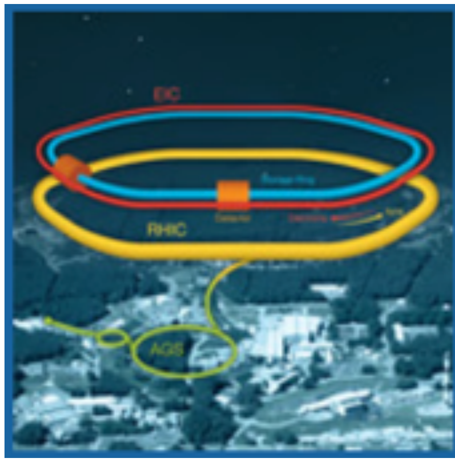


The 2025 CFNS-SURGE Summer Workshop on the Physics of the Electron-Ion Collider

Lattice QCD (selected topics)

Lecture 1

Martha Constantinou

Temple University

Center for Frontiers in Nuclear Science

Stony Brook University, USA

June 2 - 13, 2025

Let's get to know each other!

Have you ever written a code longer than 500 lines?

Have you ever stared at data at 3 a.m.?

Have you ever blamed a bug on systematic effects?

Do you believe we will ever solve QCD analytically in 4D?

Lattice QCD is THE first-principle approach for QCD.

I find it equally frustrating and beautiful...

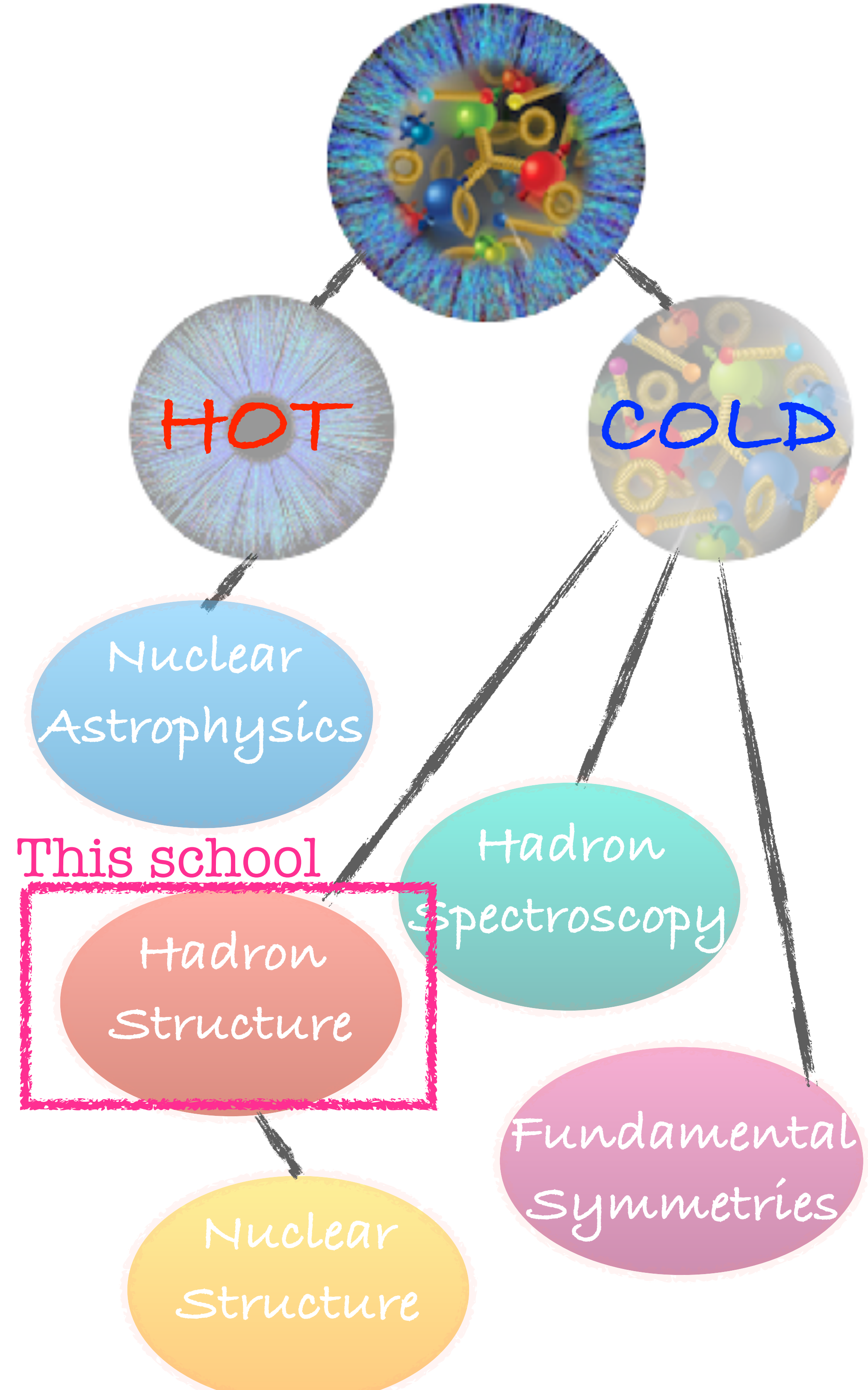
Lattice QCD* complements theory & experiments

*Lattice gauge theories also encompass unphysical theories

Overview of lattice QCD studies

★ Important Lattice QCD contributions that complement the experimental program in both Hot and Cold QCD

USQCD Nuclear Physics Program



OUTLINE OF LECTURES

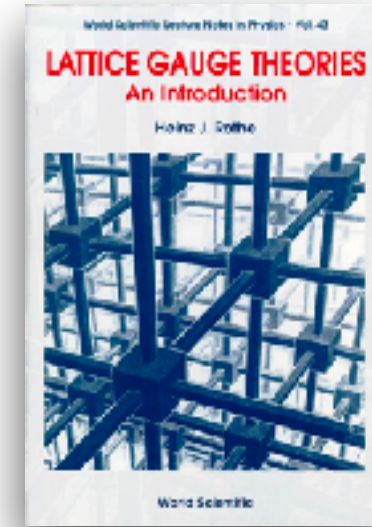
★ Tuesday, June 3:

- Motivation and Formulation of Lattice QCD
- How to extract physical information

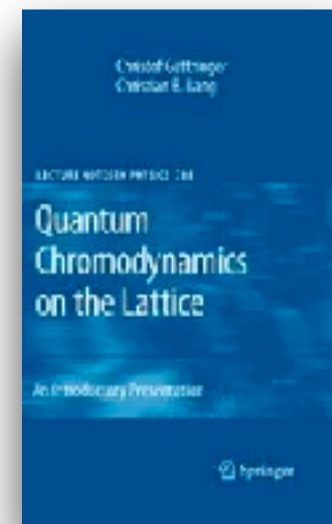
★ Thursday, June 5:

- Hadron Structure from Lattice QCD in the EIC era
- Novel methods for x-dependent distributions?
- Suggestions?
- Synergistic efforts

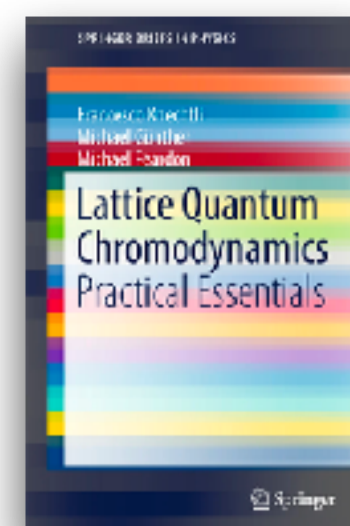
Useful Reading Material



★ Lattice Gauge Theories: An Introduction
H. J. Rothe
https://www.worldscientific.com/worldscibooks/10.1142/1268



★ Quantum Chromodynamics on the Lattice
An Introductory Presentation
C. Gattringer and C. Lang
https://www.springer.com/us/book/9783642018497



★ Lattice quantum chromodynamics: practical essentials
Knechtli, Günther & Peardon
https://link.springer.com/book/10.1007/978-94-024-0999-4

Useful Reading Material

Review article *Eur.Phys.J.C* 83 (2023) 1125

1

50 Years of Quantum Chromodynamics

Franz Gross[a,1,2], Eberhard Klempt[b,3],

Stanley J. Brodsky[c,4], Andrzej J. Buras[c,5], Volker D. Burkert[c,1], Gudrun Heinrich[c,6], Karl Jakobs[c,7], Curtis A. Meyer[c,8], Kostas Orginos[c,1,2], Michael Strickland[c,9], Johanna Stachel[c,10], Giulia Zanderighi[c,11,12],

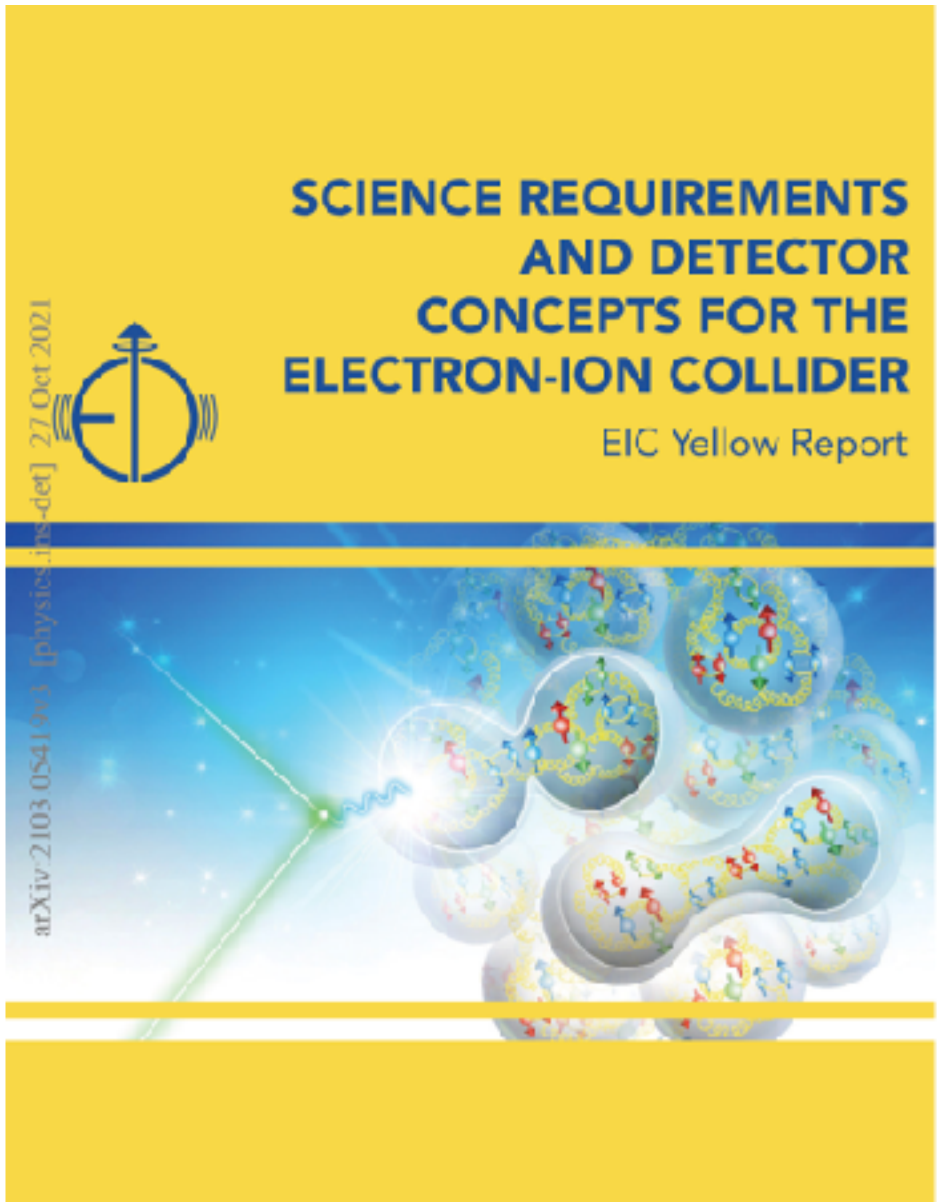
Nora Brambilla[5,12,13], Peter Braun-Munzinger[10,14], Daniel Britzger[11], Simon Capstick[15], Tom Cohen[16], Volker Crede[15], Martha Constantinou[17], Christine Davies[18], Luigi Del Debbio[19], Achim Denig[20], Carleton DeTar[21], Alexandre Deur[1], Yuri Dokshitzer[22,23], Hans Günter Dosch[10] Jozef Dudek[1,2], Monica Dunford[24], Evgeny Epelbaum[25], Miguel A. Escobedo[26], Harald Fritzsch[d,27], Kenji Fukushima[28], Paolo Gambino[11,29], Dag Gillberg[30,31], Steven Gottlieb[32], Per Grafstrom[33], Massimiliano Grazzini[34], Boris Grube[1], Alexey Guskov[35], Toru Iijima[36], Xiangdong Ji[16], Frithjof Karsch[37], Stefan Kluth[11], John B. Kogut[38,39], Frank Krauss[40], Shunzo Kumano[41,42], Derek Leinweber[43], Heinrich Leutwyler[44], Hai-Bo Li[45], Yang Li[46], Bogdan Malaescu[47], Chiara Mariotti[48], Pieter Maris[49], Simone Marzani[50], Wally Melnitchouk[1], Johan Messchendorp[51], Harvey Meyer[20], Ryan Edward Mitchell[52], Chandan Mondal[53], Frank Nerling[51,54,55], Sebastian Neubert[3], Marco Pappagallo[56], Saori Pastore[57], José R. Peláez[58], Andrew Puckett[59], Jianwei Qiu[1,2], Klaus Rabbertz[60], Alberto Ramos[61], Patrizia Rossi[1,62], Anar Rustamov[51,63], Andreas Schäfer[64], Stefan Scherer[65], Matthias Schindler[66], Steven Schramm[67], Mikhail Shifman[68], Edward Shuryak[69], Torbjörn Sjöstrand[70], George Sterman[71], Iain W. Stewart[72], Joachim Stroth[51,54,55], Eric Swanson[73], Guy F. de Téramond[74], Ulrike Thoma[3], Antonio Vairo[75], Danny van Dyk[40], James Vary[49], Javier Virto[76,77], Marcel Vos[78], Christian Weiss[1], Markus Wobisch[79] Sau Lan Wu[80], Christopher Young[81], Feng Yuan[82], Xingbo Zhao[53], Xiaorong Zhou[46]

Contents

arXiv: 2212.11107
https://inspirehep.net/literature/2617065

Useful Reading Material

EIC Yellow Report
Nucl.Phys.A 1026 (2022) 122447



arXiv:2103.05419
https://inspirehep.net/literature/1851258

Lattice QCD at the EIC:
a non-perturbative window
into hadron structure and
interactions

OUTLINE OF LECTURE 1

★ Path Integral Formalism

★ Lattice QCD formulation

★ Landscape of numerical simulations

★ Selected “objects” we calculate on the lattice

★ Key points of Lecture 1

Why is Lattice QCD essential?

QCD Lagrangian

QCD is a non-abelian gauge theory with symmetry group SU(3):

— 8 generators of SU(3) gauge group

— dimensionality of transformation space: 3

QCD Lagrangian density:

$$\mathscr{L}_{\text{QCD}} = \underbrace{\sum_f \bar{\psi}_f \left(i\gamma^\mu D_\mu - m_f \right) \psi_f}_{\mathscr{L}_{\text{fermionic}}} \underbrace{- \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}}_{\mathscr{L}_{\text{gluonic}}}$$

$$\gamma^\mu D_\mu = \gamma^\mu \partial_\mu + i g G^a_\mu \gamma^\mu T^a$$

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g f_{abc} G^b_\mu G^c_\nu$$

Quark kinetic & mass terms
quark-gluon terms

Gluon kinetic & interaction terms

f_{abc} : structure constants of SU(3)

T^α : SU(3) generators, α : 1,2,...,8

$F^\alpha_{\mu\nu}$: field tensor operator

G^α_μ : gluon field

$\psi^{s,c}_f(x)$: quark field, 4 component spinors, 3 component color, 6 flavors

Huge increase of complexity level to solve QCD (compared to QED)

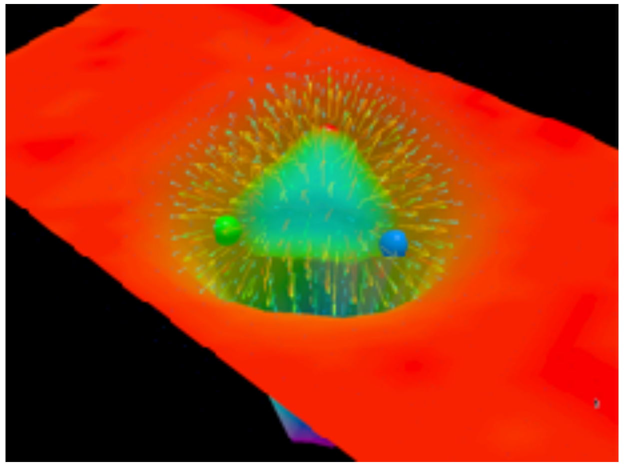
Features of QCD: The running coupling

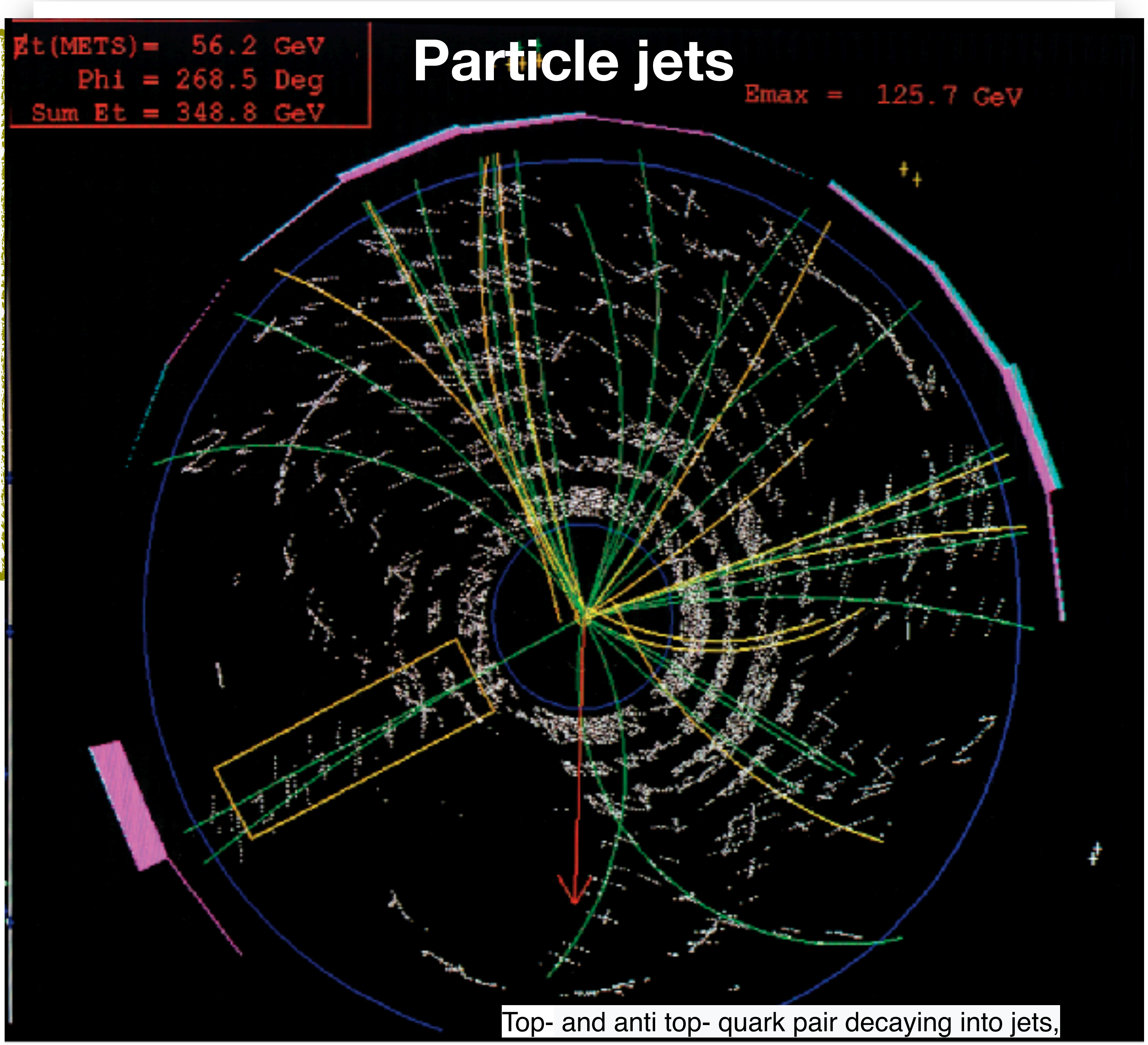
Strong coupling is not constant

"running" of coupling α_s almost resulted in abandoning QCD:

α_s too strong for perturbation theory to be of any use.

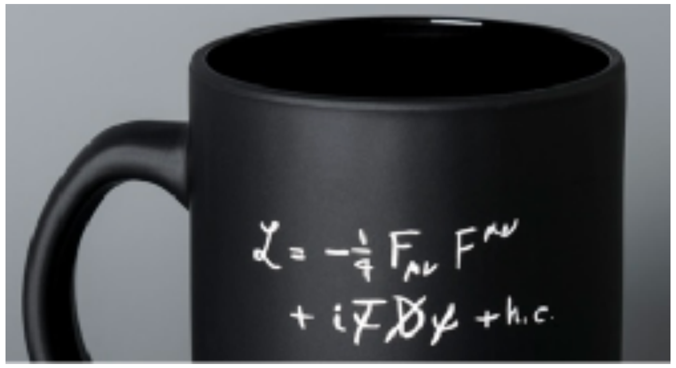
Quark confinement

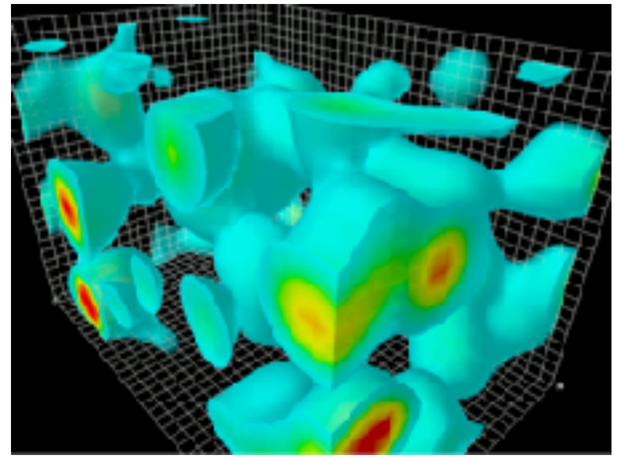




Top- and anti top- quark pair decaying into jets,

In the quest of solving complex problems



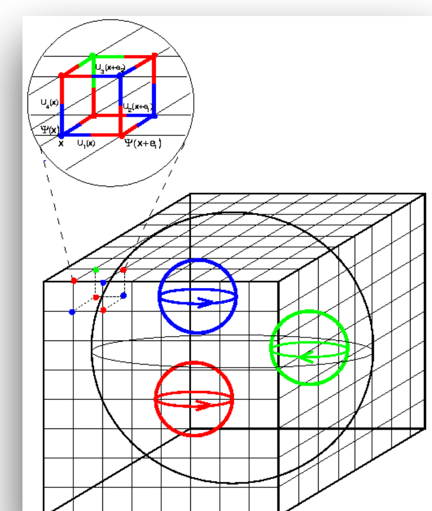


"Simulate non-analytical systems"

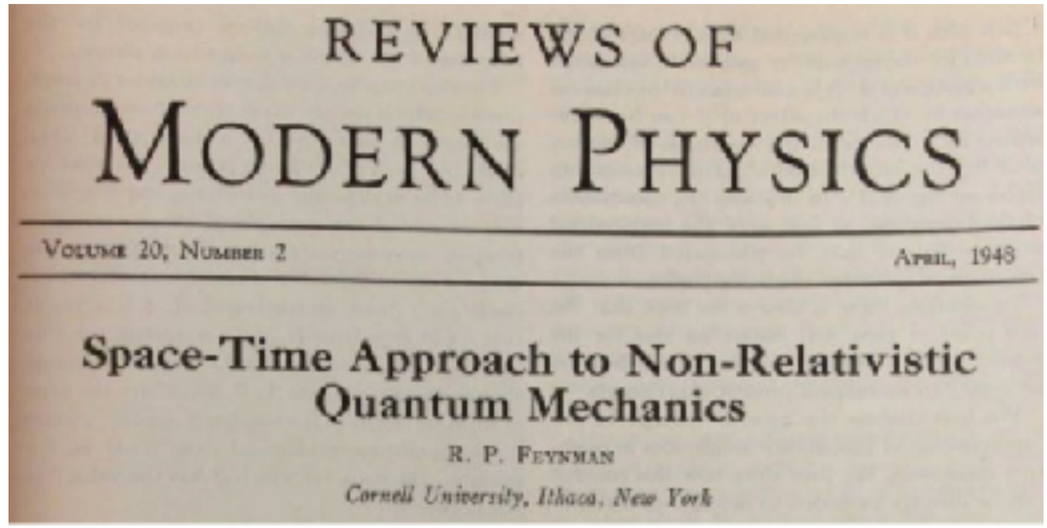
“Oτι δεν λύνεται, κόβεται”

Alexander the Great while cutting the Gordian knot

QCD and beyond



Path Integral Formalism


REVIEWS OF
MODERN PHYSICS

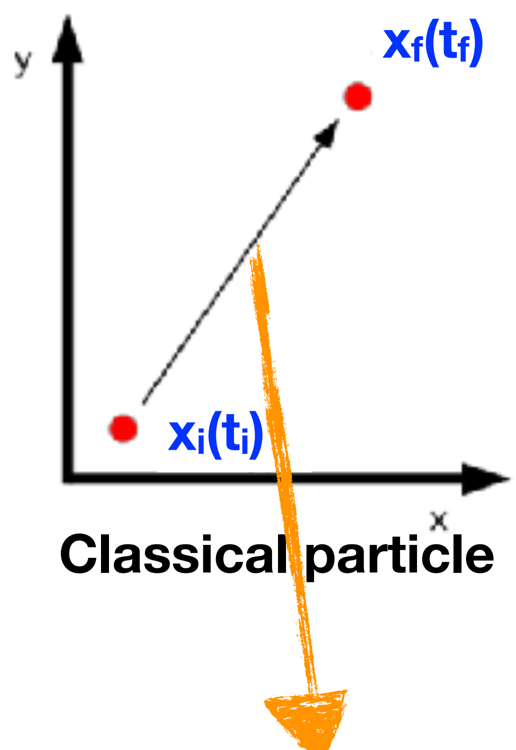
VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

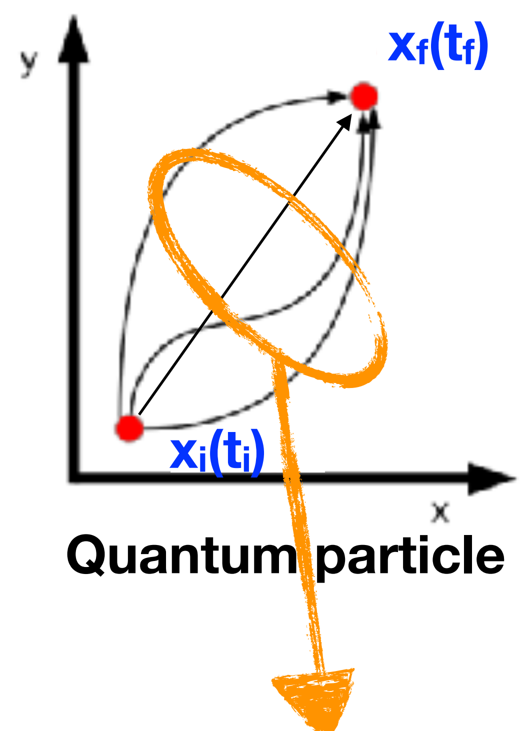
R. P. FEYNMAN

Cornell University, Ithaca, New York

Probability of getting to $x_f(t_f)$ when initially at $x_i(t_i)$



unique trajectory



All possible trajectories (weighted)

Propagator

$$U(x_f, t_f; x_i, t_i) = \langle \psi(x_f, t_f) \,|\, \psi(x_i, t_i) \rangle$$
$$= \langle \psi(x_f) \,|\, e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)} \,|\, \psi(x_i) \rangle$$

Path integral "language" (sum of all paths weighted by action)

$$U(x_f, t_f; x_i, t_i) = \int \mathscr{D}x(t) e^{\frac{i}{\hbar}S[x(t)]}$$

$\int \mathscr{D}x(t)$: measure

$S[x(t)]$: action

Path Integral Formalism

- ★ Equivalent to the Schrödinger formalism - more intuitive in interpretation
- ★ Very practical for quantum mechanics (weighted sum over all paths)
- ★ Critical for quantum field theories (weighted sum over all field values)
  Successfully applied to QCD (Lattice QCD)

- ★ Partition function

$$\mathcal{Z} = \int D[U] D[\bar{\psi}]\, D[\psi]\, e^{i\, S_{\rm QCD}[U,\bar{\psi},\psi]} = \int D[U]\, det(D[U])^{N_f}\, e^{i\, S_{\rm QCD,G}[U]}$$

Functional volume element
for corresponding fields

Fermion degrees of freedom integrated out
(anticommuting Granssmann variables)

- ★ Observables:
  (v.e.v of operator)

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int D[U]\, \mathcal{O}(D^{-1}, U)\, det(D[U])^{N_f}\, e^{i\, S_{\rm QCD}[U]}$$

Complex action problem: makes weight sampling impossible
(oscillatory phase factors)

Euclidean metric & Discretization

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int D[U]\, \mathcal{O}(D^{-1}, U)\, det(D[U])^{N_f}\, e^{i\, S_{\rm QCD}[U]}$$

★ Wick rotation to imaginary (Euclidean) time: $t \rightarrow i\tau$
(temporal and spatial components same sign in invariant length)

$$e^{i\, S_{\rm QCD}[U]} \rightarrow e^{-\, S_{\rm QCD}[U]}$$

★ Statistical mechanics methods may be utilized (Boltzmann probability)

We have not reach the lattice discretization yet!

★ Path integral has infinite degrees of freedom:

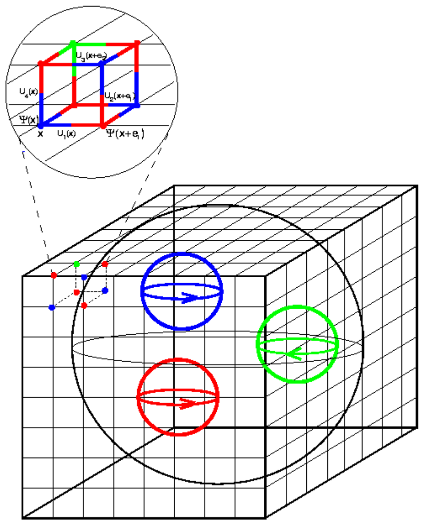
Need to introduce a space-time discretization

Lattice formulation of QCD


K. Wilson


M. Creutz

First principle (ab initial) formulation



★ Space-time discretization on a finite-size 4-D grid

★ Serves as a regulator of theory:

– UV (hard momentum) cut-off (finite integrals): inverse lattice spacing (α^{-1})
momentum and energy < $|\pi/\alpha|$

$$\int_{-\infty}^{\infty} dp \;\rightarrow\; \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi}$$

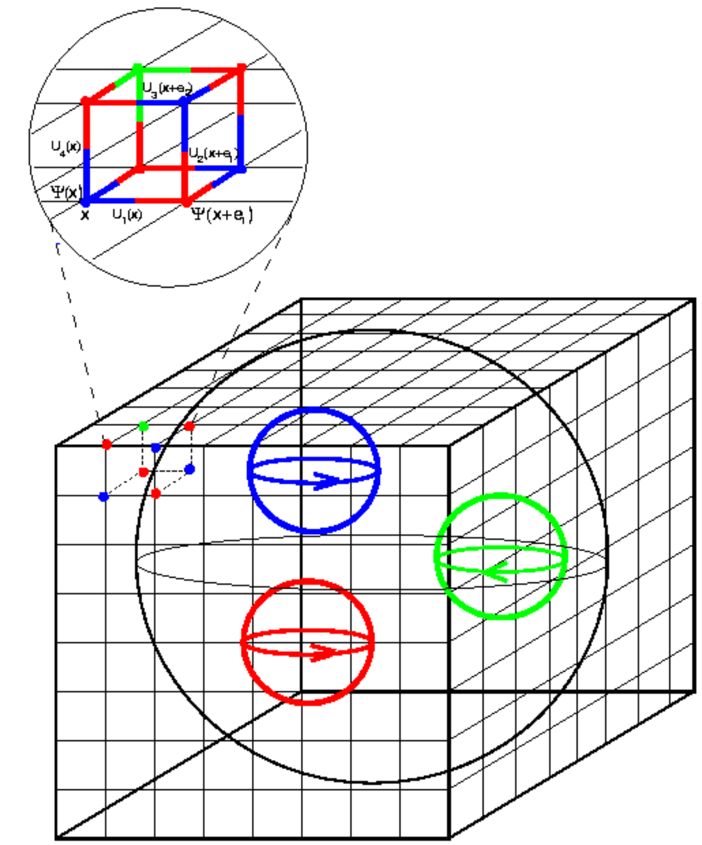
– IR cut-off (finite number of d.o.f): inverse lattice size ($V^{-1/4}$)

$$\int dpF(p) \;\rightarrow\; \sum_{n}^{N_{\max}} \frac{2\pi}{L} F(p_0 + \frac{2\pi n}{L})$$

★ Removal of regulator $L \rightarrow \infty\,,\;\; a \rightarrow 0$

Lattice formulation of QCD

Technical Aspects



★ **Parameters (define cost of simulations):**

- quark masses (aim at physical values)
- lattice spacing* (ideally fine lattices)
- lattice size (need large volumes)

★ **Discretization not unique**

- clover improved fermions
- Domain wall fermions
- Overlap fermions
- Staggered fermions
- Twisted mass fermions

* In practice, the coupling is set in simulation and α is defined by comparing lattice results and values of physical quantities, e.g., proton mass

Monte Carlo Methods for Lattice QCD

★ Direct evaluation of (finite d.o.f.) path integral is unfeasible:
One needs to invert the Dirac matrix (~ $10^8 \times 10^8$)

★ Solution: Stochastic estimation of path integral

★ Discretization in a lattice of volume:
e.g., $48^3 \times 96$: 340 Million degrees of freedom!

⬇

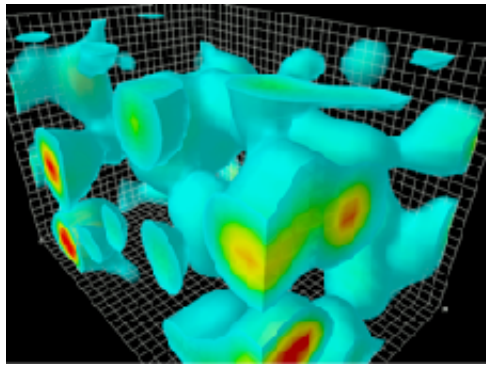
Monte Carlo Methods
for Numerical Simulations

Monte Carlo Methods for Lattice QCD

★ Representative ensemble of gauge field configurations of the vacuum with acceptance probability

$$e^{-S[U]+N_f \log(det(D[U]))}$$

- Metropolis Algorithm:
  Very slow due to sequential repetition of updating variables
- Hybrid MC, important sampling, use of Markov process:
  update all variables at once, better scaling behavior in volume



★ Expectation value of operator (correlation functions) for this distribution, which requires an inversion of sparse matrix

★ Repetition of this process N times
N: number of "measurements"

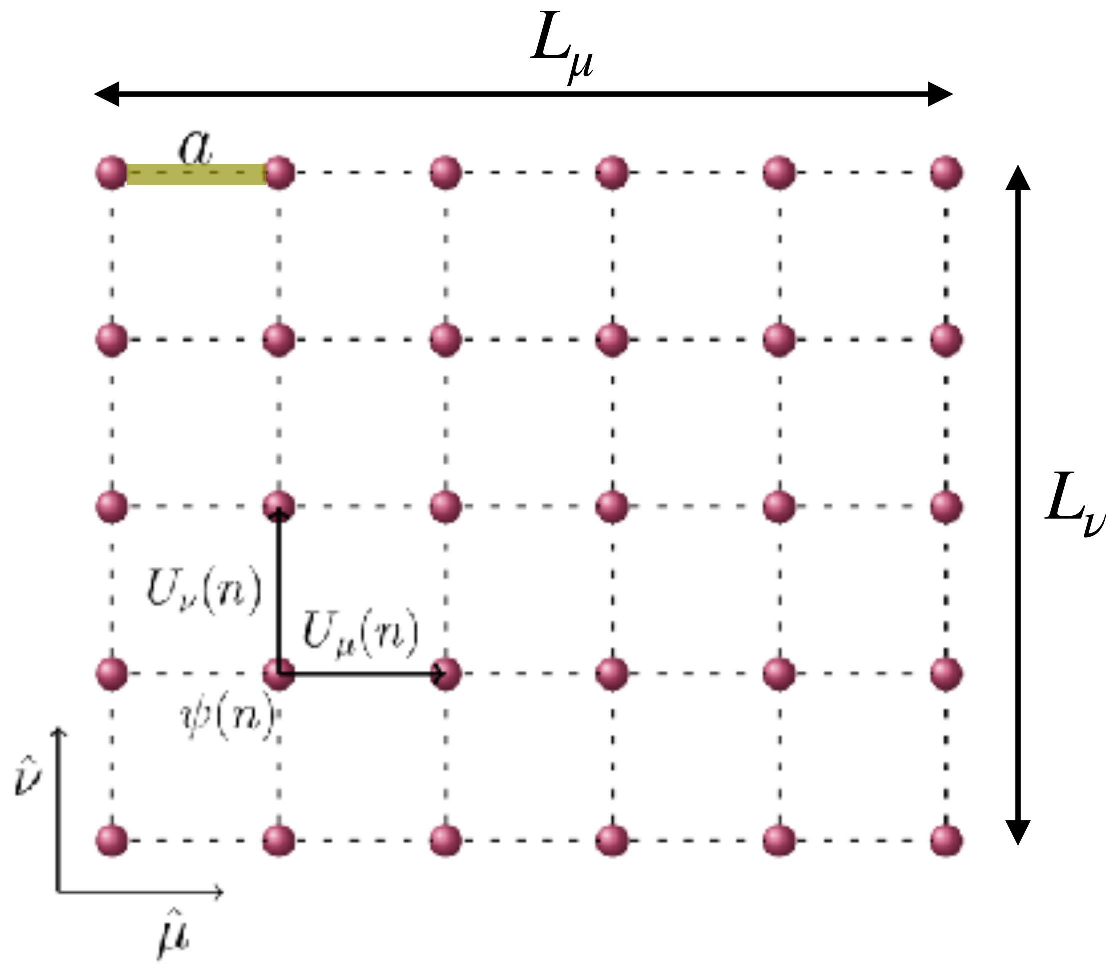
★ Average of results $\overline{O} = \frac{1}{N} \sum_N \mathcal{O}(U)$

★ Statistical errors (jackknife, bootstrap) decrease as $\sigma(\overline{O}) \propto 1/\sqrt{N}$

Theoretical aspects of lattice QCD

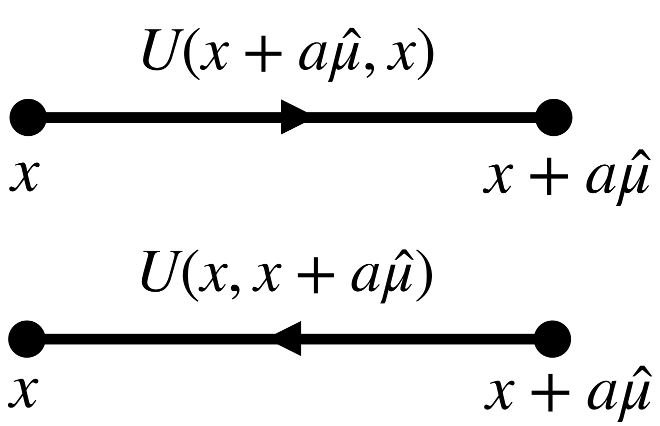
The boring stuff…

Fermions and Gluons on the Lattice



Link variable U_μ relates to gauge field G_μ

$$U(x+a\hat{\mu};x) = U_\mu(x) = \mathcal{P}e^{-ig\int_x^{x+a\hat{\mu}} dx\, G^b_\mu(x) T_b} \simeq e^{-igaG^b_\mu(x)T_b} \qquad x = na$$



$\Psi(x)$: anticommuting Grassmann variables

For more theoretical aspects see backup slides

Fermions and Gluons on the Lattice

★ **Lattice formulation "must" be invariant under SU(3) local gauge transformation**

$$\psi(x) \to V(x)\psi(x)\,, \qquad \bar\psi(x) \to \bar\psi(x)V^\dagger(x) \qquad\qquad V(x) = e^{-i\theta_a(x)\frac{\lambda_a}{2}}$$

$$U_\mu(x) \to V(x)U_\mu(x)V^\dagger(x+\hat\mu a)$$

★ **Giving up gauge invariance would create a series of problems:**

– More parameters to tune
(couplings for quark-gluon, 3- & 4-gluon interactions, the gluon mass,…)

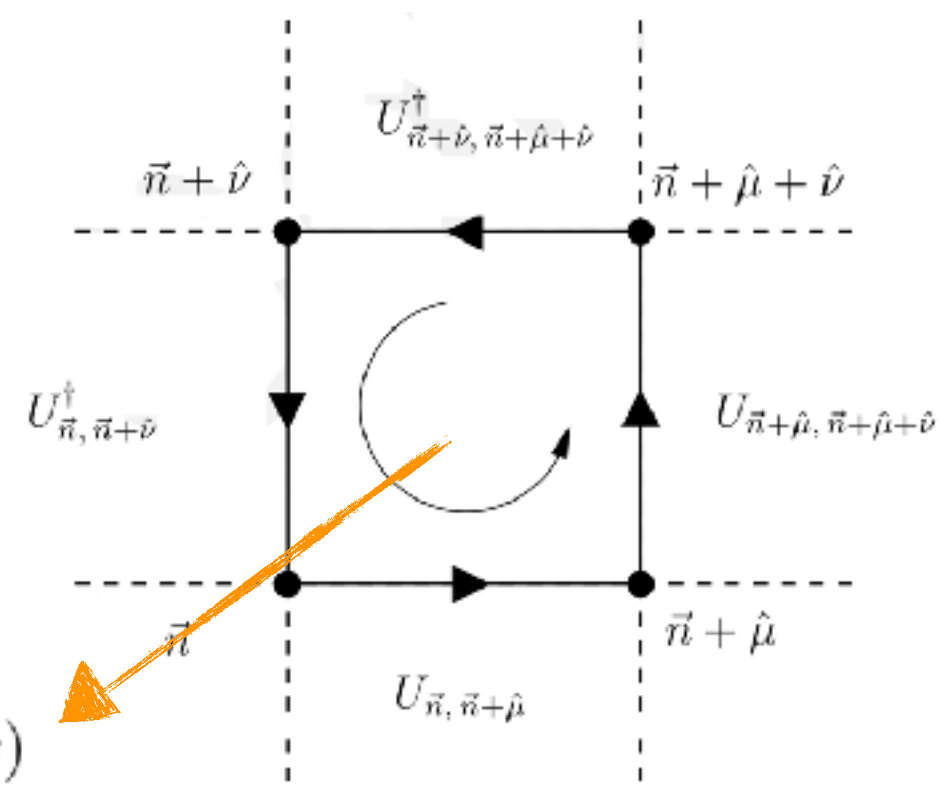
– More operators at any given order in α, thus increase of discretization errors

– Proofs of renormalizability within perturbation theory rely on strict gauge invariance
[T. Reisz & H. Rothe, Nucl.Phys. B575 (2000) 255]

★ **Gauge invariant quantities:**

– Products of Ψ(x), Ψ(x') and gauge links connecting x and x'

– Closed gluonic loops



$$P_{\mu\nu} \equiv U_\mu(x)U_\nu(x+a\hat\mu)U^\dagger_\mu(x+a\hat\nu)U^\dagger_\nu(x)$$

Example: Naive fermion discretization

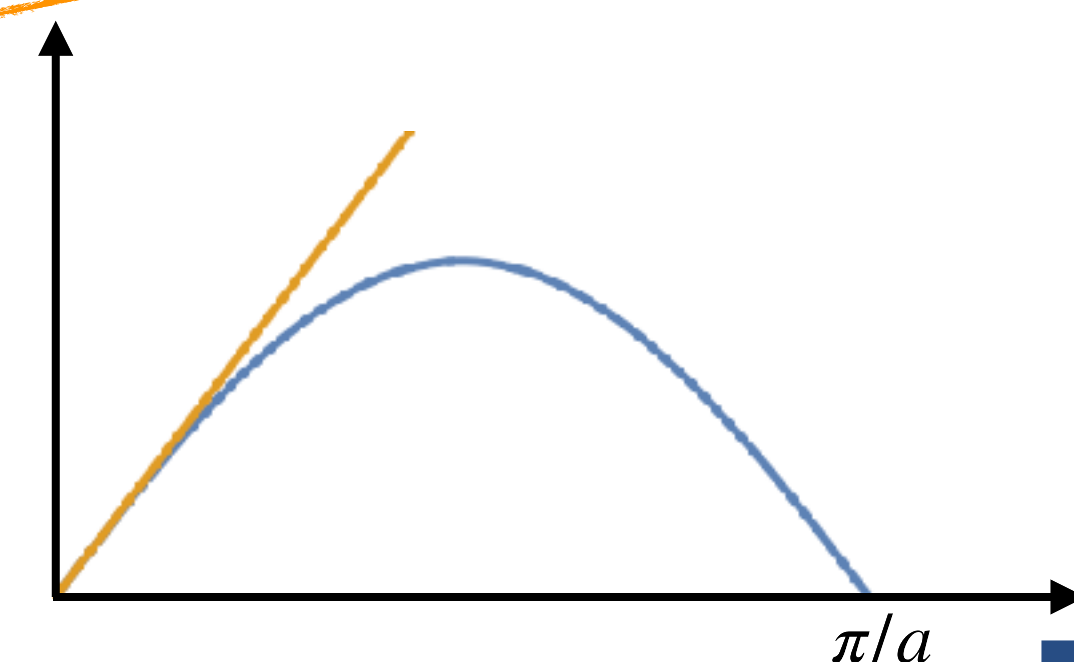
- Discretization of fermionic action complicated
- Naive discretization preserves gauge invariance, but results in fermion doubling problem: appearance of spurious states and continuum limit wrongly leads to 2^4 fermions instead of one.

$$S_F^{naive} = a^4 \sum_x \frac{1}{2a}\gamma^\mu[\bar{\psi}(x)U_\mu(x)\psi(x+a\hat{\mu}) - \bar{\psi}(x)U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu})] + m\bar{\psi}(x)\psi(x)$$

Fermion propagator (in momentum space upon Fourier Transform):

$$\langle\psi(x)\overline{\psi}(y)\rangle = \lim_{a\to 0}\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\,\frac{-i\sum_\mu \gamma_\mu sin(k_\mu) + m_0}{\sum_\mu sin^2(k_\mu) + m_0^2}$$

Additional poles:
Vanishes at the ends of Brillouin zone [-π/α,π/α].
In 4-dim these are sixteen regions instead of p~0 only, thus 16 species of fermions



Nielsen-Ninomiya (No-Go) theorem

It is not possible to define a local, translationally invariant, hermitian lattice action that preserves chiral symmetry and does not have doublers

★ Several proposals for fermion action to avoid fermion doubling
Wilson, Clover, Twisted Mass, Staggered, Overlap, Domain Wall, Mixed actions

★ Improved actions have different advantages and disadvantages:

Clover:
computationally fast
break chiral symmetry & require operator improvement

Twisted Mass:
computationally fast & automatic improvement
break chiral symmetry & violation of isospin

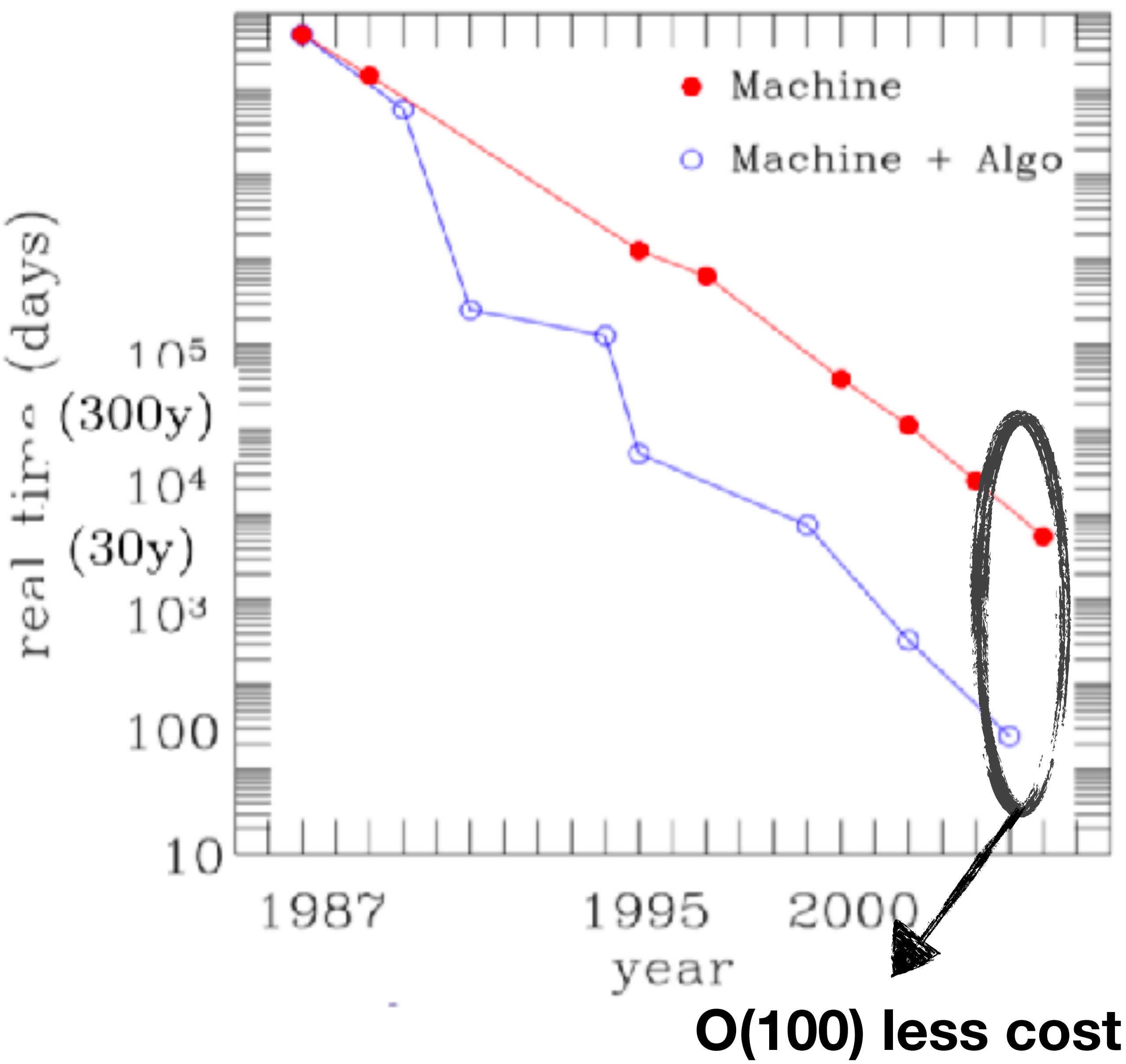
Staggered:
computationally fast
4 doublers & difficult contractions

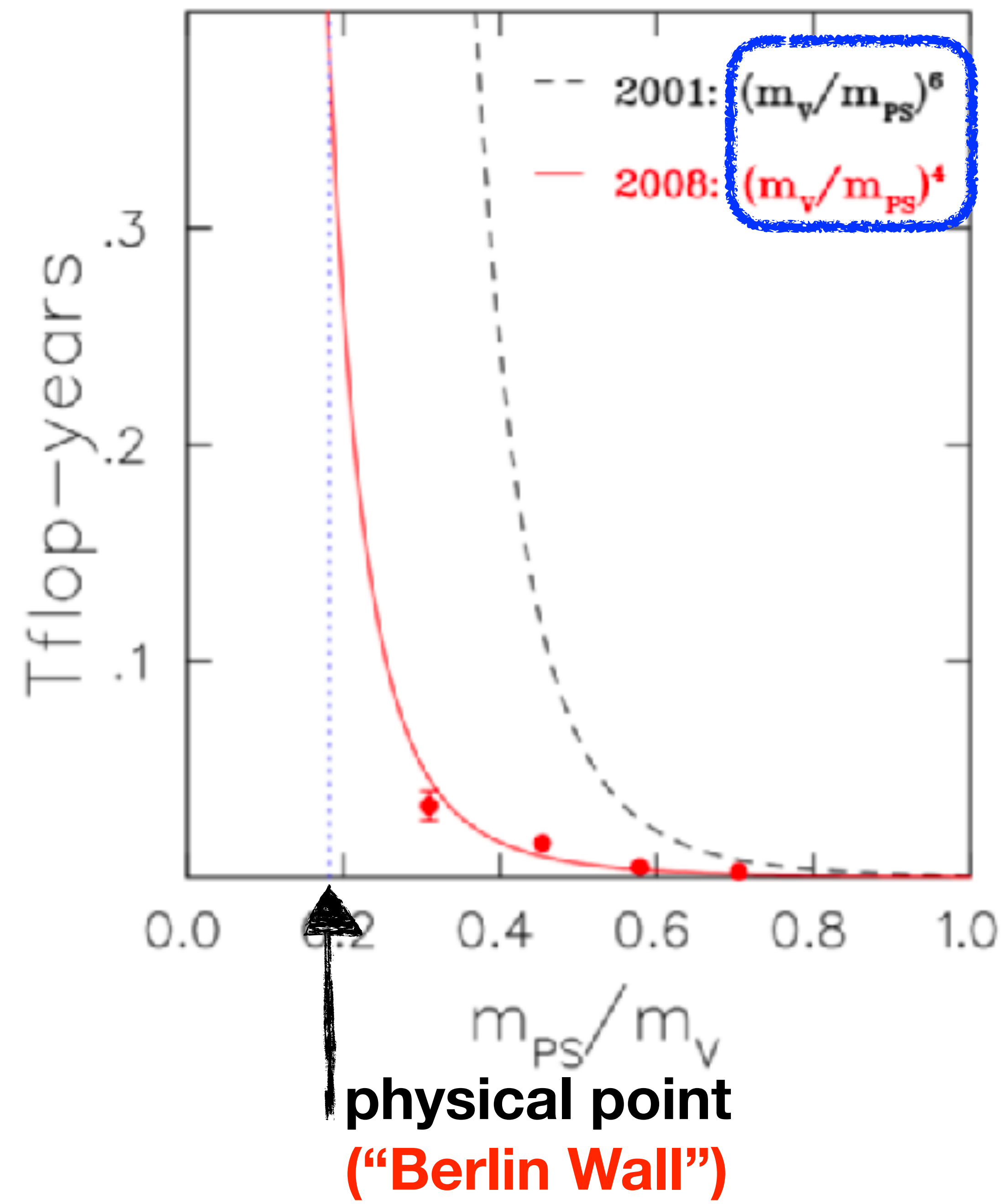
Overlap:
exact chiral symmetry
computationally expensive

Domain Wall
improved chiral symmetry
computationally demanding & require tuning

All these formulations are used to understand aspects of QCD (hadron structure, spectroscopy, etc)

Challenges of numerical simulations





★ Above benchmark is for a small-scale calculation

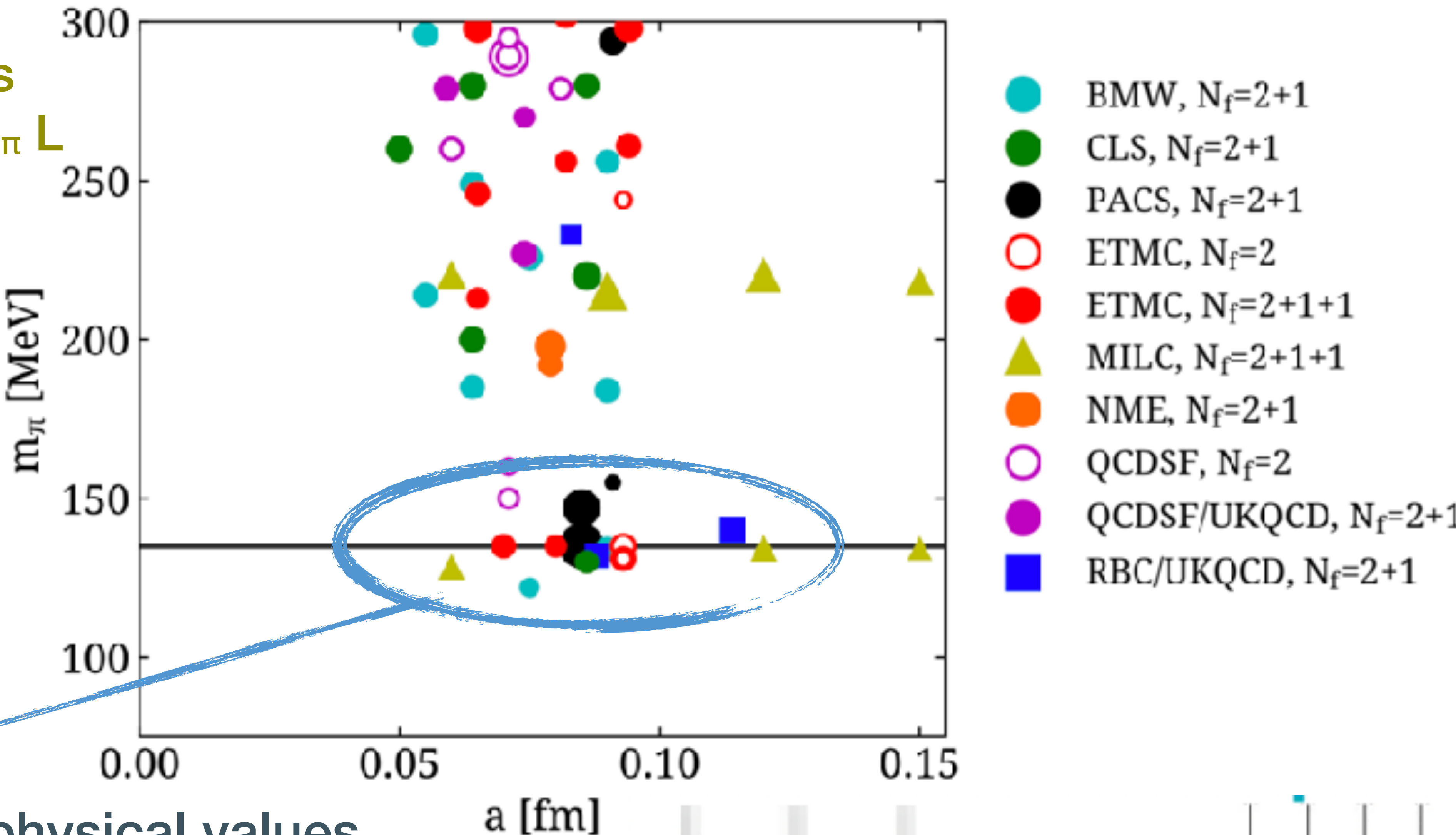
★ Modern calculations (physical parameters) require TFlops x years

Landscape of numerical simulations

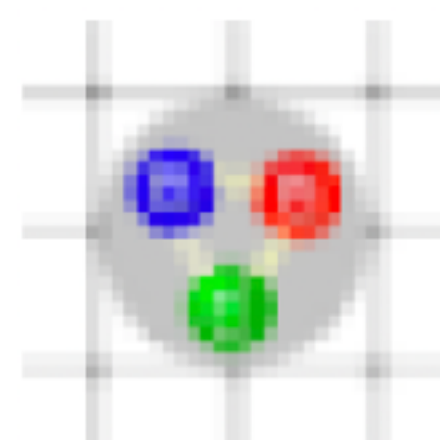
Lattice (fermion) formulations employed by various groups:

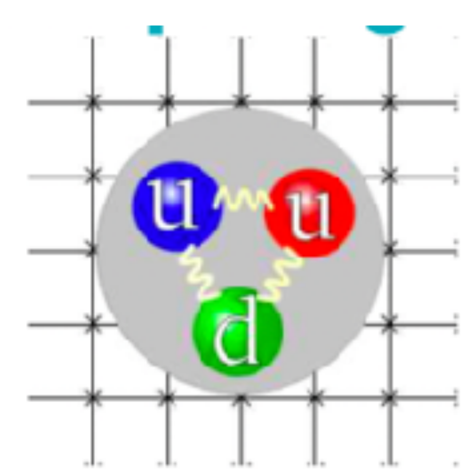
Wilson, Clover, Twisted Mass, Staggered, Overlap, Domain Wall, Mixed actions

Size of symbols proportional to $m_\pi L$



Ensembles with physical values for quark masses (physical point)





What should we first study in Lattice QCD?

Start from quantities that are (relatively) easy to compute, and can be compared against experimental data

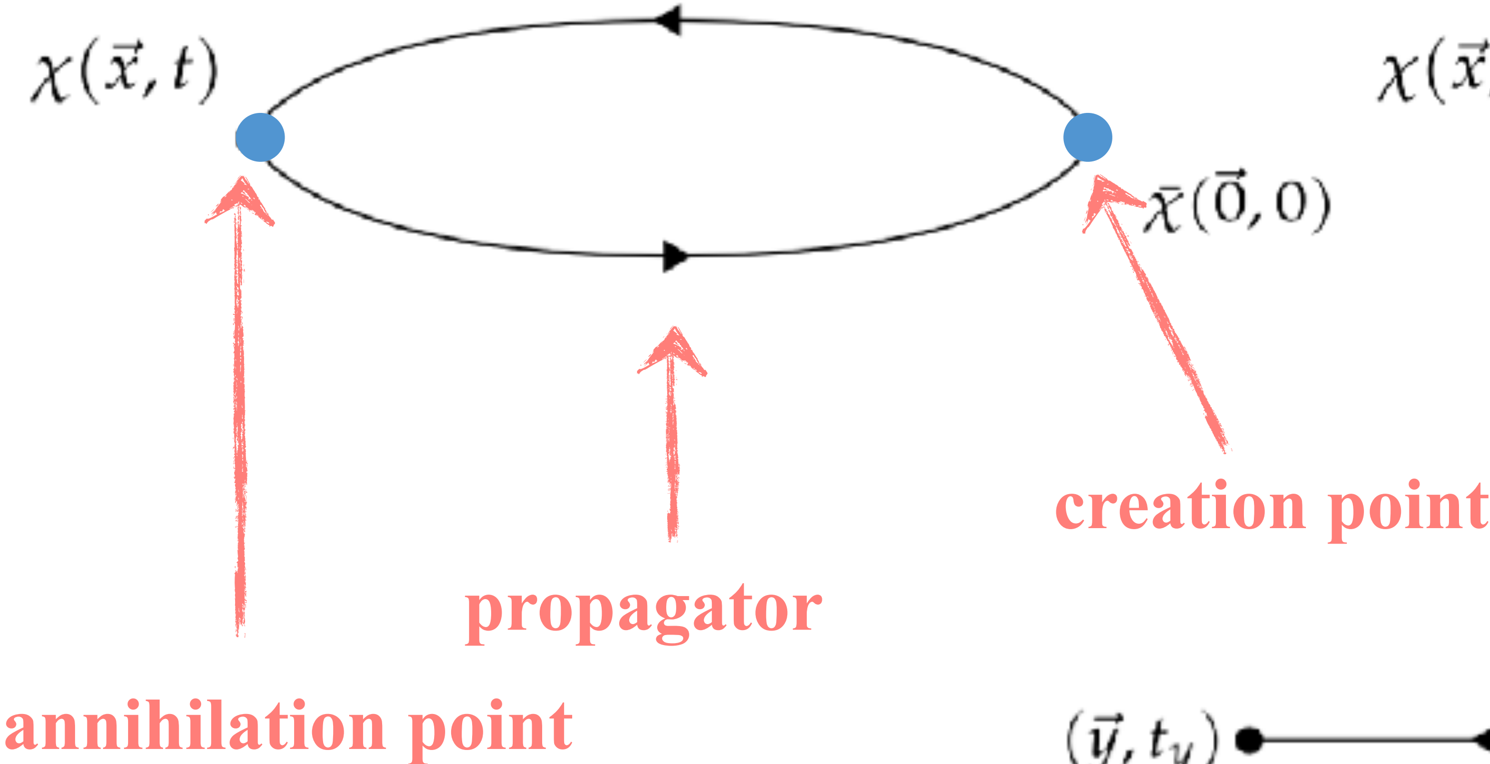
First goals of Lattice QCD

Reproduce the low-lying spectrum

$\langle H(p) | H(p) \rangle$

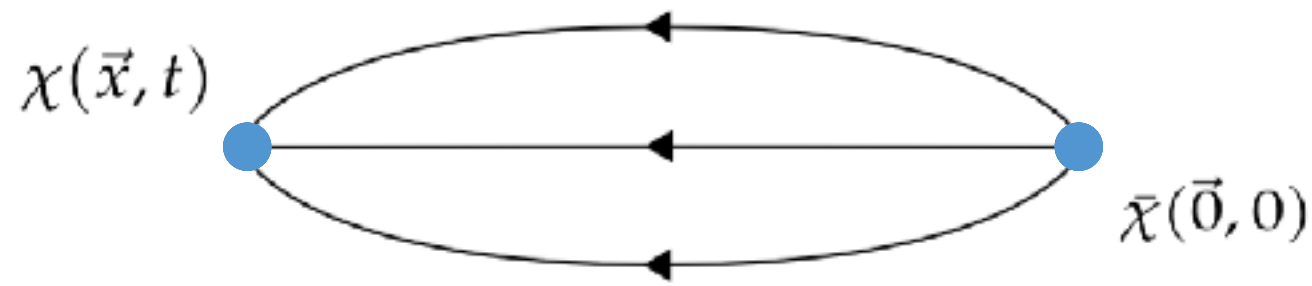
Mesons

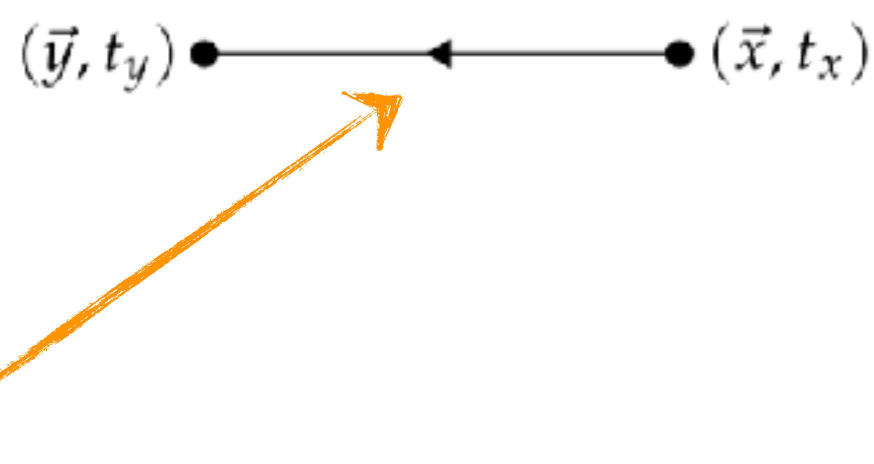
e.g. pion, kaon



Baryons

e.g. proton





$= G(\vec{y}, t_y; \vec{x}, t_x)$

Quark propagator

Most costly part of calculation

Calculation of Hadron mass

Extraction of a hadron's mass from its propagator:

★ Two-point correlator (hadron level, Heisenberg picture):

$$C(t) = \sum_{\vec{x}} \langle\Omega|\, \chi(\vec{x},t)\bar{\chi}(\vec{0},0)\,|\Omega\rangle = \sum_{\vec{x}} \langle\Omega|\, e^{-i\hat{\vec{p}}\cdot\vec{x}} e^{\hat{H}t} \chi(\vec{0},0) e^{-\hat{H}t} e^{i\hat{\vec{p}}\cdot\vec{x}} \bar{\chi}(\vec{0},0)\,|\Omega\rangle$$

Insertion of complete set of momentum and energy states:

$$\mathbb{1} = \sum_{\vec{k},n} \frac{1}{2E_n(\vec{k})} |n,\vec{k}\,\rangle\langle\, n,\vec{k}|,$$

$$C(t) = \sum_{\vec{x},n,\vec{k}} \frac{|\langle\,\Omega|\chi(\vec{0},0)|n,\vec{k}\,\rangle\,|^2}{2E_n(\vec{k})} e^{-E_n(\vec{k})t} e^{i\vec{k}\cdot\vec{x}} = \sum_n \frac{|\langle\,\Omega|\chi(\vec{0},0)|n,\vec{0}\,\rangle|^2}{2E_n(\vec{k})} e^{-m_n t}$$

Sum over x gives δ(κ), $E_n(0) = m_n$

Only terms that have same quantum numbers as χ survive

★ The mass of the hadron appears, for the n^{th} state

Calculation of Hadron mass

★ Overlap with ground state, excitations, other hadron states. Thus:

$$C(t) = \sum_{n'} \frac{1}{2E_n(\vec{k})} |\langle \Omega | \chi(\vec{0},0) | (n',\vec{0}) \rangle|^2 e^{-m_{n'} t}$$

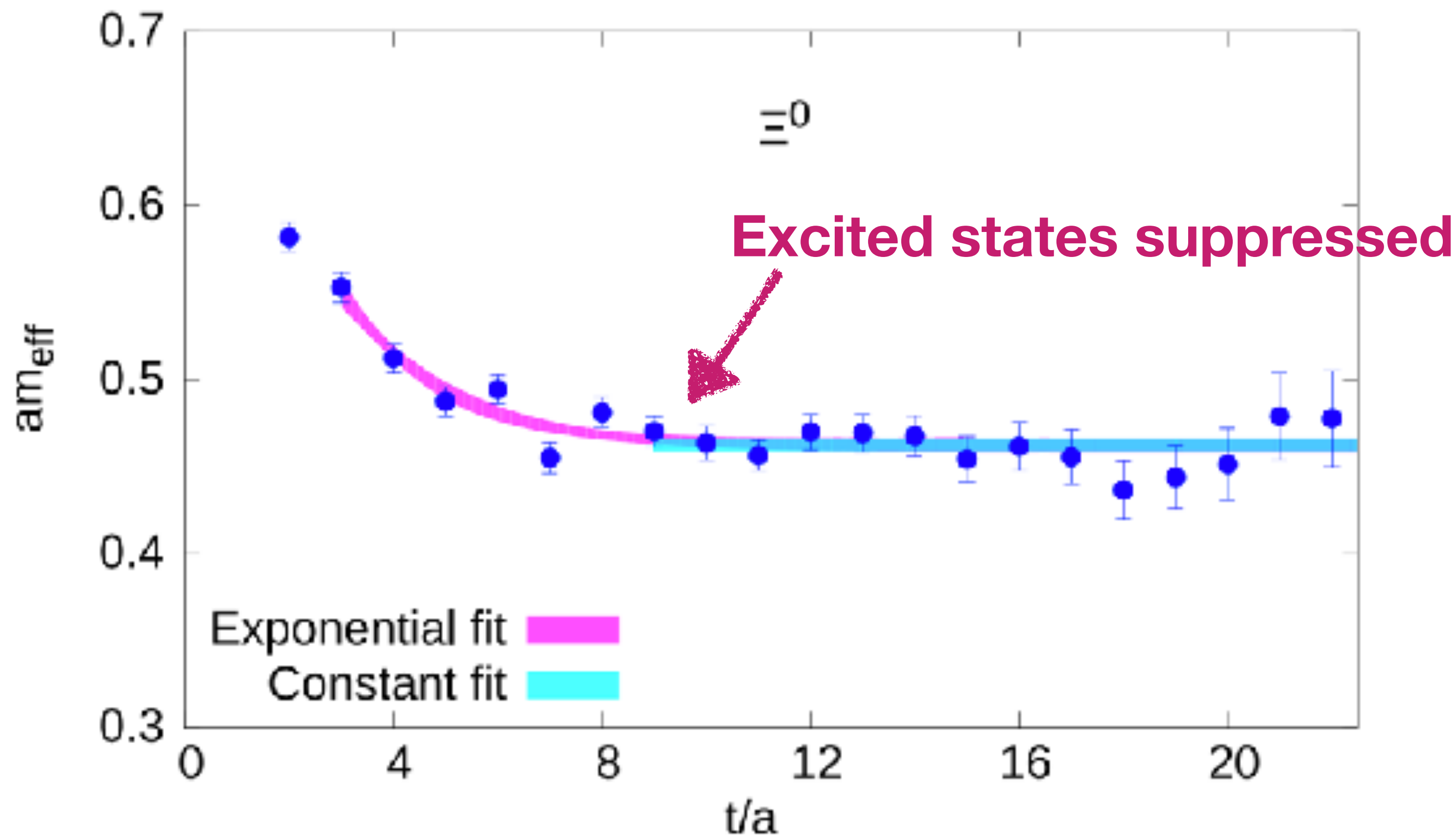
★ For large enough *t* the exponential for excited states and multi-hadron states, becomes very small, thus ground-state dominance.

$$C(t) \underset{t>>1}{=} \frac{1}{2m^H} |\langle \Omega | \chi(\vec{0},0) | H(\vec{0},0) \rangle|^2 e^{-m^H t}$$

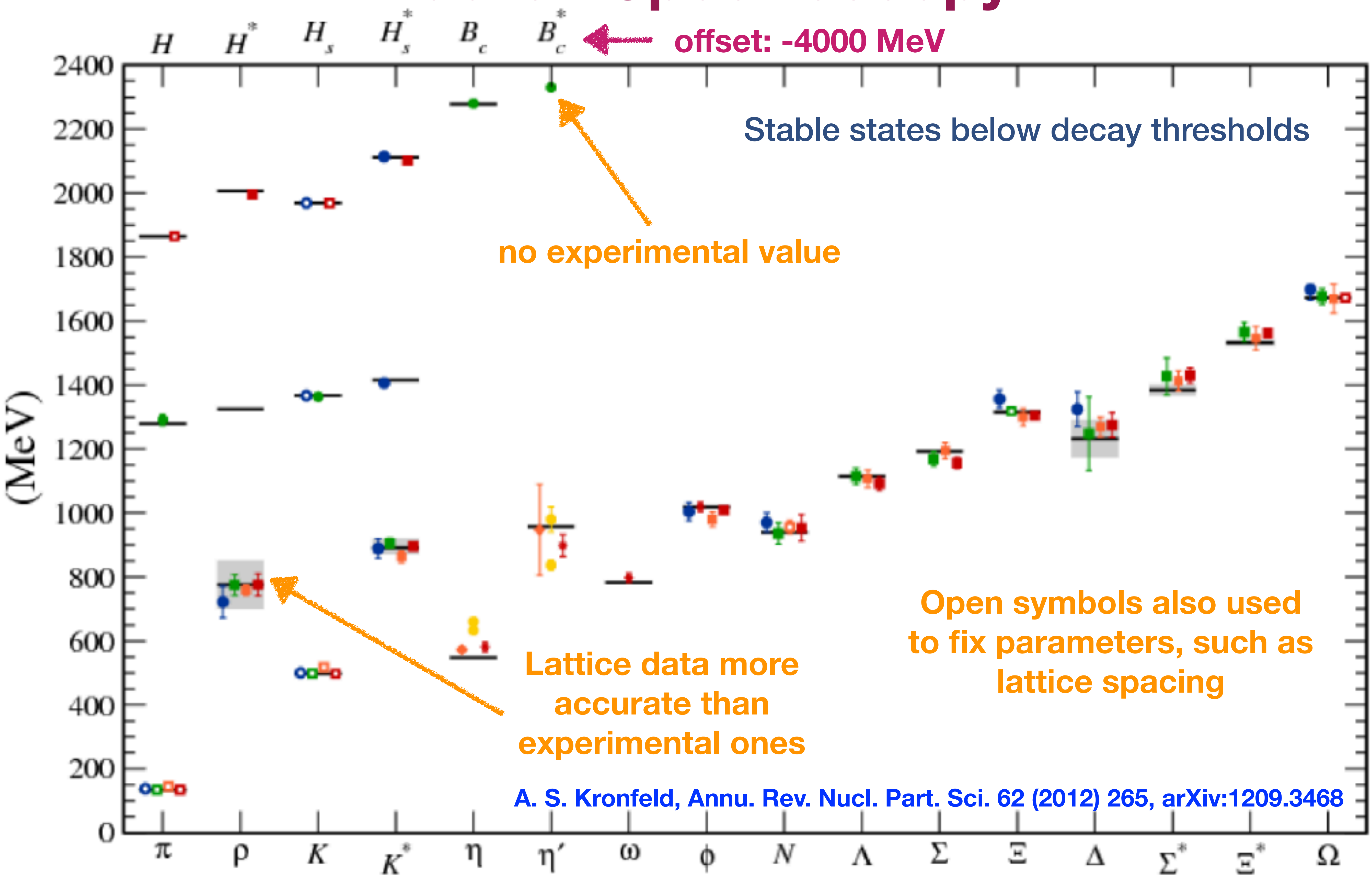
mass of ground state

$$a\, m^H_{eff}(t) = \log\left(\frac{C(t)}{C(t+1)}\right)$$

One may proceed with a constant or multi-state fit



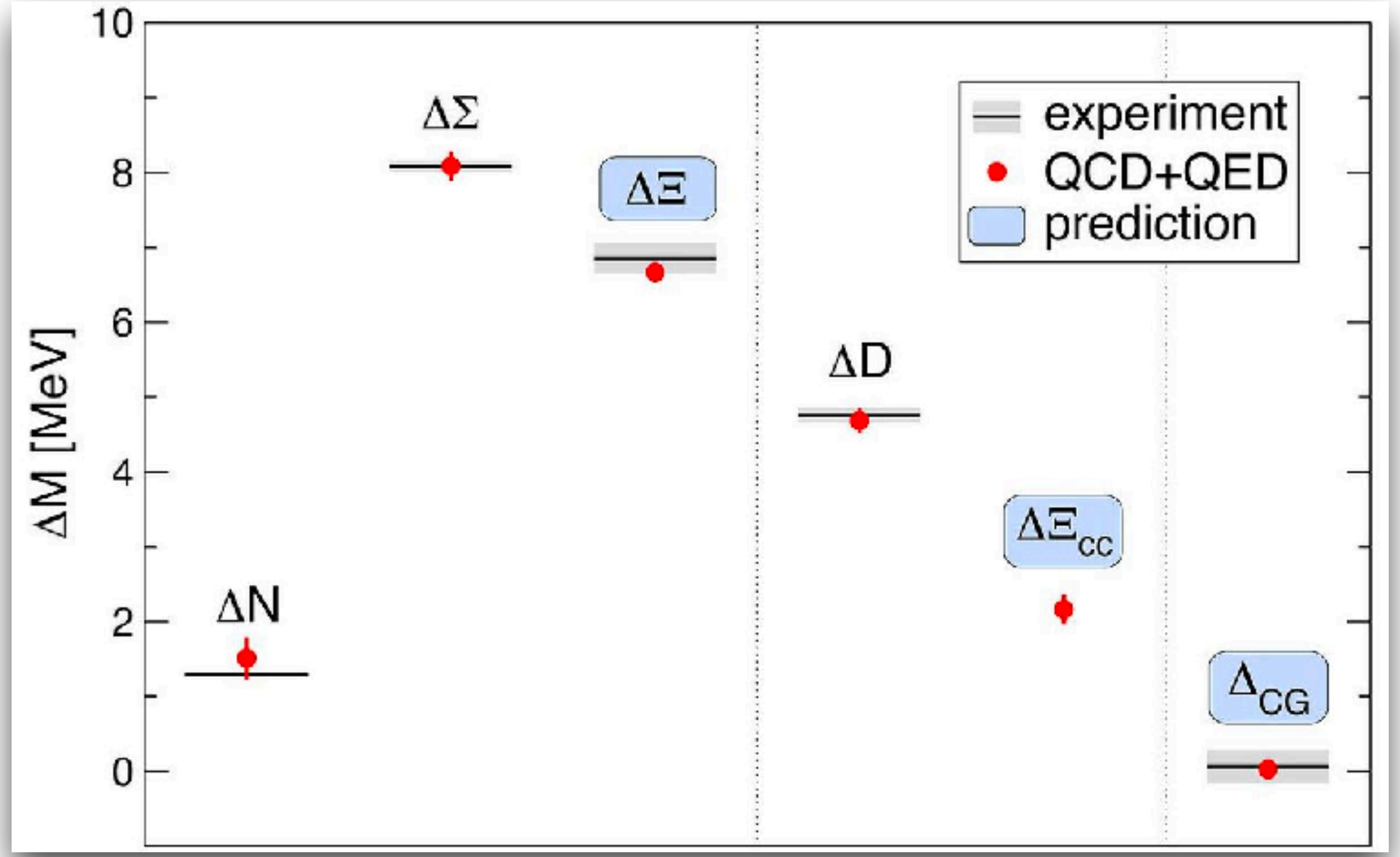
Hadron Spectroscopy



Lattice results reproduce experimental values

Hadron Spectroscopy

★ QCD + QED effects: mass splitting between, e.g., proton and neutron



Borsanyi et al, Science 347, 14521455 (2015)

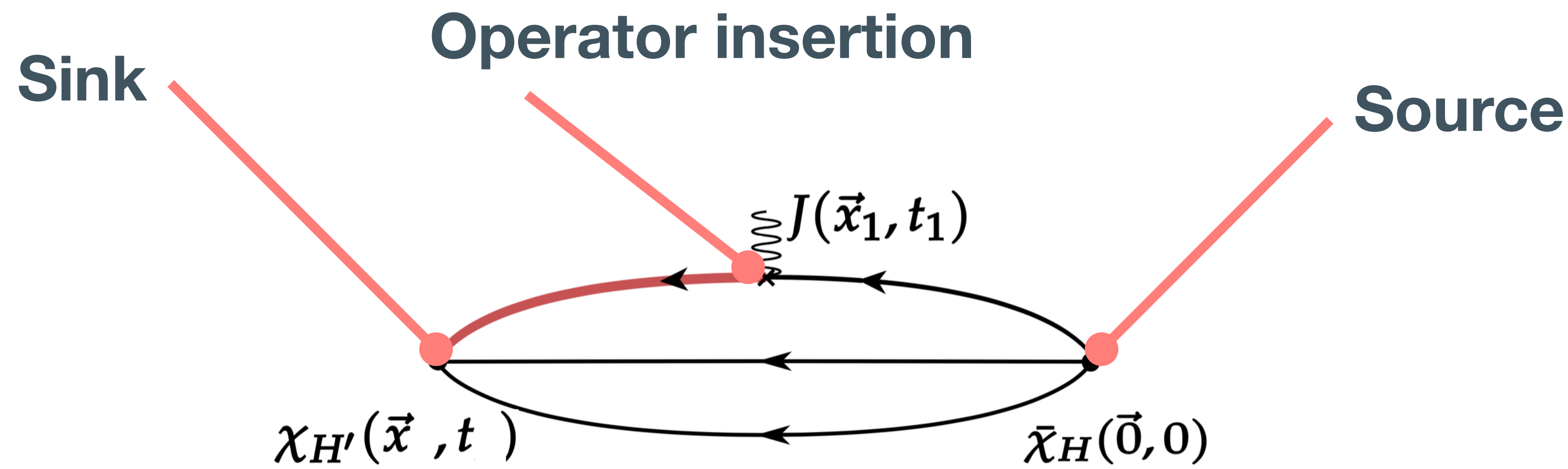
For new exciting advances in spectroscopy see lectures of 2024 CFNS School

Correlation functions related to Hadron Structure (See lecture #2)

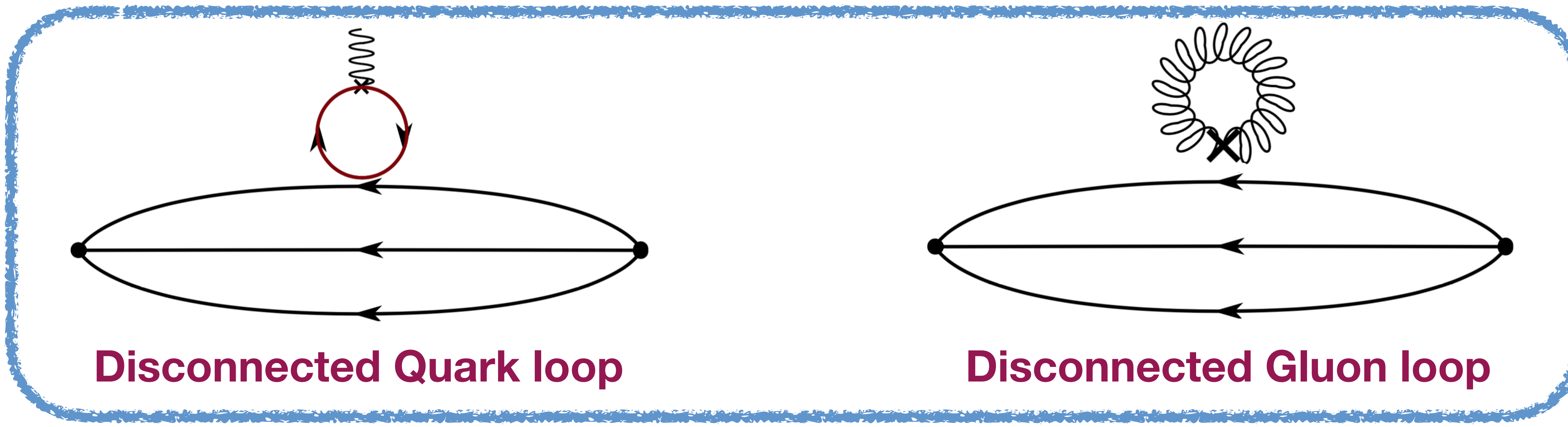
Hadrons on the Lattice

$\langle N(p_f) \,|\, \mathcal{O} \,|\, N(p_i) \rangle$

Diagrams for baryons



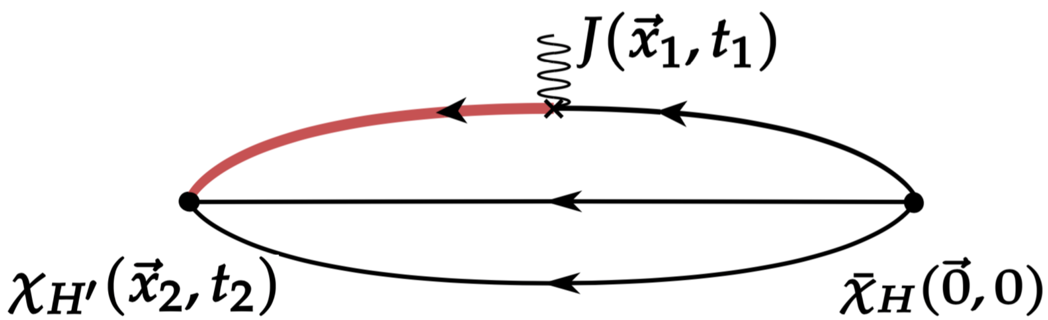
Connected



Particularly interesting for EIC physics

- ★ Separation between source and sink: excited states investigation
- ★ Type of current insertion gives different observable
- ★ Extraction of each contribution has its own challenges
  (statistical and systematic uncertainties)

Hadrons on the Lattice



Connected

3pt correlation function

$$G^{\mathcal{H}'\mathcal{J}^\mu\mathcal{H}}(x; x_1) = \langle \Omega \,|\chi_{\mathcal{H}'}(x)\mathcal{J}^\mu(x_1)\bar{\chi}_\mathcal{H}|\, \Omega\rangle$$

Fourier transform ↓ **Insertion of two set of complete eigenstate**

$$\sum_{\vec{x},\vec{x}_1} e^{-i\vec{x}\cdot\vec{p}'} G^{\mathcal{H}'\mathcal{J}^\mu\mathcal{H}}(\vec{x}, t; \vec{x}_1, t_1) e^{-i\vec{x}_1\cdot\vec{p}_1} = \sum_{\vec{x},\vec{x}_1} e^{-i\vec{x}\cdot\vec{p}'} \langle \Omega \,|\chi_{\mathcal{H}'} e^{-\hat{H}t} e^{i\vec{x}\cdot\hat{\vec{p}}'} e^{-i\vec{x}_1\cdot\hat{\vec{p}}'} e^{\hat{H}t_1} \mathcal{J}^\mu e^{-\hat{H}t_1} e^{i\vec{x}_1\cdot\hat{\vec{p}}'} \bar{\chi}_\mathcal{H}|\, \Omega\rangle e^{-i\vec{x}_1\cdot\vec{p}_1} =$$

Momentum conservation → $p_f = p', p_i = p' - p_1$

$$\sum_{\substack{\vec{x},\vec{x}_1\\ \vec{k},\vec{k}',n,n'}} \frac{\langle \Omega \,|\chi_{\mathcal{H}'}|n', \vec{k}'\,\rangle \langle\, \vec{k}, n|\bar{\chi}_\mathcal{H}|\, \Omega\rangle}{2\sqrt{E_{n'}(\vec{k}')E_n(\vec{k})}} \; e^{-E_{n'}(\vec{k}')(t-t_1)} e^{-i\vec{x}\cdot(\vec{p}'-\vec{k}')} \langle\, n', \vec{k}'|\mathcal{J}^\mu|n, \vec{k}\,\rangle e^{-E_n(\vec{k})t_1} e^{-i\vec{x}_1\cdot(\vec{k}'-\vec{k}+\vec{p}_1)}$$

↓ **Limit $t-t_1 \gg 1$, $t_1 \gg 1$: ground state dominance**

$$G^{\mathcal{H}'\mathcal{J}^\mu\mathcal{H}}(\vec{p}, \vec{p}'; t, t_1) \equiv \sum_{\vec{x},\vec{x}_1} e^{-i\vec{x}\cdot\vec{p}'} G^{\mathcal{H}'\mathcal{J}^\mu\mathcal{H}}(\vec{x}, t; \vec{x}_1, t_1) e^{-i\vec{x}_1\cdot(\vec{p}'-\vec{p})} = \frac{\langle \Omega \,|\chi_{\mathcal{H}'}|\mathcal{H}'(\vec{p}')\,\rangle \langle\, \mathcal{H}(\vec{p})|\bar{\chi}_\mathcal{H}|\, \Omega\rangle}{2\sqrt{E(\vec{p}')E(\vec{p})}} \langle\, \mathcal{H}'(\vec{p}')|\mathcal{J}^\mu|\mathcal{H}(\vec{p})\,\rangle \times e^{-E(\vec{p}')(t-t_1)} e^{-E(\vec{p})t_1}$$

- $H' \neq H$: transition amplitudes and transition form factors
- $H' = H$: hadron structure
  - $p_f = p_i$: no momentum transfer (charges)
  - $p_f \neq p_i$: momentum transfer (form factors)

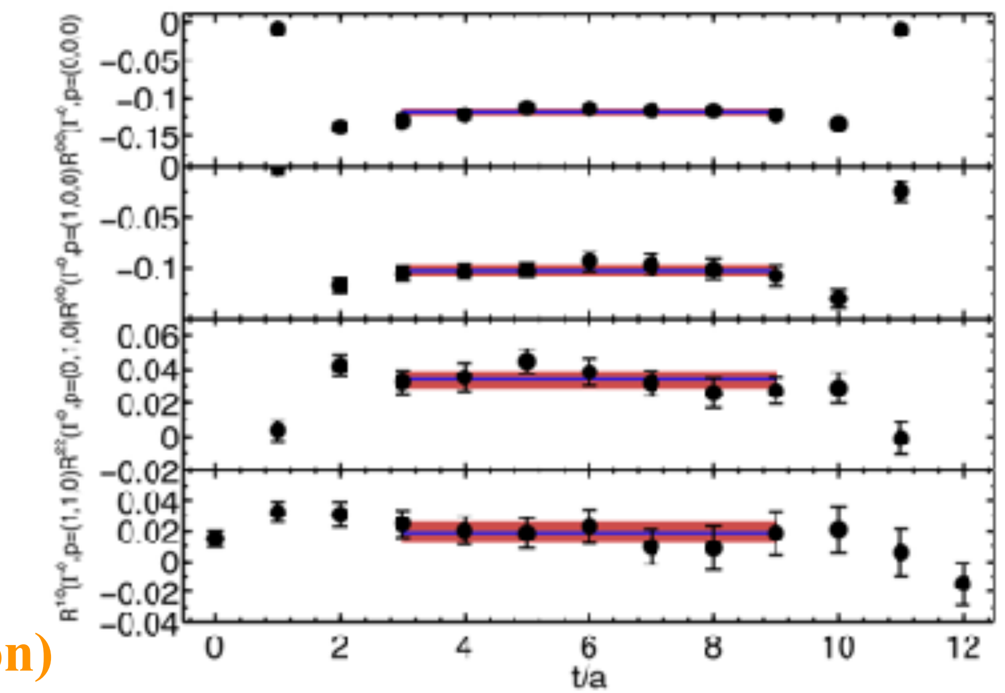
Nucleon on the Lattice

A. Calculation of matrix elements with appropriate currents for the quantities under study (e.g., vector-axial current)

$$C^{2pt} = \langle N | N \rangle \qquad C^{3pt}_{\Gamma} = \langle N | \overline{\psi}(0)\Gamma\psi(0) | N \rangle$$

B. Construction of optimized ratios and identify ground state

$$R^{\mu}_{\mathcal{O}}(\Gamma,\vec{q},t) = \frac{G_{\mathcal{O}}(\Gamma,\vec{q},t)}{G(\vec{0},t_f)}\sqrt{\frac{G(-\vec{q},t_f-t)G(\vec{0},t)G(\vec{0},t_f)}{G(\vec{0},t_f-t)G(-\vec{q},t)G(-\vec{q},t_f)}} \quad \overset{0<<\tau<<t}{=} \quad \Pi_{\Gamma}$$



Multiply analysis techniques available (single- & multi-state, summation)

C. Renormalization (usually multiplicative)

$$\Pi^R_{\Gamma} = Z\Pi_{\Gamma}$$

D. Kinematic factors based on symmetry properties, e.g.

$$A^3_{\mu} \equiv \bar{\psi}\,\gamma_{\mu}\gamma_5\frac{\tau^3}{2}\psi \Rightarrow \bar{u}_N(p')\left[G_A(q^2)\,\gamma_{\mu}\gamma_5 + G_p(q^2)\frac{q_{\mu}\gamma_5}{2\,m_N}\right]u_N(p)$$

Inherited uncertainties in lattice calculations

Statistical errors significantly increase with:

- ★ decrease of pion mass
- ★ increase of momentum transfer between initial-final state
- ★ Type of operator
- ★ increase of source-sink separation T_{sink}
- ★ ...

Sources of systematic uncertainties:

- ★ cut-off effects (finite lattice spacing)
- ★ finite volume effects
- ★ contamination from other hadron states
- ★ chiral extrapolation for unphysical pion mass
- ★ renormalization and mixing
- ★ ...

Careful error budgeting is essential for comparison to experiments

Investigation of systematic uncertainties

On a single ensemble:

★ Excited states contamination

★ Pion mass (with simulations at physical point)

★ Renormalization and mixing

★ Different methodologies to extract the same observables

Using multiple ensembles:

★ Cut-off effects due to finite lattice spacing

★ Finite volume effects

★ Pion mass dependence

} Effects reduced in single ensemble with appropriate parameters

Recap of Lecture 1

Key points of Lecture 1

★ QCD Lagrangian is compact, but extremely difficult to solve

★ Several models of QCD provide intuitive understanding, and can reliable results to high energy scales

★ Lattice QCD is the only first-principle non-perturbative formulation to study QCD from first principle

★ Lattice regularization is a well-formulated 4-D discretization

★ Several discretizations proposed for fermion and gluon action, with different advantages disadvantages

★ Computational cost is among the challenges of numerical simulations

★ Robust connection with observables

Join us at EINN 2025

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28 October – 01 November, 2025

Frontiers and Careers Workshops:

Pre-conference

Henry Klest (Argonne National Lab)
Aleksandr Pustyntsev (University of Mainz)
Abhyuday Sharda (University of Tennessee)
Natalie Wright (MIT)

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Frontiers and Careers in Photonuclear Physics 2025

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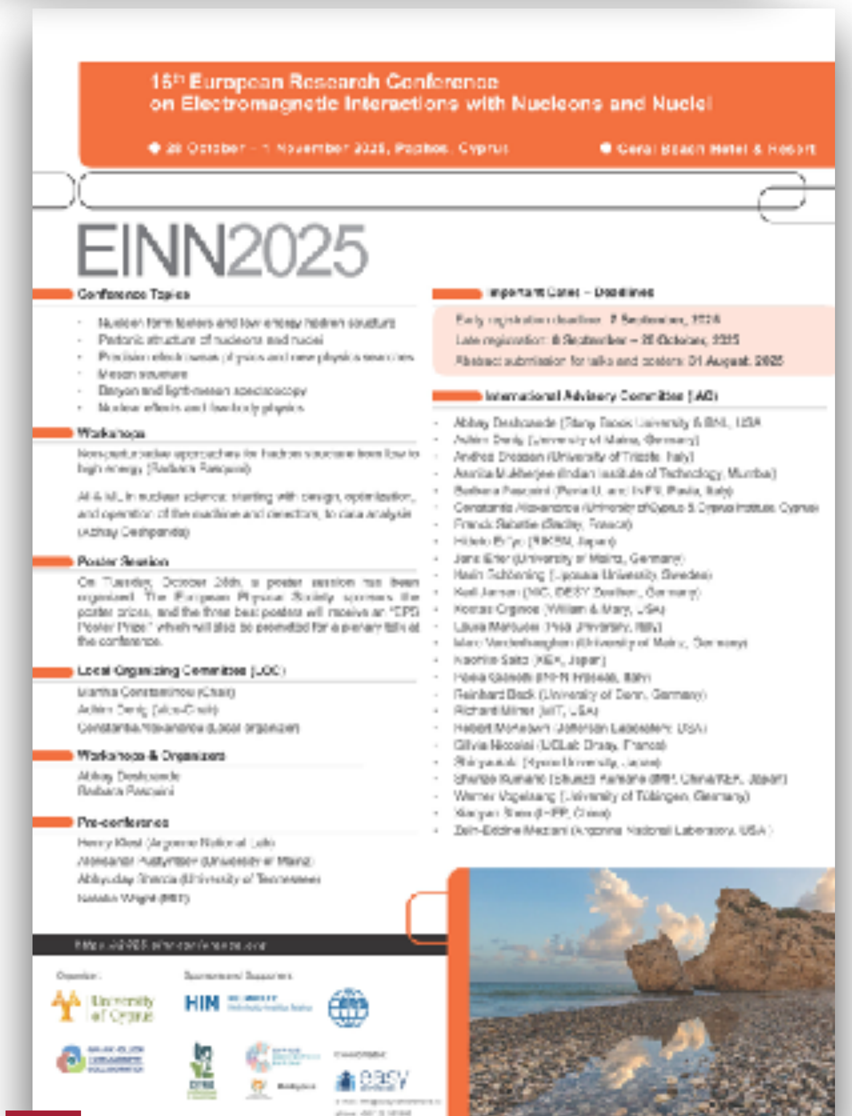


Abstract submission is Open!

Other topics relevant to EINN
Poster
Talk in workshop 1 "Non-perturbative approaches for hadron structure from low t
Talk in workshop 2: "AI & ML in nuclear science: starting with design, optimization, a

Thank you

Backup slides

Fermions and Gluons on the Lattice

★ Lattice formulation "must" be invariant under SU(3) local gauge transformation

$$\psi(x) \to V(x)\psi(x)\,, \qquad \bar{\psi}(x) \to \bar{\psi}(x)V^\dagger(x) \qquad\qquad V(x) = e^{-i\theta_a(x)\frac{\lambda_a}{2}}$$

$$U_\mu(x) \to V(x)U_\mu(x)V^\dagger(x+\hat{\mu}a)$$

★ Giving up gauge invariance would create a series of problems:

– More parameters to tune
(couplings for quark-gluon, 3- & 4-gluon interactions, the gluon mass,…)

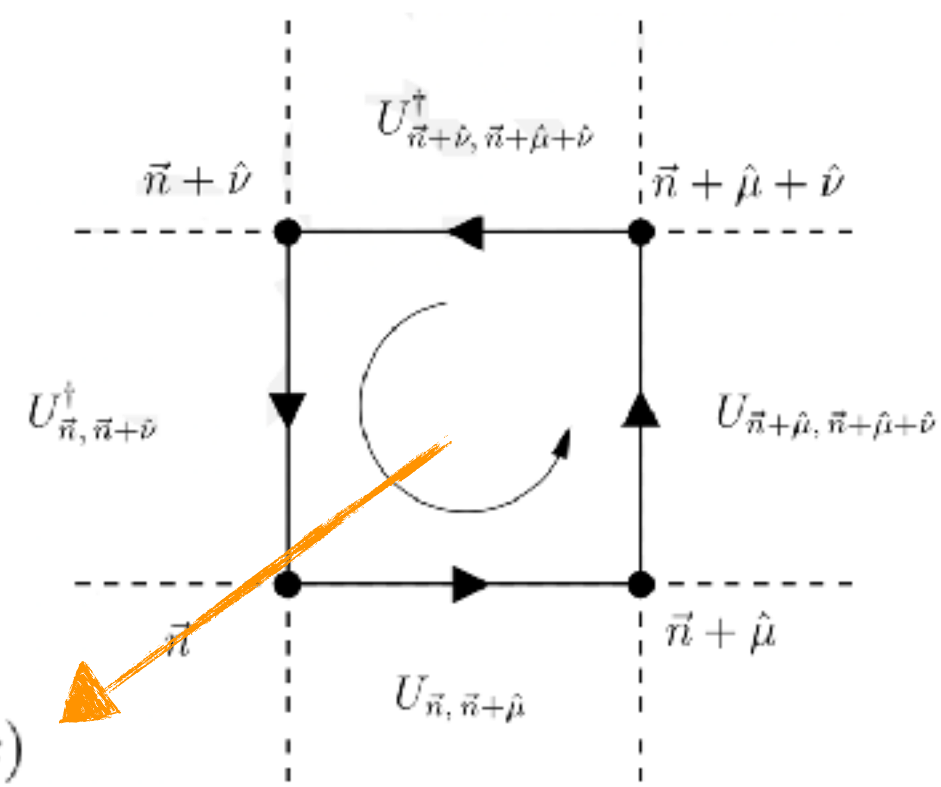
– More operators at any given order in α, thus increase of discretization errors

– Proofs of renormalizability within perturbation theory rely on strict gauge invariance
[T. Reisz & H. Rothe, Nucl.Phys. B575 (2000) 255]

★ Gauge invariant quantities:

– Products of Ψ(x), Ψ(x') and gauge links connecting x and x'

– Closed gluonic loops



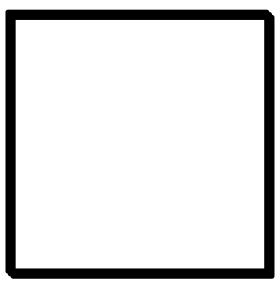
$$P_{\mu\nu} \equiv U_\mu(x)U_\nu(x+a\hat{\mu})U^\dagger_\mu(x+a\hat{\nu})U^\dagger_\nu(x)$$

Gluons on the Lattice

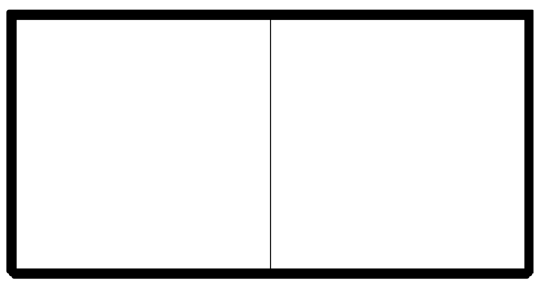
Gluon Actions:

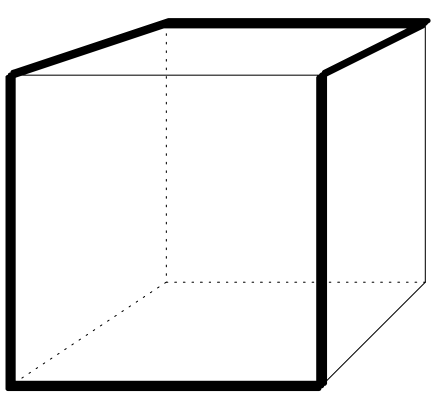
$$S_G = \frac{2}{g^2}\Big[c_0 \sum_{\text{plaquette}} \mathrm{Re\,Tr}\,\{1 - U_{\text{plaquette}}\} + c_1 \sum_{\text{rectangle}} \mathrm{Re\,Tr}\,\{1 - U_{\text{rectangle}}\}$$
$$+ c_2 \sum_{\text{chair}} \mathrm{Re\,Tr}\,\{1 - U_{\text{chair}}\} + c_3 \sum_{\text{parallelogram}} \mathrm{Re\,Tr}\,\{1 - U_{\text{parallelogram}}\}\Big]$$

plaquette

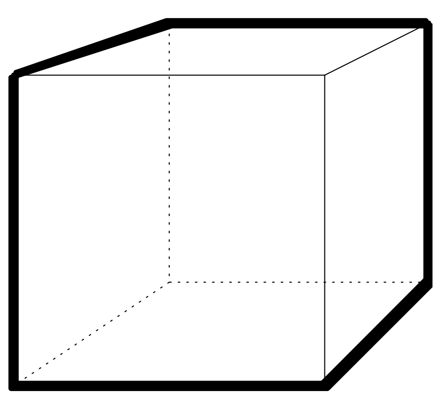


rectangle





chair



parallelogram

★ Choice of discretization not unique

Action	c_0	c_1	c_3
Plaquette	1.0	0	0
Symanzik	1.6666667	-0.083333	0
TILW, $\beta c_0 = 8.60$	2.3168064	-0.151791	-0.0128098
TILW, $\beta c_0 = 8.45$	2.3460240	-0.154846	-0.0134070
TILW, $\beta c_0 = 8.30$	2.3869776	-0.159128	-0.0142442
TILW, $\beta c_0 = 8.20$	2.4127840	-0.161827	-0.0147710
TILW, $\beta c_0 = 8.10$	2.4465400	-0.165353	-0.0154645
TILW, $\beta c_0 = 8.00$	2.4891712	-0.169805	-0.0163414
Iwasaki	3.648	-0.331	0
DBW2	12.2688	-1.4086	0

★ O(α^2) improved actions: approach better continuum limit

Fermions on the Lattice

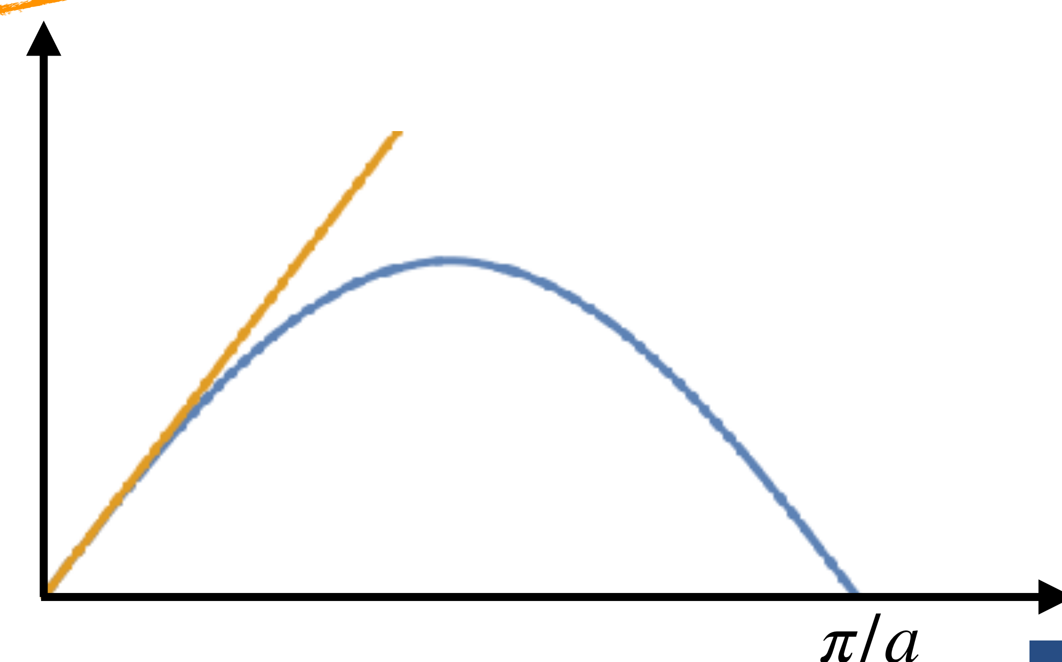
- Discretization of fermionic action complicated
- Naive discretization preserves gauge invariance, but results in fermion doubling problem: appearance of spurious states and continuum limit wrongly leads to 2^4 fermions instead of one.

$$S_F^{naive} = a^4 \sum_x \frac{1}{2a} \gamma^\mu [\bar{\psi}(x) U_\mu(x) \psi(x+a\hat{\mu}) - \bar{\psi}(x) U^\dagger_\mu(x-a\hat{\mu}) \psi(x-a\hat{\mu})] + m\bar{\psi}(x)\psi(x)$$

Fermion propagator (in momentum space upon Fourier Transform):

$$\langle \psi(x)\overline{\psi}(y)\rangle = \lim_{a\to 0} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{-i\sum_\mu \gamma_\mu sin(k_\mu) + m_0}{\sum_\mu sin^2(k_\mu) + m_0^2}$$

Additional poles:
Vanishes at the ends of Brillouin zone [-π/α,π/α].
In 4-dim these are sixteen regions instead of p~0 only, thus 16 species of fermions



Fermions on the Lattice

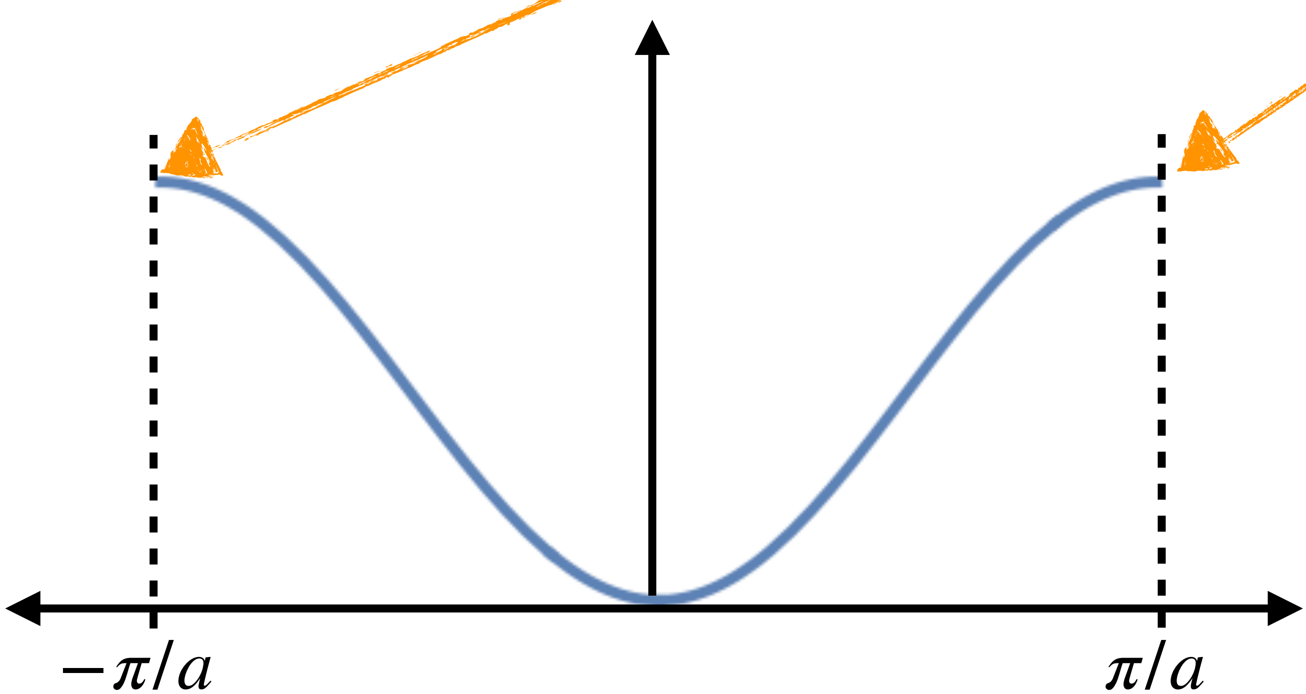
★ Wilson action to avoid doubling problem [Kenneth G. Wilson, Phys. Rev. D10 2445 (1974)]

$$S_F^W = a^4 \sum_{x,\mu} \bar{\psi}(x)\gamma^\mu D_\mu \psi(x) - a\frac{r}{2}\bar{\psi}(x) D_\mu D^\mu \psi(x) + m\bar{\psi}(x)\psi(x)$$

Wilson term, r: (0,1]

Denominator of Fermion propagator becomes

$$\frac{1}{a^2}\sum_\mu \sin^2(ak_\mu) + \Big(m + \frac{2r}{a}\sum_\mu \sin(a\frac{k_\mu}{2})\Big)^2$$



No poles at the edge of B.Z.

Properties of Wilson fermion action

★ Gauge invariant

★ Translational invariance

★ Invariant under charge conjugation (C), parity (P) and time reversal (T) transformations

★ Only nearest neighbors interactions (useful for lattice pert. theory)

★ Wilson-Dirac operator has γ_5-hermicity: $\gamma_5 D_W \gamma_5 = D^\dagger$

★ Wilson-Dirac operator, D_W+m is not protected against zero modes (quark mass: additive and multiplicative renormalization)

★ Chiral symmetry is explicitly broken at O(α) by Wilson term

★ O(α) Discretization effects

★ Axial current transformations are not exact symmetry and nonsinglet axial current requires renormalization