Necessity for orbital angular momentum

Polarized DGLAP evolution

$$\frac{d}{d\ln Q^2}\begin{pmatrix}\Delta\Sigma(x)\\ \Delta G(x)\end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dz}{z}\begin{pmatrix}\Delta P_{qq}(z) & \Delta P_{qg}(z)\\ \Delta P_{gq}(z) & \Delta P_{gg}(z)\end{pmatrix}\begin{pmatrix}\Delta\Sigma(x/z)\\ \Delta G(x/z)\end{pmatrix}$$

$$\Delta P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right),$$

$$\Delta P_{qg}(z) = n_f(2z-1),$$

$$\Delta P_{gq}(z) = C_F(2-z),$$

$$\Delta P_{gg}(z) = 6\left(\frac{1}{(1-z)_+}-2z+1\right)+\frac{\beta_0}{2}\delta(z-1)$$

Integrate over x

$$\frac{d}{d\ln Q^2}\left(\frac{1}{2}\Delta\Sigma(Q^2)+\Delta G(Q^2)\right)\neq 0$$

Helicity is not a conserved quantity!

Angular momentum conservation

Only the sum of helicity and OAM is conserved.

$$\frac{d}{d\ln Q^2}\left(\frac{1}{2}\Delta\Sigma(Q^2)+\Delta G(Q^2)+L_q(Q^2)+L_g(Q^2)\right)=0$$

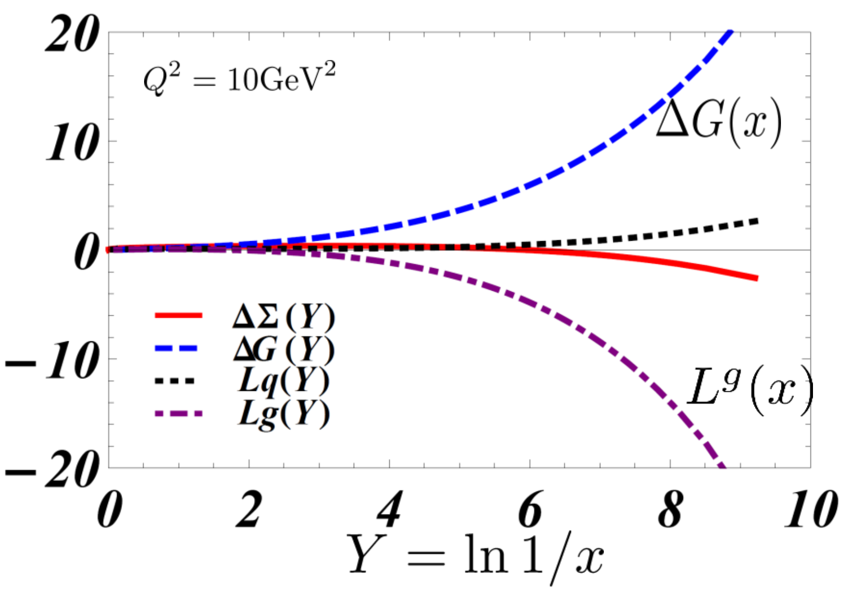
In practice, the scale dependence of $\Delta\Sigma$ very weak (starts at 2-loop in perturbation theory)

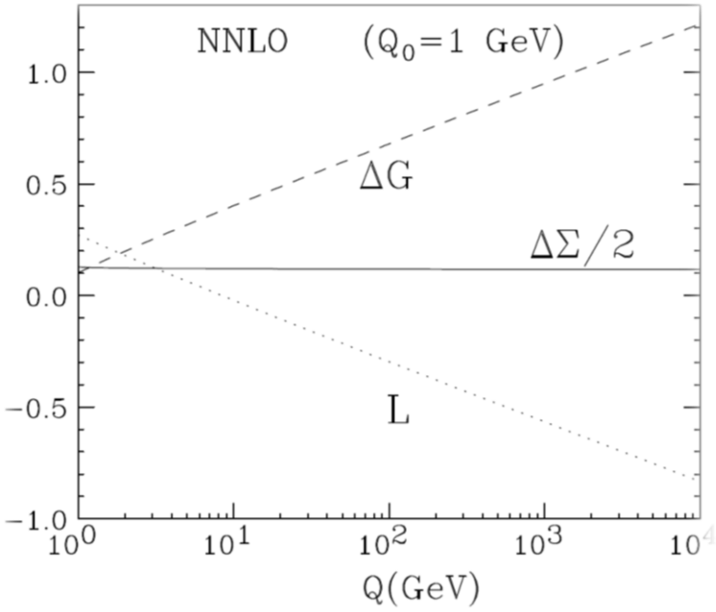
Kodaira (1980)

$$\frac{d}{d\ln Q^2}\Delta\Sigma = -12C_F T_F n_f\left(\frac{\alpha_s}{4\pi}\right)^2\Delta\Sigma$$

Can we directly measure quark/gluon OAM at the EIC?
Challenging, but several observables have been proposed.

YH, Yang 1802.02716





de Florian, Vogelsang 1902.04636

Gauge invariant completion of JM decomposition

YH (2011) see also Chen et al. 0806.3166

$$\langle PS|\epsilon^{ij}F^{i+}A^j_{phys}|PS\rangle = 2S^+\Delta G$$

$$\lim_{\Delta\to 0}\langle P'S|\bar{\psi}\gamma^+ i\overleftrightarrow{D}^i_{pure}\psi|PS\rangle = iS^+\epsilon^{ij}\Delta_{\perp j}L^q_{can}$$

$$\lim_{\Delta\to 0}\langle P'S|F^{+\alpha}\overleftrightarrow{D}^i_{pure}A^{phys}_\alpha|PS\rangle = -i\epsilon^{ij}\Delta_{\perp j}S^+L^g_{can}$$

where

$$A^\mu_{phys} = -\int_{x^-}^{\infty} dz^- W[x^-,z^-]F^{+\mu}(z^-,x_\perp)$$

$$D^\mu_{pure} = D^\mu - igA^\mu_{phys} \ (= \partial^\mu \text{ in the light cone gauge})$$

Wigner distribution in quantum mechanics

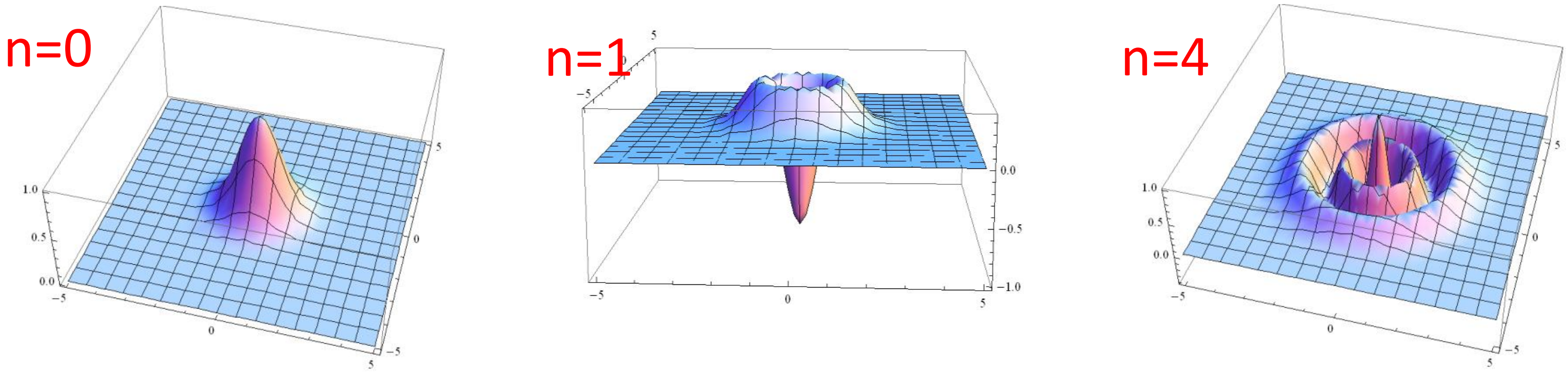
Phase space distribution in QM

$$f_W(q,p) = \int dx e^{-ipx/\hbar} \langle \psi | q - x/2 \rangle \langle q + x/2 | \psi \rangle$$

Reduces to q and p distributions upon integration

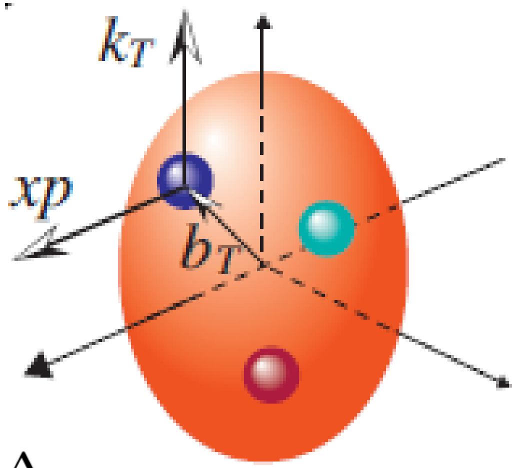
$$\int \frac{dq}{2\pi\hbar} f_W(q,p) = |\langle \psi | p \rangle|^2 \,, \qquad \int \frac{dp}{2\pi\hbar} f_W(q,p) = |\langle \psi | q \rangle|^2 \,.$$

Not positive definite, no probabilistic interpretation



n-th excited state of 1D harmonic oscillator

QCD Wigner distribution



Phase space distribution of partons in QCD—the `mother distribution'

Belitsky, Ji, Yuan (2004)

$$W(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$=\int \frac{d^2\Delta_\perp}{(2\pi)^2}\frac{dz^- d^2 z_\perp}{16\pi^3} e^{ixP^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp}\langle P-\frac{\Delta}{2}|\bar{q}(b-z/2)\gamma^+ q(b+z/2)|P+\frac{\Delta}{2}\rangle$$

$\int d\vec{b}_\perp$

$\int d\vec{k}_\perp$

TMD $f(x,\vec{k}_\perp)$

GPD $f(x,\vec{b}_\perp)$

$\int d\vec{k}_\perp$

$\int d\vec{b}_\perp$

$\int dx$

$f(x)$

PDF

$F(\vec{b}_\perp)$ Form factor

$\int dx$

Q

charge

$\int d\vec{b}_\perp$

OAM from the Wigner distribution

Lorce, Pasquini (2011);
YH (2011);

Define $$L^q = \int dx \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^q(x, \vec{b}_\perp, \vec{k}_\perp)$$

Go to momentum space $b_\perp \to \Delta_\perp$ and look for the component

$$W^{q,g} = i\frac{S^+}{P^+}\epsilon^{ij} k_\perp^i \Delta_\perp^j f^{q,g}(x, k_\perp) + \cdots$$

Then $$L^{q,g} = \int dx \int d^2k_\perp k_\perp^2 f^{q,g}(x, k_\perp)$$

Canonical OAM from the light-cone staple Wilson line

Make the Wigner distribution gauge invariant by attaching a staple-shaped Wilson line

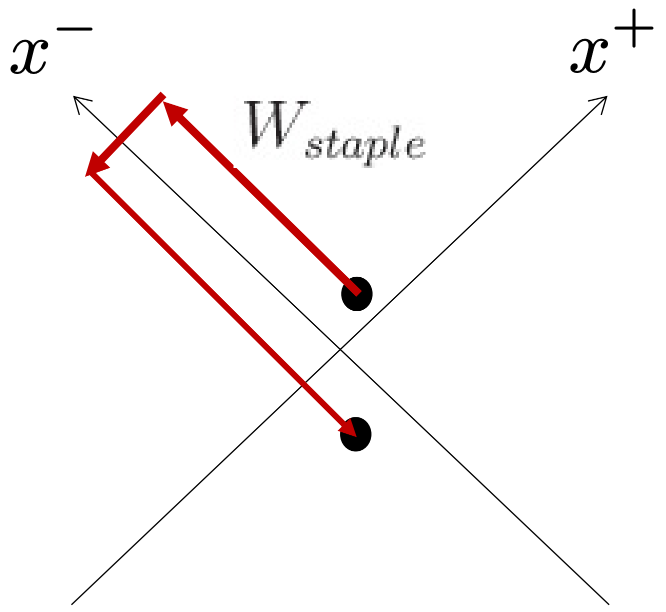
$$W^q \sim \int dz^- d^2 z_\perp e^{ixP^+ z^- - ik_\perp \cdot z_\perp} \langle \bar{\psi}(b - \frac{z}{2}) \gamma^+ W_{staple} \psi(b + \frac{z}{2}) \rangle$$

The resulting OAM is the gauge invariant canonical OAM

YH (2011)

$$\int d^2 k_\perp (b_\perp \times k_\perp) W^q(b_\perp, k_\perp) = \langle \bar{\psi} b_\perp \times i D_\perp^{pure} \psi \rangle$$

Proof: replace $k_\perp^i \to -i \frac{\partial}{\partial z_\perp^i}$



Wilson line derivative

$$\frac{\partial}{\partial z_\perp^i} W[\infty, z^-, z_\perp]\psi(z^-, z_\perp) = W\partial_{\perp i}\psi(z) - ig\int_{z^-}^{\infty} dx^- W[\infty, x^-]\underbrace{\frac{\partial A^+(x^-, z_\perp)}{\partial z_\perp^i}}_{D^+A_i - F^+{}_i} W[x^-, z^-]\psi(z)$$

Use the trick

$$\int_{z^-}^{\infty} dx^- W[\infty, x^-]D^+A_i(x^-, z_\perp)W[x^-, z^-] = \int_{z^-}^{\infty} dx^- \frac{d}{dx^-}\left(W[\infty, x^-]A_i(x^-)W[x^-, z^-]\right)$$

$$= A_i(\infty)W[\infty, x^-] - W[\infty, x^-]A_i(z^-)$$

Therefore,

$$\frac{\partial}{\partial z_\perp^i} W[\infty, z^-]\psi(z) = W[\infty, z^-]\left(D_{\perp i}\psi + ig\int_{z^-}^{\infty} dx^- W[z^-, x^-]F^+{}_i(x^-)W[x^-, z^-]\psi\right)$$

$$= W[\infty, z^-]D_{\perp i}^{pure}\psi(z)$$

Improved (Belinfante) energy momentum tensor

Return to the canonical angular momentum tensor and write

$$M^{\mu\nu\lambda}_{can} = x^\nu T^{\mu\lambda}_{can} - x^\lambda T^{\mu\nu}_{can} + H^{\mu\nu\lambda}$$

Define $$\tilde{T}^{\mu\nu} = T^{\mu\nu}_{can} + \partial_\rho G^{\rho\mu\nu}$$ ← One can add a total derivative.

where $$G^{\rho\mu\nu} = \frac{1}{2}(H^{\rho\mu\nu} - H^{\mu\rho\nu} - H^{\nu\rho\mu})$$

Exercise: Show that $\tilde{T}^{\mu\nu}$ is symmetric and conserved. Hint: use $\partial_\mu M^{\mu\nu\lambda}_{can} = 0$

Exercise: Show that in QCD, $$\tilde{T}^{\mu\nu} = \bar{\psi} i\gamma^{(\mu} \overleftrightarrow{D}^{\nu)} \psi - F^{\mu\rho} F^{\nu}{}_{\rho} - g^{\mu\nu}\mathcal{L} = \tilde{T}^{\mu\nu}_q + \tilde{T}^{\mu\nu}_g$$

$$A^{(\mu} B^{\nu)} \equiv \frac{A^\mu B^\nu + A^\nu B^\mu}{2}$$

$$\tilde{M}^{\mu\nu\lambda} = x^\nu \tilde{T}^{\mu\lambda} - x^\lambda \tilde{T}^{\mu\nu}$$

Hint: A useful identity

$$\overleftrightarrow{D}^{\mu} = \frac{D^{\mu} - \overleftarrow{D}^{\mu}}{2}$$

$$\overleftarrow{D}^{\mu} = \overleftarrow{\partial}^{\mu} - igA^{\mu}$$

From the Dirac equation $(\not{D} + iM)\psi = \bar{\psi}(\overleftarrow{\not{D}} - iM) = 0$

$$\begin{aligned}
0 &= \bar{\psi}\gamma^{\mu}\gamma^{\nu}(\not{D} + iM)\psi - \bar{\psi}(\overleftarrow{\not{D}} - iM)\gamma^{\nu}\gamma^{\mu}\psi \\
&= \bar{\psi}(g^{\mu\nu}\gamma^{\rho} + g^{\nu\rho}\gamma^{\mu} - g^{\mu\rho}\gamma^{\nu} + i\epsilon^{\mu\nu\rho\sigma}\gamma_{\sigma}\gamma_5)D_{\rho}\psi \\
&\quad -\bar{\psi}\overleftarrow{D}_{\rho}(g^{\rho\nu}\gamma^{\mu} + g^{\nu\mu}\gamma^{\rho} - g^{\rho\mu}\gamma^{\nu} + i\epsilon^{\rho\nu\mu\sigma}\gamma_{\sigma}\gamma_5)\psi + 2iMg^{\mu\nu}\bar{\psi}\psi \\
&= 2\bar{\psi}(\gamma^{\mu}\overleftrightarrow{D}^{\nu} - \gamma^{\nu}\overleftrightarrow{D}^{\mu})\psi + i\epsilon^{\rho\mu\nu\sigma}\partial_{\rho}(\bar{\psi}\gamma_{\sigma}\gamma_5\psi)
\end{aligned}$$

Derivation from general relativity

If you are only interested in the symmetric form, there is a much quicker derivation.
Write down the action in curved space

$$S = \int d^4x \mathcal{L}[\psi, A] \to \int d^4x \sqrt{-g}\mathcal{L}[g^{\mu\nu}, \psi, A]$$

In my convention, $g^{\mu\nu} = (+1, -1, -1, -1)$ in the flat limit.

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{-g}\mathcal{L} \qquad T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int \sqrt{-g}\mathcal{L}$$

beware the sign difference

Don't do this:

$$\bar{\psi}\gamma^\mu\partial_\mu\psi \to \bar{\psi}g^{\mu\nu}\gamma_\mu\partial_\nu\psi \qquad \frac{\delta}{\delta g^{\mu\nu}}\bar{\psi}g^{\mu\nu}\gamma_\mu\partial_\nu\psi = \bar{\psi}\gamma_\mu\partial_\nu\psi$$

Why is this incorrect?

Ji decomposition

Belinfante energy momentum tensor

$$\langle P|J^z_{q,g}|P\rangle = \frac{1}{V}\langle P|\epsilon^{ij}\int d^3x x^i T^{0j}_{q,g}(x)|P\rangle$$
$$= \frac{1}{V}\lim_{P'\to P}\langle P'|\epsilon^{ij}\int d^3x x^i T^{0j}_{q,g}(x)|P\rangle$$
$$= -i\lim_{\Delta\to 0}\epsilon^{ij}\frac{\partial}{\partial\Delta^i}\langle P'|T^{0j}_{q,g}(0)|P\rangle$$

$$\hat{O}(x) = e^{i\hat{P}x}\hat{O}(0)e^{-i\hat{P}x}$$

$$\Delta = P' - P$$
$$\bar{P} = \frac{P+P'}{2}$$

Parametrization (gravitational form factors)

$$\langle P'|T^{\alpha\beta}_q|P\rangle = \bar{u}(P')\left[A_q(t)\gamma^{(\alpha}\bar{P}^{\beta)} + B_q(t)\frac{\bar{P}^{(\alpha}i\sigma^{\beta)\lambda}\Delta_\lambda}{2m_N} + D_q(t)\frac{\Delta^\alpha\Delta^\beta - g^{\alpha\beta}\Delta^2}{4m_N} + \bar{C}_q(t)m_N g^{\alpha\beta}\right]u(P)$$

Use $\bar{u}(P+\Delta)\gamma^i u(P) \approx -i\epsilon^{ijk}\Delta^j\xi'\sigma^k\xi$

$$\Rightarrow \quad \frac{1}{2} = \sum_q J_q + J_g \qquad\qquad J_{q,g} = \frac{1}{2}(A_{q,g}(0) + B_{q,g}(0))$$

$A_q(0), A_g(0)$ Momentum fraction of proton carried by quarks/gluons

$$\sum_q A_q(0) + A_g(0) = 1 \quad \Rightarrow \quad \sum_q B_q(0) + B_g(0) = 0$$

Further decomposition in the quark part (but not in the gluon part)

$$\bar{\psi} i \gamma^{(\mu} \overleftrightarrow{D}^{\nu)} \psi = \bar{\psi} i \gamma^{\mu} \overleftrightarrow{D}^{\nu} \psi - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \partial_\rho (\bar{\psi} \gamma_\sigma \gamma_5 \psi)$$

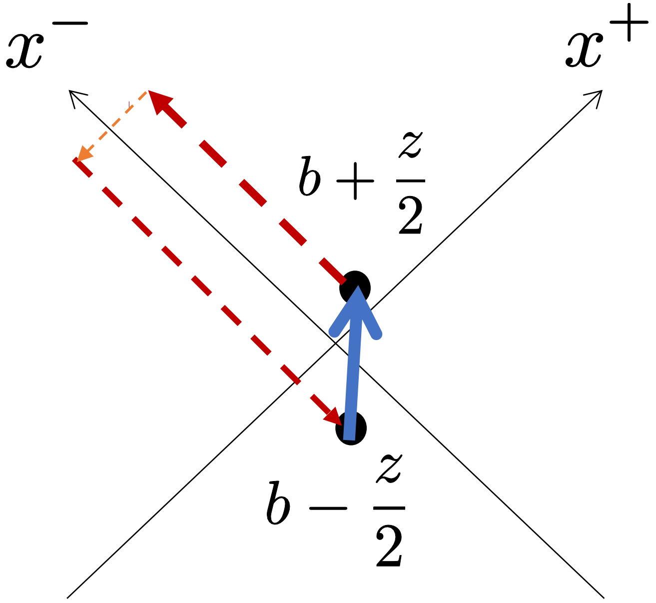
$$J_q = \frac{1}{2}\Delta\Sigma + L^q_{kin}$$ kinetic OAM (features covariant derivative)

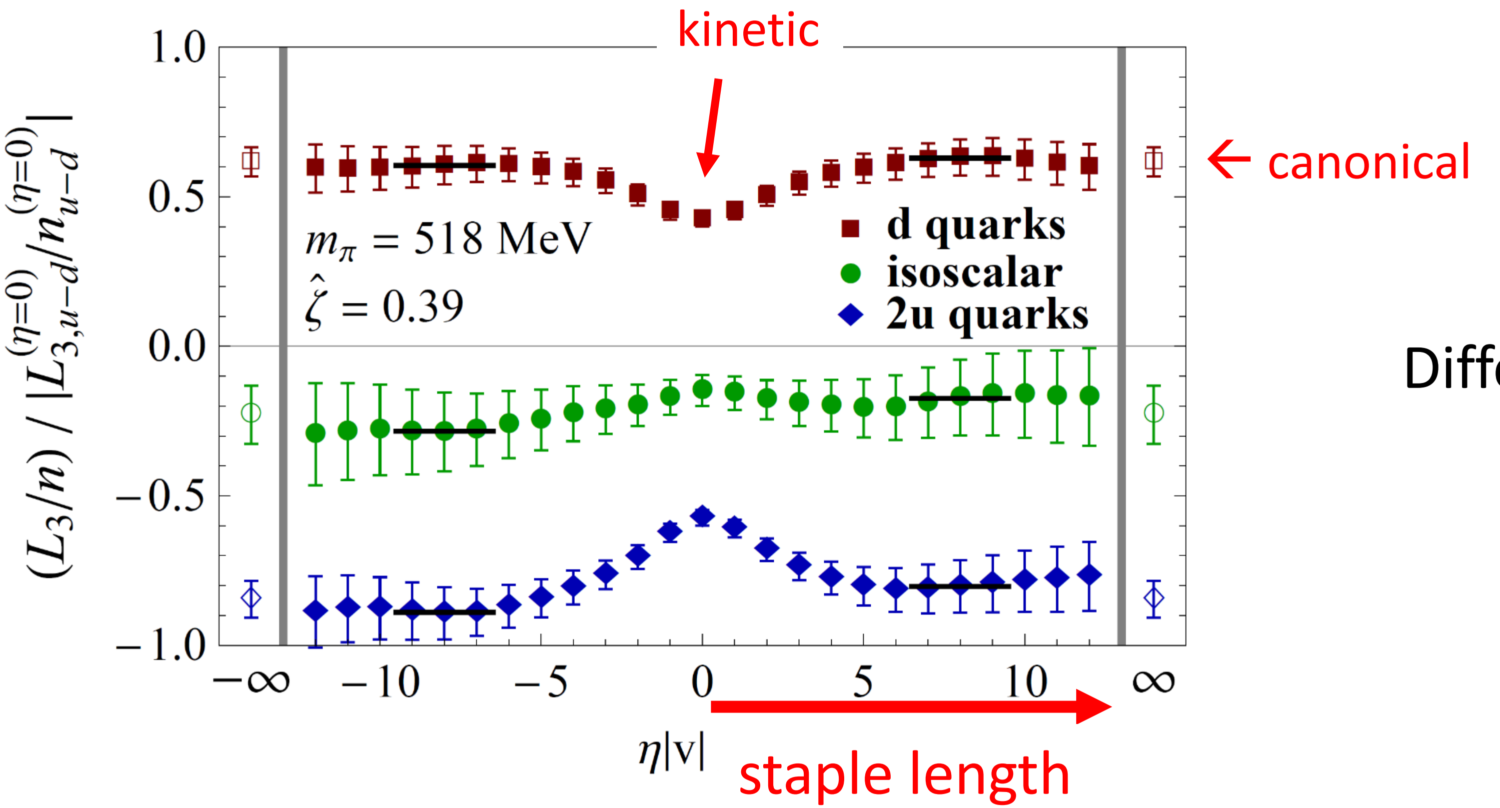
All the operators involved are local and gauge invariant → calculable on a lattice

Kinetic (Ji's) OAM from the straight Wilson line

Ji, Xiong, Yuan (2012)

$$\int d^2k_\perp (b_\perp \times k_\perp) W_{straight}(b_\perp, k_\perp) = \langle \bar{\psi} b_\perp \times iD_\perp \psi \rangle$$





Difference seen in lattice QCD

Engelhardt (2017)

Generalized Parton Distribution

Diehl, hep-ph/0307382
Belitsky, Radyushkin, hep-ph/0504030

Unpolarized

non-forward matrix element $\Delta^\mu = P'^\mu - P^\mu$

$$\bar{P}^+ \int \frac{dz^-}{2\pi} e^{ix\bar{P}^+ z^-} \langle P'|\bar{q}(-z/2)\gamma^+ q(z/2)|P\rangle = \bar{u}(P')\left[\gamma^+ H_q(x,\eta,t) + \frac{i\sigma^{+\nu}\Delta_\nu}{2m_N} E_q(x,\eta,t)\right] u(P)$$

Polarized

$$\bar{P}^+ \int \frac{dz^-}{2\pi} e^{ix\bar{P}^+ z^-} \langle P'|\bar{q}(-z/2)\gamma^+\gamma_5 q(z/2)|P\rangle = \bar{u}(P')\left[\gamma^+\gamma_5 \tilde{H}_q(x,\eta,t) + \frac{\gamma_5\Delta^+}{2m_N} \tilde{E}_q(x,\eta,t)\right] u(P)$$

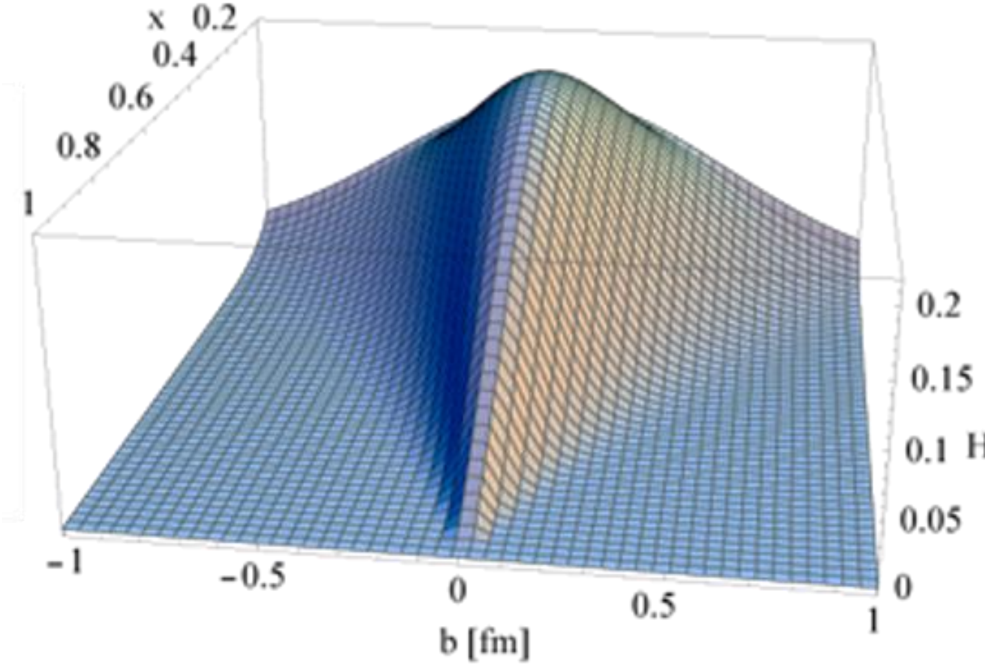
skewness $$\eta = \frac{-\Delta^+}{2\bar{P}^+} = \frac{P^+ - P'^+}{P^+ + P'^+}$$

Different definitions of skewness exist.
They differ by power-suppressed corrections.

$$t = \Delta^2 = -\frac{4\eta^2 M^2}{1-\eta^2} - \frac{\vec{\Delta}_\perp^2}{1-\eta^2}$$

Exercise: Derive this

GPD and 3D tomography



Set $\eta = 0$ and Fourier transform $\Delta_\perp \leftrightarrow b_\perp$

Distribution of quarks in impact parameter $\vec{b}_\perp$ space

$$H_q(x, t = -\vec{\Delta}_\perp^2) \to H_q(x, \vec{b}_\perp)$$

For a transversely polarized nucleon, $\bar{u}\sigma^{+i}u \approx -\frac{2P^+}{m}\epsilon^{ij}S_j$

Get the linear combination

$$H_q(x,t) - \frac{i\epsilon^{ij}\Delta_i S_j}{2m^2}E_q(x,t) \quad \Longrightarrow \quad H_q(x,b_\perp) - \frac{\epsilon^{ij}S_j}{2m^2}\frac{\partial}{\partial b_i}E_q(x,b_\perp)$$

deformation in xy plane Burkardt (2002)

Multiply by x and integrate over x .

$$\int_{-1}^{1} dx\,x(P^+)^2 \int \frac{dy^-}{2\pi} e^{ixP^+y^-} \langle \bar{\psi}(0)\gamma^+\psi(y^-)\rangle = \langle P'|\bar{\psi}\gamma^+ iD^+\psi|P\rangle = \langle P'|T_q^{++}|P\rangle$$

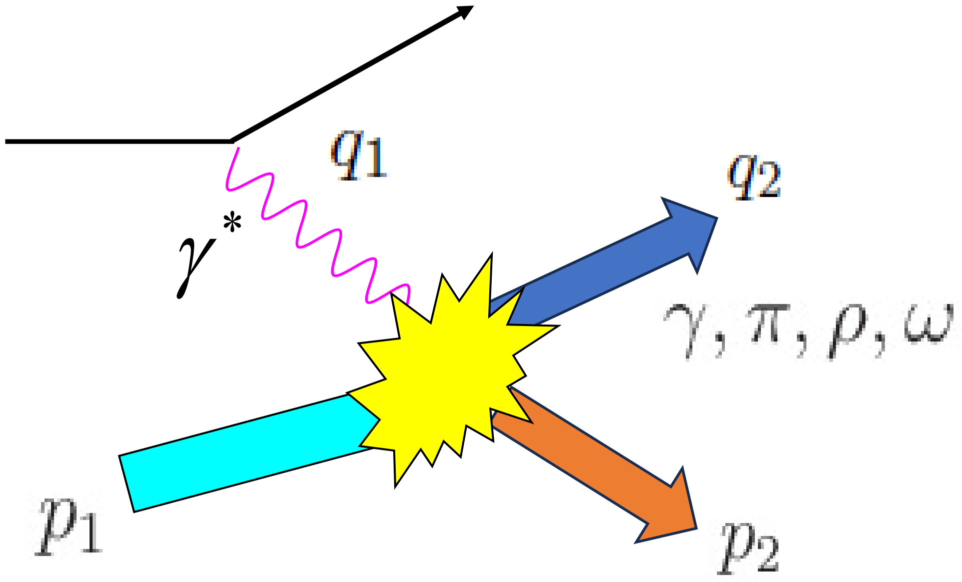
$$\langle P'|T_q^{++}|P\rangle = \bar{u}(P')\left[A_q(t)\gamma^+\bar{P}^+ + B_q(t)\frac{\bar{P}^+ i\sigma^{+\lambda}\Delta_\lambda}{2M} + D_q(t)\frac{(\Delta^+)^2}{4M}\right]u(P)$$

$$A_q(0) = \int dx\,x H_q(x,0,0) \qquad B_q(0) = \int dx\,x E_q(x,0,0)$$

Ji sum rule (GPD version) $\quad J_q = \frac{1}{2}\int dx\,x(H_q(x,0,0) + E_q(x,0,0))$

Deeply Virtual Compton Scattering (DVCS)
Deeply Virtual Meson Production (DVMP)

GPD can be probed in exclusive processes
Small cross section, need high luminosity → JLab/EIC



$$q = \frac{q_1 + q_2}{2} \qquad P = \frac{p_1 + p_2}{2} \qquad Q^2 = -q^2$$

$$\Delta = p_2 - p_1 = q_1 - q_2$$

Bjorken variable $\quad x_B = \dfrac{-q_1^2}{2p_1 \cdot q_1}$ $\qquad$ generalized Bjorken variable $\quad \xi = \dfrac{Q^2}{2P \cdot q}$

Skewness $\quad \eta = \dfrac{-\Delta \cdot q}{2P \cdot q}$

Exercise: Show that when $q_2^2 = 0$ (DVCS) and $Q^2 \gg |\Delta^2|$, $\quad \eta \approx \xi \approx \dfrac{x_B}{2 - x_B}$

Virtual Compton amplitude

$$T^{\mu\nu}(\xi,\eta,t) = \frac{i}{2\pi}\int d^4y e^{iq\cdot y}\langle p_2|\mathrm{T}\{\bar{\psi}\gamma^\mu\psi(y/2)\bar{\psi}\gamma^\nu\psi(-y/2)\}|p_1\rangle$$

$$\psi(y/2)\bar{\psi}(-y/2) = \frac{i\not{y}}{2\pi^2(y^2-i\epsilon)^2}$$

$$e^{-iq\cdot y}\gamma_\mu\not{y}\gamma_\nu + e^{iq\cdot y}\gamma_\nu\not{y}\gamma_\mu$$
$$= (e^{-iq\cdot y}+e^{iq\cdot y})(g_{\mu\rho}g_{\nu\tau}+g_{\nu\rho}g_{\mu\tau}-g_{\mu\nu}g_{\rho\tau})y^\rho\gamma^\tau + (e^{-iq\cdot y}-e^{iq\cdot y})i\epsilon_{\mu\rho\nu\lambda}y^\rho\gamma^\lambda\gamma_5$$

$$\langle p_2|\bar{\psi}(-y/2)\gamma^\tau\psi(y/2)|p_1\rangle \approx \int dx e^{-ixP\cdot y}\int\frac{P^+dy'^-}{2\pi}e^{ixP^+y'^-}\langle p_2|\bar{\psi}(-y'^-/2)\gamma^\tau\psi(y'^-/2)|p_1\rangle$$

Keep only $\tau = +$ (twist-2)

Do the d^4y integration

$$\int d^4y \frac{y^\rho}{2\pi^2 y^4}(e^{-iq\cdot y}+e^{iq\cdot y})e^{-ixP\cdot y} = \frac{-q^\rho - xP^\rho}{(-q-xP)^2} + \frac{q^\rho - xP^\rho}{(q-xP)^2}$$

$$= \frac{\xi P^\rho - \frac{Q^2}{2\xi}n^\rho - xP^\rho}{-Q^2+2xq\cdot P+i\epsilon} + \frac{-\xi P^\rho + \frac{Q^2}{2\xi}n^\rho - xP^\rho}{-Q^2-2xq\cdot P+i\epsilon}$$

$$= -\frac{n^\rho}{2}\left(\frac{1}{x-\xi+i\epsilon}+\frac{1}{x+\xi-i\epsilon}\right)$$

where I introduced a light-like vector

$$n^\mu = \frac{1}{P^+}\delta^\mu_+ \qquad P\cdot n = 1$$

Final result (leading order formula)

unpolarized GPD

$$T^{\mu\nu}(\xi,\eta,t) = -(g^{\mu\rho}g^{\nu-}+g^{\mu-}g^{\nu\rho}-g^{\mu\nu}g^{\rho-})n_\rho \int \frac{dx}{4\pi}\left(\frac{1}{x+\xi-i\epsilon}+\frac{1}{x-\xi+i\epsilon}\right)\bar{u}(p_2)\left[H\gamma^+ + E\frac{i\sigma^{+\sigma}\Delta_\sigma}{2m}\right]u(p_1)$$
$$-i\epsilon^{\mu\nu\rho-}n^\rho\int\frac{dx}{4\pi}\left(\frac{1}{x+\xi-i\epsilon}-\frac{1}{x-\xi+i\epsilon}\right)\bar{u}(p_2)\left[\tilde{H}\gamma^+\gamma_5+\tilde{E}\frac{\gamma_5\Delta^+}{2m}\right]u(p_1)$$

polarized GPD

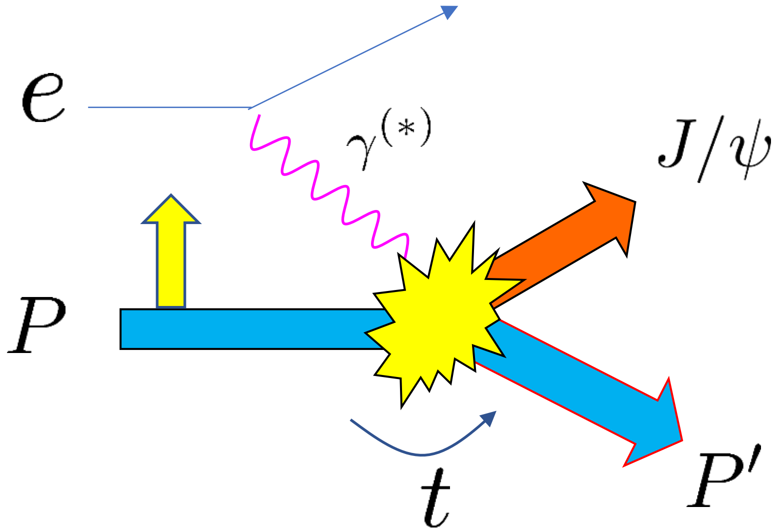
State-of-the-art: next-to-next-to-leading order (NNLO) coefficient functions
Braun, Ji, Schoenleber (2022)

GPD challenges

- More variables → more difficult to extract from experiments
- Many GPDs. `Polarized' GPDs contribute to unpolarized processes
- Severe inverse problem. How can one reconstruct $H(x,\xi,t)$ if one only knows $\int_{-1}^{1} dx \frac{1}{x-\xi+i\epsilon} H(x,\xi,t)$

In contrast, PDF is directly related to the structure functions $g_1(x) = \frac{1}{2}\sum_f e_f^2(\Delta q_f(x) + \Delta \bar{q}_f(x)) + \cdots$

- Evolution equation complicated.
- Difficult to access gluon GPDs
- Yet, many recent progress in theory and lattice!



Gluon GPD E from exclusive single spin asymmetry