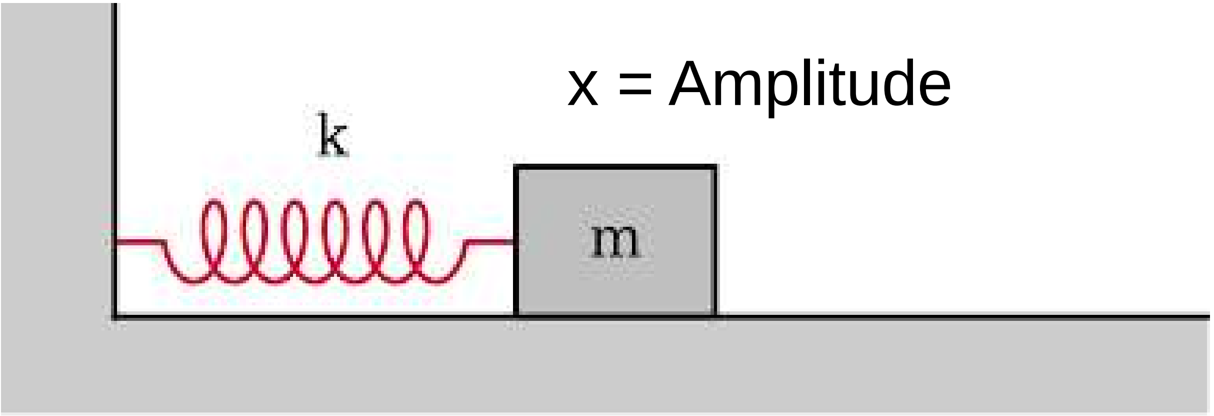
# Dimensional Analysis

2025 CFNS-SURGE School
2 June 2025
Fred Olness

# Simple Harmonic Oscillations (SHO)




Hooke's Law:

$F = - k x$ and $F = m a$

so, dimensionally,

$k = F/x = (m a)/x = (Kg)(m/s^2)/(m)$

$k = Kg/s^2$

x = meters

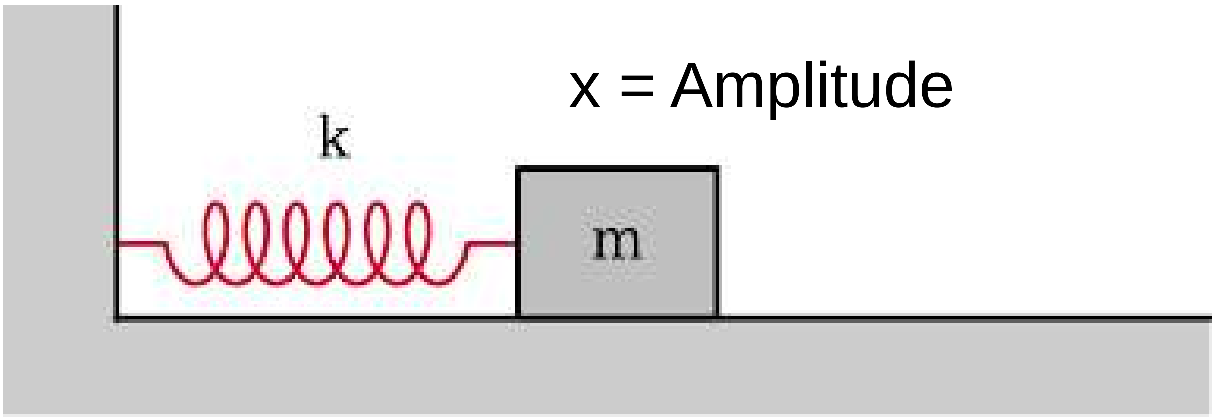
m = kilograms

k = $Kg/sec^2$

Period: (sec)

T=

# Simple Harmonic Oscillations (SHO)




x = Amplitude

x = meters

m = kilograms

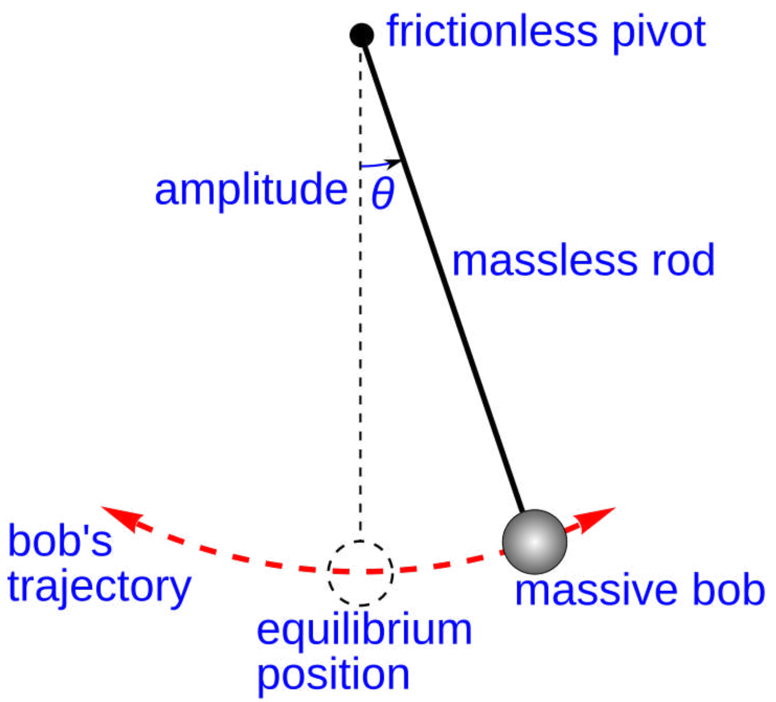
k = Kg/sec$^2$

Period: (sec)

$$T = 2\pi\sqrt{\frac{m}{k}}$$

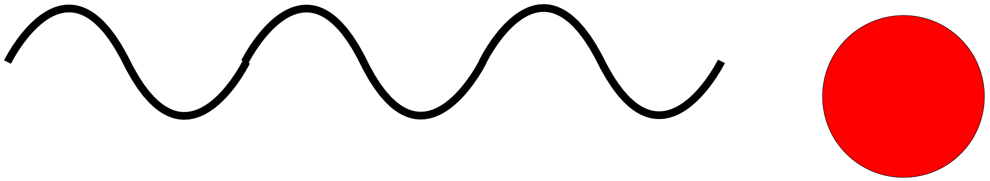
**Independent of amplitude**

**Independent of mass**

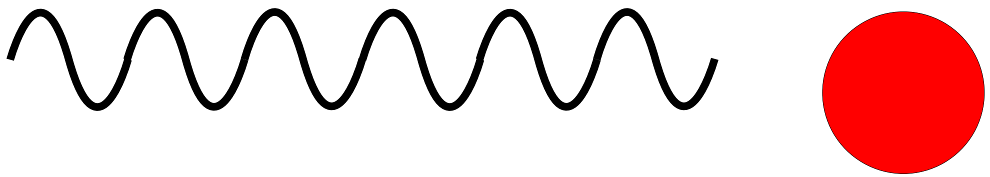


# What if the structure does not depend on RESOLUTION???

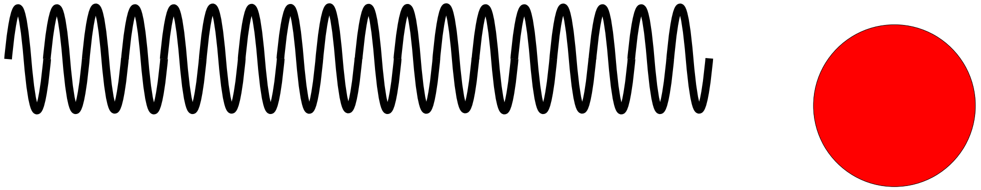




$$d\sigma \sim \sigma_0 \times 1$$



$$d\sigma \sim \sigma_0 \times 1$$



$$d\sigma \sim \sigma_0 \times 1$$

**Dimensional considerations**

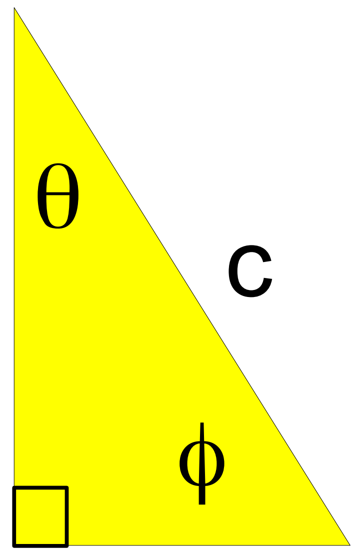
**Structure Function**

# Pythagorean Theorem

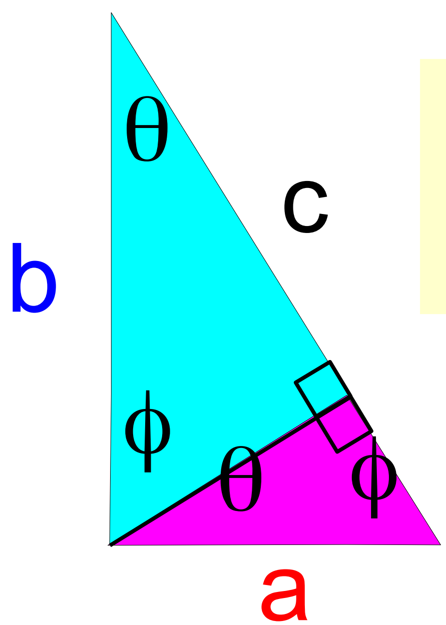


GOAL:
Pythagorean Theorem

METHOD:
Dimensional Analysis




$$A_c = c^2 f(\theta, \phi)$$




$$A_a + A_b =$$
$$a^2 f(\theta, \phi) + b^2 f(\theta, \phi)$$

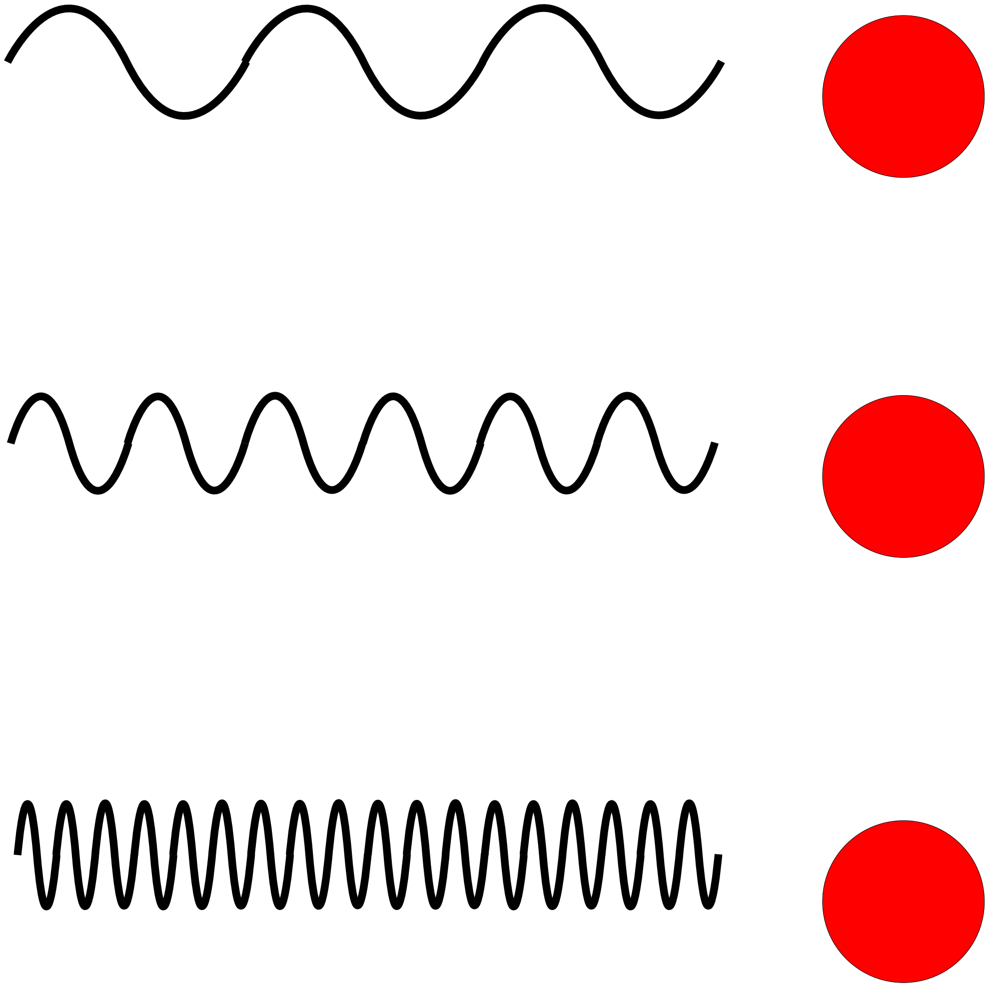
$$A_a + A_b = A_c$$

$$a^2 + b^2 = c^2$$

# Scaling

# What if the structure does not depend on RESOLUTION???





$$d\sigma \sim \sigma_0 \times 1$$

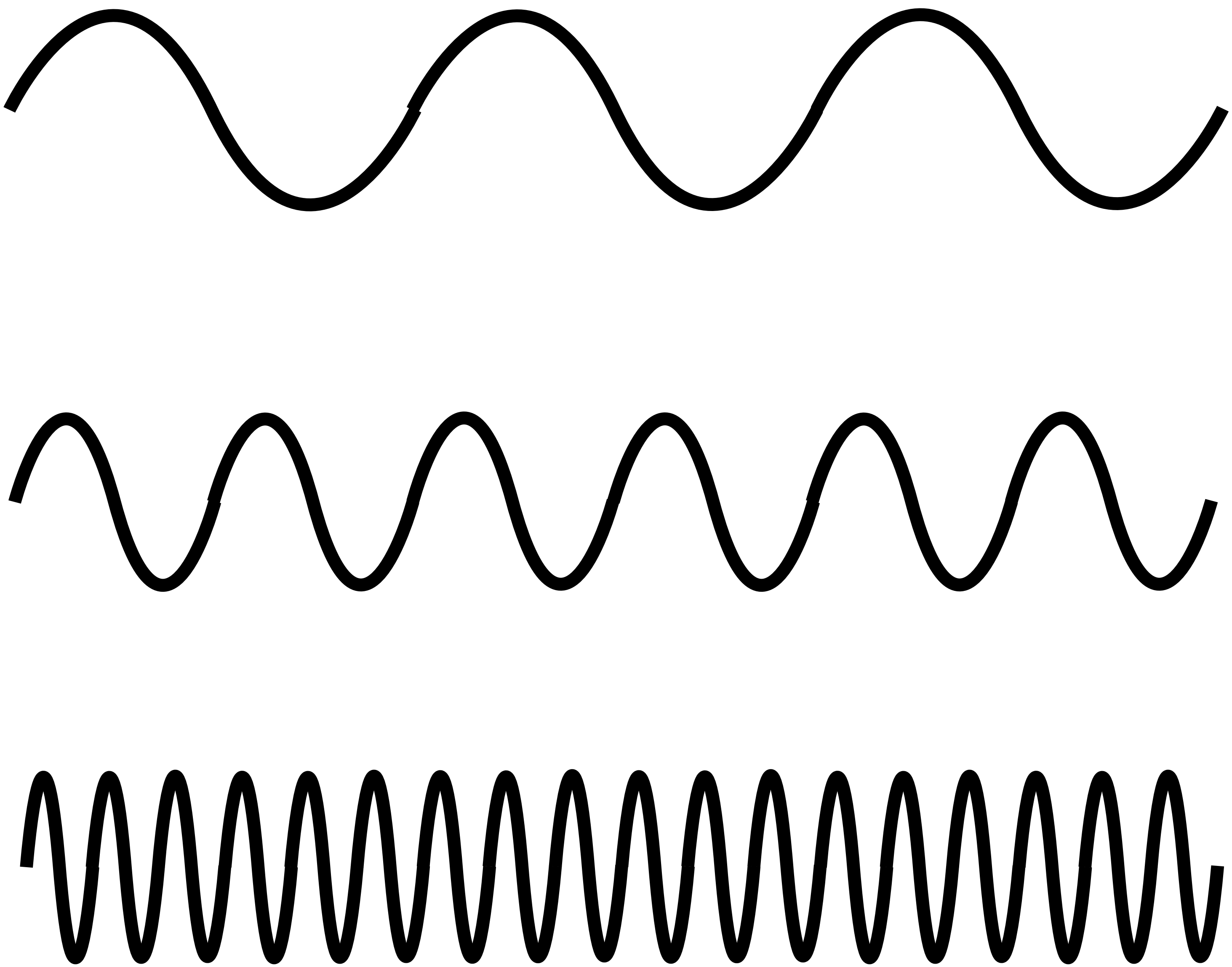
$$d\sigma \sim \sigma_0 \times 1$$

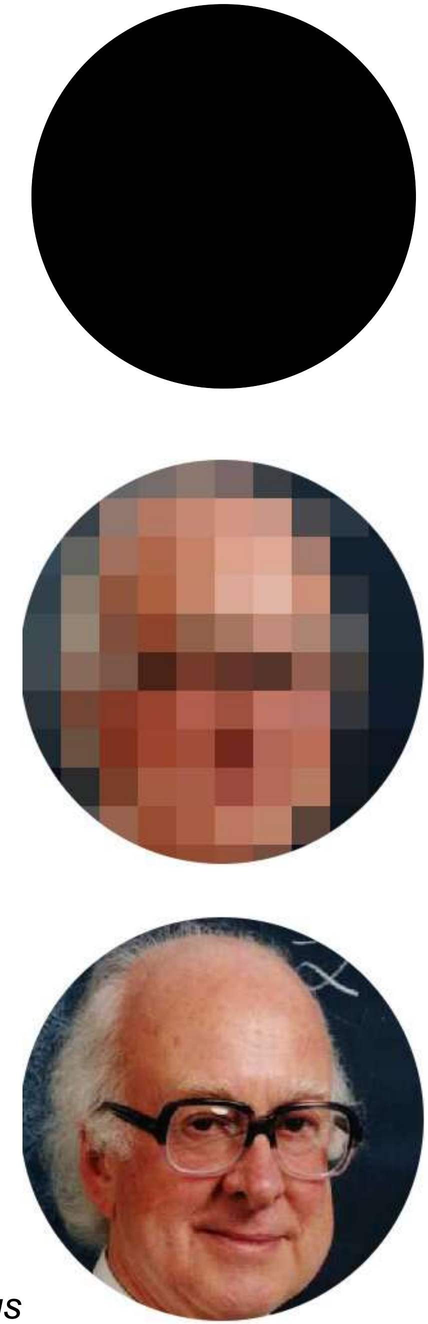
$$d\sigma \sim \sigma_0 \times 1$$

**Dimensional considerations**

**Structure Function**

# Is this a point like particle ???

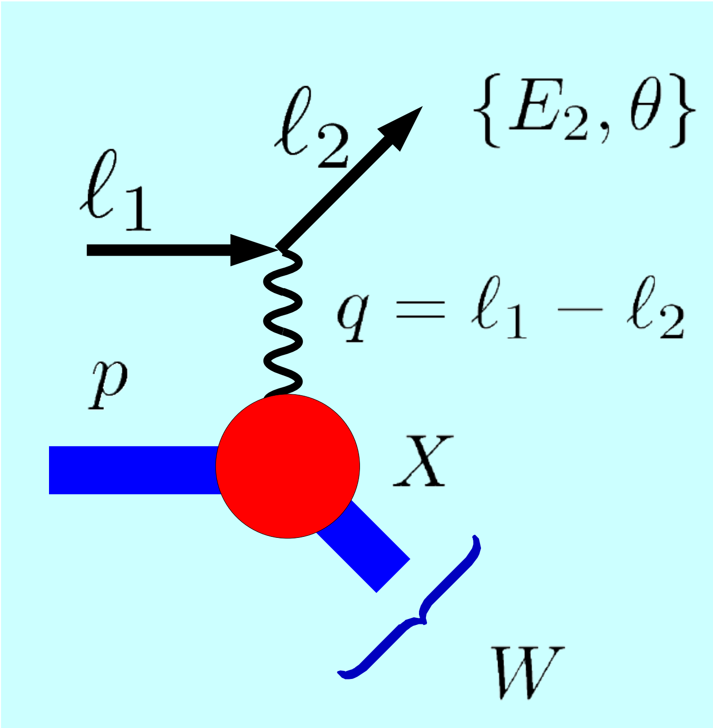




*We found the Higgs*

# Inclusive Deeply Inelastic Scattering (DIS)



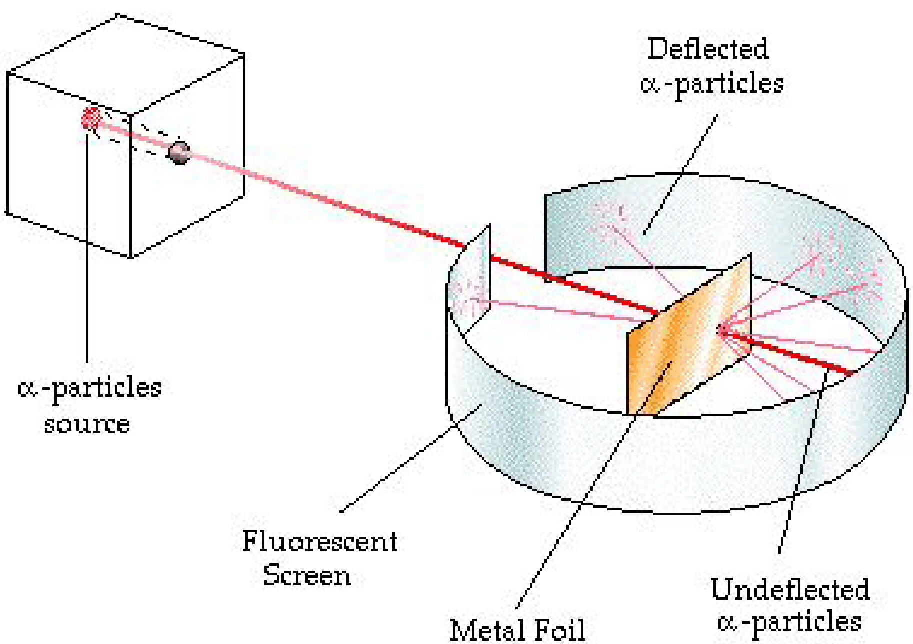


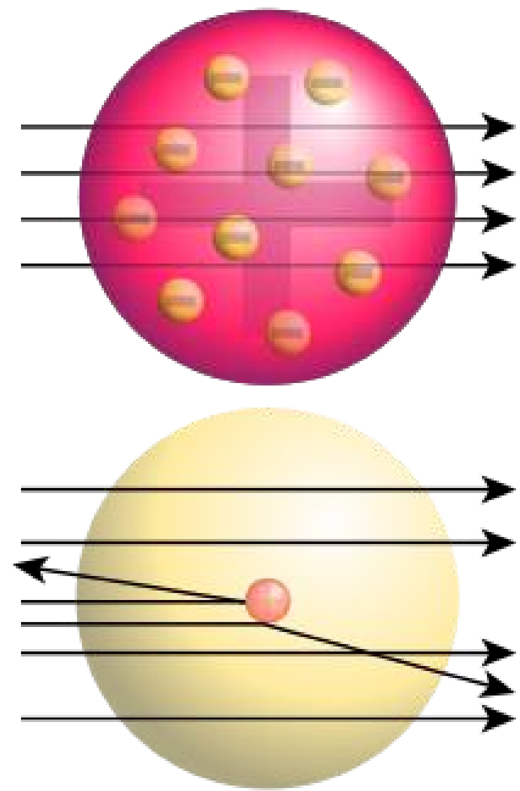

Measure $\{E_2, \theta\} \Leftrightarrow \{x, Q^2\}$ Inclusive

Deep: $Q^2 \geq 1 GeV^2$

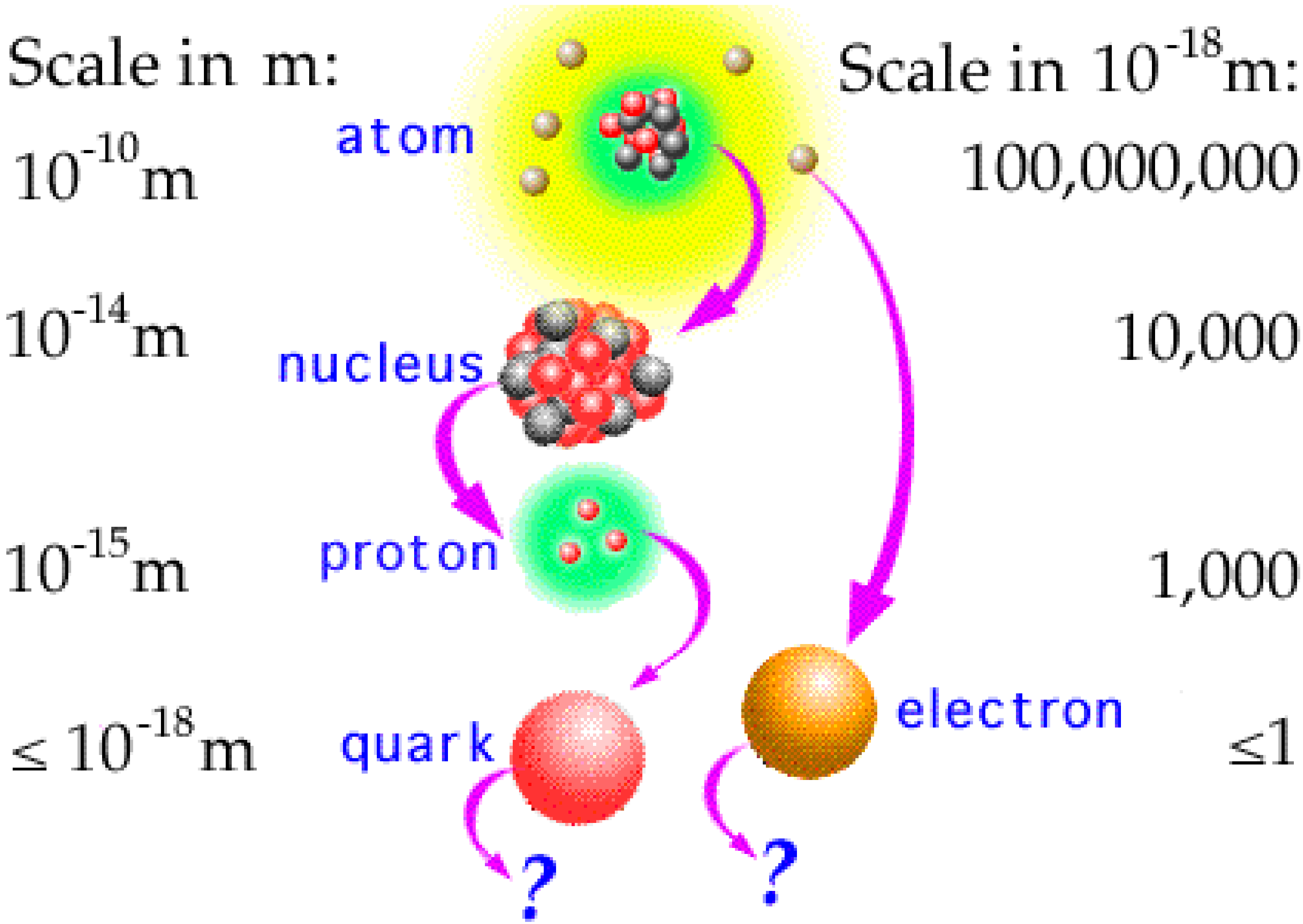
Inelastic: $W^2 \geq M_p^2$

Analogue of Rutherford scattering



Small distance ~ High Energy




Going to smaller scale, we get to <u>simpler</u>, more fundamental objects

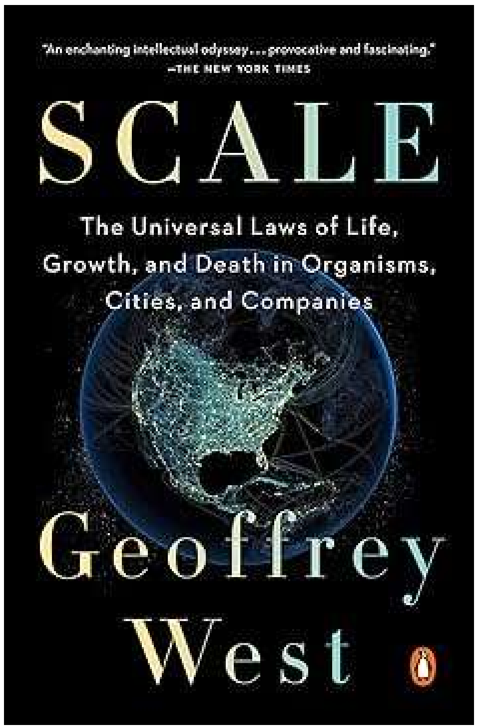
# More

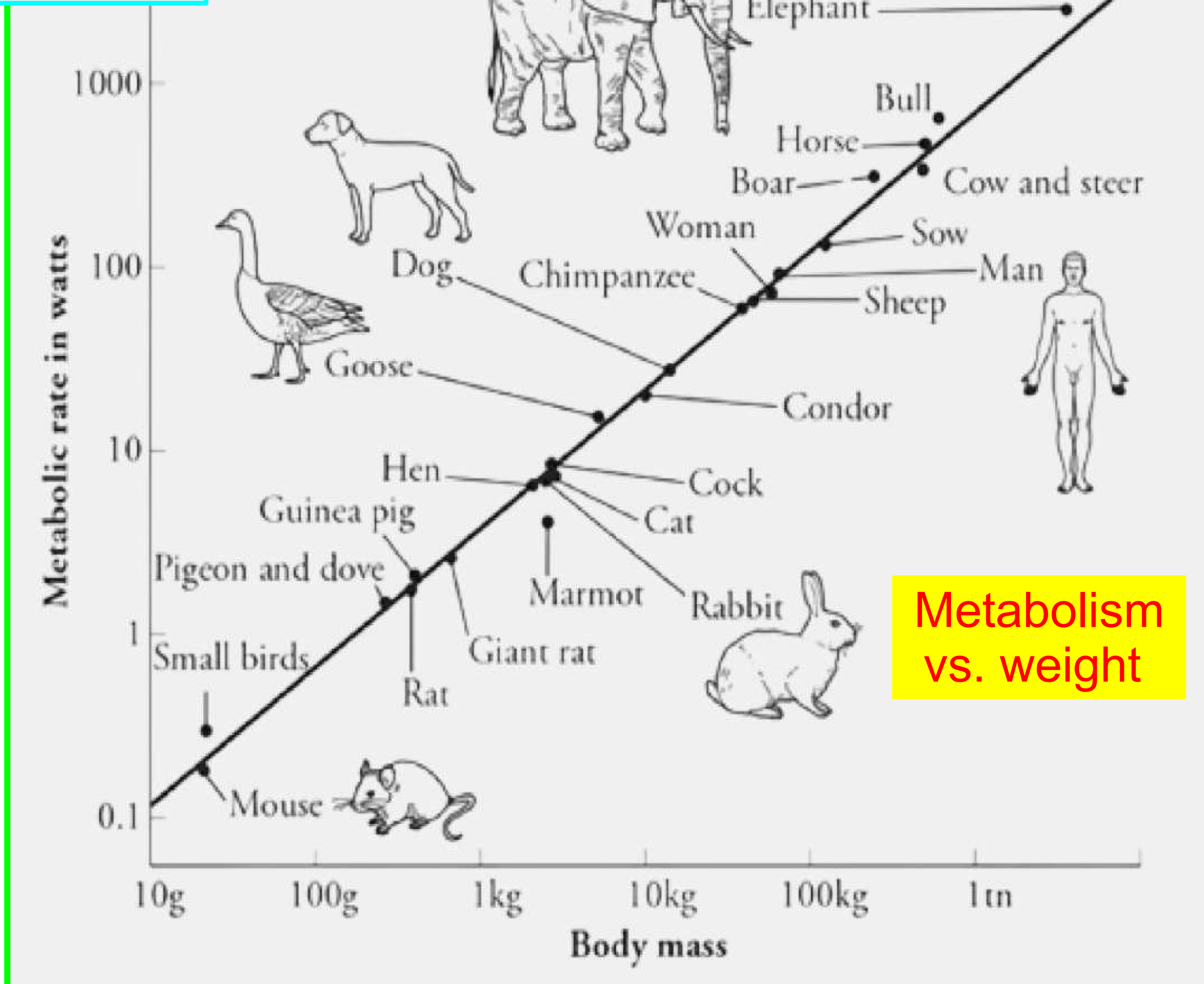
# Scaling

$$A_c = c^2 f(\theta, \phi)$$

# Kleiber's law

$$metabolic\ rate \approx k \cdot (mass)^{\frac{3}{4}}$$




Metabolism vs. weight

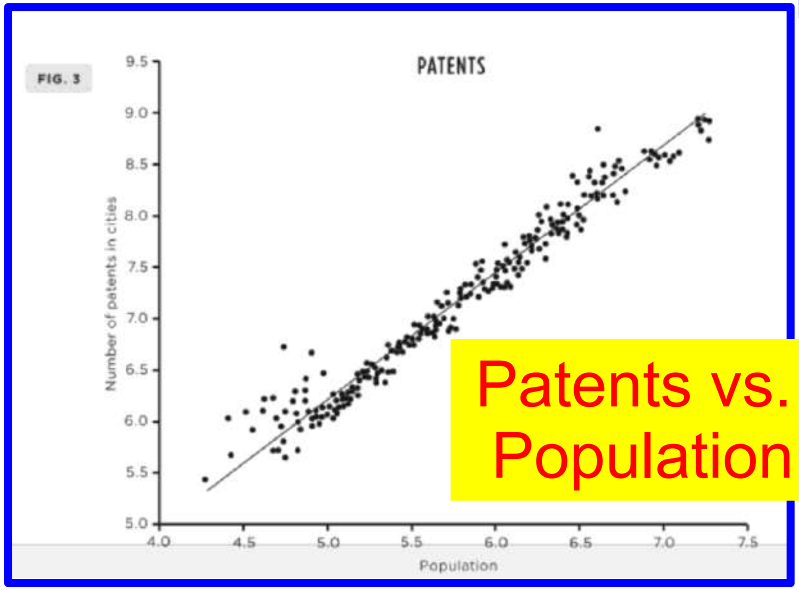
The Reynolds number is defined as:[6]

$$\boxed{\mathrm{Re} = \frac{uL}{\nu} = \frac{\rho u L}{\mu}}$$

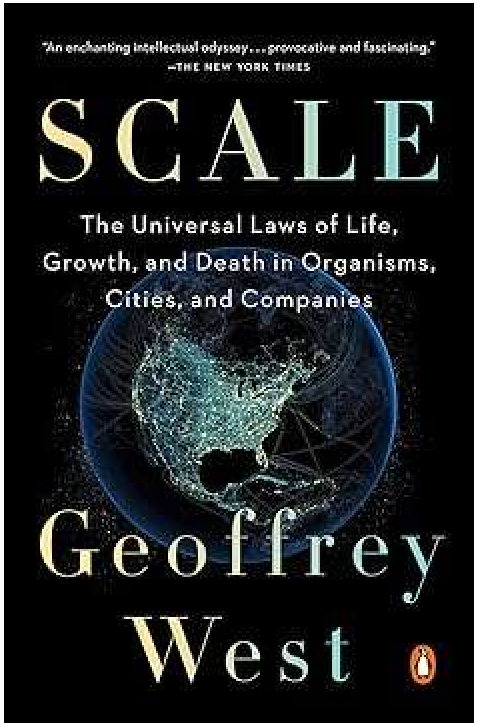
where:

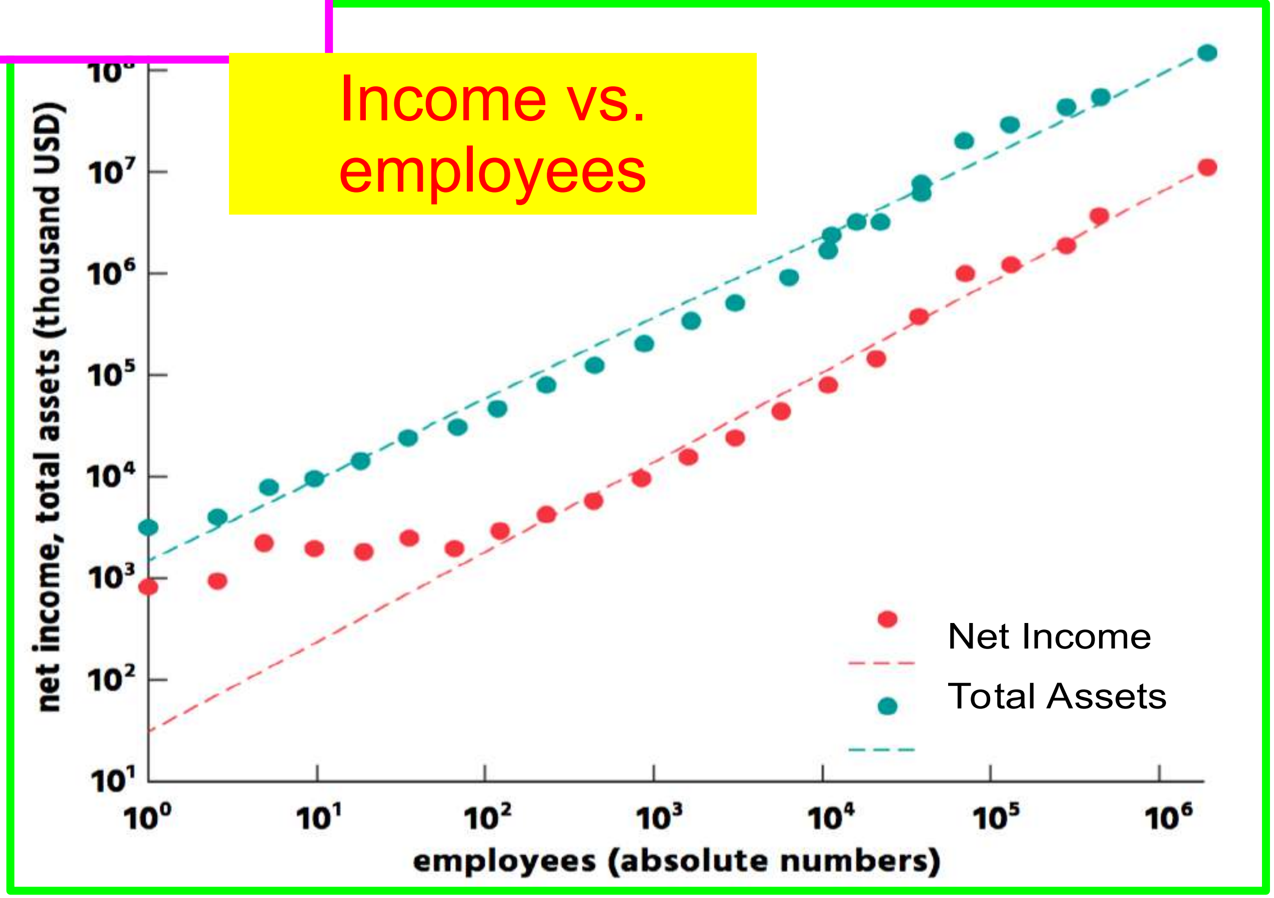
- $\rho$ is the density of the fluid (SI units: kg/m$^3$)
- $u$ is the flow speed (m/s)
- $L$ is a characteristic length (m)
- $\mu$ is the dynamic viscosity of the fluid (Pa·s or N·s/m$^2$ or kg/(m·s))
- $\nu$ is the kinematic viscosity of the fluid (m$^2$/s).

Fluid Flow




Patents vs. Population



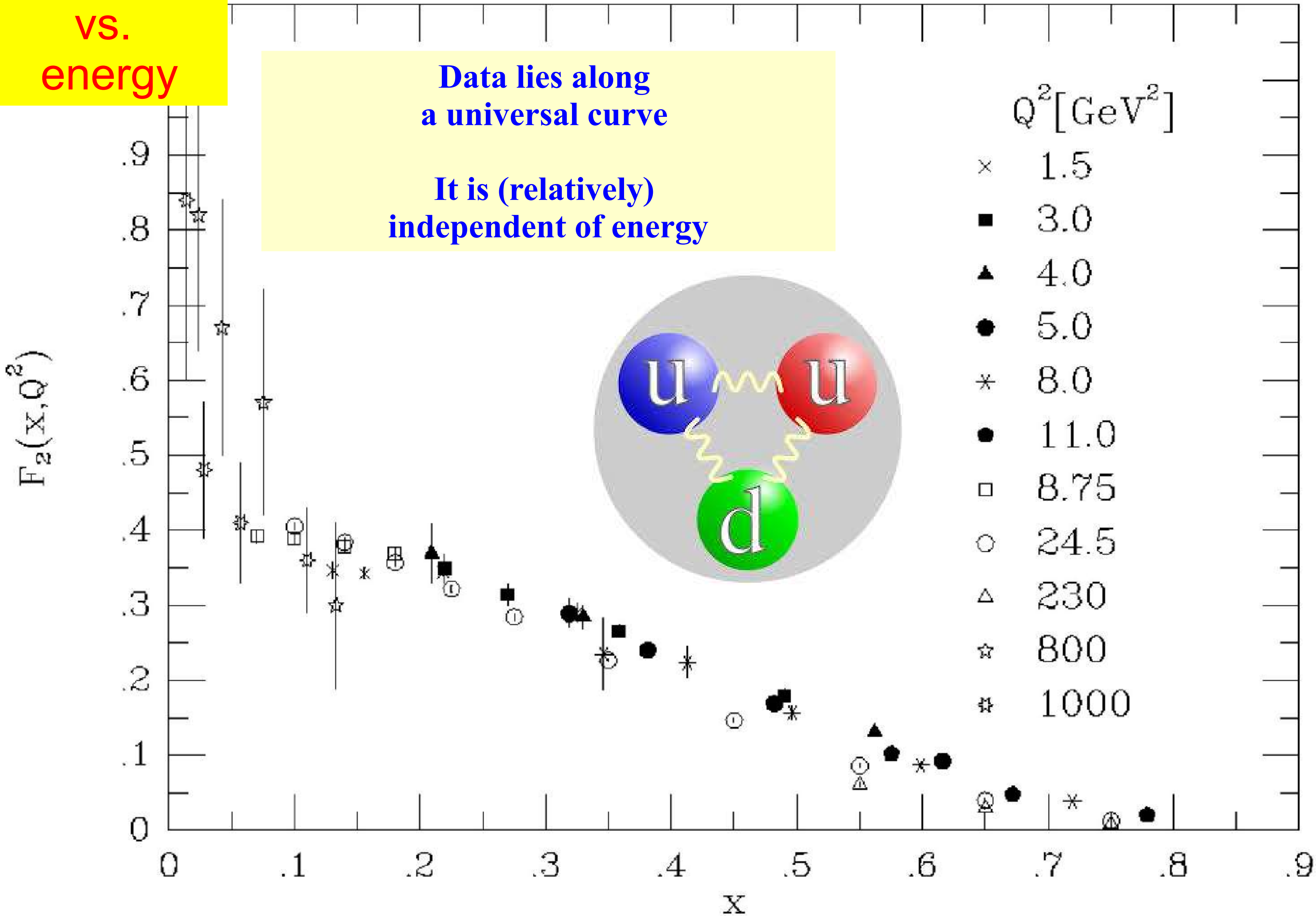

Income vs. employees

# The Scaling of the Proton Structure Function

**Structure vs. energy**

**Data lies along a universal curve**

**It is (relatively) independent of energy**



# Dimensional Analysis, multi-scale problems

...

*where to the logs come from*

$$A_c = c^2 f(\theta, \phi)$$

$$
\begin{aligned}
\sigma(Q^2)=\sigma_0 \Bigg\{ 1 &+ \frac{\alpha_s(Q^2)}{4\pi}(3C_F) + \left[\frac{\alpha_s(Q^2)}{4\pi}\right]^2 \left[ -C_F^2\left[\frac{3}{2}\right] + C_F C_A\left[\frac{123}{2} - 44\zeta(3)\right] + C_F T n_f(-22+16\zeta(3)) \right] \\
&+ \left[\frac{\alpha_s(Q^2)}{4\pi}\right]^3 \Bigg[ C_F^3\left[-\frac{69}{2}\right] + C_F^2 C_A(-127-572\zeta(3)+880\zeta(5)) \\
&+ C_F C_A^2\left[\frac{90445}{54} - \frac{10948}{9}\zeta(3) + \frac{440}{3}\zeta(5)\right] \\
&+ C_F^2 T n_f(-29+304\zeta(3)-320\zeta(5)) + C_F C_A T n_f\left[-\frac{31040}{27} + \frac{7168}{9}\zeta(3) + \frac{160}{3}\zeta(5)\right] \\
&+ C_F T^2 n_f^2\left[\frac{4832}{27} - \frac{1216}{9}\zeta(3)\right] - C_F \pi^2\left[\frac{11}{3}C_A - \frac{4}{3}T n_f\right]^2 + \frac{\left[\sum_f Q_f\right]^2}{(N\sum Q_f^2)}\frac{D}{16}\left[\frac{176}{3} - 128\zeta(3)\right] \Bigg] \Bigg\} .
\end{aligned}
$$

**e+e- Total $\sigma$ at 3 Loops**

$$
\begin{aligned}
H_{q\bar q}^{(2),S+V}(z) = \left[\frac{\alpha_s}{4\pi}\right]^2 \delta(1-z) \Bigg\{ & C_A C_F\left[ [\tfrac{193}{3} - 24\zeta(3)]\ln\left[\frac{Q^2}{M^2}\right] - 11\ln^2\left[\frac{Q^2}{M^2}\right] - \tfrac{12}{5}\zeta(2)^2 + \tfrac{592}{9}\zeta(2) + 28\zeta(3) - \tfrac{1535}{12}\right] \\
& + C_F^2\Bigg[ [18-32\zeta(2)]\ln^2\left[\frac{Q^2}{M^2}\right] + [24\zeta(2)+176\zeta(3)-93]\ln\left[\frac{Q^2}{M^2}\right] \\
& \quad + \tfrac{8}{5}\zeta(2)^2 - 70\zeta(2) - 60\zeta(3) + \tfrac{511}{4}\Bigg] \\
& + n_f C_F\left[ 2\ln^2\left[\frac{Q^2}{M^2}\right] - \tfrac{34}{3}\ln\left[\frac{Q^2}{M^2}\right] + 8\zeta(3) - \tfrac{112}{9}\zeta(2) + \tfrac{127}{6}\right]\Bigg\} \\
+ C_A C_F \Bigg[ & -\tfrac{44}{3}\mathcal{D}_0(z)\ln^2\left[\frac{Q^2}{M^2}\right] + \{[\tfrac{536}{9} - 16\zeta(2)]\mathcal{D}_0(z) - \tfrac{176}{3}\mathcal{D}_1(z)\}\ln\left[\frac{Q^2}{M^2}\right] \\
& - \tfrac{176}{3}\mathcal{D}_2(z) + [\tfrac{1072}{9} - 32\zeta(2)]\mathcal{D}_1(z) + [56\zeta(3) + \tfrac{176}{3}\zeta(2) - \tfrac{1616}{27}]\mathcal{D}_0(z)\Bigg] \\
+ C_F^2 \Bigg[ & [64\mathcal{D}_1(z) + 48\mathcal{D}_0(z)]\ln^2\left[\frac{Q^2}{M^2}\right] + \{192\mathcal{D}_2(z) + 96\mathcal{D}_1(z) - [128+64\zeta(2)]\mathcal{D}_0(z)\}\ln\left[\frac{Q^2}{M^2}\right] \\
& + 128\mathcal{D}_3(z) - (128\zeta(2)+256)\mathcal{D}_1(z) + 256\zeta(3)\mathcal{D}_0(z)\Bigg] \\
+ n_f C_F \Bigg[ & \tfrac{8}{3}\mathcal{D}_0(z)\ln^2\left[\frac{Q^2}{M^2}\right] + [\tfrac{32}{3}\mathcal{D}_1(z) - \tfrac{80}{9}\mathcal{D}_0(z)]\ln\left[\frac{Q^2}{M^2}\right] + \tfrac{32}{3}\mathcal{D}_2(z) - \tfrac{160}{9}\mathcal{D}_1(z) + [\tfrac{224}{27} - \tfrac{32}{3}\zeta(2)]\mathcal{D}_0(z)\Bigg] .
\end{aligned}
$$

**Drelly-Yan at 2 Loops:**

Two mass scales: $\{Q^2, M^2\}$. Logarithms!!!

# Dimensional Regularization

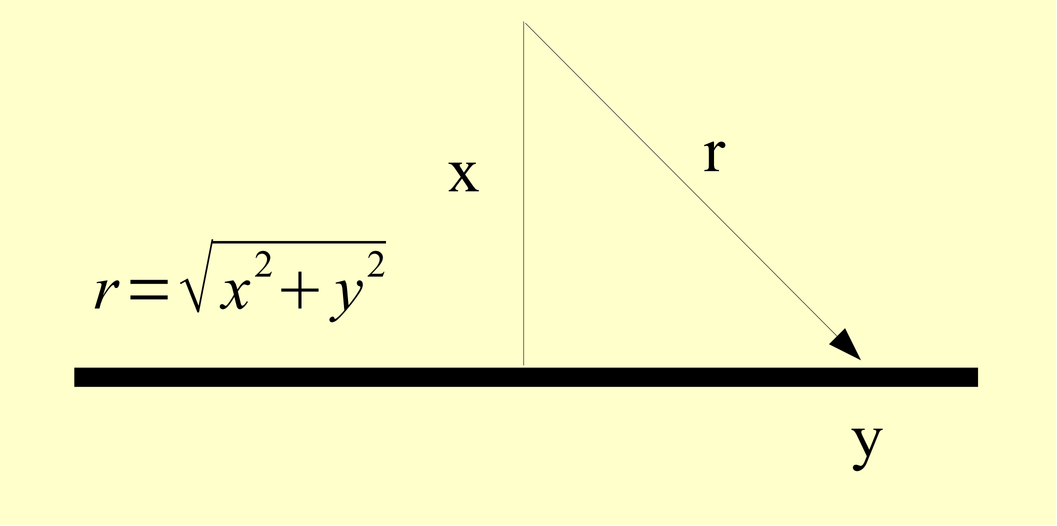
meets

# Freshman E&M

M. Hans, Am.J.Phys. 51 (8) August (1983). p.694
C. Kaufman, Am.J.Phys. 37 (5), May (1969) p.560
B. Delamotte, Am.J.Phys. 72 (2) February (2004) p.170

Regularization, Renormalization, and Dimensional Analysis:
Dimensional Regularization meets Freshman E&M.
Olness & Scalise, arXiv:0812.3578 [hep-ph]

# Infinite Line of Charge

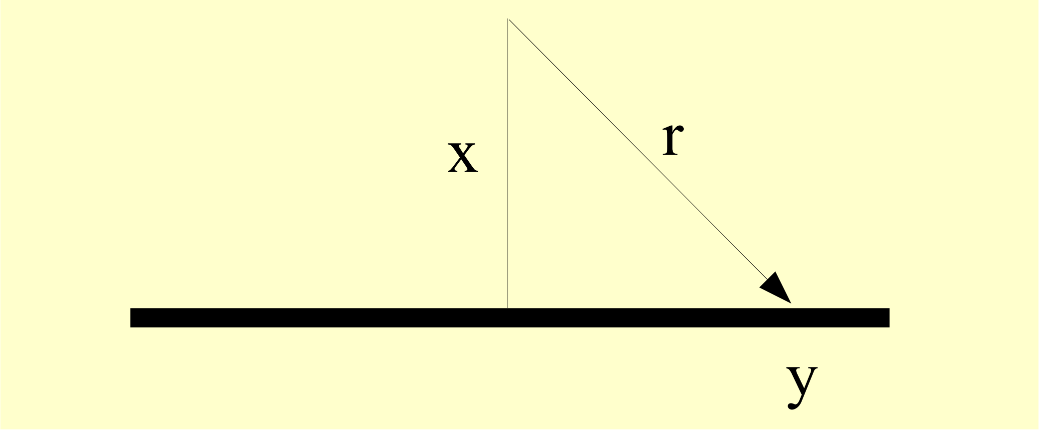



$$dV = \frac{1}{4\pi\epsilon_0} \frac{dQ}{r} \qquad \lambda = \frac{Q}{y}$$

$$V = \underbrace{\frac{\lambda}{4\pi\epsilon_0}}_{\text{potential}} \underbrace{\int_{-\infty}^{+\infty} dy \frac{1}{\sqrt{x^2 + y^2}}}_{\text{dimensionless } f(x)} = \infty$$

*Note: $\infty$ can be very useful*

# Scale Invariance




$$V(k\,x) =$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \; \frac{1}{\sqrt{(kx)^2 + y^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} d\left(\frac{y}{k}\right) \; \frac{1}{\sqrt{x^2 + (y/k)^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dz \; \frac{1}{\sqrt{x^2 + z^2}}$$
$$= V(x)$$

$$V(k\,x) = V(x)$$

*Naively Implies:*
*$V(k\,x) - V(x) = 0$*

*Note:* $\infty + c = \infty$

$\therefore \quad \infty - \infty = c$

*How do we distinguish this from*

$\infty - \infty = c + 17$

***need to regulate***

# Regularization Method #1: Cutoff Method

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{+L} dy \; \frac{1}{\sqrt{x^2 + y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{+L + \sqrt{L^2 + x^2}}{-L + \sqrt{L^2 + x^2}}\right]$$

V(x) depends on artificial regulator L

We cannot remove the regulator L

*Argument of Log is dimensionless*

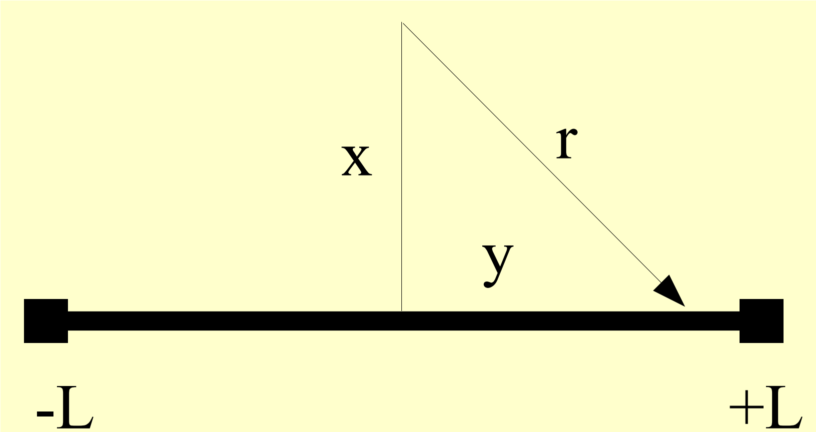
All physical quantities are independent of the regulator:

Electric Field $$E(x) = \frac{-dV}{dx} = \frac{\lambda}{2\pi\epsilon_0 \; x} \; \frac{L}{\sqrt{L^2 + x^2}} \;\to\; \frac{\lambda}{2\pi\epsilon_0 \; x}$$

Energy $$\delta V = V(x_1) - V(x_2) \underset{L\to\infty}{\to} \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{x_2^2}{x_1^2}\right]$$

Problem solved at the expense of an extra scale L
**AND** we have a broken symmetry: translation invariance

# Broken Translational Symmetry




Shift: $y \rightarrow y' = y - c$

$y=[+L+c, -L+c]$

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-L+c}^{+L+c} dy \; \frac{1}{\sqrt{x^2+y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{+(L+c)+\sqrt{(L+c)^2+x^2}}{-(L-c)+\sqrt{(L-c)^2+x^2}}\right]$$

V(r) depends on “y” coordinate!!!

***In QFT, gauge symmetries are important. E.g., Ward identies***

# Regularization Method #2: Dimensional Regularization

Compute in n-dimensions

$$dy \to d^n y = \frac{d\Omega_n}{2} \; y^{n-1} dy$$

$$\Omega_n = \int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \qquad \Omega_{1,2,3,4} = \{2, 2\pi, 4\pi, 2\pi^2\}$$

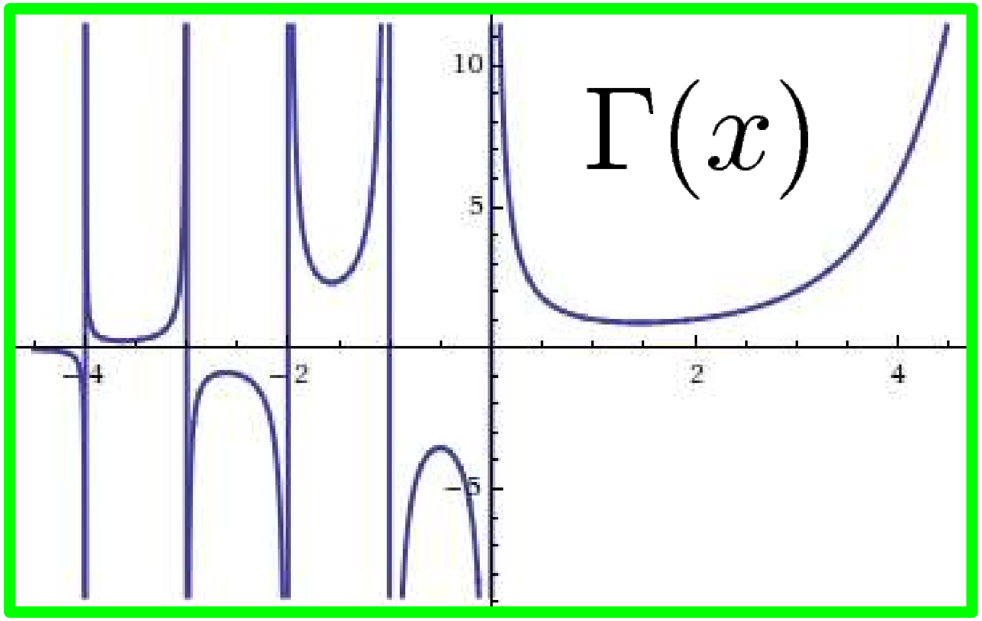
$$V = \frac{\lambda}{4\pi\epsilon_0} \int_0^{+\infty} d\Omega_n \; \frac{y^{n-1}}{\mu^{n-1}} \; \frac{dy}{\sqrt{x^2+y^2}}$$

Each term is individually dimensionless

New scale $\mu$

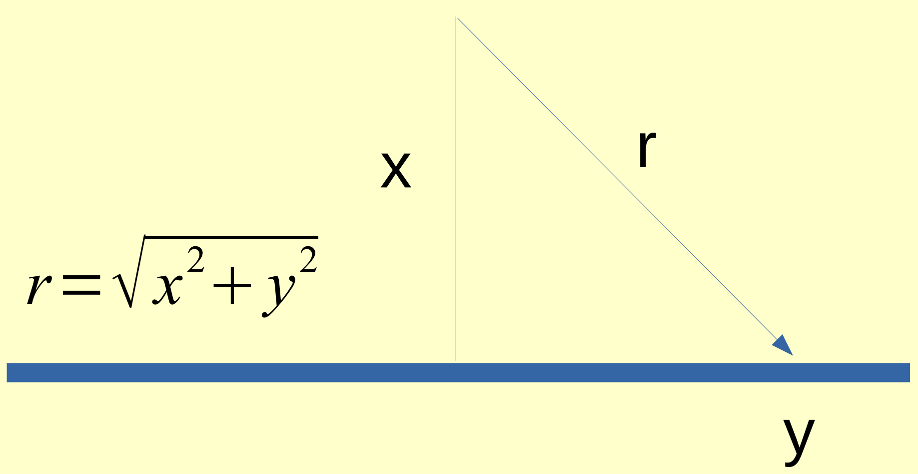
$$n = 1 - 2\epsilon$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \left( \frac{\mu^{2\epsilon}}{x^{2\epsilon}} \frac{\Gamma[\epsilon]}{\pi^{\epsilon}} \right)$$



# Why do we need an extra scale $\mu$ ???






$$r = \sqrt{x^2 + y^2}$$

$$dV = \frac{1}{4\pi\epsilon_0}\,\frac{dQ}{r} \qquad \lambda = \frac{Q}{y}$$

$$\boxed{V = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}}\,\frac{\Gamma[\epsilon]}{\pi^{\epsilon}}\right)}$$

$$V = \underbrace{\frac{\lambda}{4\pi\epsilon_0}}_{\textit{potential}} \times \underbrace{f\left(\frac{x}{\mu}\right)}_{\textit{dimensionless } f(x/\mu)}$$

$$V = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}}\frac{\Gamma[\epsilon]}{\pi^{\epsilon}}\right)$$

All physical quantities are independent of the regulators:

Electric Field $$E(x) = \frac{-dV}{dx} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2\epsilon\mu^{2\epsilon}\Gamma[\epsilon]}{\pi^{\epsilon}x^{1+2\epsilon}}\right] \underset{\epsilon\to 0}{\to} \frac{\lambda}{2\pi\epsilon_0}\frac{1}{x}$$

Energy $$\delta V = V(x_1) - V(x_2) \underset{\epsilon\to 0}{\to} \frac{\lambda}{4\pi\epsilon_0}\log\left[\frac{x_2^2}{x_1^2}\right]$$

Problem solved at the expense of an extra scale μ **AND** regulator ε

Translation invariance is preserved!!!

**Dimensional Regularization respects symmetries**

# Connection to QFT

$$V = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}}\frac{\Gamma[\epsilon]}{\pi^{\epsilon}}\right)$$

Expand in Taylor series in $\varepsilon$

The was the potential from our "Toy" calculation:

$$V \to \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{1\ \pi}\right] + \ln\left[\frac{\mu^2}{x^2}\right]\right]$$

This is a partial result from
a <u>real</u> NLO Drell-Yan Calculation:
*Cf., B. Potter*

$$\frac{D(\epsilon)}{\epsilon} = \left(\frac{4\pi\mu^2}{Q^2}\right)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \to \left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{4\pi}\right] + \ln\left[\frac{\mu^2}{Q^2}\right]\right]$$

# Renormalization

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$

Original

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$

MS

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$

MS-Bar

Physical quantities are independent of renormalization scheme!

$$V_{\overline{MS}}(x_1)-V_{\overline{MS}}(x_2)=\delta V=V_{MS}(x_1)-V_{MS}(x_2)$$

But only if performed consistently:

$$V_{\overline{MS}}(x_1)-V_{MS}(x_2)\neq\delta V\neq V_{MS}(x_1)-V_{\overline{MS}}(x_2)$$

Regulator provides unique definition of V, f, ω

Cutoff regulator L:

simple, but does NOT respect symmetries

Dimensional regulator ε:

respects symmetries: translation, Lorentz, Gauge invariance

introduces new scale μ

All physical quantities (E, dV, σ) are independent of the regulator

AND the new scale μ

Renormalization group equation: $d\sigma/d\mu=0$

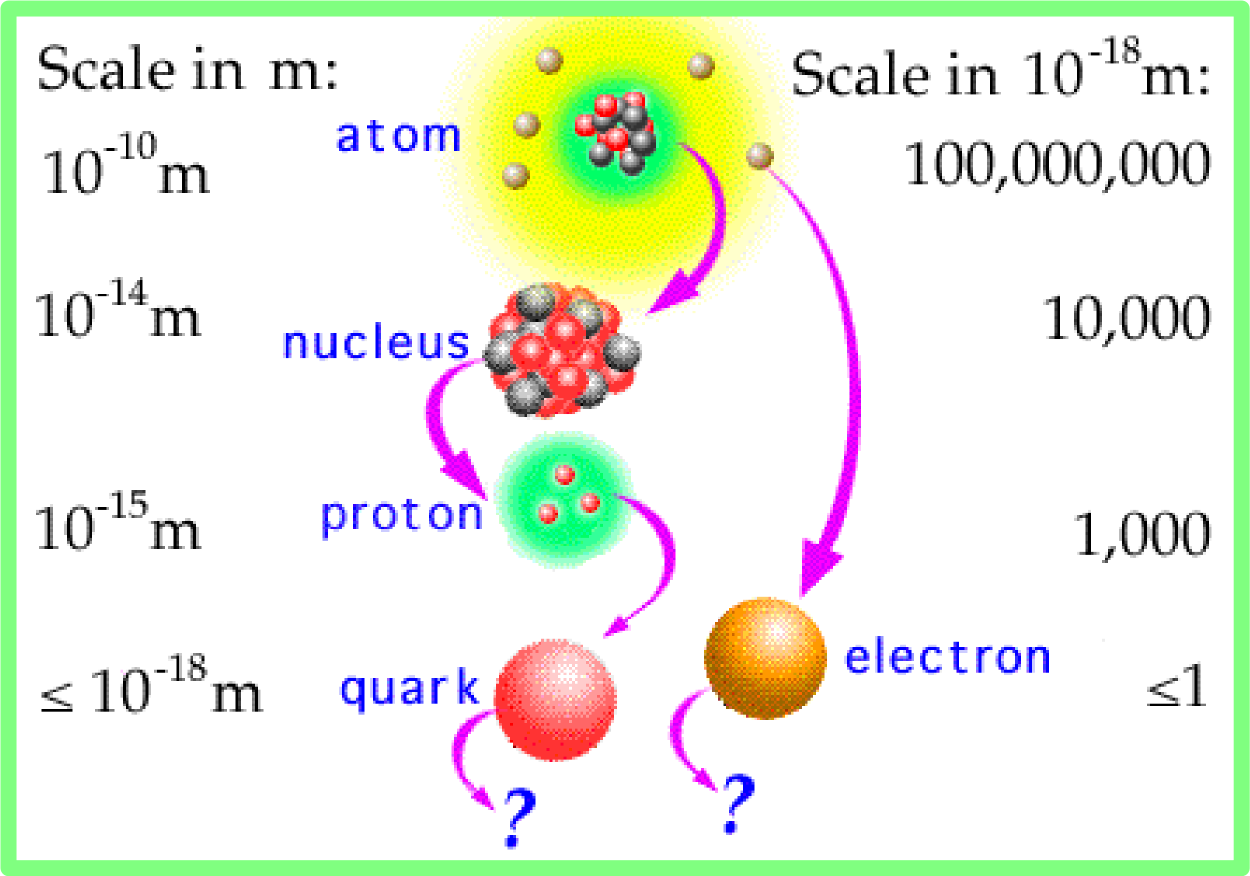
We can define renormalized quantities (V,f,ω)

Renormalized (V,f,ω) are scheme dependent and arbitrary

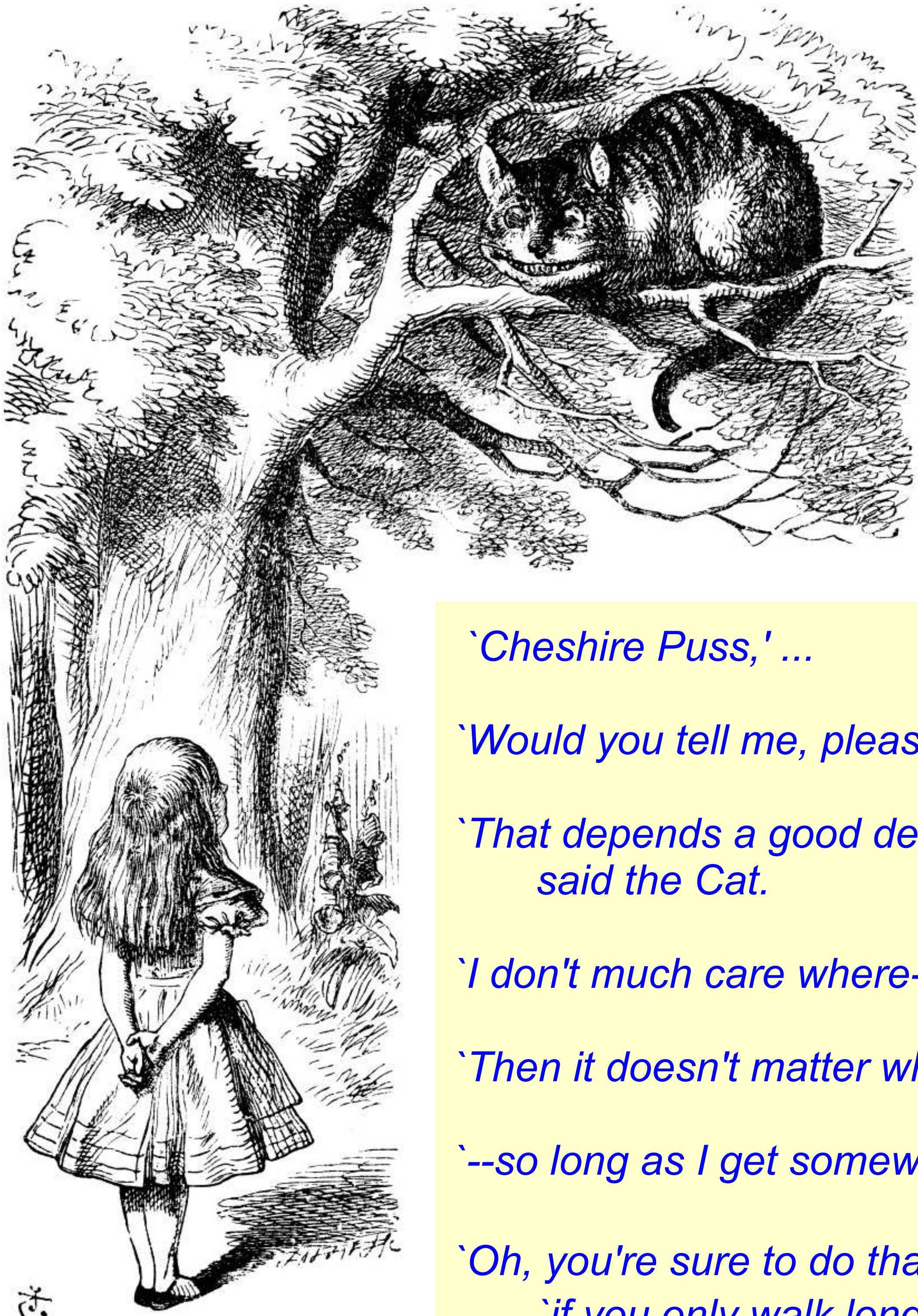
Physical quantities (E,dV, σ) are unique and scheme independent

if we apply the scheme consistently

When we do our calculations,
where does the mysterious
$\mu$ does renormalization scale
come from???




What is inside the proton/nucleon???
The answer depends on how closely you look.



The answer is dependent upon the question

... an old preprint by Charles Dodgson

`Cheshire Puss,' ...

`Would you tell me, please, which way I ought to go from here?'

`That depends a good deal on where you want to get to,'
said the Cat.

`I don't much care where--' said Alice.

`Then it doesn't matter which way you go,' said the Cat.

`--so long as I get somewhere,' Alice added as an explanation.

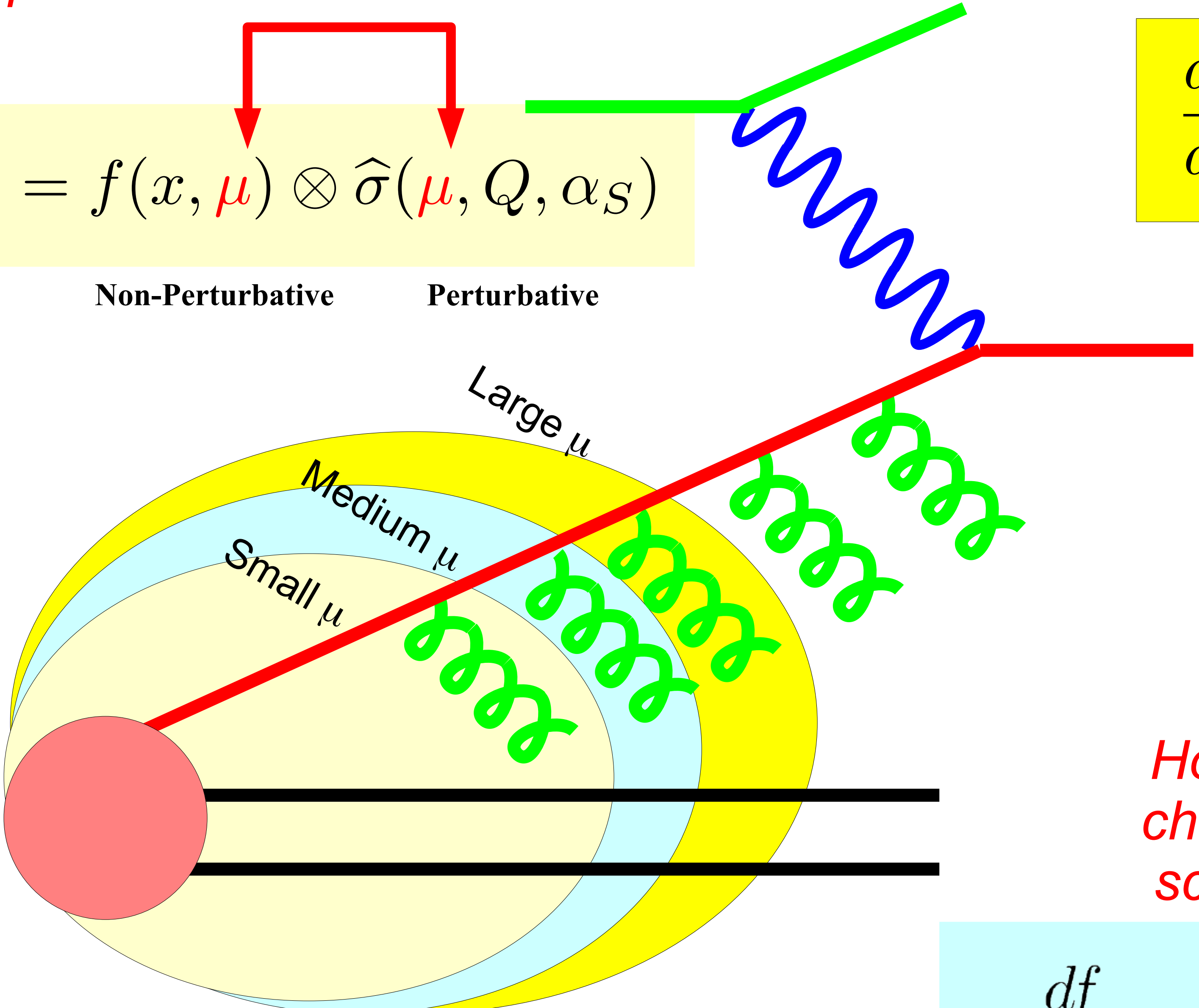
`Oh, you're sure to do that,' said the Cat,
`if you only walk long enough.'

*$\mu$ dependence must balance*

$$\sigma(Q,x) = f(x,\mu) \otimes \widehat{\sigma}(\mu, Q, \alpha_S)$$

**Observable** **Non-Perturbative** **Perturbative**

$$\frac{d\sigma}{d\mu} = 0$$




*How does f change with scale $\mu$???*

$$\frac{df}{d\ \ln[\mu]} = ???$$

# Renormalization Group Equation

Parton Model

$$\sigma = f(\mu) \otimes \omega(\mu)$$

$$\omega \equiv \hat{\sigma}$$

Not physical!
Poor notation

Renormalization Group Equation

$$\frac{d\sigma}{d\mu} = 0 = \frac{d\tilde{f}}{d\mu}\,\tilde{\omega} + \tilde{f}\,\frac{d\tilde{\omega}}{d\mu}$$

Take Mellin Transform

Separation of variables

$$\frac{1}{\tilde{f}}\,\frac{d\tilde{f}}{d\ln[\mu]} = -\gamma = -\frac{1}{\tilde{\omega}}\,\frac{d\tilde{\omega}}{d\ln[\mu]}$$

DGLAP Equation

DGLAP

$$\frac{d\tilde{f}}{d\ln[\mu]} = -\tilde{f}\,\gamma \qquad \frac{df}{d\ln[\mu]} = P \otimes f$$

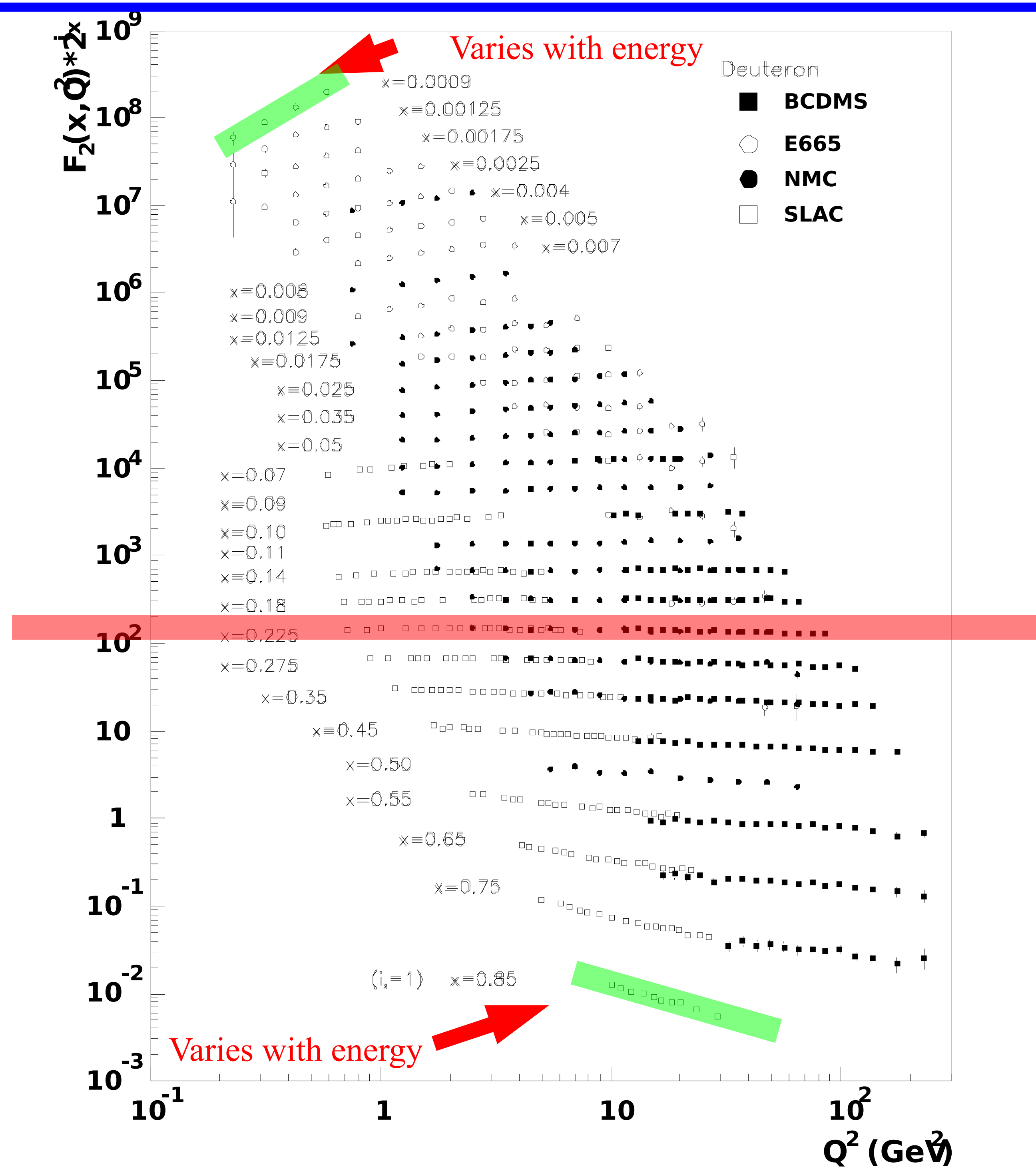
$$\tilde{f} \sim \mu^{-\gamma}$$

Anomalous Dimension

If "f" scaled,
γ would vanish

It is the dimension of
the mass scaling

# The Scaling of the Proton Structure Function




**Data is (relatively) independent of energy**

Scaling Violations observed at extreme x values

# Homework: Mellin Transform



$$\widetilde{f}(n) = \int_0^1 dx\, x^{n-1}\, f(x)$$

$$\sigma = f \otimes \omega$$

$$f(x) = \frac{1}{2\pi i} \int_C dn\, x^{-n}\, \widetilde{f}(n)$$

$$\widetilde{\sigma} = \widetilde{f}\, \widetilde{\omega}$$

$C$ is parallel to the imaginary axis, and to the right of all singularities

1) Take the Mellin transform of $f(x) = \sum_{m=1}^{\infty} a_m x^m$, and verify the inverse transform of $\widetilde{f}$ regenerates $f(x)$

2) Take the Mellin transform of $\sigma = f \otimes \omega$ to demonstrate that the Mellin transform separates a convolution yields $\widetilde{\sigma} = \widetilde{f}\, \widetilde{\omega}$.

A useful reference:

*Courant, Richard and Hilbert, David. Methods of Mathematical Physics, Vol. 1. New York: Wiley, 1989. 561 p.*

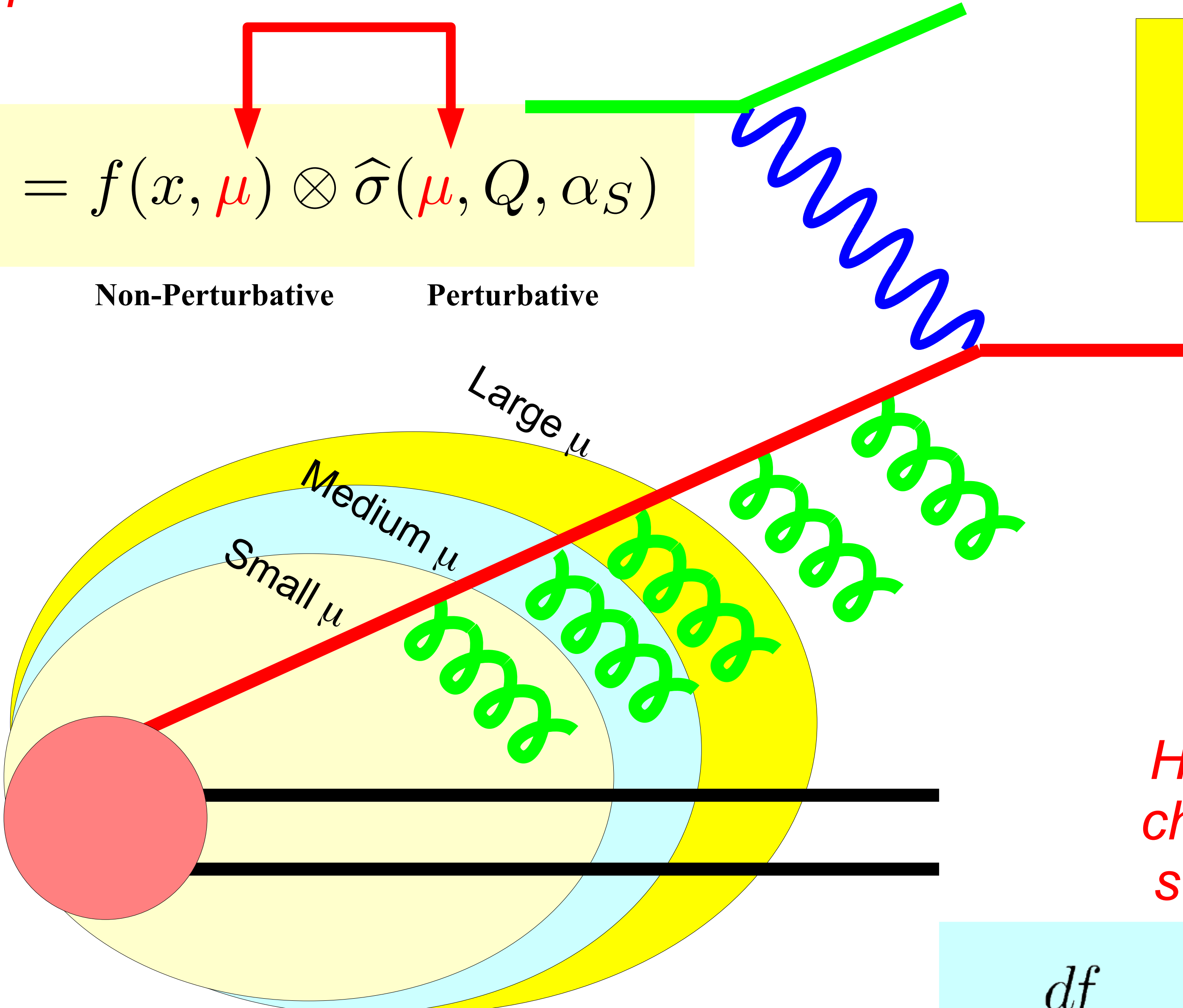
# What is inside the proton/nucleon??? ... *it depends ...*

*$\mu$ dependence must balance*

$$\sigma(Q,x) = f(x,\mu) \otimes \widehat{\sigma}(\mu, Q, \alpha_S)$$

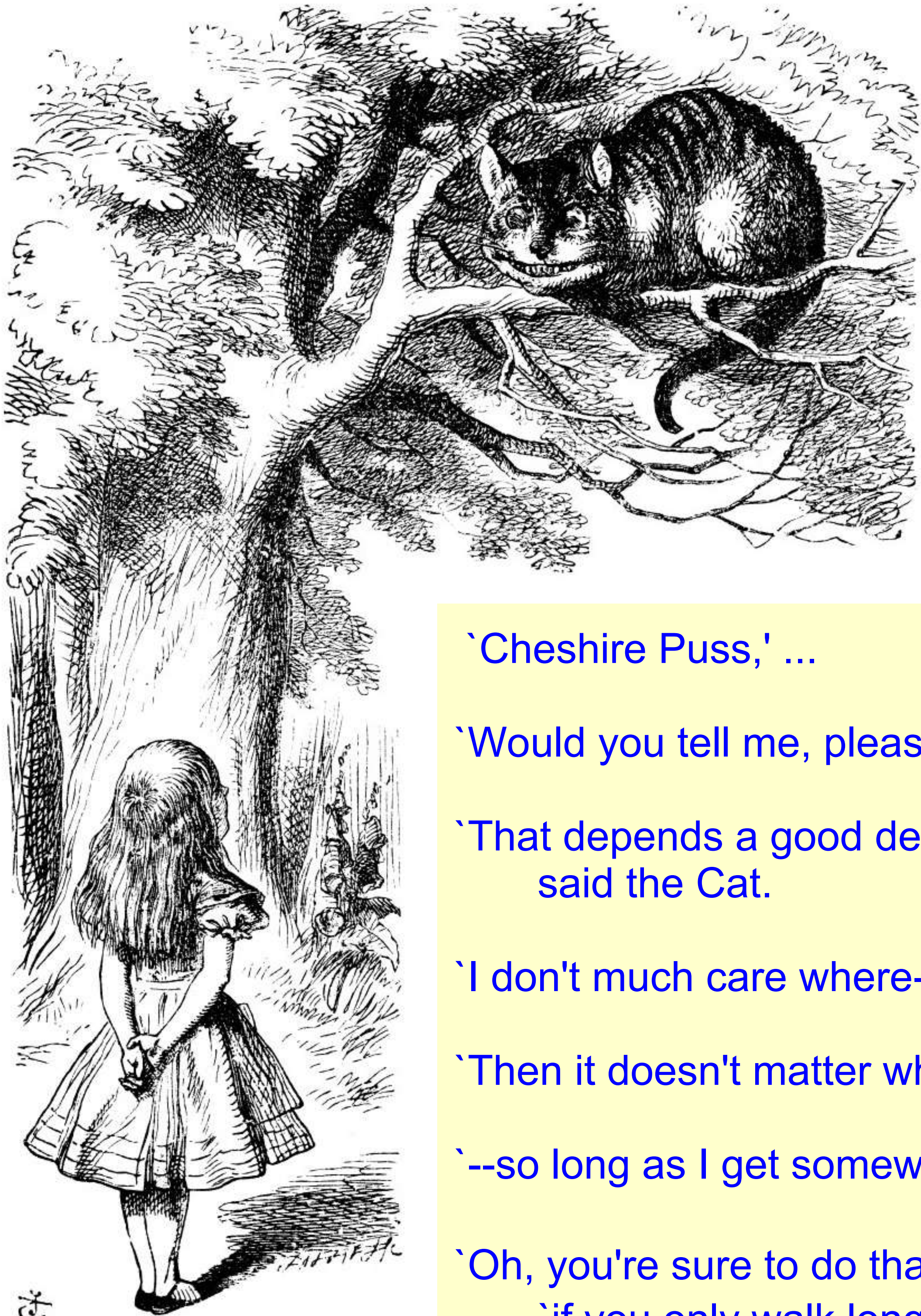
**Observable** **Non-Perturbative** **Perturbative**

$$\frac{d\sigma}{d\mu} = 0$$




*How does f change with scale $\mu$???*

$$\frac{df}{d\,\ln[\mu]} = ???$$



The answer is dependent upon the question

`Cheshire Puss,' ...

`Would you tell me, please, which way I ought to go from here?'

`That depends a good deal on where you want to get to,'
said the Cat.

`I don't much care where--' said Alice.

`Then it doesn't matter which way you go,' said the Cat.

`--so long as I get somewhere,' Alice added as an explanation.

`Oh, you're sure to do that,' said the Cat,
`if you only walk long enough.'

# LEFTOVER

# Where do the Logs come from?

# Putting it all together: $\sigma_{TOT} = \sigma_{RESUM} + \sigma_{PERT} - \sigma_{ASYM}$

Let's expand out the resummed expression:

$$\frac{d\sigma}{dq_T^2} \sim \frac{\alpha_s L}{q_T^2} \; e^{\alpha_s (L^2+L)} \sim \frac{1}{q_T^2} \left\{ \alpha_s L \; + \; \alpha_s^2 (L^3 + L^2) + \ldots \right\}$$

Compare the above with the perturbative and asymptotic results:

$$d\sigma_{resum} \sim \left\{ \alpha_s L \; + \; \alpha_s^2 (L^3 + L^2 + \; 0 + 0) \; + \; \alpha_s^3 (L^5 + L^4) + \ldots \right\}$$
$$d\sigma_{pert} \sim \left\{ \alpha_s L \; + \; \alpha_s^2 (L^3 + L^2 + L^1 + 1) \; + \; \alpha_s^3 (0 + 0) \right\}$$
$$d\sigma_{asym} \sim \left\{ \alpha_s L \; + \; \alpha_s^2 (L^3 + L^2 + \; 0 + 0) \; + \; \alpha_s^3 (0 + 0) \right\}$$

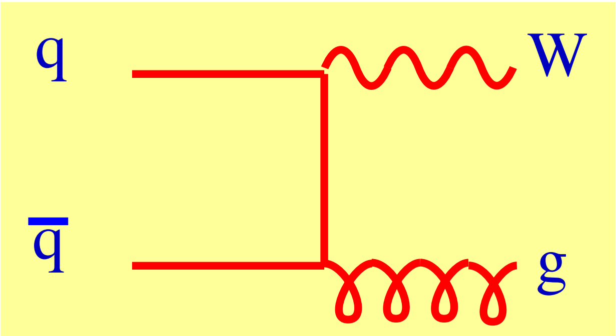
Note that $\sigma_{ASYM}$ removes overlap between $\sigma_{RESUM}$ and $\sigma_{PERT}$.

We expect:

$\sigma_{RESUM}$ is a good representation for $q_T \sim 0$

$\sigma_{PERT}$ is a good representation for $q_T \sim M_W$

# NLO $P_T$ distribution for the W boson




In the limit $P_T \to 0$

$$\frac{d\sigma}{d\tau\, dy\, dp_T^2} \approx \left(\frac{d\sigma}{d\tau\, dy}\right)_{Born} \times \frac{4\alpha_s}{3\pi} \frac{\ln s/p_T^2}{p_T^2}$$

finite

singular

$$\int_0^s \frac{d\sigma}{d\tau\, dy\, dp_T^2}\, dp_T^2 = \left(\frac{d\sigma}{d\tau\, dy}\right)_{Born} + O(\alpha_s)$$

$$\int_0^{p_T^2} \frac{d\sigma}{d\tau\, dy\, dp_T^2}\, dp_T^2 = \left(\frac{d\sigma}{d\tau\, dy}\right)_{Born} \times \left\{1 - \int_{p_T^2}^{s} \frac{4\alpha_s}{3\pi} \frac{\ln s/p_T^2}{p_T^2}\, d\, p_T^2\right\}$$

$$= \left(\frac{d\sigma}{d\tau\, dy}\right)_{Born} \times \left\{1 - \frac{2\alpha_s}{3\pi} \ln^2 \frac{s}{p_T^2}\right\}$$

effect of gluon emission

$$= \left(\frac{d\sigma}{d\tau\, dy}\right)_{Born} \times \exp\left\{\frac{2\alpha_s}{3\pi} \ln^2 \frac{s}{p_T^2}\right\}$$

assume this exponentiates

Parisi & Petronzio, NP B154, 427 (1979)
Dokshitzer, D'yakanov, Troyan, Phy. Rep. 58, 271 (1980)

## Resummation of soft gluons: Step #1

Differentiating the previous expression for $d^2\sigma/d\tau\, dy$

Sudakov Form Factor

$$\frac{d\sigma}{d\tau\, dy\, dp_T^2} \approx \left(\frac{d\sigma}{d\tau\, dy}\right)_{Born} \times \frac{4\alpha_s}{3\pi}\,\frac{\ln s/p_T^2}{p_T^2} \times \exp\left\{-\frac{2\alpha_s}{3\pi}\ln^2\frac{s}{p_T^2}\right\}$$

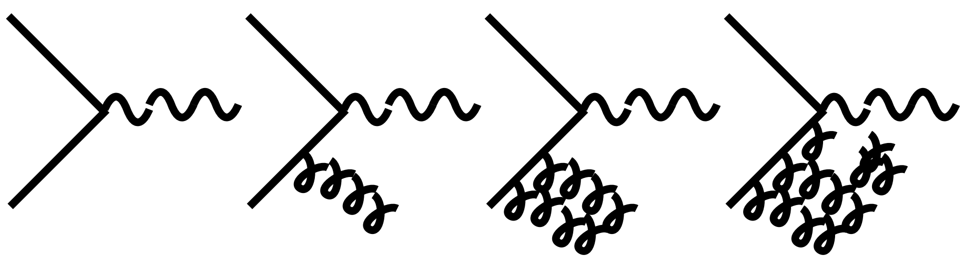
*finite at $p_T=0$*

We just resummed (exponentiated) an infinite series of soft gluon emissions

$$e^{-\alpha_s L^2} \approx 1 - \alpha_s L^2 + \frac{(\alpha_s L^2)^2}{2!} - \frac{(\alpha_s L^2)^3}{3!} + \ldots$$

$$L = \ln\frac{s}{p_T^2}$$

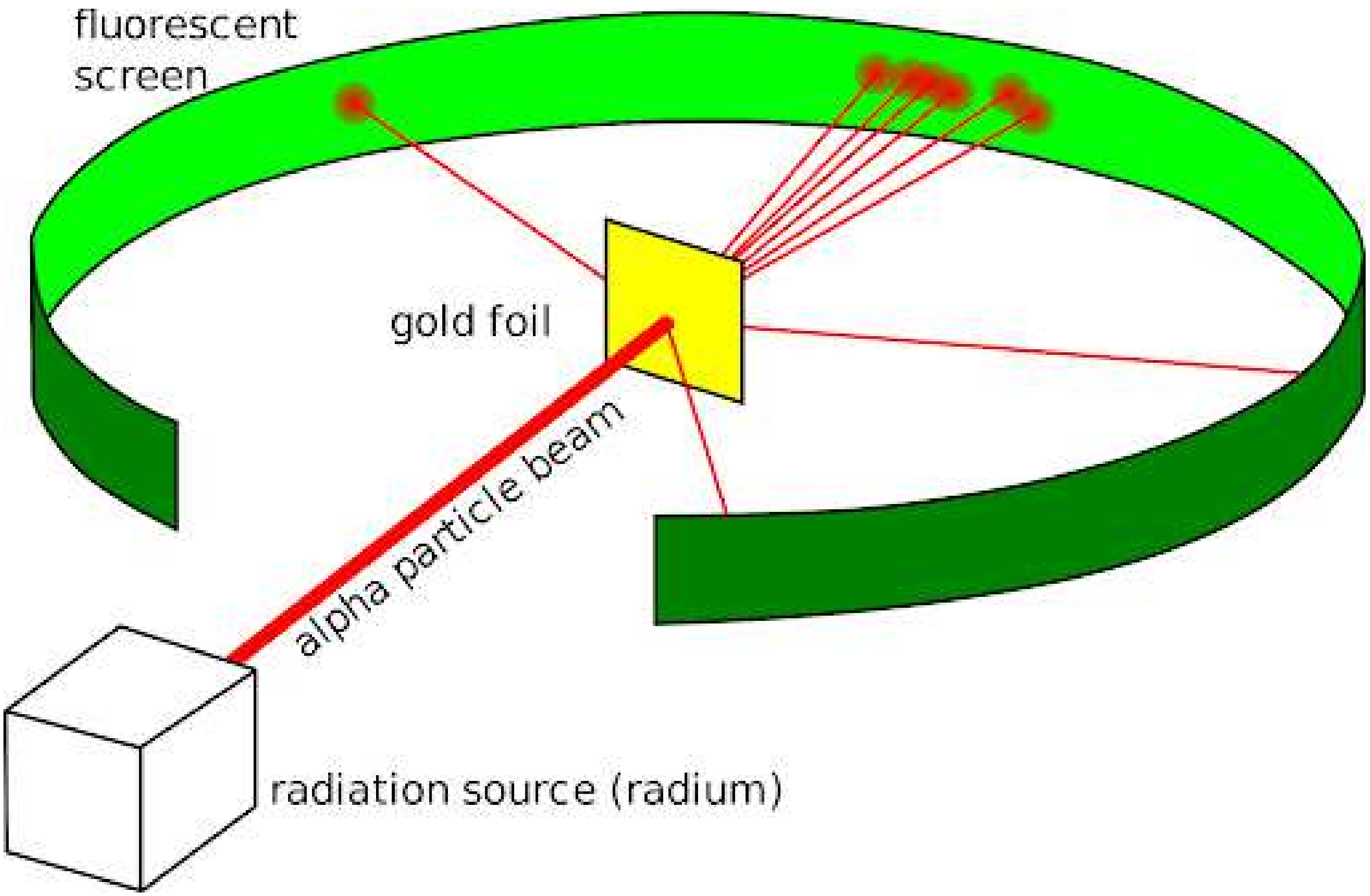
Soft gluon emissions treated as uncorrelated



*I've skipped over some details ..*

Parisi & Petronzio, NP B154, 427 (1979)
Dokshitzer, D'yakanov, Troyan, Phy. Rep. 58, 271 (1980)

Curci, Greco, Srivastava, PRL 43, 834 (1979); NP B159, 451 (1979)
Jeff Owens, 2000 CTEQ Summer School Lectures


fluorescent
screen
gold foil
alpha particle beam
radiation source (radium)

# Total Cross Section: σ($e^+e^-$) at 3 Loops

$$\sigma(Q^2)=\sigma_0\Bigg\{1+\frac{\alpha_s(Q^2)}{4\pi}(3C_F)+\left[\frac{\alpha_s(Q^2)}{4\pi}\right]^2\left[-C_F^2\left[\frac{3}{2}\right]+C_FC_A\left[\frac{123}{2}-44\zeta(3)\right]+C_FTn_f(-22+16\zeta(3))\right]$$

$$+\left[\frac{\alpha_s(Q^2)}{4\pi}\right]^3\Bigg[C_F^3\left[-\frac{69}{2}\right]+C_F^2C_A(-127-572\zeta(3)+880\zeta(5))$$

$$+C_FC_A^2\left[\frac{90445}{54}-\frac{10948}{9}\zeta(3)+\frac{440}{3}\zeta(5)\right]$$

$$+C_F^2Tn_f(-29+304\zeta(3)-320\zeta(5))+C_FC_ATn_f\left[-\frac{31040}{27}+\frac{7168}{9}\zeta(3)+\frac{160}{3}\zeta(5)\right]$$

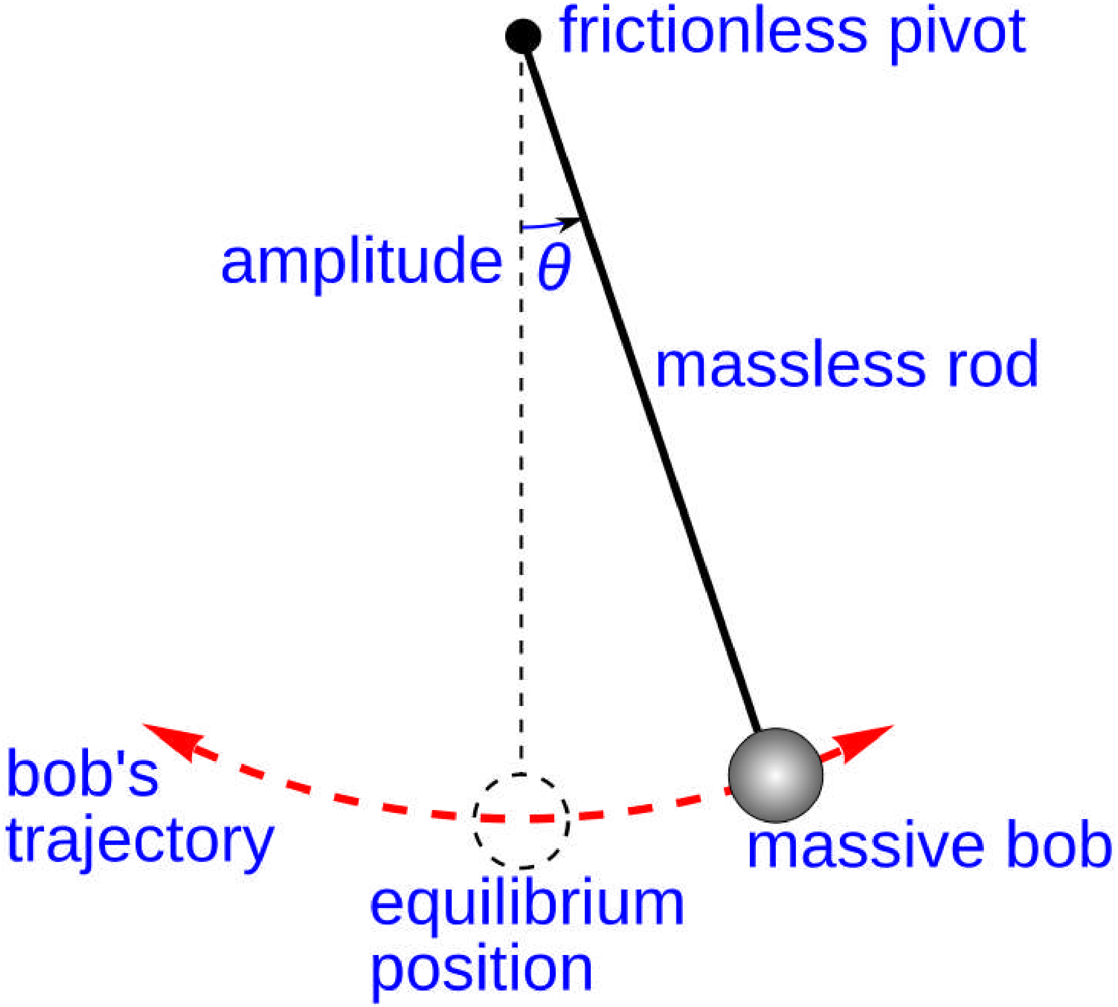
$$+C_FT^2n_f^2\left[\frac{4832}{27}-\frac{1216}{9}\zeta(3)\right]-C_F\pi^2\left[\frac{11}{3}C_A-\frac{4}{3}Tn_f\right]^2+\frac{\left[\sum_f Q_f\right]^2}{(N\sum_f Q_f^2)}\frac{D}{16}\left[\frac{176}{3}-128\zeta(3)\right]\Bigg]\Bigg\}.$$

One mass scale: $Q^2$. No logarithms!!!

# Drelly-Yan at 2 Loops:

$$
\begin{aligned}
H_{q\bar{q}}^{(2),S+V}(z) = \left[\frac{\alpha_s}{4\pi}\right]^2 \delta(1-z)\Bigg\{ & C_A C_F\left[\left[\tfrac{193}{3}-24\zeta(3)\right]\ln\left[\frac{Q^2}{M^2}\right]-11\ln^2\left[\frac{Q^2}{M^2}\right]-\tfrac{12}{5}\zeta(2)^2+\tfrac{592}{9}\zeta(2)+28\zeta(3)-\tfrac{1535}{12}\right] \\
& + C_F^2\left[[18-32\zeta(2)]\ln^2\left[\frac{Q^2}{M^2}\right]+[24\zeta(2)+176\zeta(3)-93]\ln\left[\frac{Q^2}{M^2}\right]\right. \\
& \left. + \tfrac{8}{5}\zeta(2)^2-70\zeta(2)-60\zeta(3)+\tfrac{511}{4}\right] \\
& + n_f C_F\left[2\ln^2\left[\frac{Q^2}{M^2}\right]-\tfrac{34}{3}\ln\left[\frac{Q^2}{M^2}\right]+8\zeta(3)-\tfrac{112}{9}\zeta(2)+\tfrac{127}{6}\right]\Bigg\} \\
& + C_A C_F\left[-\tfrac{44}{3}\mathcal{D}_0(z)\ln^2\left[\frac{Q^2}{M^2}\right]+\left\{\left[\tfrac{536}{9}-16\zeta(2)\right]\mathcal{D}_0(z)-\tfrac{176}{3}\mathcal{D}_1(z)\right\}\ln\left[\frac{Q^2}{M^2}\right]\right. \\
& \left. -\tfrac{176}{3}\mathcal{D}_2(z)+\left[\tfrac{1072}{9}-32\zeta(2)\right]\mathcal{D}_1(z)+\left[56\zeta(3)+\tfrac{176}{3}\zeta(2)-\tfrac{1616}{27}\right]\mathcal{D}_0(z)\right] \\
& + C_F^2\left[[64\mathcal{D}_1(z)+48\mathcal{D}_0(z)]\ln^2\left[\frac{Q^2}{M^2}\right]+\{192\mathcal{D}_2(z)+96\mathcal{D}_1(z)-[128+64\zeta(2)]\mathcal{D}_0(z)\}\ln\left[\frac{Q^2}{M^2}\right]\right. \\
& \left. +128\mathcal{D}_3(z)-(128\zeta(2)+256)\mathcal{D}_1(z)+256\zeta(3)\mathcal{D}_0(z)\right] \\
& + n_f C_F\left[\tfrac{8}{3}\mathcal{D}_0(z)\ln^2\left[\frac{Q^2}{M^2}\right]+\left[\tfrac{32}{3}\mathcal{D}_1(z)-\tfrac{80}{9}\mathcal{D}_0(z)\right]\ln\left[\frac{Q^2}{M^2}\right]+\tfrac{32}{3}\mathcal{D}_2(z)-\tfrac{160}{9}\mathcal{D}_1(z)+\left[\tfrac{224}{27}-\tfrac{32}{3}\zeta(2)\right]\mathcal{D}_0(z)\right] .
\end{aligned}
$$

Two mass scales: $\{Q^2, M^2\}$. Logarithms!!!



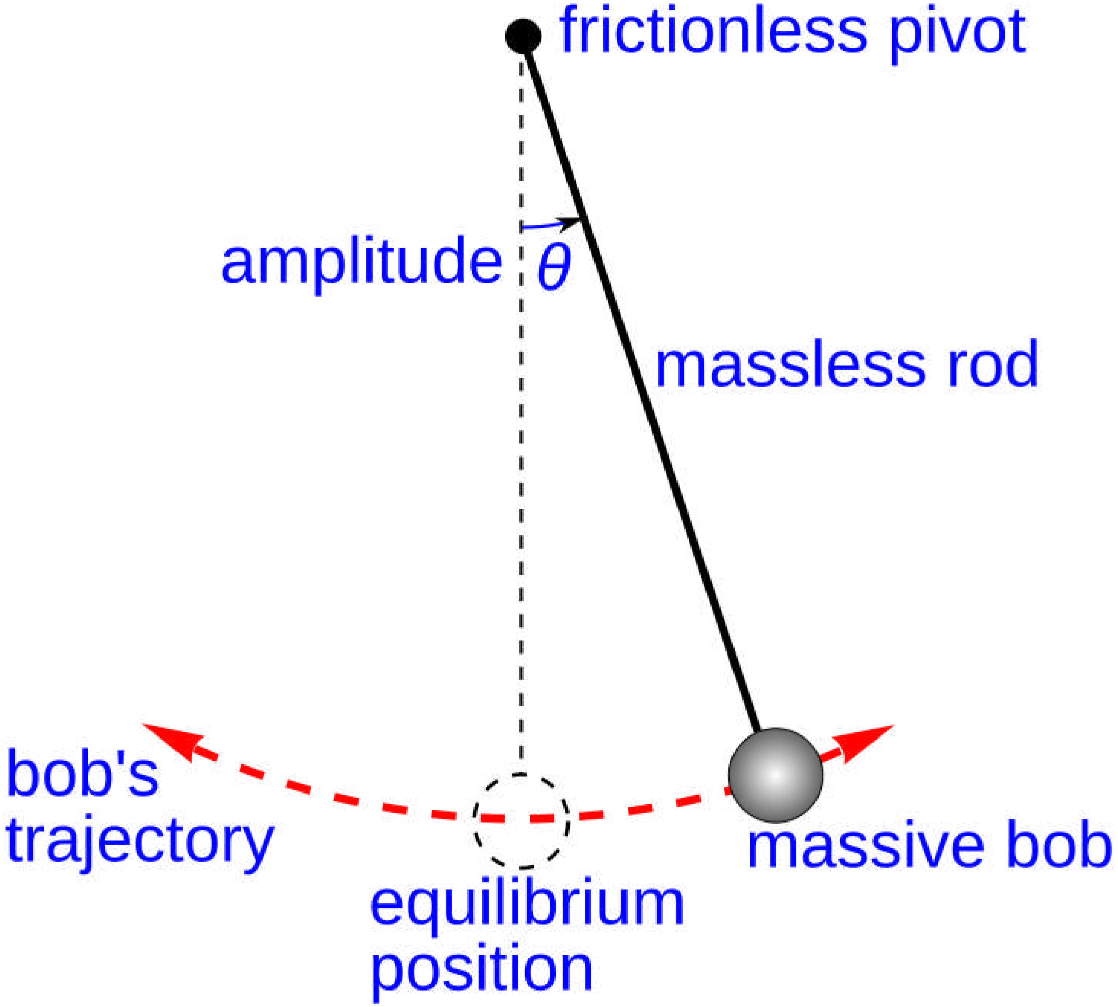

L = meters

g = meters/sec$^2$

m = kilograms

Period: (sec)

T=




L = meters

g = meters/sec$^2$

m = kilograms

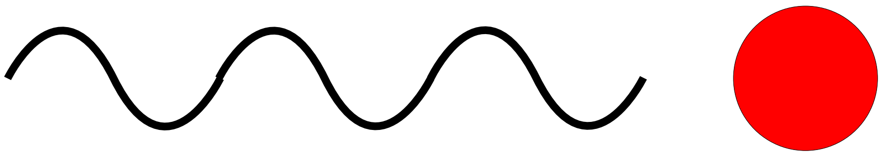
Period: (sec)

T= $2\pi\sqrt{\dfrac{L}{g}}$

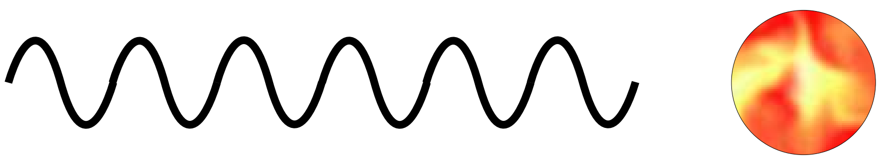
**Independent of mass**

# Structure of the Proton



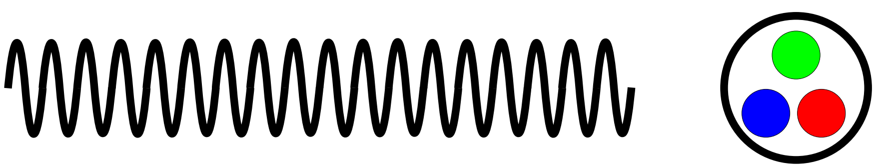


$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times F\left(\frac{Q^2}{\Lambda^2}\right)$$

$\Lambda$ of order of the proton mass scale



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times \sum_i e_i^2$$