Quantum Simulation of Fragmentation functions

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arXiv:2406.03147

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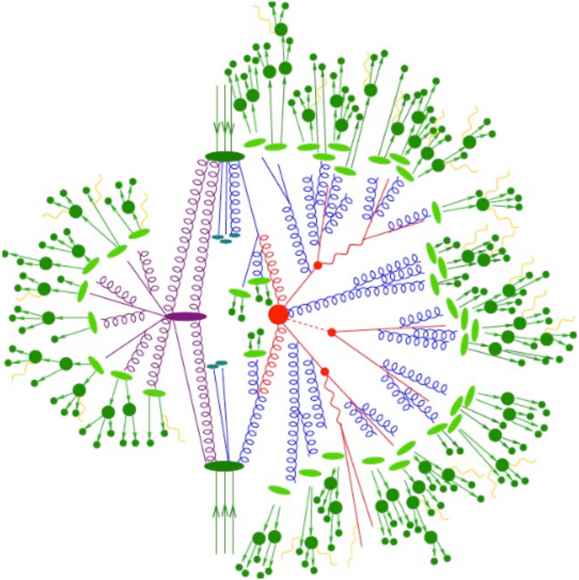
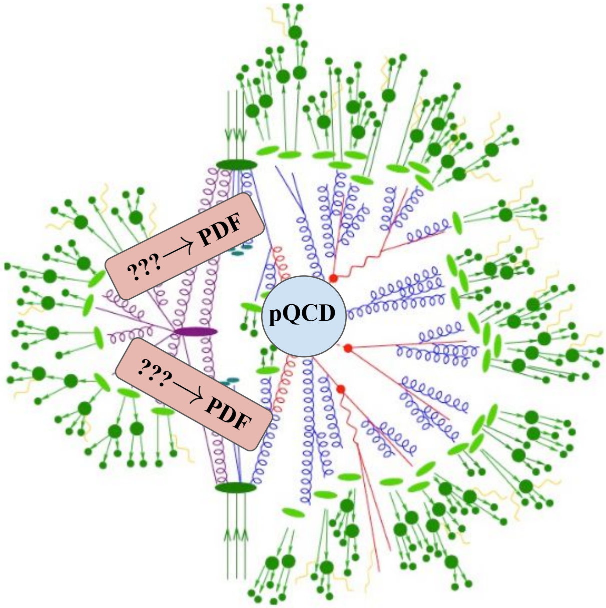
Stony Brook University 07/06/2025

Fragmentation function




???⟶ PDF
pQCD
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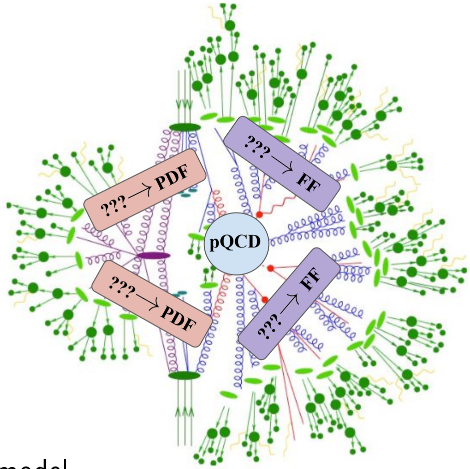
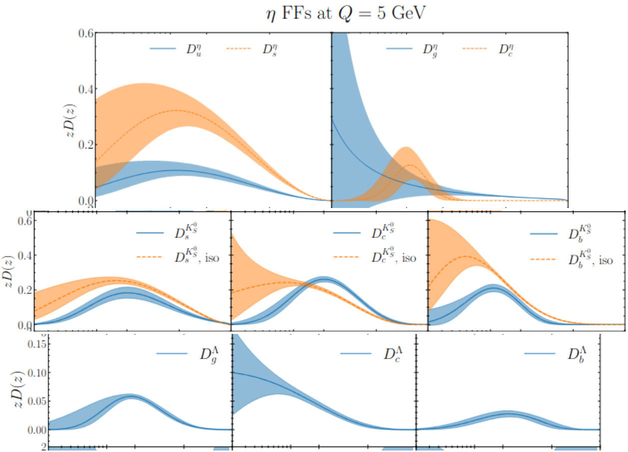


In the parton model

$$d_{h/j}(z) = \frac{\mathrm{Tr}_{\mathrm{color}}}{N_{c,j}} \sum_{X} \langle j, k | \, h, X, out \rangle \langle h, X, out \, | j, k \rangle$$

Some FF



From recent global analysis 2503.21311

QCD in the LF - nutshell format

From the QCD Lagrangian density

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a} + \bar{\psi} \left(i\not{\partial} - m \right) \psi + g\, A^a_\mu J^{\mu,a}_D$$

take $A^+ = 0$ (LF gauge) to obtain

$$\begin{aligned} P^- =& \frac{1}{2} \int dx_+ d^2 x_\perp \Bigg[\bar{\psi} \gamma^+ \frac{m^2 - \nabla^2_\perp}{i\partial^+} \psi - A^\mu_r \nabla^2_\perp A_{r,\mu} \\ & + g J^\mu_r A_{r,\mu} + \frac{g^2}{4} B^{\mu\nu}_r B_{r,\mu\nu} \\ & - \frac{g^2}{2} J^\mu_r \frac{1}{(\partial^+)^2} J^\mu_r - i \frac{g^2}{2} \bar{\psi} \gamma^\mu T_r A_{r,\mu} \frac{\gamma^+}{\partial^+} \left(\gamma^\mu T_s A_{s,\nu} \psi \right) \Bigg] \end{aligned}$$

Expand the fields in terms of plane waves

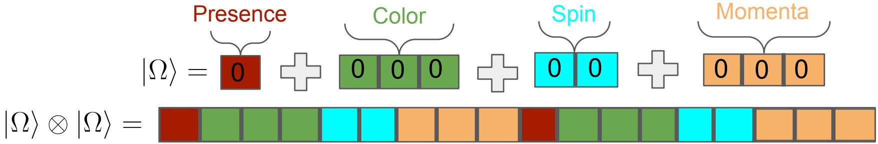
$$\psi_{\alpha,r}(x) = \sum_{\sigma} \int [p] \left(u_{\alpha\sigma}(p) b_{p\sigma r} e^{-ipx} + v_{\alpha\sigma}(p) e^{ipx} d^{\dagger}_{p\sigma r} \right)$$

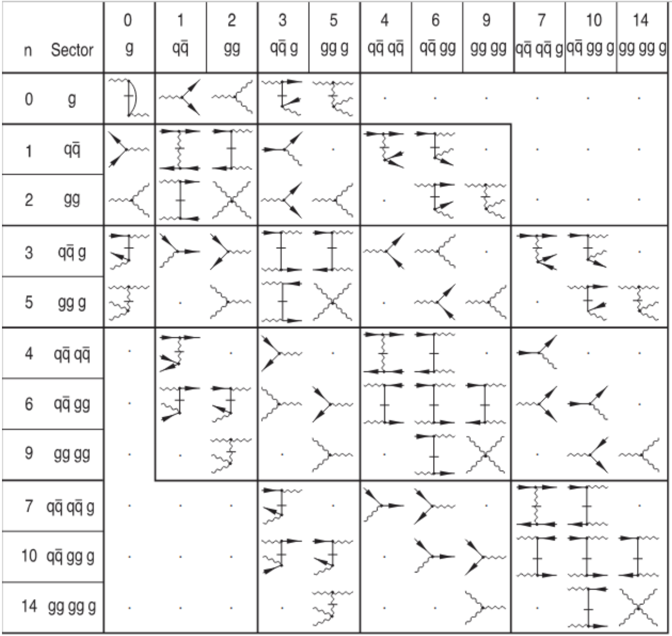
$$A^{\mu}_{a}(x) = \int [p] \left[\epsilon^{\mu}_{\sigma}(p) a_{p,\sigma,a} e^{ipx} + \epsilon^{\mu *}_{\sigma}(p) a^{\dagger}_{p,\sigma,a} e^{-ipx} \right],$$

$p^2 = m^2$ & **Schrödinger** picture $\Rightarrow$ **3-vectors** !!
1-particle states are $|p, \sigma, a\rangle = a^{\dagger}_{p,\sigma,a} |\Omega\rangle$ and:

$$\left[a_{p\sigma a}, a^{\dagger}_{k\sigma' b} \right] = \delta \left(k^{+} - p^{+} \right) \delta^{2} \left(p^{\perp} - k^{\perp} \right) \delta^{\sigma'}_{\sigma} \delta^{a}_{b}$$

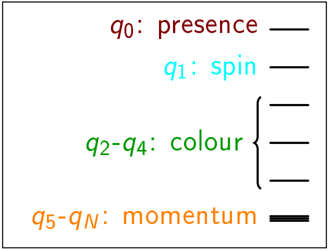
Memory divided in **particle registers:**




n Sector
0 g
1 qq̄
2 gg
3 qq̄ g
5 gg g
4 qq̄ qq̄
6 qq̄ gg
9 gg gg
7 qq̄ qq̄ g
10 qq̄ gg g
14 gg gg g

One-particle encoding

How to write a and $a^\dagger$ in terms of quantum gates?



Gluon register

Empty register

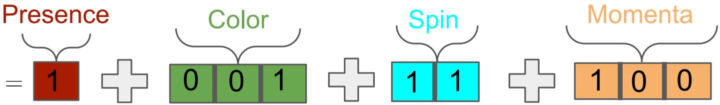
$$|\Omega\rangle = |0\rangle \otimes |0\rangle \otimes |00\rangle \otimes |0\dots0\rangle$$

Pack/unpack with operators

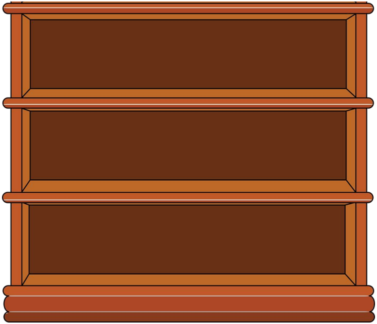
$$\mathfrak{s}^\dagger_{s,c,p}|\Omega\rangle = |1\rangle \otimes |s\rangle \otimes |c\rangle \otimes |p\rangle$$

$$\mathfrak{s}_{s,c,,p}|1\rangle \otimes |s\rangle \otimes |c\rangle \otimes |p\rangle = |\Omega\rangle$$

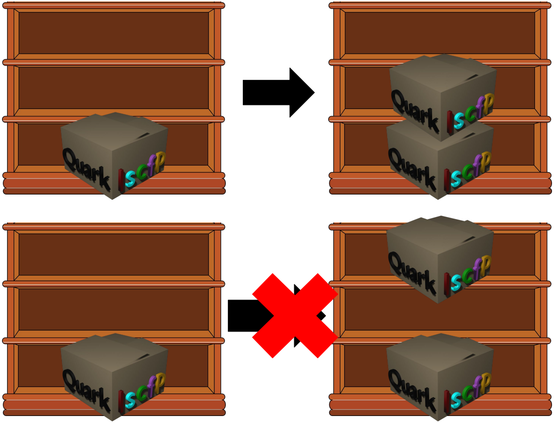
$$\mathfrak{C}_{10} \otimes \mathfrak{s}^\dagger_{4,3,1}\,|\Omega\rangle =$$



Think of memory as



Add particles only downn-up



Antisymmetrize:

Boson operators

Field-theory like $b_q^{(n)\dagger} = \sum_j b_{q,j}^{(n)\dagger}$:

$$b_q^{(n)\dagger} = \sum_{j=1}^{n} \mathcal{S}_{j\leftarrow j-1} \cdot \mathbb{P}_0^{(n-j)} \otimes \left(\mathfrak{C}_{10} \otimes \mathfrak{s}_q^\dagger\right)_j \otimes \mathbb{P}_{j-1}^{(j-1)}$$

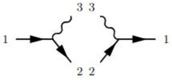
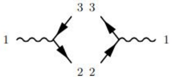
Adjoint to find annihilator. They fulfil

Commutation relations

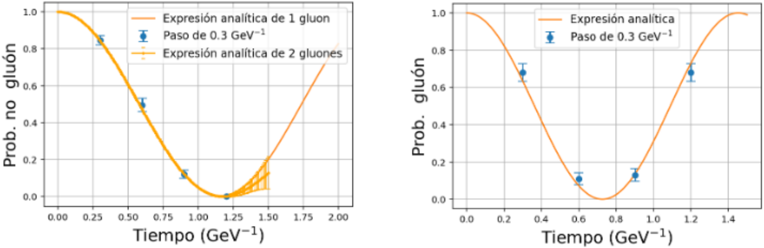
$$\left[b_{q_1}^{(n)}, b_{q_2}^{(n)\dagger}\right] = \overbrace{\delta_{q_1,q_2} \left(\mathfrak{C}_{00} \otimes \mathbb{I}\right)_n \otimes \mathbb{I}^{(n-1)}}^{\text{canonical}}$$
$$- \underbrace{\mathcal{S}_{n\leftarrow n-1} \cdot \left(\mathfrak{C}_{11} \otimes \mathfrak{s}_{q_1}^\dagger \mathfrak{s}_{q_2}\right)_n \otimes \mathbb{P}_{n-1}^{(n-1)} \cdot \mathcal{S}_{n\leftarrow n-1}}_{\text{boundary term}}$$

Hamiltonian exponentiation

Using the above equations, exponentiate

$$U_{\Delta t} = \exp\Bigg[$$ 
+ 
$$\Bigg]$$

and calculate $\langle\psi| U_{\Delta t} |\psi\rangle$

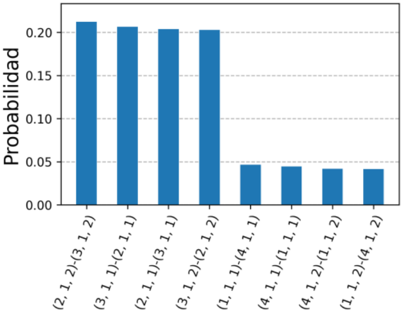


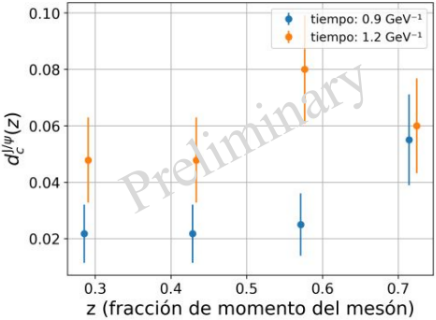
Front Master thesis of **Nicolás A. Martínez-Arenaza**

FF

Let time pass and measure "meson" ($q\bar{q}$ state)

$$\psi^m(x,s,c) = \sqrt{N}\,[x(1-x)]^\alpha\,\sigma_s^m \eta_c$$

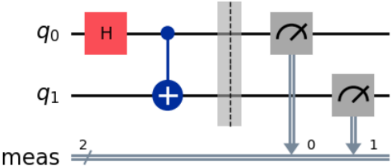


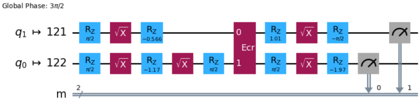


Hello quantum world

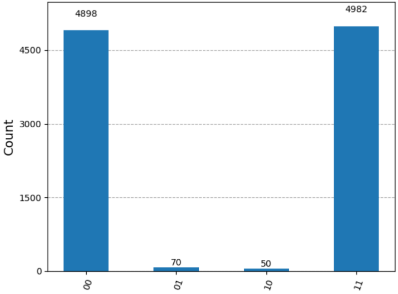
Quantum memory advances by **single-qubit** gates and at least an **entangling** gate:

- Single-qubit: **H**, Pauli **X,Y,Z**, Rots $R_x(\theta) = \exp(-i\theta X)$, etc.
- Entangling: **CNOT**, **Toffoli**





Preps $\Phi_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



Pauli strings

Four basic (non-unitary) qubit transformations:

$$\mathfrak{C}_{00} \equiv |0\rangle\langle 0| = \frac{I+Z}{2} \qquad \mathfrak{C}_{10} \equiv |1\rangle\langle 0| = \frac{X-iY}{2}$$

$$\mathfrak{C}_{01} \equiv |0\rangle\langle 1| = \frac{X+iY}{2} \qquad \mathfrak{C}_{11} \equiv |1\rangle\langle 1| = \frac{I+Z}{2}$$

From these:

$$\mathfrak{s}_3^\dagger \equiv |11\rangle\langle 00| = \mathfrak{C}_{10} \otimes \mathfrak{C}_{10} = \frac{1}{4}\left(\underbrace{XX}_{\text{Pauli string}} - iXY - iYX - YY \right)$$

Encoding: Discretization of Hilbert space, $|\psi\rangle \approx \sum_{i=0}^{2^n-1} c_i |i\rangle$.
General transformation is then

$$e^{-it|\psi_f\rangle\langle\psi_0|} = e^{-it\sum_{i,j=0}^{2^n-1} c_j c_i^* |j\rangle\langle i|} = e^{-it\sum_{i,j=0}^{2^n-1} c_j c_i^* \mathfrak{s}_j^\dagger \mathfrak{s}_i}$$

Add more registers for multi-particle states

How to match 1-particle operators $\mathfrak{s}_p^\dagger$, $\mathfrak{s}_p$ to

$$a_{p_1}^\dagger |\Omega\rangle = |p_1\rangle\,, \quad a_{p_2}^\dagger |p_1\rangle = |p_2\rangle\,|p_1\rangle\,, \ \text{etc.?}$$

Divide a:

$$a_{p_1}^\dagger = a_{p_1,1}^\dagger + a_{p_1,2}^\dagger + a_{p_1,3}^\dagger$$

and use **presence** qubits as controls:

$$\mathfrak{C}_{00}|0\rangle = |0\rangle\,, \quad \mathfrak{C}_{11}|0\rangle = 0, \ \text{etc}$$

so that memory is filled **in order**.

Particle-statistics & unitarity

$$\left[b^\dagger_{p_1}, b_{p_2}\right] = \delta_{p_1 p_2}\mathbb{I} \rightarrow \text{Add symmetrizers } \mathcal{S}:$$

1 01 ⊗ 1 10 + 1 10 ⊗ 1 01

Compatible with unitarity?

- $\mathbb{P}^{(n)}_i$, $\mathfrak{C}_{10}$ ... are not unitary $\rightarrow$ **exponentiate** or **decompose**
- $\mathcal{S}$ is not unitary $\rightarrow$ scrap qubits

Decompose $S_n = S_{n\leftarrow n-1} \ldots S_{2\leftarrow 1}$ with $\mathcal{S}_{j\leftarrow j-1}$:

$$\mathcal{S}_{j\leftarrow j-1} \equiv \frac{1}{\sqrt{j}}\left(\mathbb{I}^{\otimes j} + \mathcal{P}_{j(j-1))} + \ldots + \mathcal{P}_{j2} + \mathcal{P}_{j1}\right).$$

and define:

$$b^{(n)\dagger}_{p,j} = \mathcal{S}_{j\leftarrow j-1} \cdot \mathbb{P}^{(n-j)}_0 \otimes \left(\mathbb{C}_{10} \otimes \mathfrak{s}^\dagger_p\right)_j \otimes \mathbb{P}^{(j-1)}_{j-1}$$

Hamiltonian exponentiation

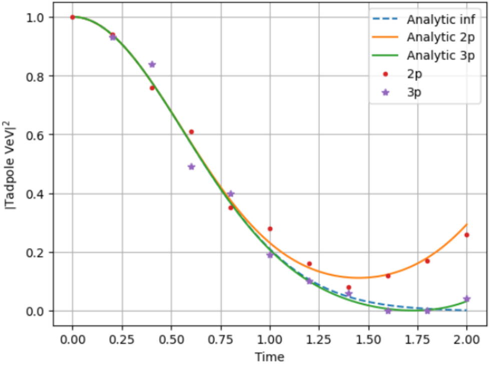
Using the above equations, exponentiate

$$U_{10}(\Delta t) = e^{-i\Delta t \sum \lambda_q \left(a_p^\dagger + a_p\right)}$$

and calculate $\langle 0 | U_{10}(\Delta t) | 0 \rangle$

Simulation setup:

- 2 and 3 particles with 3 modes
- Start with empty registers
- Measure presence qubits, 100 shots/point



Asymptotic costs

Operator	CNOT & single qubit costs
Kinetic U_{11}	$\mathcal{O}\left(nN_p\right)$
Potential U_{22}	$\mathcal{O}\left(n^2 N_p^3 \log^2 N_p\right)$
Tadpole U_{10}	$\mathcal{O}\left(N_p \log^2 N_p\right)$
Splitting 1 U_{21}	$\mathcal{O}\left(n_b n_f^2 N_p^2 \log^2 N_p\right)$
Splitting 2 U'_{21}	$\mathcal{O}\left(n_b n_f N_p^3 \log N_p\right)$

- N_p: # of modes on each register
- n_b: # of gluon registers
- n_f: # of fermion registers
- n: total # of registers

Fixed particle number

We can now get unitaries by exponentiation:

$$\hat{H}_{11}^{(n)} = \sum_p E_p \, a_p^{(n)\dagger} a_p^{(n)} = \sum_p E_p \sum_{j',j} a_{p,j}^{(n)\dagger} a_{p,j'}^{(n)}$$

$$= \sum_j \mathcal{A}_{j\leftarrow j-1} \cdot \mathbb{P}_0^{(n-j)} \otimes \left(\mathfrak{C}_{11} \otimes \sum_p E_q \, \mathfrak{s}_p^\dagger \mathfrak{s}_p \right)_j \otimes \mathbb{P}_{j-1}^{(j-1)} \cdot \mathcal{A}_{j\leftarrow j-1}$$

Fixed particle # $\rightarrow$ antisymmetrizers can be conmmuted:

$$\hat{H}_{11}^{(n)} = \sum_{j,k} \mathbb{P}_0^{(n-j)} \otimes \mathbb{P}_k^{(k)} \otimes \left(\mathfrak{C}_{11} \otimes \sum_p E_p \mathfrak{s}_p^\dagger \mathfrak{s}_p \right)_{j-k} \otimes \mathbb{P}_{j-1-k}^{(j-1-k)} \cdot \frac{\mathcal{A}_{j\leftarrow j-1}}{\sqrt{j}}$$

Antisymmetrizers to the right just simplify!

For Pauli exlusion principle, define fermion interchanger:

$$\mathtt{fX}_j(p) = (\mathfrak{C}_{00})_j \otimes \mathtt{fS}_j(0,p) + (\mathfrak{C}_{11})_j \otimes \mathtt{fS}_j(L,p)$$

Empty register

$$(\mathfrak{C}_{00})_j \otimes \mathtt{fS}_j(0,p)\,|\Omega\rangle_j \underbrace{(\ldots|1p\rangle_i\ldots)_A}_{\text{p there}} = \underbrace{|0p\rangle_j(\ldots|10\ldots0\rangle_i\ldots)_A}_{\text{creation avoided}}$$

$$(\mathfrak{C}_{00})_j \otimes \mathtt{fS}_j(0,p)\,|\Omega\rangle_j \underbrace{(\ldots|1q\rangle_i\ldots)_A}_{\text{no p}} = \underbrace{|\Omega\rangle_j(\ldots|1q\rangle_i\ldots)_A}_{\text{creation allowed}}$$

Occupied register

$$(\mathfrak{C}_{11})_j \otimes \mathtt{fS}_j(L,p)\,|1P\rangle_j \underbrace{(\ldots|1p\rangle_i\ldots)_A}_{\text{p there, swap}} = \underbrace{|1p\rangle_j(\ldots|1P\rangle_i\ldots)_A}_{\text{p on } j\text{, annihilation}}$$

$$(\mathfrak{C}_{11})_j \otimes \mathtt{fS}_j(L,p)\,|1P\rangle_j \underbrace{(\ldots|1q\rangle_i\ldots)_A}_{\text{no p no swap}} = \underbrace{|1P\rangle_j(\ldots|1q\rangle_i\ldots)_A}_{\text{no p on } j\text{, no annihilation}}$$