gamma^+ H_q(x,\eta,t) + \frac{i\sigma^{+\nu}\Delta_\nu}{2m_N} E_q(x,\eta,t) \right] u(P)$$

Polarized

$$\bar{P}^+ \int \frac{dz^-}{2\pi} e^{ix\bar{P}^+ z^-} \langle P' | \bar{q}(-z/2)\gamma^+\gamma_5 q(z/2) | P \rangle = \bar{u}(P') \left[\gamma^+\gamma_5 \tilde{H}_q(x,\eta,t) + \frac{\gamma_5 \Delta^+}{2m_N} \tilde{E}_q(x,\eta,t) \right] u(P)$$

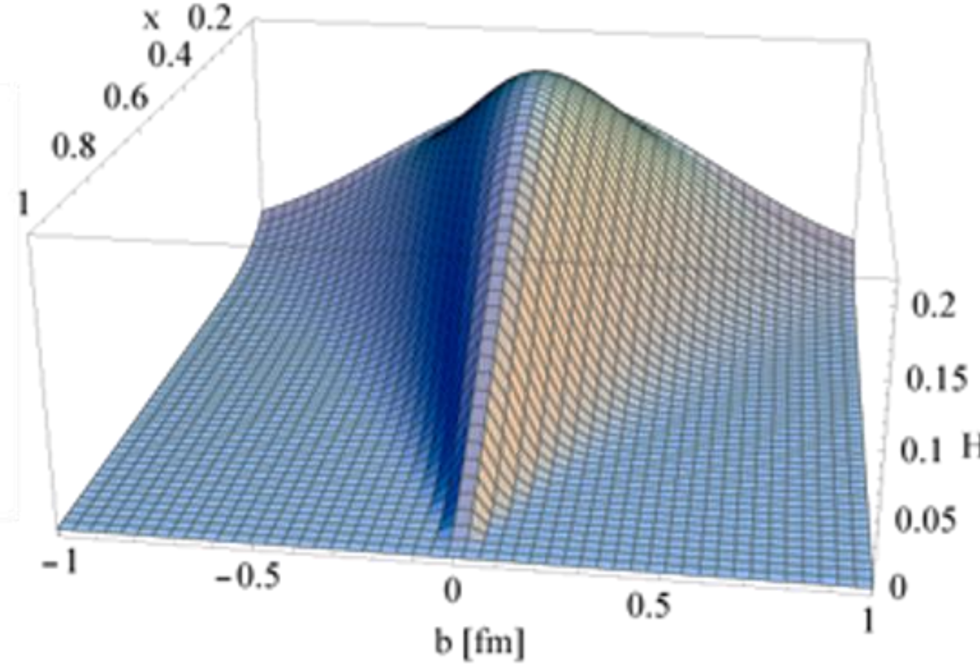
skewness $$\eta = \frac{-\Delta^+}{2\bar{P}^+} = \frac{P^+ - P'^+}{P^+ + P'^+}$$

Different definitions of skewness exist.
They differ by power-suppressed corrections.

$$t = \Delta^2 = -\frac{4\eta^2 M^2}{1-\eta^2} - \frac{\vec{\Delta}_\perp^2}{1-\eta^2}$$

Exercise: Derive this

GPD and 3D tomography



Set $\eta = 0$ and Fourier transform $\Delta_\perp \leftrightarrow b_\perp$

Distribution of quarks in impact parameter $\vec{b}_\perp$ space

$$H_q(x, t = -\vec{\Delta}_\perp^2) \to H_q(x, \vec{b}_\perp)$$

For a transversely polarized nucleon, $\bar{u}\sigma^{+i}u \approx -\frac{2P^+}{m}\epsilon^{ij}S_j$

Get the linear combination

$$H_q(x,t) - \frac{i\epsilon^{ij}\Delta_i S_j}{2m^2}E_q(x,t) \Rightarrow H_q(x,b_\perp) - \frac{\epsilon^{ij}S_j}{2m^2}\frac{\partial}{\partial b_i}E_q(x,b_\perp)$$

deformation in xy plane Burkardt (2002)

Multiply by x and integrate over x .

$$\int_{-1}^{1} dx\, x (P^+)^2 \int \frac{dy^-}{2\pi} e^{ixP^+y^-} \langle \bar{\psi}(0)\gamma^+\psi(y^-)\rangle = \langle P'|\bar{\psi}\gamma^+ iD^+\psi|P\rangle = \langle P'|T_q^{++}|P\rangle$$

Gravitational form factors

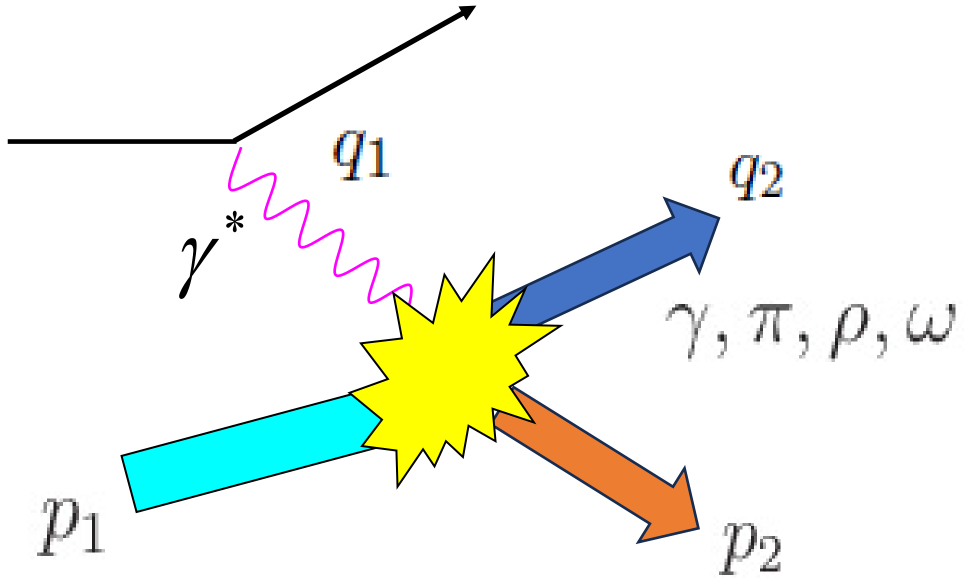
$$\langle P'|T_q^{++}|P\rangle = \bar{u}(P')\left[A_q(t)\gamma^+\bar{P}^+ + B_q(t)\frac{\bar{P}^+ i\sigma^{+\lambda}\Delta_\lambda}{2M} + D_q(t)\frac{(\Delta^+)^2}{4M}\right]u(P)$$

$$A_q(0) = \int dx\, x H_q(x,0,0) \qquad B_q(0) = \int dx\, x E_q(x,0,0)$$

Ji sum rule (GPD version) $$J_q = \frac{1}{2}\int dx\, x (H_q(x,0,0) + E_q(x,0,0))$$

Deeply Virtual Compton Scattering (DVCS)
Deeply Virtual Meson Production (DVMP)

GPD can be probed in exclusive processes
Small cross section, need high luminosity → JLab/EIC



$$q = \frac{q_1 + q_2}{2} \qquad P = \frac{p_1 + p_2}{2} \qquad Q^2 = -q^2$$

$$\Delta = p_2 - p_1 = q_1 - q_2,$$

Bjorken variable $x_B = \frac{-q_1^2}{2p_1 \cdot q_1}$ generalized Bjorken variable $\xi = \frac{Q^2}{2P \cdot q}$

Skewness $\eta = \frac{-\Delta \cdot q}{2P \cdot q}$

Exercise: Show that when $q_2^2 = 0$ (DVCS) and $Q^2 \gg |\Delta^2|$, $\eta \approx \xi \approx \frac{x_B}{2 - x_B}$

Compton amplitude

$$\mathcal{M}^{\mu\nu'}(\xi,\eta,t) \quad = \quad \frac{i}{2\pi}\int d^4y e^{iq\cdot y}\langle p_2|\mathrm{T}\{\bar{\psi}\gamma^\mu\psi(y/2)\bar{\psi}\gamma^\nu\psi(-y/2)\}|p_1\rangle$$

$$\psi(y/2)\bar{\psi}(-y/2) = \frac{i\not{y}}{2\pi^2(y^2-i\epsilon)^2}$$

$$e^{-iq\cdot y}\gamma_\mu\not{y}\gamma_\nu + e^{iq\cdot y}\gamma_\nu\not{y}\gamma_\mu$$
$$= (e^{-iq\cdot y}+e^{iq\cdot y})(g_{\mu\rho}g_{\nu\tau}+g_{\nu\rho}g_{\mu\tau}-g_{\mu\nu}g_{\rho\tau})y^\rho\gamma^\tau + (e^{-iq\cdot y}-e^{iq\cdot y})i\epsilon_{\mu\rho\nu\lambda}y^\rho\gamma^\lambda\gamma_5$$

$$\langle p_2|\bar{\psi}(-y/2)\gamma^\tau\psi(y/2)|p_1\rangle \approx \int dx e^{-ixP\cdot y}\int\frac{P^+dy'^-}{2\pi}e^{ixP^+y'^-}\langle p_2|\bar{\psi}(-y'^-/2)\gamma^\tau\psi(y'^-/2)|p_1\rangle$$

Keep only $\tau=+$ (twist-2)

Do the d^4y integration

$$\int d^4y \frac{y^\rho}{2\pi^2 y^4}(e^{-iq\cdot y}+e^{iq\cdot y})e^{-ixP\cdot y} = \frac{-q^\rho - xP^\rho}{(-q-xP)^2} + \frac{q^\rho - xP^\rho}{(q-xP)^2}$$

$$= \frac{\xi P^\rho - \frac{Q^2}{2\xi}n^\rho - xP^\rho}{-Q^2 + 2xq\cdot P + i\epsilon} + \frac{-\xi P^\rho + \frac{Q^2}{2\xi}n^\rho - xP^\rho}{-Q^2 - 2xq\cdot P + i\epsilon}$$

where I introduced a light-like vector

$$n^\mu = \frac{1}{P^+}\delta^\mu_+ \qquad P\cdot n = 1$$

$$= -\frac{n^\rho}{2}\left(\frac{1}{x-\xi+i\epsilon} + \frac{1}{x+\xi-i\epsilon}\right)$$

Final result (leading order formula)

unpolarized GPD

$$\mathcal{M}^{\mu\nu}(\xi,\eta,t) = -(g^{\mu\rho}g^{\nu-} + g^{\mu-}g^{\nu\rho} - g^{\mu\nu}g^{\rho-})n_\rho \int \frac{dx}{4\pi}\left(\frac{1}{x+\xi-i\epsilon} + \frac{1}{x-\xi+i\epsilon}\right)\bar{u}(p_2)\left[H\gamma^+ + E\frac{i\sigma^{+\sigma}\Delta_\sigma}{2m}\right]u(p_1)$$
$$-i\epsilon^{\mu\nu\rho-}n^\rho \int \frac{dx}{4\pi}\left(\frac{1}{x+\xi-i\epsilon} - \frac{1}{x-\xi+i\epsilon}\right)\bar{u}(p_2)\left[\tilde{H}\gamma^+\gamma_5 + \tilde{E}\frac{\gamma_5\Delta^+}{2m}\right]u(p_1)$$

polarized GPD

State-of-the-art: next-to-next-to-leading order (NNLO) coefficient functions
Braun, Ji, Schoenleber (2022)

GPD challenges

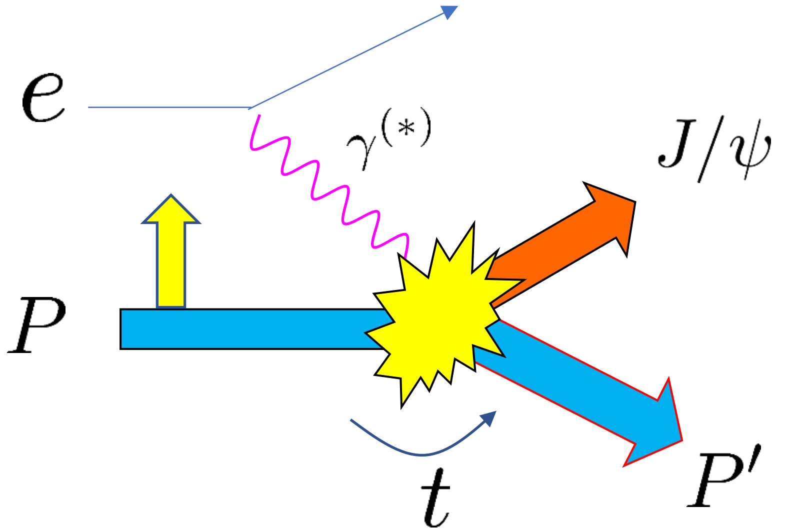
- More variables → more difficult to extract from experiments
- Many GPDs. `Polarized' GPDs contribute to unpolarized processes
- Severe inverse problem. How can one reconstruct $H(x,\xi,t)$ if one only knows

$$\int_{-1}^{1} dx \frac{1}{x-\xi+i\epsilon} H(x,\xi,t)$$

In contrast, PDF is directly related to the structure functions

$$g_1(x) = \frac{1}{2}\sum_f e_f^2(\Delta q_f(x) + \Delta\bar{q}_f(x)) + \cdots$$

- Evolution equation complicated.
- Difficult to access gluon GPDs
- Yet, many recent progress in theory and lattice!



Gluon GPD E from exclusive single spin asymmetry

Form factor

Proton electromagnetic form factors

$$\langle p'|J^{\mu}_{em}|p\rangle = \bar{u}(p')\left[F_1\gamma^{\mu} + \frac{i\sigma^{\mu\rho}\Delta_{\rho}}{2m}(F_2 + i\gamma_5 F_3) + \frac{1}{m^2}(\not{\Delta}\Delta^{\mu} - \Delta^2\gamma^{\mu})\gamma_5 F_a\right]u(p)$$

Total electric charge $F_1(0) = 1$

Anomalous magnetic moment $F_2(0) = \frac{g-2}{2}$

Electric dipole moment (EDM) $\vec{d} = \frac{2F_3(0)}{2m}\vec{s}$ violates P, CP

Anapole moment $\vec{a} = \frac{2F_a}{m^2}\vec{s}$ violates P

Exercise: Derive the relation between $F_3(0)$ and EDM

Solution:

$$
\begin{aligned}
\mathcal{T} &= -\int d^4x \langle P'|J^\mu_{em}(x)A_\mu(x)|P\rangle = -\langle P'|J^\mu_{em}(0)|P\rangle \int d^4x A_\mu(x) e^{i\Delta\cdot x} \\
&\sim -\bar{u}(P')\left[\frac{i\sigma^{\mu\rho}\Delta_\rho}{2m} i\gamma_5 F_3\right] u(P) \int d^4x A_\mu(x) e^{i\Delta\cdot x} \\
&= \frac{F_3}{2m}\bar{u}(P')\sigma^{\mu\rho} i\gamma_5 u(P) \int d^4x \partial_\rho A_\mu(x) e^{i\Delta\cdot x} \\
&\approx -F_3 \frac{V_4}{m^2} P^\mu S^\rho \partial_\rho A_\mu \\
&\approx F_3 \frac{V_4}{m} S_k \partial_k A^0 \\
&= F_3 \frac{V_4}{m} \vec{E}\cdot\vec{S}
\end{aligned}
$$

$$P^\mu = \delta^\mu_0 m, \; S^{\rho'} = \delta^\rho_i S^i$$

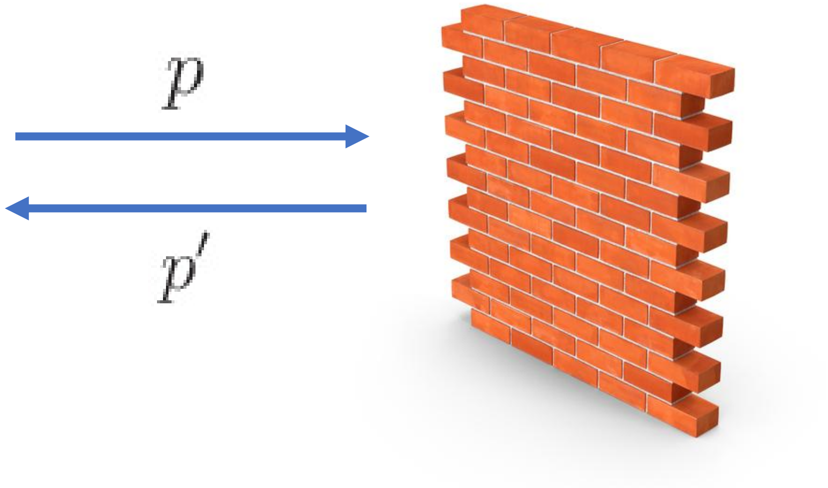
nonrelativistic approximation

$$\vec{S} = 2m\vec{s} \text{ so that } |\vec{s}| = \tfrac{1}{2}$$

Interaction potential

$$V = -\frac{\mathcal{T}}{2mV_4} = -\vec{E}\cdot\vec{d}$$

Breit (brick wall) frame



$$p^\mu = \left(\sqrt{m^2 + \frac{\vec{\Delta}^2}{4}}, \frac{\vec{\Delta}}{2}\right) \qquad p'^\mu = \left(\sqrt{m^2 + \frac{\vec{\Delta}^2}{4}}, -\frac{\vec{\Delta}}{2}\right)$$

In this frame,

$$\begin{aligned} \bar{u}(p's')\gamma^\mu u(ps) &= \frac{1}{2(p^0+m)}(\xi'^\dagger, \xi'^\dagger)\left(p^0\gamma^0 - \frac{\vec{\Delta}\cdot\vec{\gamma}}{2} + m\right)\gamma^\mu\left(p^0\gamma^0 + \frac{\vec{\Delta}\cdot\vec{\gamma}}{2} + m\right)\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\ &= \left(2m\xi'^\dagger\xi, i\xi'^\dagger\vec{\sigma}\xi \times \vec{\Delta}\right) \end{aligned}$$

$$\bar{u}(p's')u(ps) = \sqrt{4m^2 - t}\,\xi'^\dagger\xi$$

Electric & magnetic form factors

Exercise: Show that, in the Breit frame,

$$\langle p's'|J^0(0)|ps\rangle \quad = \quad 2m\left(F_1 - \frac{\vec{\Delta}^2}{4m^2}F_2\right)\xi'\xi$$

$$\langle p'|J^i(0)|p\rangle \quad = \quad i(\xi'\vec{\sigma}\xi \times \vec{\Delta})^i(F_1 + F_2)$$

Electric form factor $\quad G_E(t) = F_1(t) + \frac{t}{4M^2}F_2(t)$

Magnetic form factor $\quad G_M(t) = F_1(t) + F_2(t)$

Proton charge radius

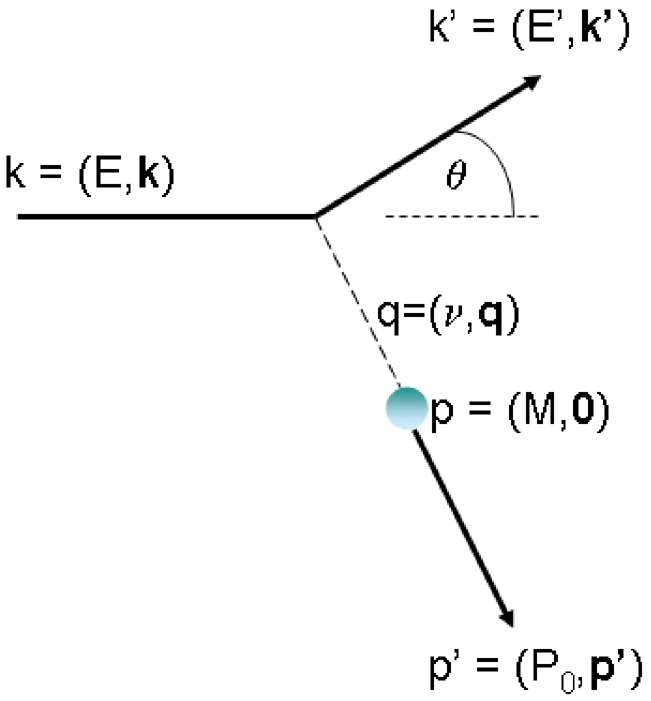
$$G_E(t=-\vec{k}^2) = \int d^3\vec{r}\rho(r)e^{i\vec{k}\cdot\vec{r}} \quad = \quad \int d^3\vec{r}\rho(r)\left(1 + i\vec{k}\cdot\vec{r} - \frac{1}{2}r^2\vec{k}^2\cos^2\theta + \cdots\right)$$

Electric charge density

Proton charge radius (squared) $\langle r^2 \rangle = \int d^3r\rho(r)r^2 = 6\frac{dG_E(t)}{dt}\bigg|_{t=0}$

Caveat: This is just one definition of radius.
A definition tied to the Breit frame.

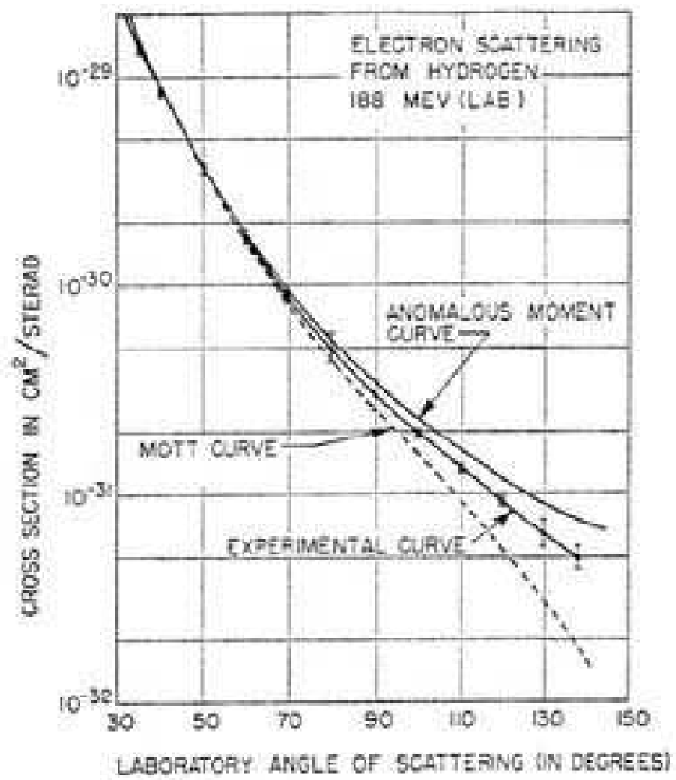
Charge radius from electron scattering (1950s~)



Rosenbluth formula

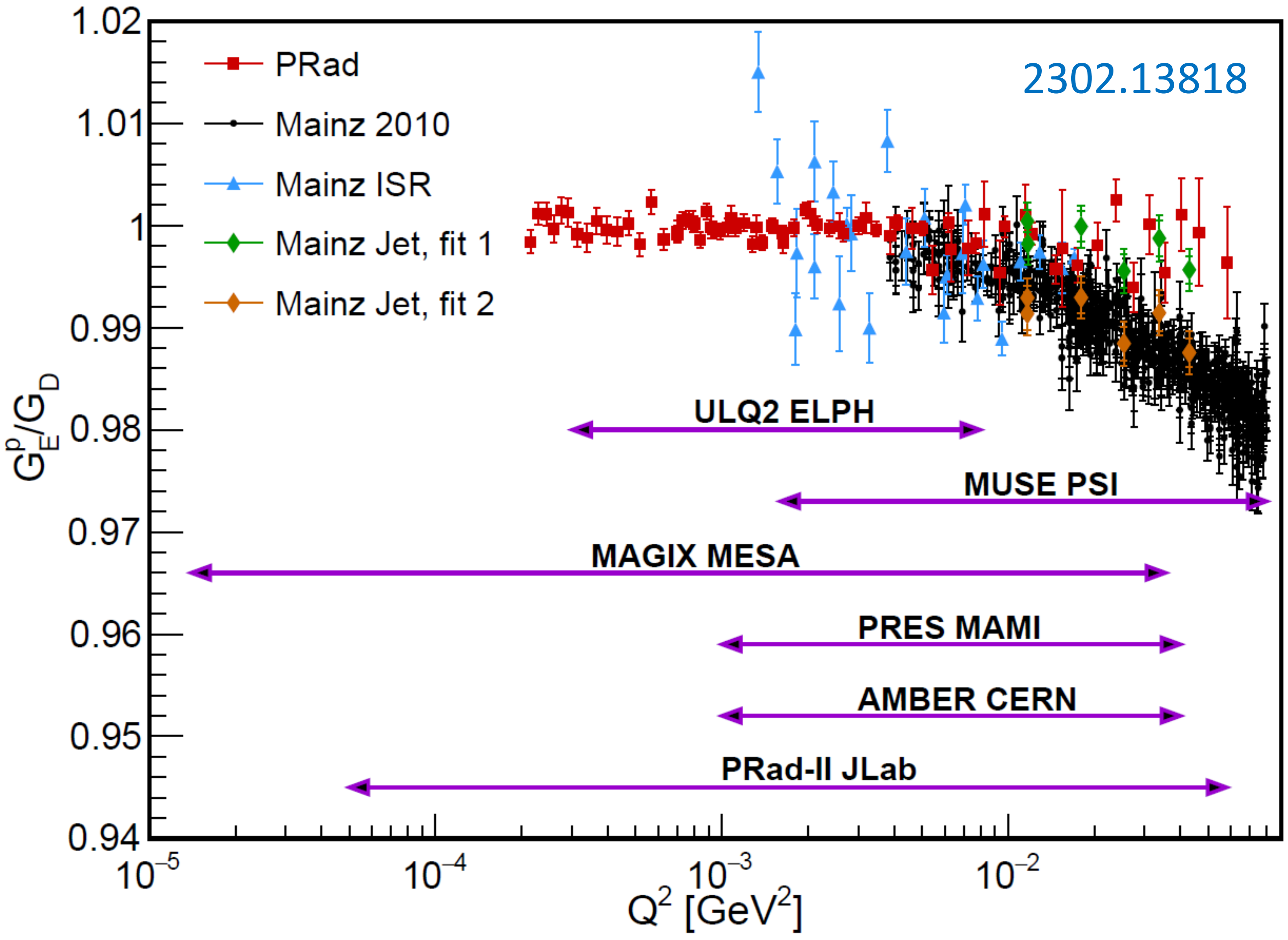
$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2E^2\left[1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right]\sin^4\frac{\theta}{2}}$$

$$\langle r^2\rangle = 6\frac{dG_E(t)}{dt}\bigg|_{t=0} \sim (0.8\ \mathrm{fm})^2$$

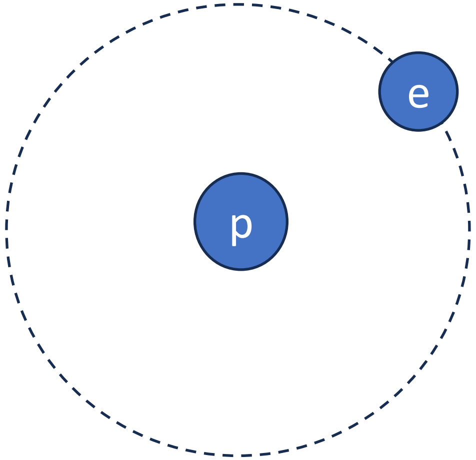


Hofstadter

Elastic scattering 70 years later



Charge radius from hydrogen atom spectrum



Coulomb potential modified in a hydrogen atom

$$V(r) = \frac{-e^2}{4\pi r} \quad \Rightarrow \quad -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{G_E(k^2) e^{-i\vec{k}\cdot\vec{r}}}{k^2}$$

$$\delta V(r) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(G_E(k^2) - 1) e^{-i\vec{k}\cdot\vec{r}}}{k^2} \approx \frac{4\pi\alpha}{6} \langle r^2 \rangle \delta^{(3)}(\vec{r})$$

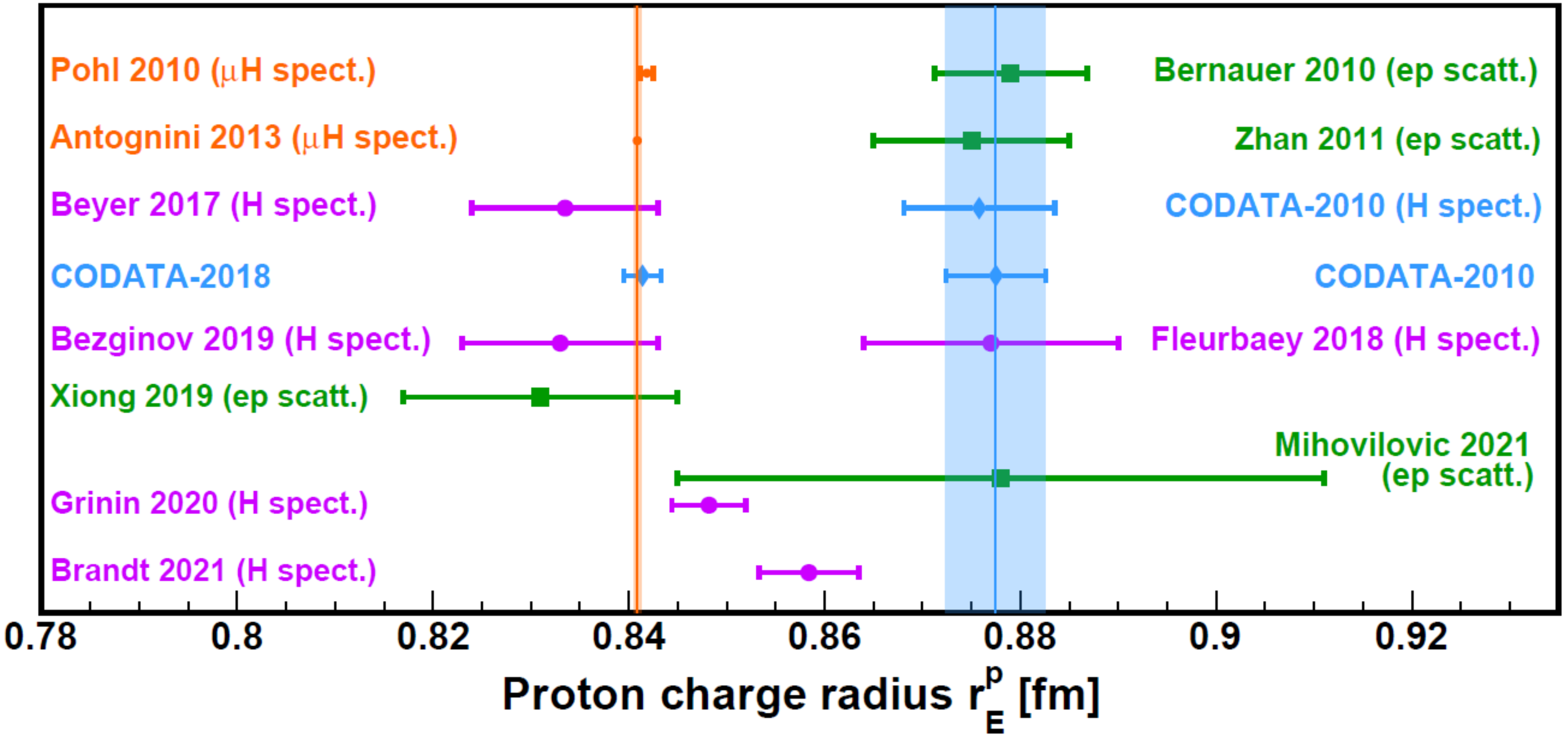
Proton charge radius!

$$\Delta E = \int d^3r \psi^*(r) \delta V(r) \psi(r) = \frac{4\pi\alpha}{6} \langle r^2 \rangle |\psi(0)|^2 = \frac{2\alpha^4 m^3}{3n^3} \langle r^2 \rangle$$

Part of the Lamb shift.

Enhanced in the muonic hydrogen! $m_\mu \approx 200 m_e$

Proton radius puzzle?



PRad (2019) $$r_p = 0.831 \pm 0.007_{\text{stat}} \pm 0.012_{\text{syst}}$$

Both CODATA and PDG now recommend the smaller value ~0.84fm.

Gravitational form factors

QCD energy momentum tensor $$T^{\mu\nu} = \sum_f \bar{\psi}_f \gamma^{(\mu} iD^{\nu)} \psi_f - F^{\mu\rho} F^{\nu}{}_{\rho} + \frac{g^{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}$$

Forward matrix element $\langle P | T^{\mu\nu} | P \rangle = 2P^\mu P^\nu$

Frequently asked question: Why isn't there a term proportional to $g^{\mu\nu}$?

Nonforward matrix element

$$\langle P' | T^{\mu\nu} | P \rangle = \bar{u}(P') \left[A(t) \gamma^{(\mu} \bar{P}^{\nu)} + B(t) \frac{\bar{P}^{(\mu} i\sigma^{\nu)\alpha} \Delta_\alpha}{2M} + D(t) \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{4M} \right] u(P)$$

Take the trace $$\langle P' | T^\mu_\mu | P \rangle = M \left(A(t) + \frac{B(t)}{4M^2} t - \frac{3D(t)}{4M^2} t \right) \bar{u}(P') u(P)$$

$A(0) = 1, B(0) = 0$ but $D(0)$ is unconstrained (and unknown).

Radius zoo

2312.12984

Charge radius

$$\langle r^2\rangle_c = \frac{\int d\boldsymbol{x} x^2 \rho_c(\boldsymbol{x})}{\int d\boldsymbol{x} \rho_c(\boldsymbol{x})} = \frac{6}{G_E(0)} \frac{dG_E(t)}{dt}\bigg|_{t=0}$$

Magnetic radius

$$\langle r^2\rangle_M = \frac{6}{G_M(0)} \frac{dG_M(t)}{dt}\bigg|_{t=0}$$

Baryon number radius

$$\langle r^2\rangle_B = \frac{\int d\boldsymbol{x} x^2 \rho_B(\boldsymbol{x})}{\int d\boldsymbol{x} \rho_B(\boldsymbol{x})}$$

Mass radius

$$\langle r^2\rangle_m = \frac{\int d\boldsymbol{x}\, x^2 T^{00}(\boldsymbol{x})}{\int d\boldsymbol{x} T^{00}(\boldsymbol{x})} = 6\frac{dA(t)}{dt}\bigg|_{t=0} - \frac{3D(0)}{2M^2}$$

Scalar radius

$$\langle r^2\rangle_s = \frac{\int d\boldsymbol{x}\, x^2 T^{\mu}_{\mu}(\boldsymbol{x})}{\int d\boldsymbol{x} T^{\mu}_{\mu}(\boldsymbol{x})} = 6\frac{dA(t)}{dt}\bigg|_{t=0} - \frac{9D(0)}{2M^2}$$

Tensor radius

$$\langle r^2\rangle_t \equiv \frac{\int d\boldsymbol{x}\, x^2 \left(T^{00}(\boldsymbol{x}) + \frac{1}{2}T_{ii}(\boldsymbol{x})\right)}{\int d\boldsymbol{x}\, \left(T^{00} + \frac{1}{2}T_{ii}\right)} = 6\frac{dA(t)}{dt}\bigg|_{t=0}$$

Mechanical radius

$$\langle r^2\rangle_{mech} = \frac{\int d\boldsymbol{x} x^2 \frac{x_i x_j}{x^2} T_{ij}(\boldsymbol{x})}{\int d\boldsymbol{x} \frac{x_i x_j}{x^2} T_{ij}(\boldsymbol{x})} = \frac{6D(0)}{\int_{-\infty}^0 dt D(t)}$$

…

Quark and gluon components

Energy momentum tensor consists of quark and gluon parts

$$T^{\mu\nu} = \underbrace{-F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{\eta^{\mu\nu}}{4}F^2}_{T_g^{\mu\nu}} + \underbrace{i\bar{q}\gamma^{(\mu}D^{\nu)}q}_{T_q^{\mu\nu}}$$

$$\langle P'|T^{\mu\nu}_{q,g}|P\rangle = \bar{u}(P')\Big[A_{q,g}\gamma^{(\mu}\bar{P}^{\nu)} + B_{q,g}\frac{\bar{P}^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{4M} + \bar{C}_{q,g}M\eta^{\mu\nu}\Big]u(P)$$

4th form factor

$$\sum_q \bar{C}_q(t) + \bar{C}_g(t) = 0$$

because the total energy momentum tensor is conserved.

GPD and form factors

Form factors are moments of GPDs

$$\int_{-1}^{1} dx H_q(x,\eta,t) = F_1^q(t) \qquad \int_{-1}^{1} dx E_q(x,\eta,t) = F_2^q(t)$$

independent of η (why?)

From Gordon identity,

$$\begin{aligned}\langle P'|T_q^{++}|P\rangle &= \bar{P}^+\bar{u}(P')\left[A_q(t)\gamma^+ + B_q(t)\frac{i\sigma^{+\lambda}\Delta_\lambda}{2m_N} + D_q(t)\frac{(\Delta^+)^2}{4m_N\bar{P}^+}\right]u(P) \\ &= \bar{P}^+\bar{u}(P')\left[\left(A_q(t)+\eta^2 D_q(t)\right)\gamma^+ + \left(B_q(t)-\eta^2 D_q(t)\right)\frac{i\sigma^{+\lambda}\Delta_\lambda}{2m_N}\right]u(P)\end{aligned}$$

$$\int_{-1}^{1} dx\, x H_q(x,\eta,t) = A_2^q(t) + \eta^2 D^q(t) \qquad \int_{-1}^{1} dx\, x E_q(x,\eta,t) = B_2^q(t) - \eta^2 D^q(t)$$

Appendix: Gordon identities

For any matrix Γ in Dirac space,

$$\bar{u}(p')\Gamma u(p) = \frac{1}{2m}\bar{u}'\left(\{\not{P},\Gamma\} + \frac{1}{2}[\not{\Delta},\Gamma]\right)u(p)$$

$P = \frac{p+p'}{2}$, $\Delta = p' - p$, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$, $\xi = -\Delta^+/2P^+$

From this one can derive

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{P^\mu}{m} + \frac{i\sigma^{\mu\nu}\Delta_\nu}{2m}\right]u(p)$$

$$\bar{u}(p')\gamma^\mu\gamma_5 u(p) = \bar{u}(p')\left[\frac{\Delta^\mu\gamma_5}{2m} + \frac{i\sigma^{\mu\nu}P_\nu\gamma_5}{m}\right]u(p)$$

$$\frac{\Delta^\mu}{2}\bar{u}(p')u(p) = -\bar{u}(p')i\sigma^{\mu\nu}P_\nu u(p)$$

$$\bar{u}(p')P^\mu\gamma_5 u(p) = -\bar{u}(p')\frac{i\sigma^{\mu\nu}\Delta_\nu}{2}\gamma_5 u(p)$$

In particular,

$$\bar{u}(p')i\sigma^{+-}u(p) = -\frac{\Delta^+}{2P^+}\bar{u}(p')u(p) = \xi\bar{u}(p')u(p)$$

D-term—the last global unknown

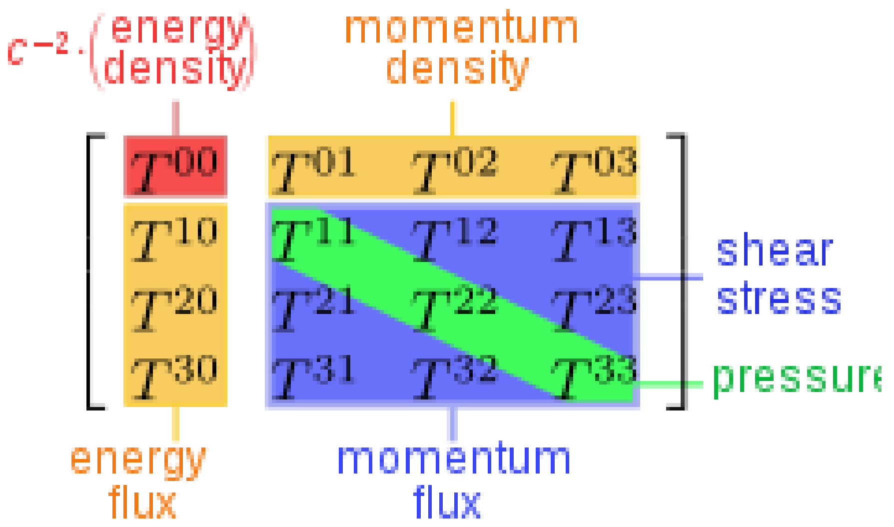
$$\langle P'|T^{ij}|P\rangle \sim (\Delta^i\Delta^j - \Delta^2\delta^{ij})D(t)$$

$D(t=0)$ is a conserved charge of the nucleon, similar to the magnetic moment

Fourier transform $\vec{\Delta} \to \vec{r}$ can be interpreted as `pressure' inside a nucleon Polyakov (2003)

$$T^{ij}(r) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right)s(r) + \delta^{ij}p(r)$$

$$p(r) = \frac{1}{6M}\int \frac{d\boldsymbol{\Delta}}{(2\pi)^3} e^{i\boldsymbol{\Delta}\cdot\boldsymbol{r}} tD(t) \qquad D = M\int d^3r r^2 p(r)$$

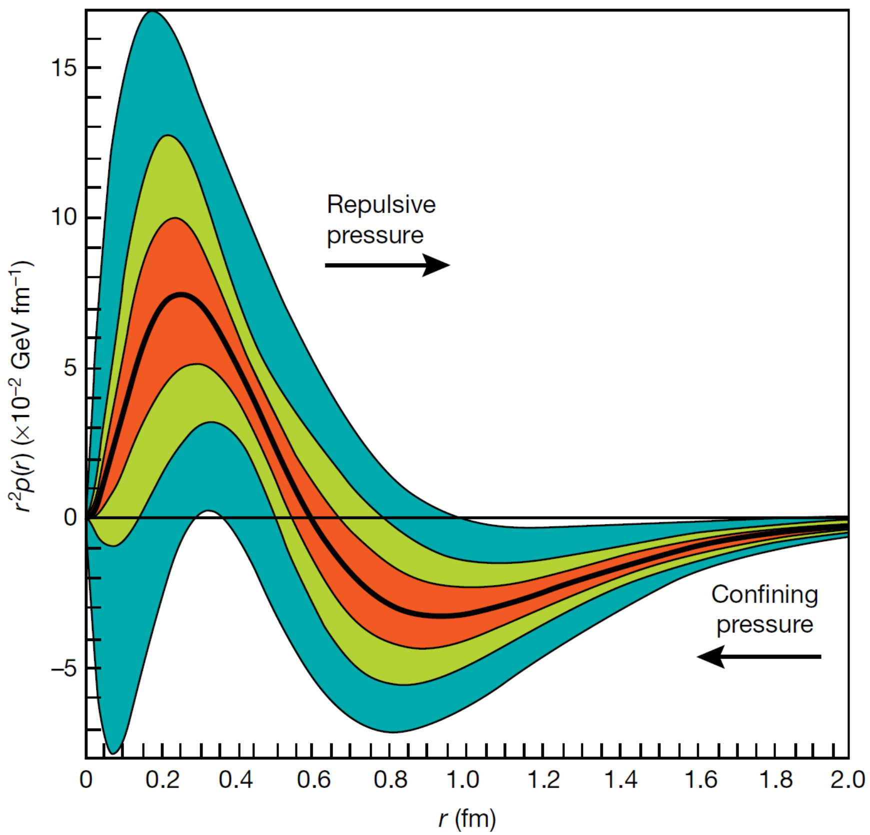


Conjecture: All stable hadrons must have $D < 0$

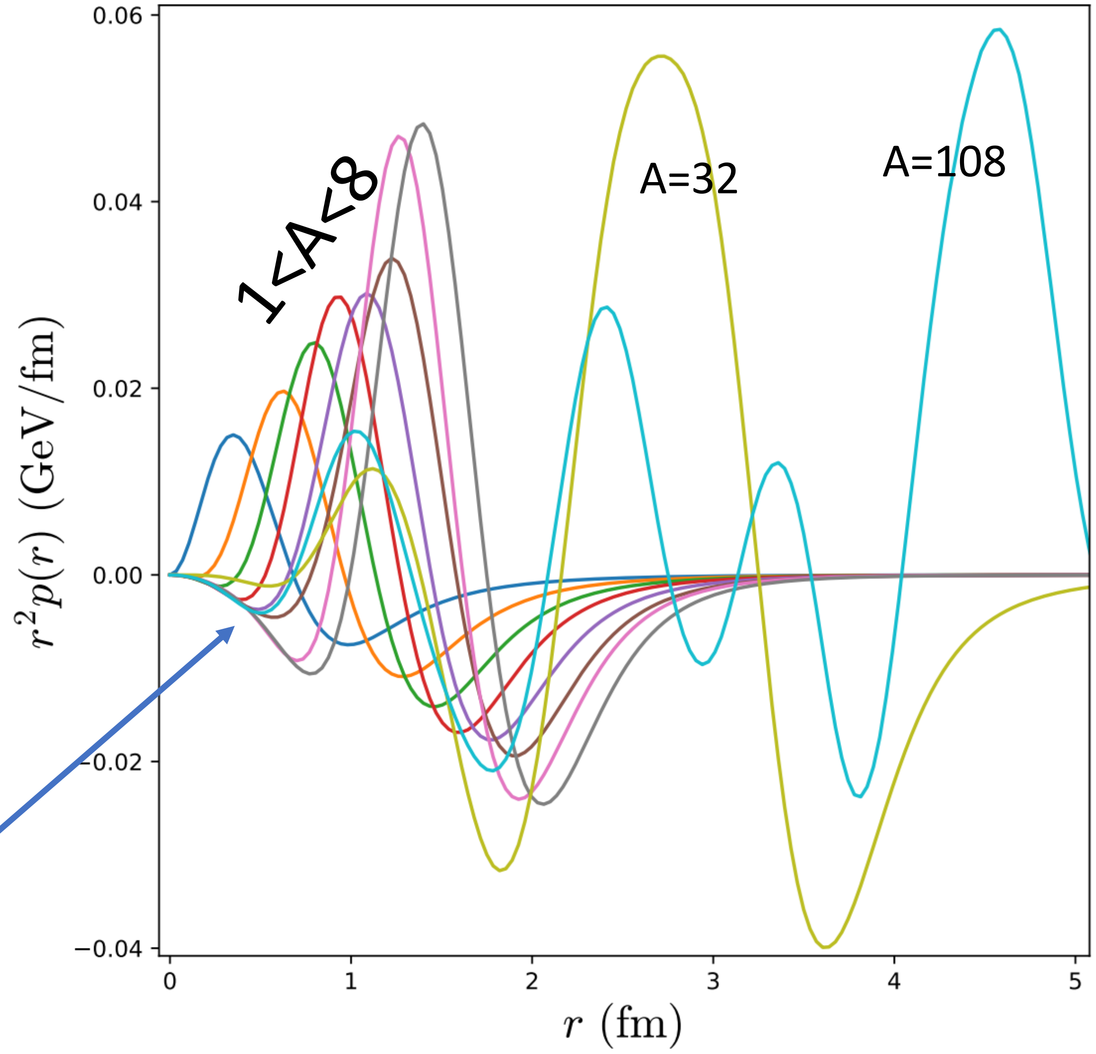
Analogy with a continuous medium should be taken with a grain of salt.

`Pressure' inside nucleon and nuclei

Burkert, Elouadrhiri, Girod (2018)



Martin-Caro, Huidobro, YH, 2312.12984



Negative pressure near the core for nuclei A>1
see also, Freese, Cosyn (2022), He, Zahed (2023)

Pion GFFs

$$P^\mu = \frac{p^\mu + p'^\mu}{2}, \quad \Delta^\mu = p'^\mu - p^\mu$$

Spin-0 hadron → 2 GFFs $\quad \langle p'|T^{\mu\nu}|p\rangle = 2A(t)P^\mu P^\nu + \frac{D(t)}{2}(\Delta^\mu\Delta^\nu - \Delta^2 g^{\mu\nu})$

In the chiral limit of QCD, $\quad D(0) = -1$

Proof: take the limit $p'^\mu \to 0$.

Then $\quad \Delta^\mu = -p^\mu, \quad P^\mu = \frac{p^\mu}{2}, \quad t = \Delta^2 = 0$

Right hand side becomes $\quad \frac{p^\mu p^\nu}{2}(A(0) + D(0))$

Left hand side vanishes due to soft pion theorem

$$\lim_{p'\to 0}\langle \pi_a(p')|T^{\mu\nu}|p\rangle = \frac{i}{f_\pi}\langle 0|[Q_5^a, T^{\mu\nu}]|p\rangle = 0$$

$$Q_5^a = \int d^3x \bar{\psi}\gamma^0\gamma_5 \frac{\tau^a}{2}\psi$$ generator of chiral rotation

$$D(0) = -A(0) = -1$$

In real QCD with massive pion $D = -1 + \mathcal{O}\left(\frac{m_\pi^2}{f_\pi^2}\right)$

Soft pion theorem: A quick derivation

Pion decay constant definition $\langle 0|J^{\mu}_{5a}(x)|\pi_b(p)\rangle = -if_\pi p^\mu e^{-ip\cdot x}\delta_{ab}$

Take the divergence $\langle 0|\partial_\mu J^{\mu}_{5a}|\pi_b(p)\rangle = -f_\pi m_\pi^2\delta_{ab}$

Pion interpolating field $\pi_a \sim -\dfrac{1}{f_\pi m_\pi^2}\partial_\mu J^\mu_{5a}$

LSZ reduction formula

$$\begin{aligned}\langle f,\pi_a(p)|\mathcal{O}(0)|i\rangle &= i\int d^4x e^{ip\cdot x}(\Box_x + m_\pi^2)\langle f|\mathrm{T}\{\pi_a(x)\mathcal{O}(0)\}|i\rangle \\ &= \frac{-i}{f_\pi m_\pi^2}\int d^4x e^{ip\cdot x}(\Box_x + m_\pi^2)\Big(\partial^x_\mu\langle f|\mathrm{T}\{J^\mu_{5a}(x)\mathcal{O}(0)\}|i\rangle - \delta(x^0)\langle f|[J^0_{5a}(x),\mathcal{O}(0)]|i\rangle\Big)\end{aligned}$$

Take the limit $p^\mu \to 0$

Can we measure GFFs in experiments?

- Introduced theoretically in the 60s.
- Received far less attention than EM form factors, not because they are less interesting/important.
- The obvious reason: We cannot measure them directly!

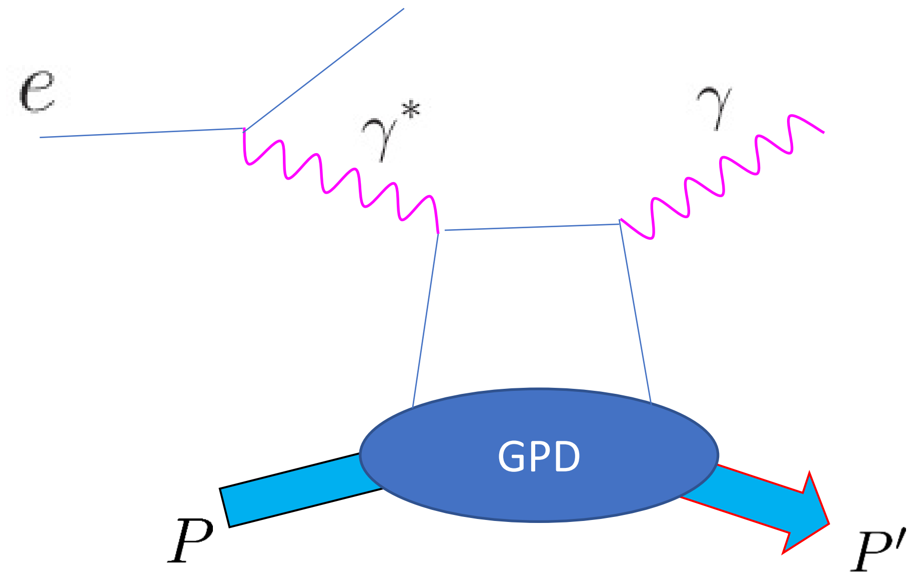
One-graviton exchange cross section $$\frac{d\sigma}{dt} \sim G_N^2 \frac{s^2}{t^2}$$

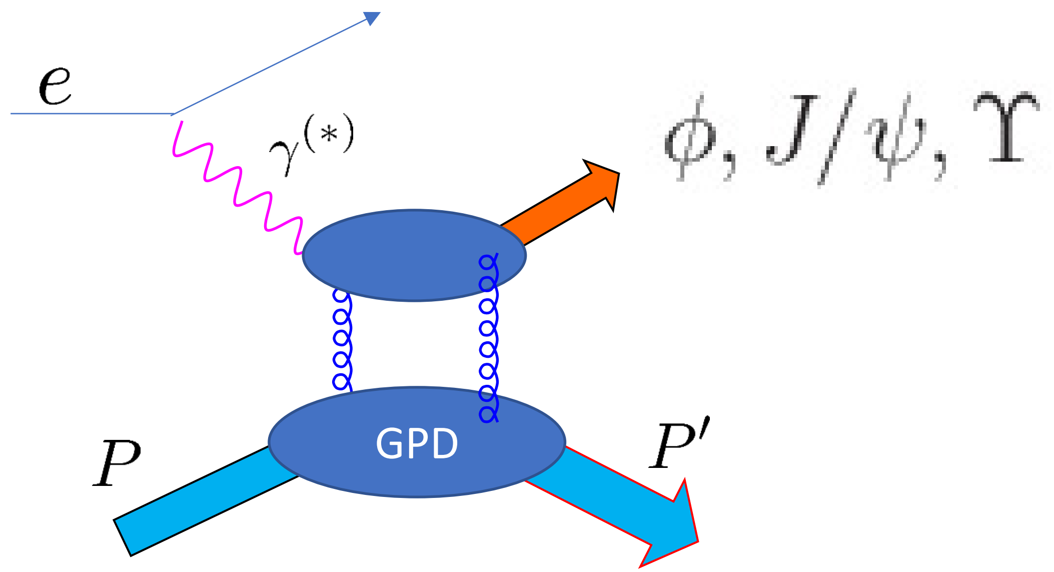
Newton constant $G_N \sim 1/M_P^2$ Planck mass $M_P \sim 10^{19}\ \mathrm{GeV}$

- There are, however, indirect ways to measure them.

1 graviton $\approx$ 2 photons or 2 gluons ?

1+1=2





Processes like DVCS and heavy meson production involve two photons/gluons
→ Can mimic a spin-2 graviton exchange!?

In reality, two-photon/gluon state can couple to operators with arbitrary spins.
Extraction of GFFs highly nontrivial. Active area of research now.

Pion GFFs at the EIC and Jlab

YH, Schoenleber, 2502.12061

Realize a pion beam in Sullivan process

Produce a heavy vector meson near threshold.
Strong sensitivity to the pion GFFs