Homotopy approach for scattering amplitude for running QCD coupling

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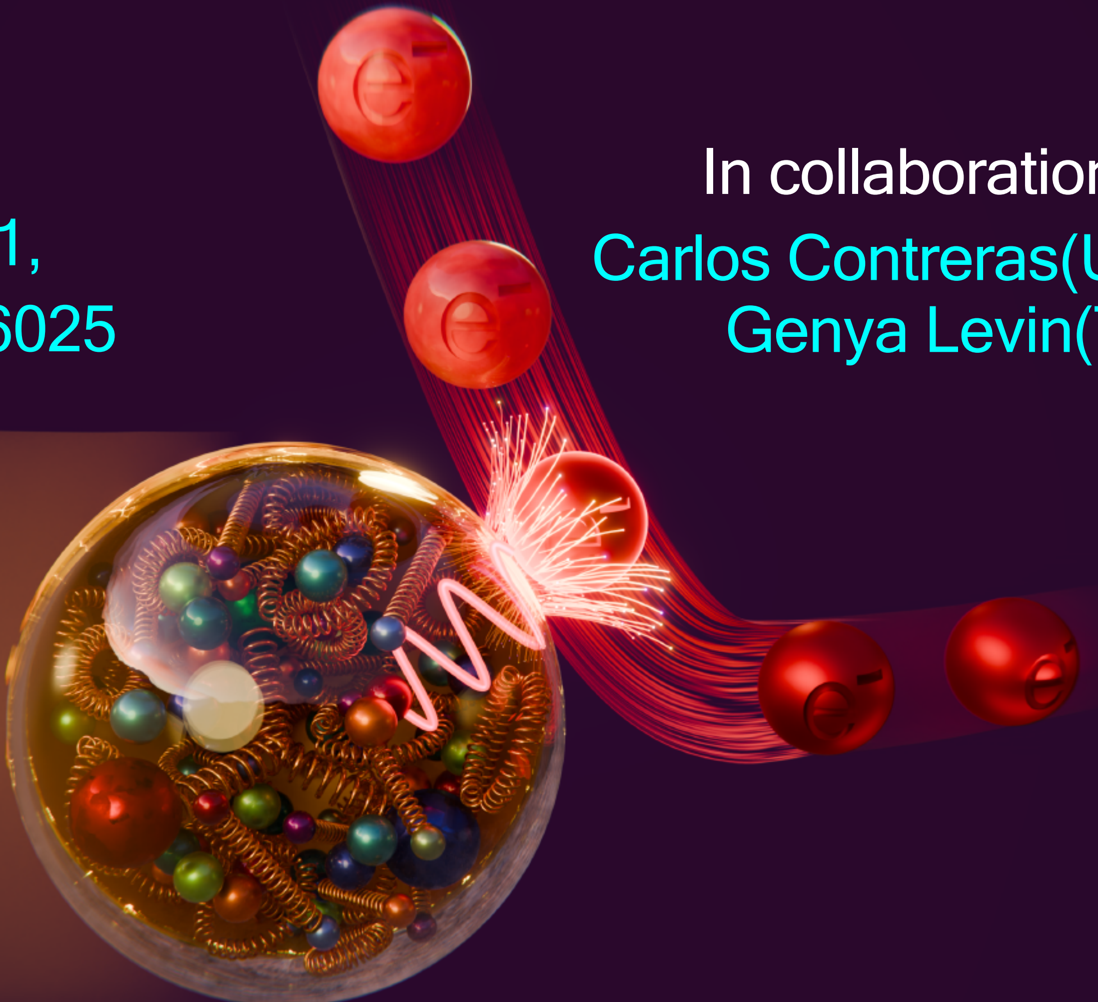

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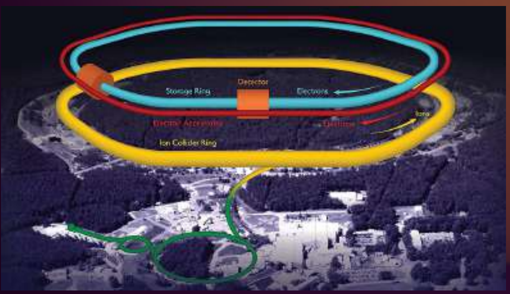

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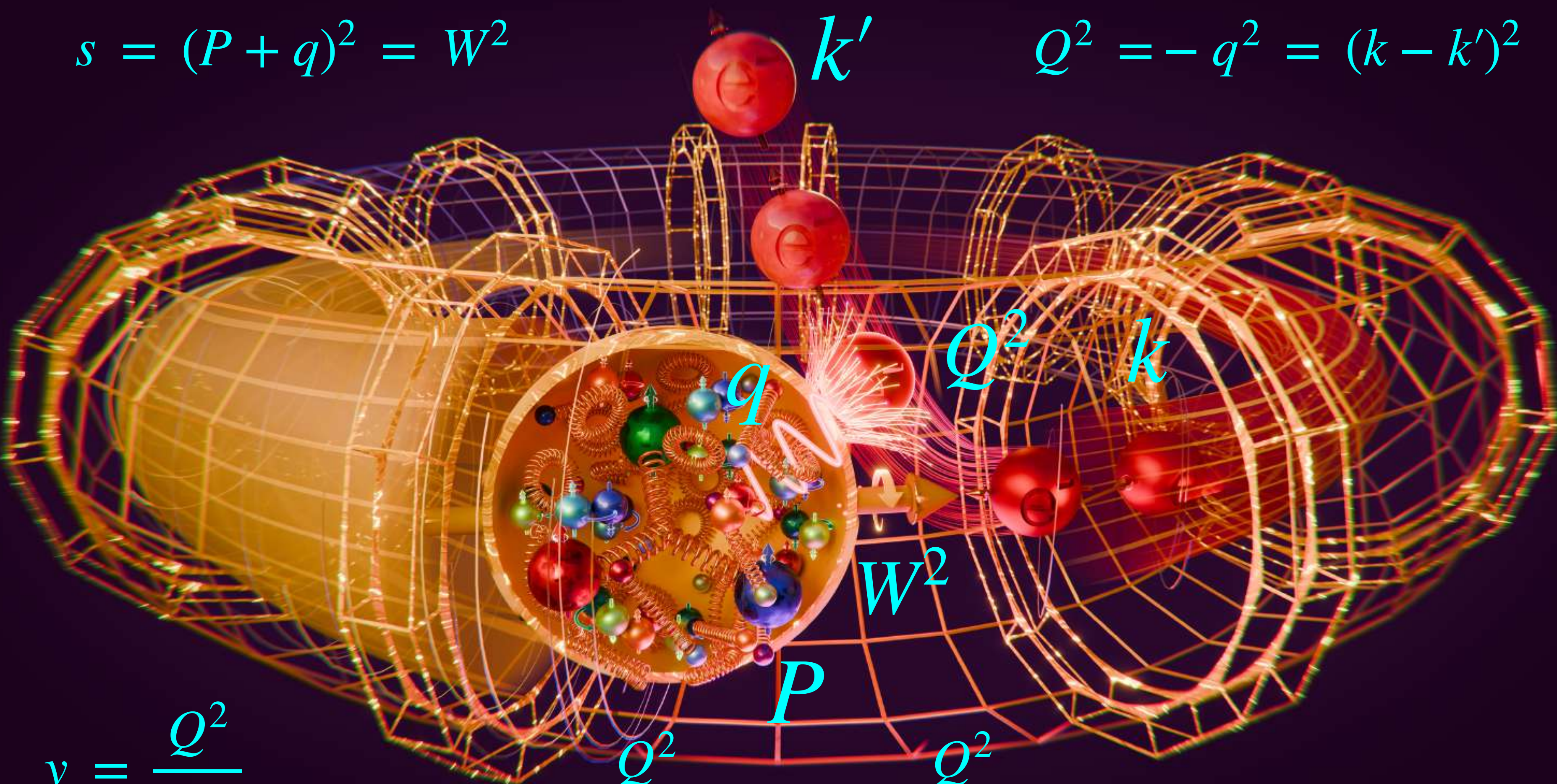
Deep Inelastic Scattering

$$s = (P+q)^2 = W^2 \qquad Q^2 = -q^2 = (k-k')^2$$

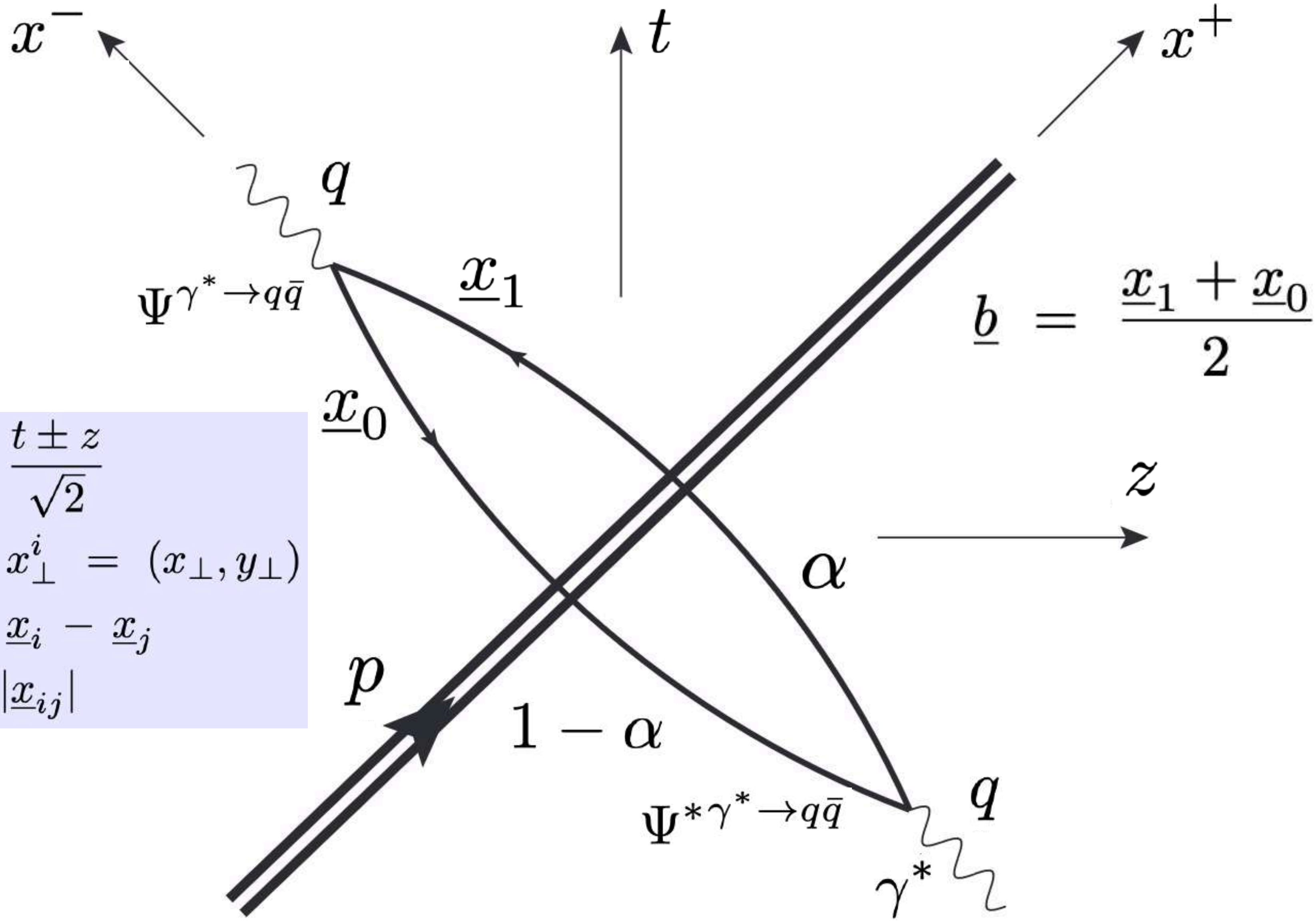


$$y = \frac{Q^2}{s x_{Bj}} \qquad x_{Bj} = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{Q^2+W^2-m_p^2}$$

Since the discovery of quarks, Deep Inelastic Scattering (DIS) has been instrumental to our understanding of the smallest building blocks of our universe.

Deep Inelastic Scattering

DIS process at small-x in the space-time representation



$$x^{\pm} = \frac{t \pm z}{\sqrt{2}}$$

$$\underline{x} = x_{\perp}^{i} = (x_{\perp}, y_{\perp})$$

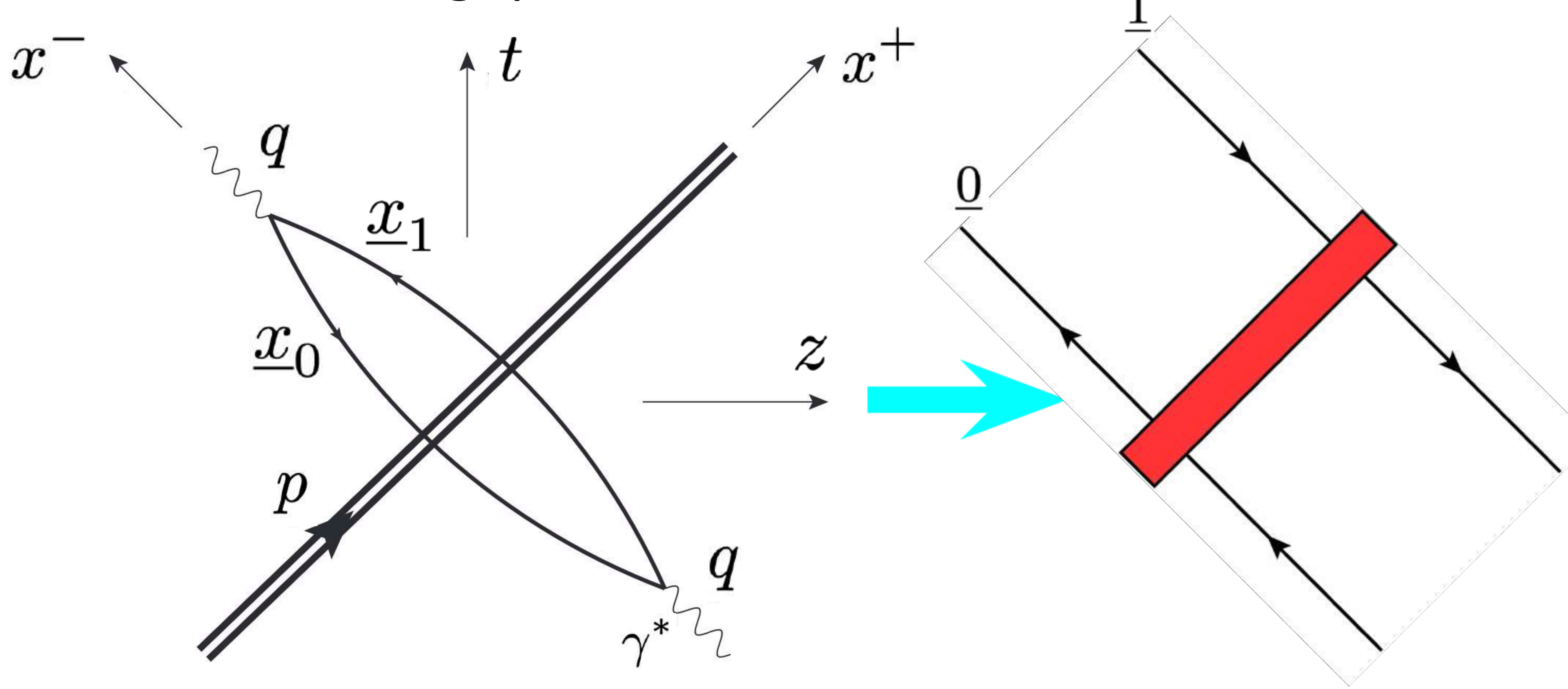
$$\underline{x}_{ij} = \underline{x}_i - \underline{x}_j$$

$$x_{ij} = |\underline{x}_{ij}|$$

aka the “shock wave”

Deep Inelastic Scattering

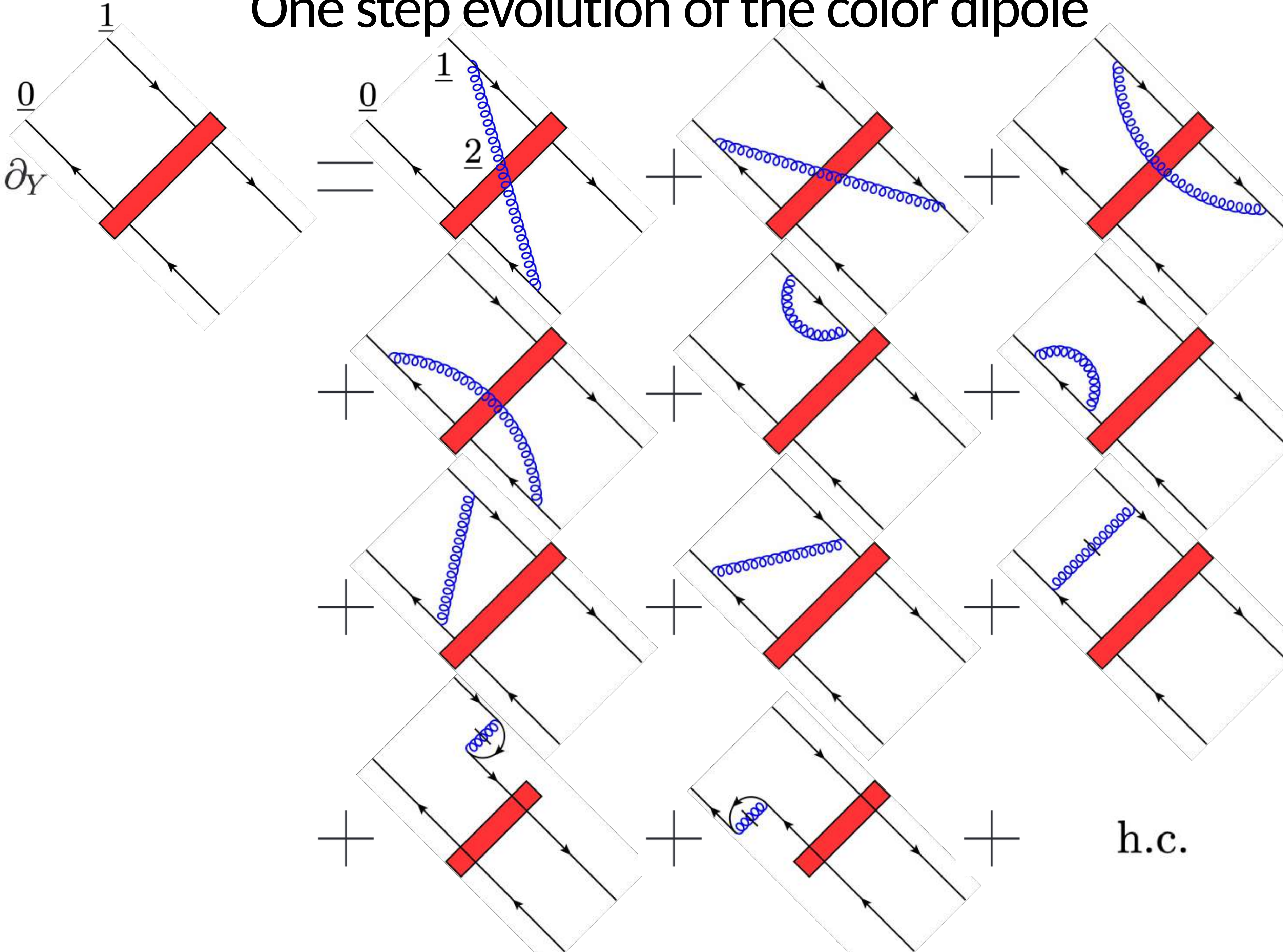
For a fast moving quark



$$\sigma_{T,L}^{\gamma^* A}(x_{Bj}, Q^2) = \int \frac{d^2 x_{10}}{4\pi} \int_0^1 \frac{dz}{z(1-z)} \left|\Psi_{L,T}^{\gamma^* \to q\bar{q}}(\underline{x}_{10}, z, Q^2)\right|^2 \sigma_{tot}^{q\bar{q}A}(\underline{x}_{10}, Y)$$

where $\sigma_{tot}^{q\bar{q}A}(\underline{x}_{10}, Y) = 2 \int d^2 b\, N(\underline{x}_{10}, Y; \underline{b})$, $\quad Y = \ln 1/x_{Bj}$

One step evolution of the color dipole



The Balitsky-Kovchegov equation

We find (Balitsky 1996 & Kovchegov 1999)

$$\frac{\partial N\left(\underline{x}_{10}, Y; \underline{b}\right)}{\partial Y} = \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \frac{x_{10}^2}{x_{20}^2 x_{21}^2} \times \left\{ N\left(\underline{x}_{21}, Y; \underline{b} - \frac{\underline{x}_{20}}{2}\right) + N\left(\underline{x}_{20}, Y; \underline{b} - \frac{\underline{x}_{21}}{2}\right) - N\left(\underline{x}_{10}, Y; \underline{b}\right) - N\left(\underline{x}_{21}, Y; \underline{b} - \frac{\underline{x}_{20}}{2}\right) N\left(\underline{x}_{20}, Y; \underline{b} - \frac{\underline{x}_{21}}{2}\right) \right\}$$

where $\bar{\alpha}_S = \alpha_S N_c / \pi$.

Its convenient to redefine the dipole sizes as

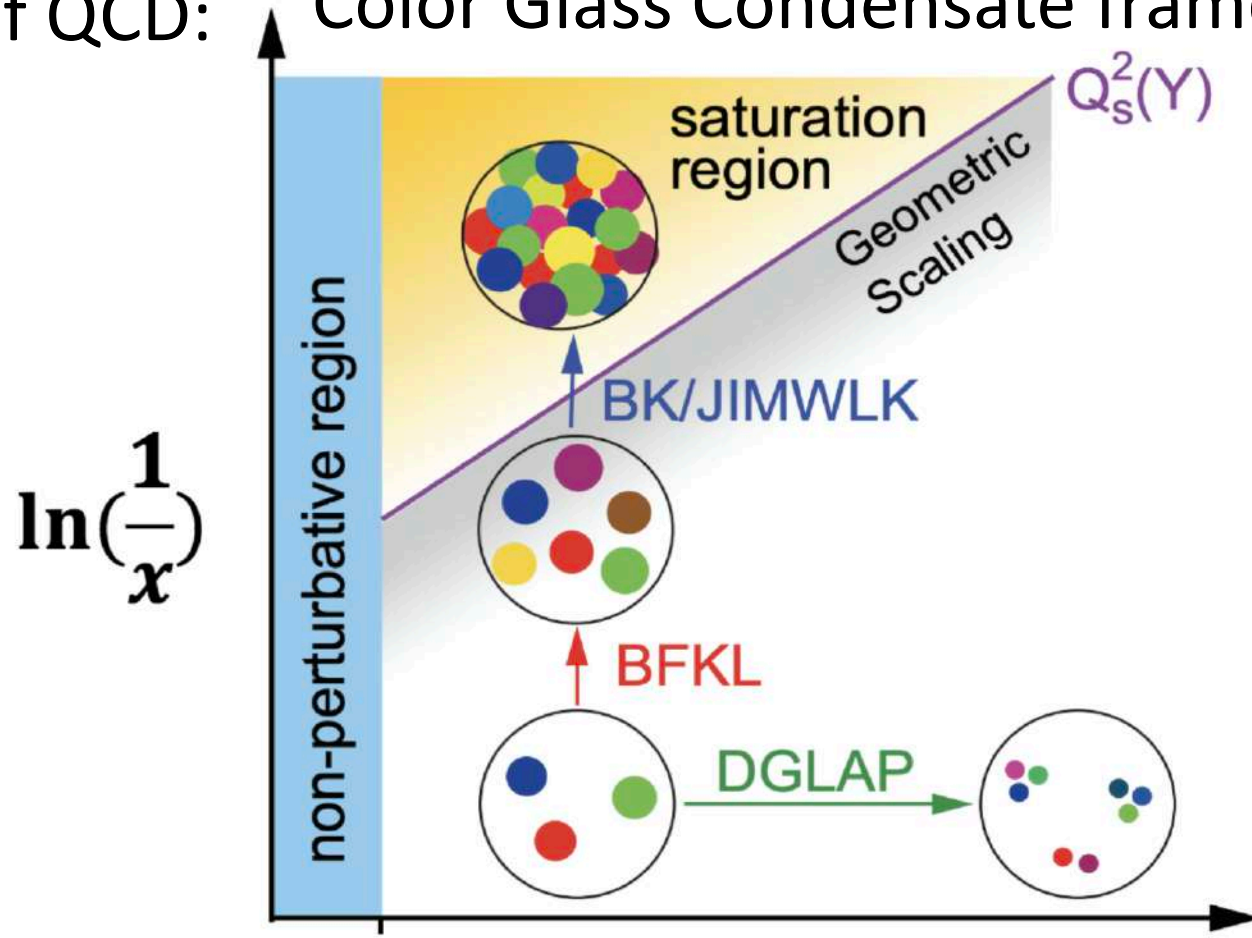
$$x_{10} \equiv r \qquad x_{20} \equiv r_1 \qquad x_{21} \equiv r_2 = r - r_1$$

and take $b \gg r$. We obtain

$$\frac{\partial N\left(\underline{r}, Y; \underline{b}\right)}{\partial Y} = \frac{\bar{\alpha}_S}{2\pi} \int d^2 r_1 \frac{r^2}{r_1^2 r_2^2} \left\{ N\left(\underline{r}_1, Y; \underline{b}\right) + N\left(\underline{r}_2, Y; \underline{b}\right) - N\left(\underline{r}, Y; \underline{b}\right) - N\left(\underline{r}_1, Y; \underline{b}\right) N\left(\underline{r}_2, Y; \underline{b}\right) \right\}$$

The Balitsky-Kovchegov equation

Map of QCD: Color Glass Condensate framework



Kinematic regions:

$r^2 < 1/Q_s^2$ pQCD

$r^2 \sim 1/Q_s^2$ Vicinity of the saturation scale

$r^2 > 1/Q_s^2$ Saturation region (sQCD)

The BFKL equation

In the region of $r^2 < 1/Q_s^2$ solution is given by

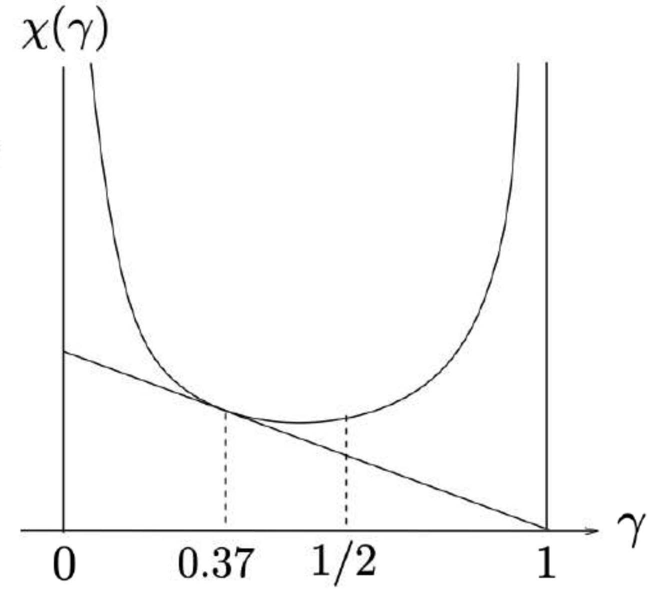
$$\frac{\partial N^{\mathrm{BFKL}}\left(\underline{r}, Y; \underline{b}\right)}{\partial Y} = \frac{\bar{\alpha}_S}{2\pi} \int d^2 r_1 \frac{r^2}{r_1^2 r_2^2} \times \left\{ N^{\mathrm{BFKL}}\left(\underline{r}_1, Y; \underline{b}\right) + N^{\mathrm{BFKL}}\left(\underline{r}_2, Y; \underline{b}\right) - N^{\mathrm{BFKL}}\left(\underline{r}, Y; \underline{b}\right) \right\}$$

using the Mellin representation of N

$$N^{\mathrm{BFKL}}\left(r, Y; \underline{b}\right) = \int d\nu\, r^{2\gamma} \tilde{N}^{\mathrm{BFKL}}(\nu, Y; \underline{b}) \quad \text{with} \quad \gamma \equiv \frac{1}{2} + i\nu$$

we obtain

$$\int d^2 r_1 \frac{r^2}{r_1^2 r_2^2} \left[(r_1^2)^\gamma + (r_2^2)^\gamma - (r^2)^\gamma \right] = 2\pi \chi(\gamma) (r^2)^\gamma$$



where

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)$$

The BFKL equation

Therefore

$$\frac{d\tilde{N}^{\rm BFKL}(\nu, Y; \underline{b})}{dY} \,=\, \bar{\alpha}_S \chi(\gamma) \tilde{N}^{\rm BFKL}(\nu, Y; \underline{b}) \Rightarrow \tilde{N}^{\rm BFKL}(\nu, Y; \underline{b}) \,=\, \exp\left(\bar{\alpha}_S \chi(\gamma) Y\right)$$

which leads to

$$N^{\rm BFKL}(r, Y; \underline{b}) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\,\pi\,i}\, e^{\bar{\alpha}_S \chi(\gamma) Y \,+\, \gamma \xi} \,\propto\, N_0 \left(r^2 Q_s^2(Y, b)\right)^{1-\gamma_{cr}} \,=\, N_0 e^{\bar{\gamma} z}$$

where

$$\xi \,=\, \ln r^2 Q_s^2(Y=0, b), \quad \xi_s \,=\, \ln\left(Q_s^2(Y, b)/Q_s^2(Y=Y_0, b)\right)$$

$$z \,=\, \ln(r^2 Q_s^2(Y, b)) \,=\, \bar{\alpha}_S \kappa Y \,+\, \xi \,=\, \xi_s \,+\, \xi$$

$$Q_s^2(Y, b) \,=\, Q_s^2(Y=0; b) e^{\bar{\alpha}_S \kappa Y}, \quad \kappa \,=\, \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}}$$

$$\gamma_{cr} \,=\, 0.37 \qquad \bar{\gamma} \,=\, 1 - \gamma_{cr}$$

The boundary condition for the nonlinear equation are

$$N(Y, \xi = -\xi_s; b) \,=\, N_0\,, \qquad \frac{\partial N(Y, \xi = -\xi_s; b)}{\partial \xi} \,=\, \bar{\gamma}\, N_0$$

The Balitsky-Kovchegov equation

Levin-Tuchin solution: Simplify the kernel by taking into account only log contributions (leading-twist approach)

$$\chi(\gamma) \approx \frac{1}{\gamma} + \text{higher twist contributions}$$

in this approximation, the BK equation reads

$$\frac{\partial^2 \widetilde{N}(Y,\xi;\underline{b})}{\partial Y\,\partial \xi} = \bar{\alpha}_S \left\{ \left(1 - \frac{\partial \widetilde{N}(Y,\xi;\underline{b})}{\partial \xi}\right) \widetilde{N}(Y,\xi;\underline{b}) \right\}$$

where $\widetilde{N}(Y,\xi;\boldsymbol{b}) = \int_{-\xi_s}^{\xi} d\xi'\, N(Y,\xi';\underline{b})$

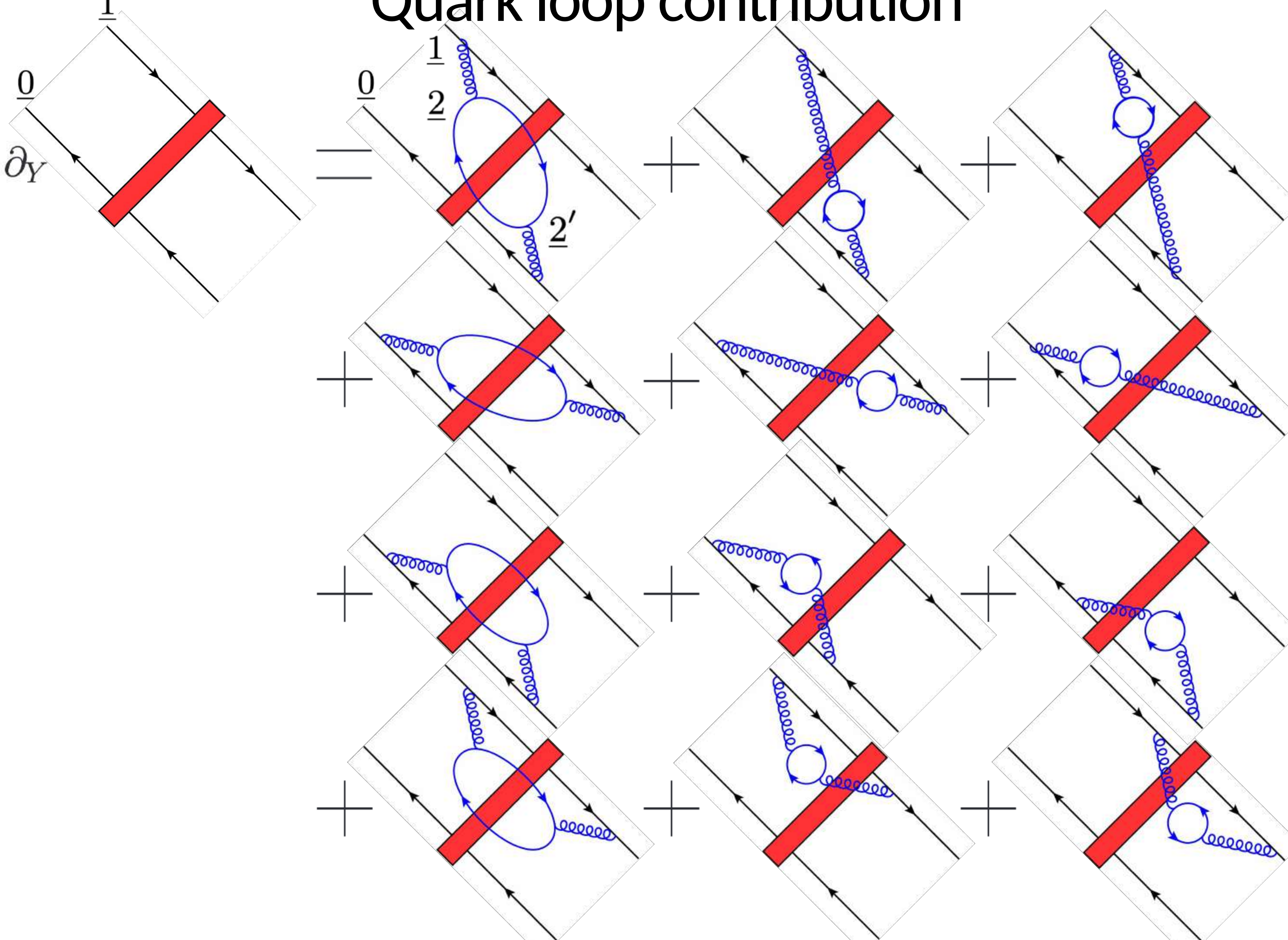
Introducing $N(r,Y;b) = 1 - e^{-\Omega(r,Y;b)}$, sol. for $\Omega(r,Y;\underline{b}) \equiv \Omega(z)$

$$\kappa \frac{d^2\Omega(z)}{dz^2} = 1 - e^{-\Omega(z)} \quad \Longrightarrow \quad \int_{\Omega_0}^{\Omega} \frac{d\Omega'}{\sqrt{-1+\Omega'+e^{-\Omega'}}} = \sqrt{\frac{2}{\kappa}}\, z$$

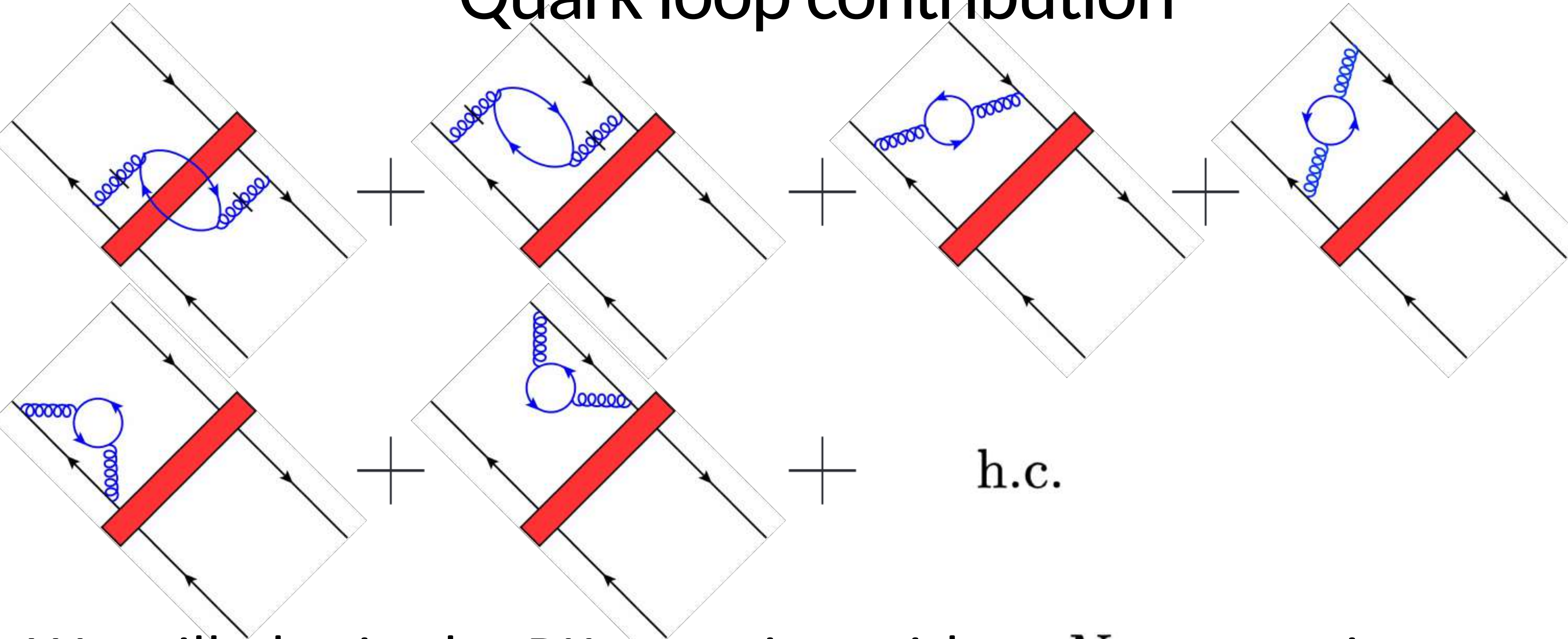
For small Ω the scattering amplitude is given by

$$N^{DIS}(r,Y;b) = 1 - \exp\left(-\frac{z^2}{2\kappa}\right)$$

Quark loop contribution



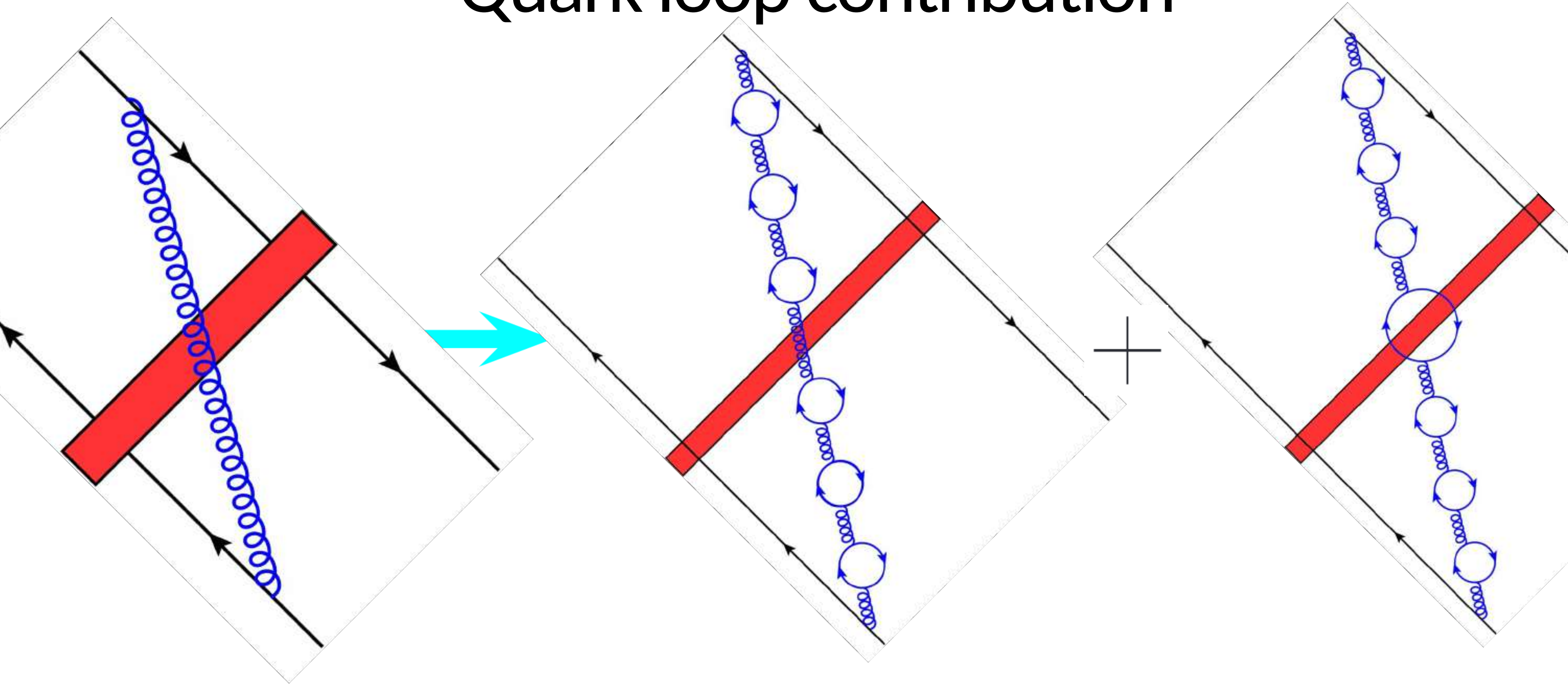
Quark loop contribution



We will obtain the BK equation with $\alpha_s N_f$ corrections.

To obtain higher order $\alpha_s N_f$ corrections to the kernel of the BK small-x evolution equation, insert infinite chains of quark loops onto the gluon lines

Quark loop contribution



Brodsky, Lepage, McKenzie (BLM) prescription:

$$N_f \;\to\; -6\pi\beta_2,\;\; \beta_2 \;=\; \frac{11N_c - 2N_f}{12\pi},\;\; \mu^2\frac{d\alpha_\mu}{d\mu^2} \;=\; -\beta_2\alpha_\mu^2 + \mathcal{O}(\alpha_\mu^3$$

$$\alpha_S(Q^2) \;=\; \frac{\alpha_\mu}{1+\alpha_\mu\beta_2\ln(Q^2/\mu^2)} \;=\; \frac{1}{\beta_2\ln(Q^2/\Lambda^2_{QCD})}$$

And take the leading log approximation (LLA) with $\alpha_s \ll 1$

Quark loop contribution

Balitsky prescription

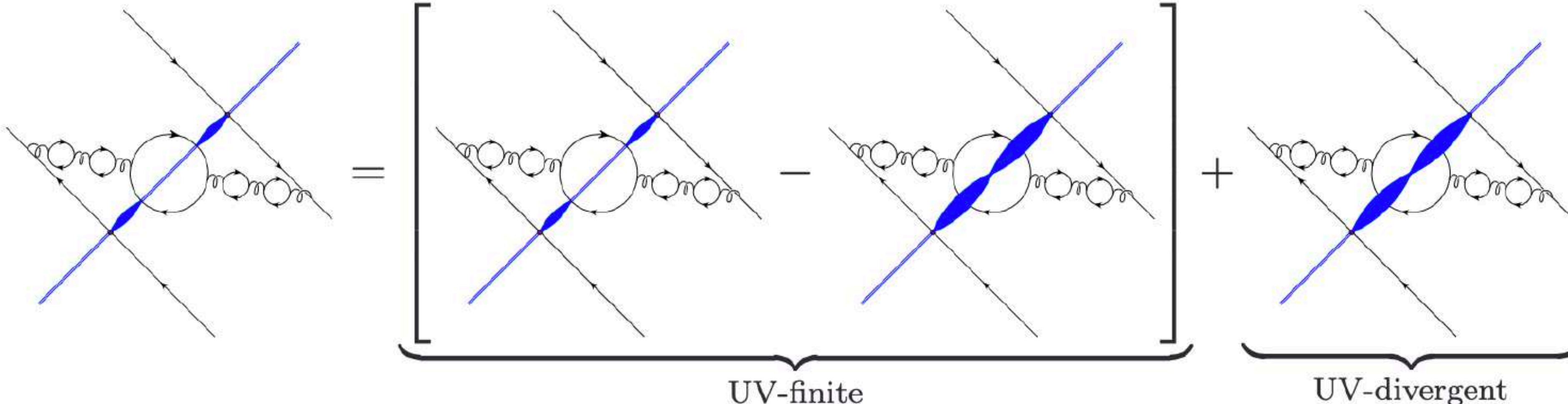
$$\begin{aligned}\frac{\partial N\left(\underline{x}_{10},Y;\underline{b}\right)}{\partial Y} &= \frac{\bar{\alpha}_S(1/x_{10}^2)}{2\pi}\int d^2x_{20}\Big\{N\left(\underline{x}_{21},Y;\underline{b}-\frac{\underline{x}_{20}}{2}\right)+N\left(\underline{x}_{20},Y;\underline{b}-\frac{\underline{x}_{21}}{2}\right)\\ &- N\left(\underline{x}_{10},Y;\underline{b}\right)-N\left(\underline{x}_{21},Y;\underline{b}-\frac{\underline{x}_{20}}{2}\right)N\left(\underline{x}_{20},Y;\underline{b}-\frac{\underline{x}_{21}}{2}\right)\Big\}\\ &\times \left[\frac{x_{10}^2}{x_{20}^2x_{21}^2}+\frac{1}{x_{20}^2}\left(\frac{\bar{\alpha}_S(1/x_{20}^2)}{\bar{\alpha}_S(1/x_{21}^2)}-1\right)+\frac{1}{x_{21}^2}\left(\frac{\bar{\alpha}_S(1/x_{21}^2)}{\bar{\alpha}_S(1/x_{20}^2)}-1\right)\right]\end{aligned}$$

Kovchegov-Weigert prescription

$$\begin{aligned}\frac{\partial N\left(\underline{x}_{10},Y;\underline{b}\right)}{\partial Y} &= \frac{1}{2\pi}\int d^2x_{20}\Big\{N\left(\underline{x}_{21},Y;\underline{b}-\frac{\underline{x}_{20}}{2}\right)+N\left(\underline{x}_{20},Y;\underline{b}-\frac{\underline{x}_{21}}{2}\right)-\\ &- N\left(\underline{x}_{10},Y;\underline{b}\right)-N\left(\underline{x}_{21},Y;\underline{b}-\frac{\underline{x}_{20}}{2}\right)N\left(\underline{x}_{20},Y;\underline{b}-\frac{\underline{x}_{21}}{2}\right)\Big\}\\ &\times \left[\frac{\bar{\alpha}_S(1/x_{21}^2)}{x_{21}^2}+\frac{\bar{\alpha}_S(1/x_{20}^2)}{x_{20}^2}\right.\\ &- \left. 2\frac{\bar{\alpha}_S(1/x_{21}^2)\bar{\alpha}_S(1/x_{20}^2)}{\bar{\alpha}_S\left(1/\left(x_{20}\,x_{21}\left(\frac{x_{21}}{x_{20}}\right)^{\frac{x_{20}^2+x_{21}^2}{x_{20}^2-x_{21}^2}-2\frac{x_{21}^2x_{20}^2}{\underline{x}_{21}\cdot\underline{x}_{20}}\frac{1}{x_{20}^2-x_{21}^2}}\right)\right)}\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2x_{20}^2}\right]\end{aligned}$$

Quark loop contribution

They differ as both models neglect different contributions (subtraction term), but they agree when added this term



We will take the Balitsky prescription in what follows.

In the vicinity of the saturation scale where

$$r^2 \approx r_1^2 \approx r_2^2 \approx 1/Q_s^2$$

we simplify as follows

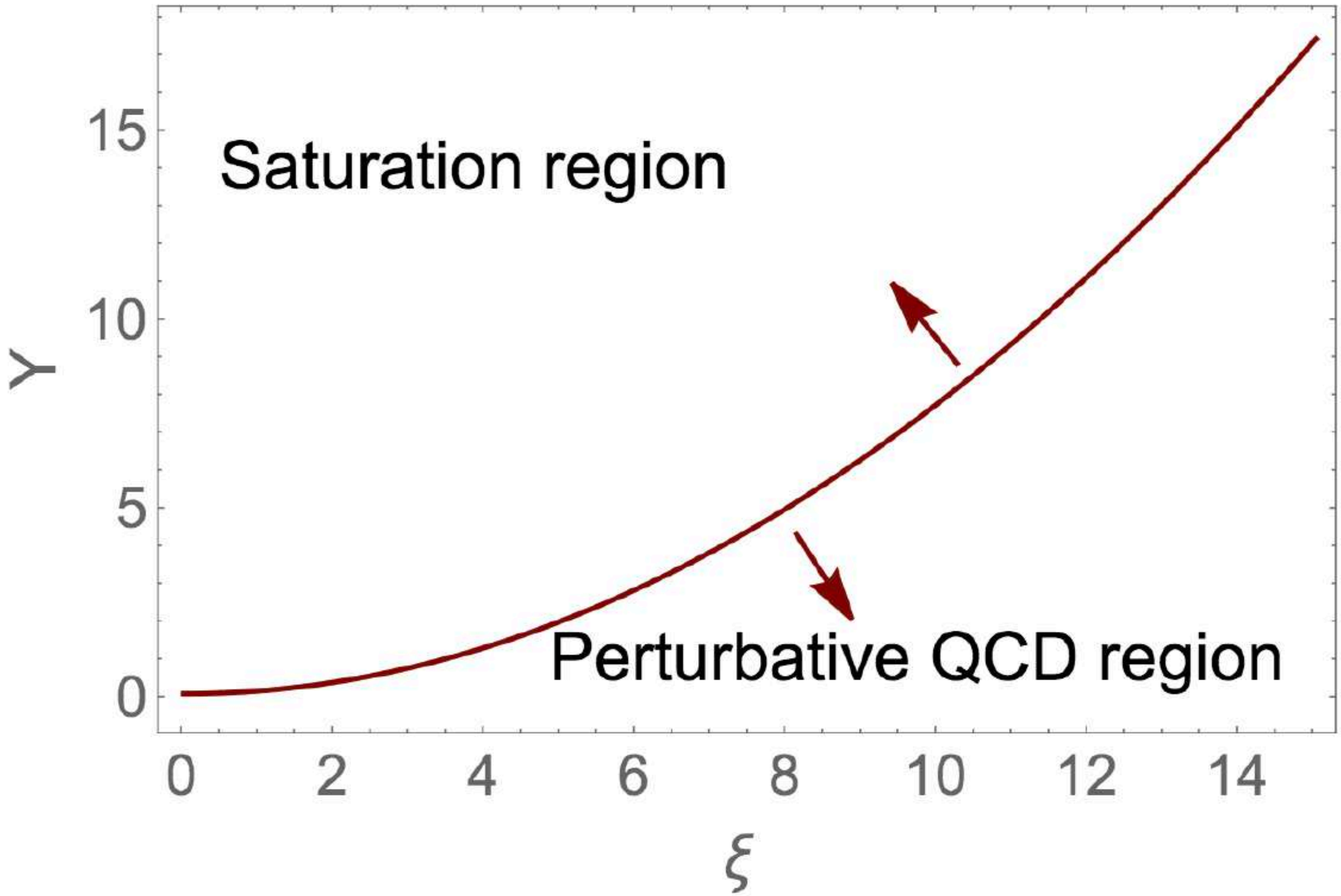
$$\alpha_S\left(r_i^2\right) = \frac{\alpha_S\left(r^2\right)}{1 + \frac{\alpha_S\left(r^2\right)}{4\pi b}\ln\left(r^2/r_i^2\right)} \xrightarrow{\ln\left(r^2/r_i^2\right) \ll \ln\left(r^2\,\Lambda^2_{QCD}\right)} \alpha_S\left(r^2\right)$$

$$K\left(r;r_1,r_2\right) = \bar{\alpha}_S\left(r^2\right)\,\frac{r^2}{r_1^2\,r_2^2}$$

The Balitsky-Kovchegov equation

This results in a saturation momentum of the form

$$Q_s^2\,(Y,b) \;=\; Q_s^2\,(Y = Y_0, b)\; e^{\sqrt{\frac{8N_c}{b_0}\,\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,Y}}$$



Such simple modification affects a lot to our equations

The Balitsky-Kovchegov equation

The rcBK equation in the leading twist approach is

$$\frac{\partial N\left(r,Y;b\right)}{\partial Y} = \tilde{N}\left(r,Y;b\right)\Big(1 - N\left(r,Y;b\right)\Big)$$

where $\tilde{N}\left(r,Y;b\right) = \int\limits_{1/Q_s^2}^{r^2} dr'^2\,\frac{\bar{\alpha}_S\left(r'^2\right)}{r'^2}\,N\left(r',Y;b\right)$

Introducing $N\left(r,Y;b\right) = 1 - e^{-\Omega(r,Y;b)}$

$$\frac{\partial^2\Omega\left(r,Y;b\right)}{\partial Y\,\partial l} = 1 - e^{-\Omega(r,Y;b)}$$

where

$$l = \int^{r^2} dr'^2\frac{\bar{\alpha}_S\left(r'^2\right)}{r'^2} = -\frac{4N_c}{b_0}\ln\Big(4N_c/\left(b_0\,\bar{\alpha}_S\left(r^2\right)\right)\Big) = -\frac{4N_c}{b_0}\ln\left(\bar{\xi}\right)$$

with $\bar{\xi} = -\ln\left(r^2\,\Lambda^2_{QCD}\right) \equiv -\xi$

The Balitsky-Kovchegov equation

Eq. for $\Omega(r, Y; \underline{b}) \equiv \Omega(z)$, with $z = \xi_s - \bar{\xi}$ gives

$$\sqrt{\frac{16N_c}{b_0}} \frac{\tilde{z}}{\sqrt{2Y}} \frac{d^2\Omega\left(\tilde{z}; b\right)}{d\,\tilde{z}^2} + \frac{d^2\Omega\left(\tilde{z}; b\right)}{d\,\tilde{z}^2} = 1 - e^{-\Omega(\tilde{z}; b)}$$

where $\tilde{z} = \sqrt{\frac{16N_c}{b}}\, z$. We cannot solve this equation.

Therefore we search a solution in the form $\Omega\left(Y, l; b\right) = \Omega\left(\zeta; b\right)$ with $\zeta = Y\, l$. It leads to

$$\zeta \frac{d^2\Omega\left(\zeta; b\right)}{d\zeta^2} + \frac{d\,\Omega\left(\zeta; b\right)}{d\zeta} = 1 - e^{-\Omega(\zeta; b)}$$

with boundary conditions

$$\Omega\left(\zeta = Y\, l_s; b\right) = \Omega_0;$$

$$\frac{d\Omega\left(\zeta = Y\, l_s; b\right)}{d\zeta} = -\frac{1}{2}\Omega_0/\xi_s = -\frac{1}{2}\Omega_0 \Big/ \sqrt{\frac{32N_c}{b}\, Y}$$

Again, we cannot solve the equation with those boundary conditions (not constant).

The Balitsky-Kovchegov equation

Therefore we search a solution in the form

$$.\Omega\left(\zeta, l - l_s b\right) \;=\; \underbrace{\Omega^{(0)}\left(\zeta, b\right) \;+\; \Omega^{(1)}\left(\zeta, b\right)}_{\Omega_\zeta(\zeta)} \;+\; \Omega'\left(\zeta, l - l_s; b\right)$$

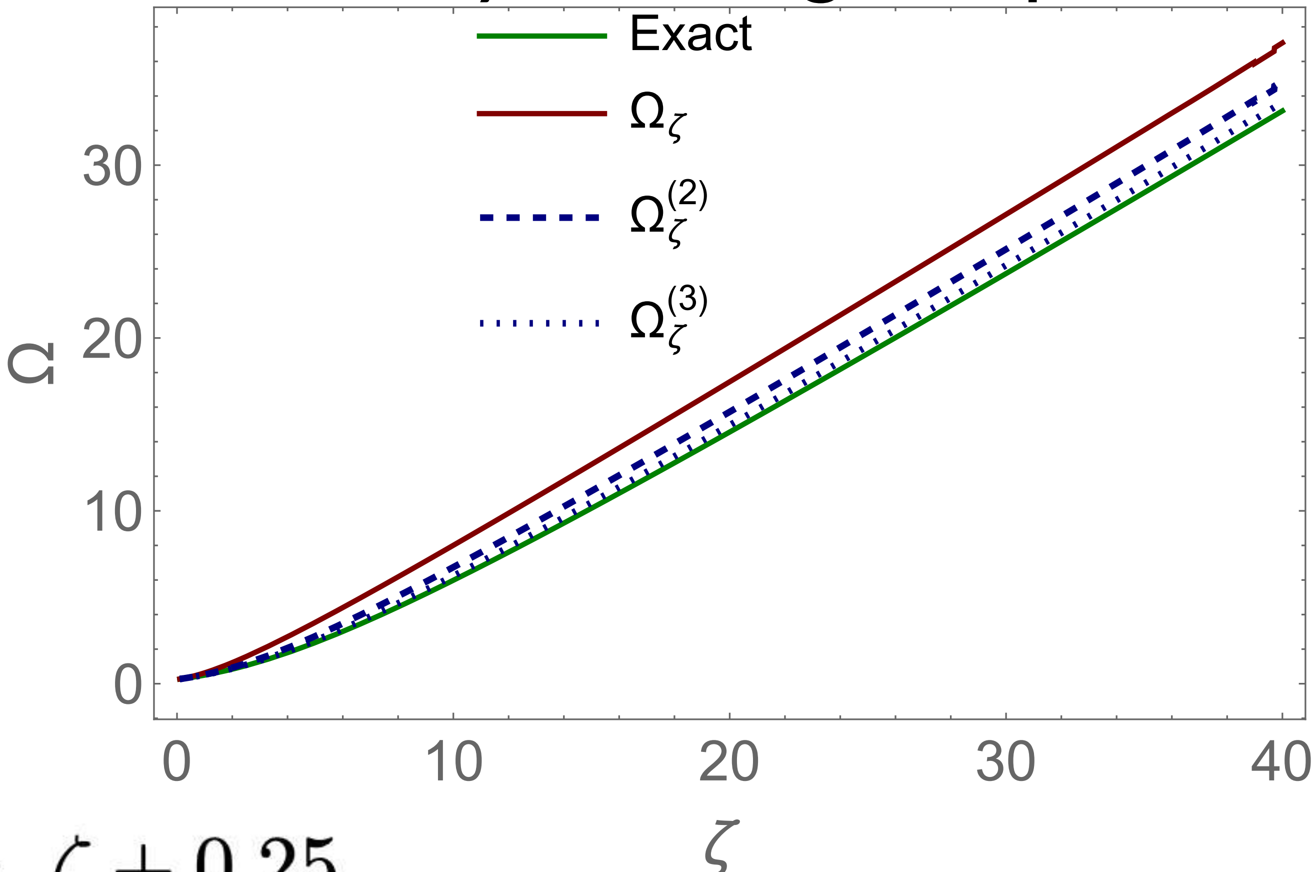
Where Ω_ζ satisfies that self-similar equation but for large Y, i.e

$$\frac{d\Omega\left(\zeta = 0; b\right)}{d\zeta} \;=\; 0;$$

with ζ redefined as $\zeta = Y\left(l - l_s\right)$ and Ω' satisfies the general equation but with dependance of new variable:

$$l \,-\, l_s \;=\; \int\limits_{1/Q_s^2}^{r^2} dr'^2 \frac{\bar{\alpha}_S\left(r'^2\right)}{r'^2} \;=\; -\frac{4\,N_c}{b_0} \ln\left(\frac{-\xi}{\xi_s}\right)$$

The Balitsky-Kovchegov equation

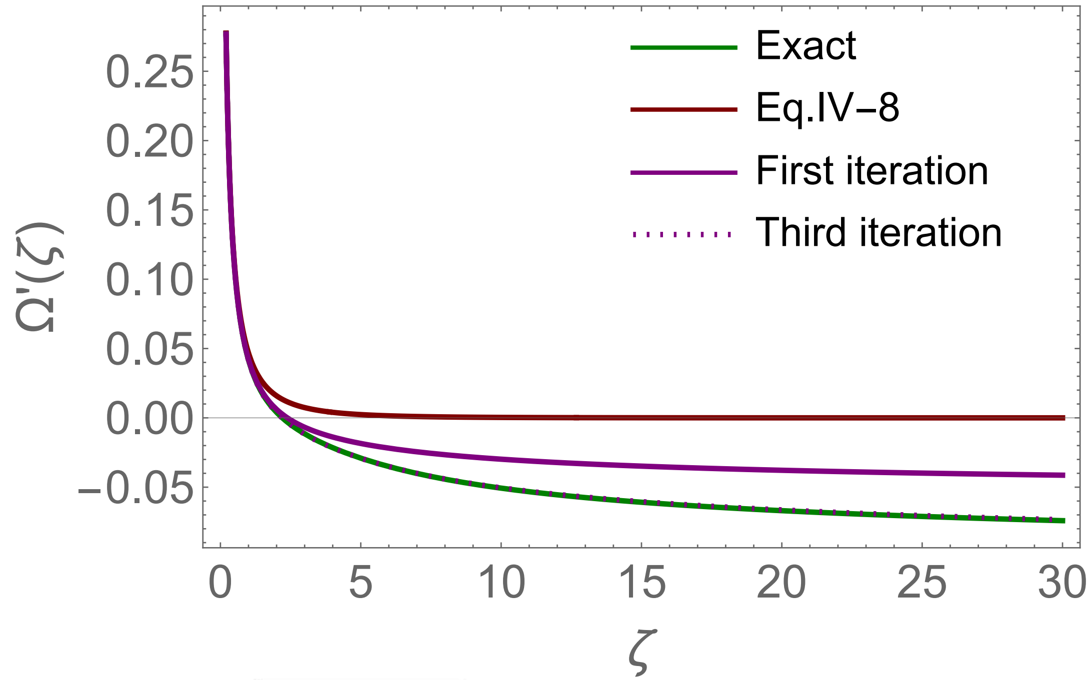


$$\Omega^{(0)} = \zeta + 0.25$$

$$\Omega^{(1)} = -e^{-\Omega^{(0)}} \left(-\mathrm{Chi}(\zeta) + \ln(\zeta) + \mathrm{Shi}(\zeta) + \gamma_{\mathrm{E}}\right)$$

$$\Omega^{(i+1)}(\zeta; b) = \int_0^{\zeta} \ln\left(\frac{\zeta}{t}\right) \exp\left(-\sum_{l=0}^{i-1} \Omega^{(l)}(t; b)\right) \left(1 - e^{-\Omega^{(i)}(t;b)}\right)$$

The Balitsky-Kovchegov equation



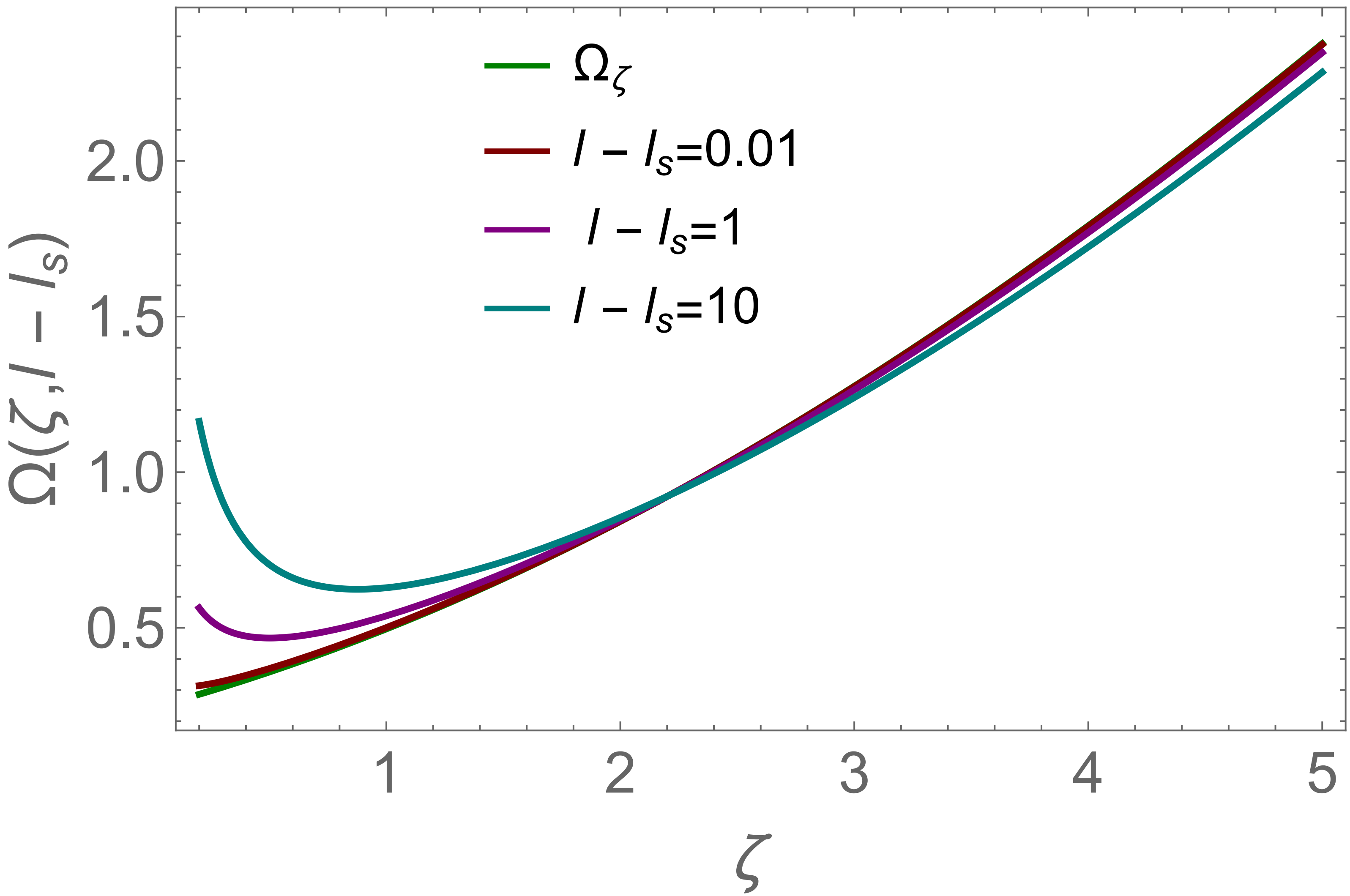
$$\Omega'\left(\zeta, l-l_s; b\right) = \underbrace{\left(\bar{\gamma}\Omega_0 e^{\Omega_0/2}\frac{\sqrt{\frac{8\,N_c}{b}\,\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}}}{\sqrt{\pi}\,\frac{4\,N_c}{b}}\right)}_{\text{Const}} \sqrt{l-l_s}K_{-\frac{1}{2}}\left(2\sqrt{\exp\left(-\Omega_0\right)\zeta}\right)/\left(\exp\left(-\Omega_0\right)\zeta\right)^{1/4}$$

$$\Omega'\left(\zeta, l\right) = \Omega'\left(\zeta, l-l_s; Eq.\ (\boxed{IV.12})\right) + \Delta\Omega'\left(\zeta, l-l_s\right) \text{ with } \Delta\Omega'\left(\zeta, l\right) = \Delta\mathcal{Z}'\left(l-l_s\right)^{\frac{1}{2}}$$

$$\Delta\mathcal{Z}\left(\zeta\right) = -\int_0^\zeta \frac{dt}{t^{3/2}}\int_0^t dt'\sqrt{t'}\left(e^{-\Omega_0} - e^{-\Omega_\zeta(t')}\right)\mathcal{Z}\left(t'; Eq.\ (\boxed{IV.12})\right) = 2\int_0^\zeta dt\left(\sqrt{\frac{t}{\zeta}}-1\right)\left(e^{-\Omega_0} - e^{-\Omega_\zeta(t)}\right)\mathcal{Z}\left(t; Eq.\ (\boxed{IV.12})\right)$$

$$\Delta^{(i)}\mathcal{Z}\left(\zeta\right) = \int_0^\zeta \frac{dt}{t^{3/2}}\int_0^t dt'\sqrt{t'}\,e^{-\Omega_\zeta(t')}\Delta^{(i-1)}\mathcal{Z}\left(t'\right) = -2\int_0^\zeta dt\left(\sqrt{\frac{t}{\zeta}}-1\right)e^{-\Omega_\zeta(t)}\Delta^{(i-1)}\mathcal{Z}\left(t\right)$$

The Balitsky-Kovchegov equation



$$\Omega^{(0)}\left(\zeta, l - l_s\right) \equiv \Omega_\zeta^{(3)}\left(\zeta\right) + \Omega'\left(\zeta, l - l_s\right)$$

Homotopy approach

Let us consider the following nonlinear differential equation $\mathscr{L}[u] + \mathscr{N}_{\mathscr{L}}[u] = 0$

Integro-differential operator

Non-linear part; arbitrary form

Introducing the homotopy function $\mathscr{H}(p,u)$:

$$\mathscr{H}(p,u) = \mathscr{L}[u_p] + p\,\mathscr{N}_{\mathscr{L}}[u_p] = 0$$

with $p \in [0,1]$ a small parameter. The solution is

$$\begin{aligned} u_p(r,Y;\boldsymbol{b}) &= u_0(r,Y;\boldsymbol{b}) + p\,u_1(r,Y;\boldsymbol{b}) + p^2\,u_2(r,Y;\boldsymbol{b}) + \ldots \\ &= u_0(r,Y;\boldsymbol{b})\left(1 + p\,\frac{u_1}{u_0}(r,Y;\boldsymbol{b}) + p^2\,\frac{u_2}{u_0}(r,Y;\boldsymbol{b}) + \ldots\right) \end{aligned}$$

Homotopy approach

At p=1 it gives the solution to the nonlinear equation

$$u = \lim_{p\to 1} u_p = u_0 + u_1 + u_2 + \cdots$$

the convergence of this method has been proved (J.H. He, 1999).

We apply the homotopy approach to the BK equation

$$\begin{aligned}
N\left(r,Y;\boldsymbol{b}\right) &= N^{(0)}\left(r,Y;\boldsymbol{b}\right) + p\,N^{(1)}\left(r,Y;\boldsymbol{b}\right) + p^2\,N^{(2)}\left(r,Y;\boldsymbol{b}\right) + \ldots \\
&= N^{(0)}\left(r,Y;\boldsymbol{b}\right)\left(1 + p\,\frac{N^{(1)}}{N^{(0)}}\left(r,Y;\boldsymbol{b}\right) + p^2\,\frac{N^{(2)}}{N^{(0)}}\left(r,Y;\boldsymbol{b}\right) + \ldots\right) \\
&\overset{p\to 1}{=} N^{(0)}\left(r,Y;\boldsymbol{b}\right)\left(1 + \frac{N^{(1)}}{N^{(0)}}\left(r,Y;\boldsymbol{b}\right) + \frac{N^{(2)}}{N^{(0)}}\left(r,Y;\boldsymbol{b}\right) + \ldots\right)
\end{aligned}$$

where

$$\begin{aligned}
N^{(0)}\left(Y,l-l_s\right) &= 1 - e^{-\Omega^{(0)}\left(Y,l-l_s\right)} \\
\frac{N^{(i)}\left(Y,l-l_s\right)}{N^{(0)}\left(Y,l-l_s\right)} &= \frac{\Omega^i\left(Y,l-l_s\right)e^{-\Omega^{(0)}\left(Y,l-l_s\right)}}{1 - e^{-\Omega^{(0)}\left(Y,l-l_s\right)}}
\end{aligned}$$

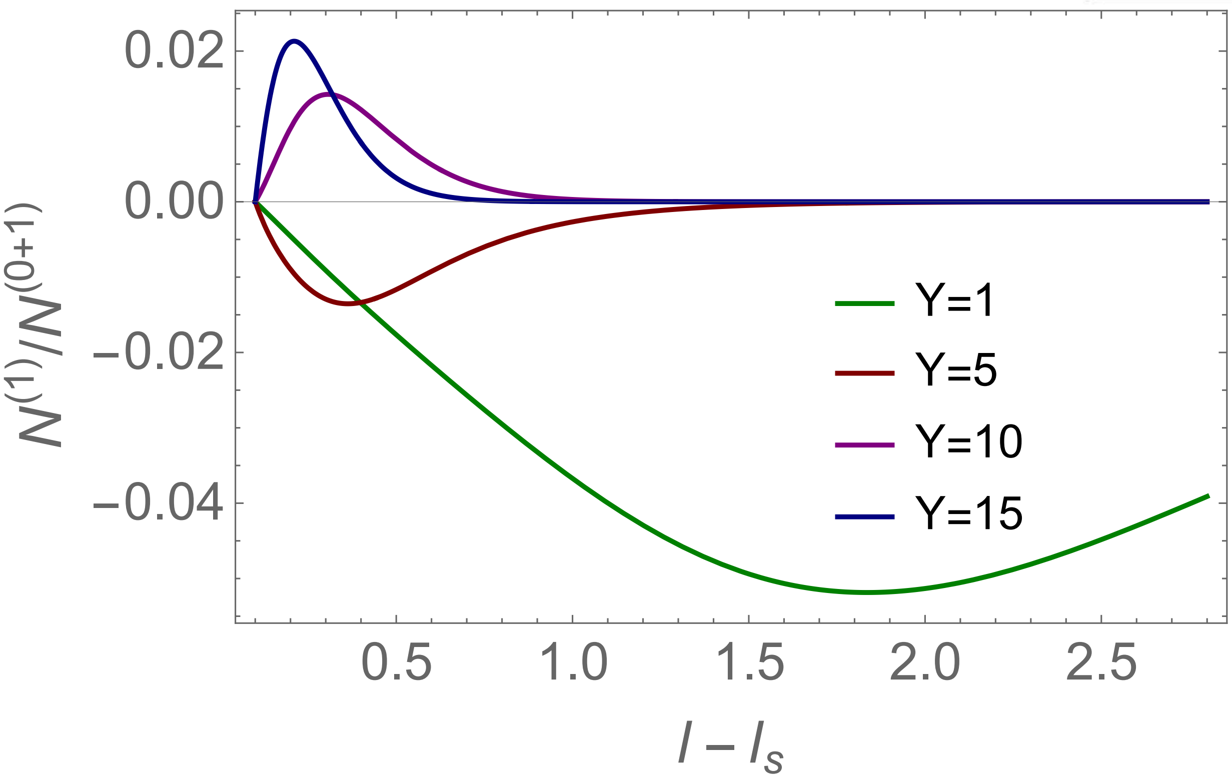
Homotopy approach

We define

$$\mathscr{L}[u_p] = 0 \qquad \mathscr{L}[\Omega] = \frac{\partial^2 \Omega\,(r, Y; b)}{\partial Y\, \partial l} - 1 + e^{-\Omega(r,Y;b)}$$

We already solved the zero iteration. The next homotopy iteration is $\mathscr{H}\left(p, \Omega^{(0)} + p\Omega^{(1)}\right) = \mathscr{L}[\Omega^{(0)} + p\,\Omega^{(1)}] + p\,\mathscr{N}_{\mathscr{L}}[\Omega^{(0)}] = 0$

$$\frac{\partial^2 \Omega^{(1)}\,(Y, l - l_s)}{\partial Y\, \partial l} = \left(1 - e^{-\Omega^{(1)}(Y, l - l_s)}\right) e^{-\Omega^{(0)}(Y, l - l_s)} - \underbrace{\frac{\partial}{\partial l}\left(e^{\Omega^{(0)}(Y, l - l_s)}\mathscr{N}_{\mathscr{L}}[\Omega^{(0)}]\right)}_{DH^{(0)}(Y, l - l_s)}$$

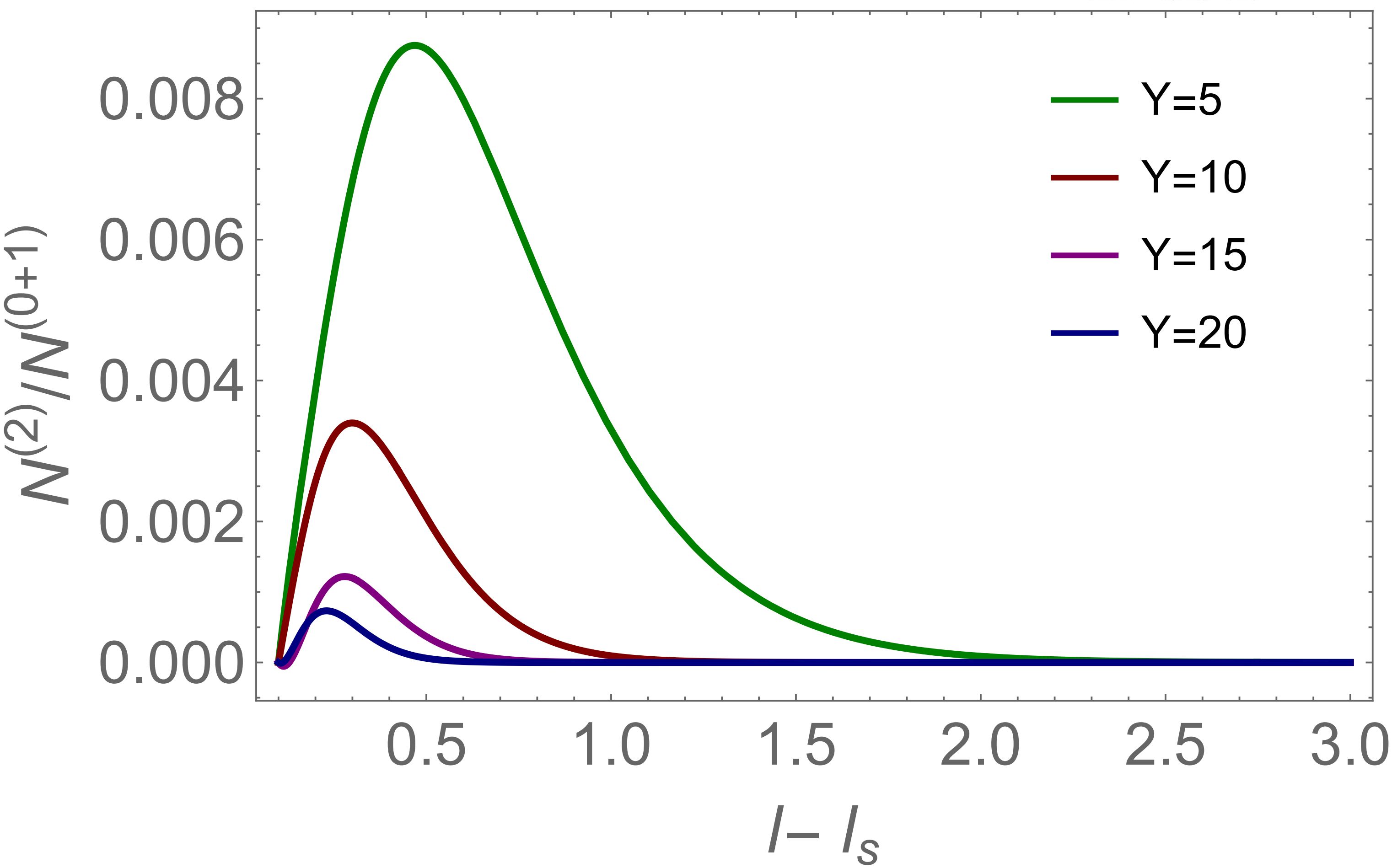


Homotopy approach

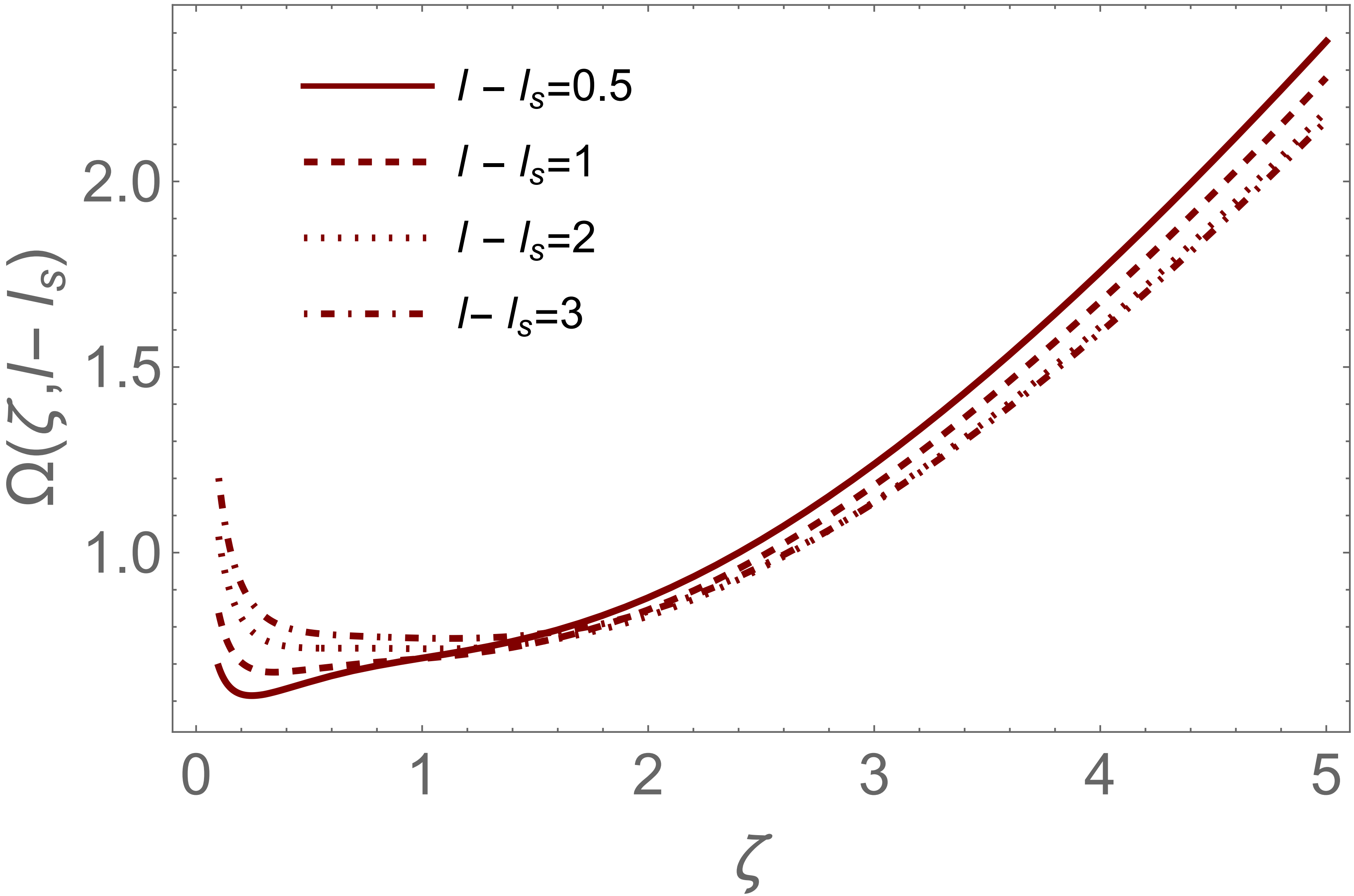
The third homotopy iteration is given by

$$\mathscr{H}\left(p, \Omega^{(0)} + p\,\Omega^{(1)} + p\,\Omega^{(2)}\right) \;=\; \mathscr{L}[\Omega^{(0)} + p\,\Omega^{(1)} + p\,\Omega^{(2)}] + p\,\mathscr{N}_{\mathscr{L}}[\Omega^{(0)} + p\,\Omega^{(1)}] \;=\; 0$$

$$\frac{\partial^2 \Omega^{(2)}\left(Y, l - l_s\right)}{\partial Y\,\partial(l - l_s)} \;=\; \Omega^{(2)}\left(Y, l - l_s\right)\; e^{-\Omega^{(0)}(Y,l-l_s) - \Omega^{(1)}(Y,l-l_s)} \;-\; \underbrace{\frac{\partial}{\partial\,(l - l_s)}\left(e^{\Omega^{(0)}(Y,l-l_s) + \Omega^{(1)}(Y,l-l_s)}\,\mathscr{N}_{\mathscr{L}}[\Omega^{(0)} \,+\, \Omega^{(1)}]\right)}_{DH^{(1)}(Y,l-l_s)}$$

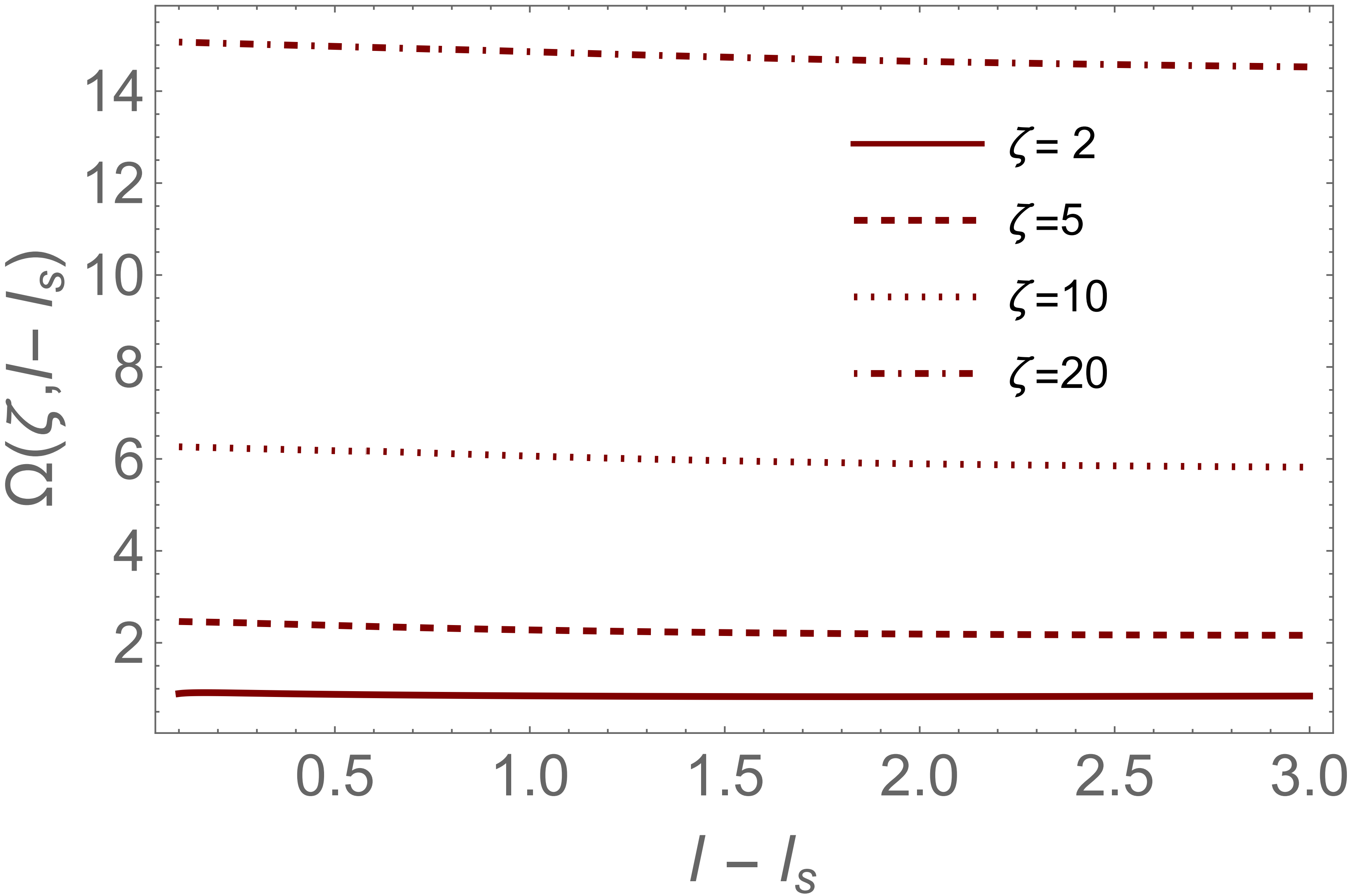


Results



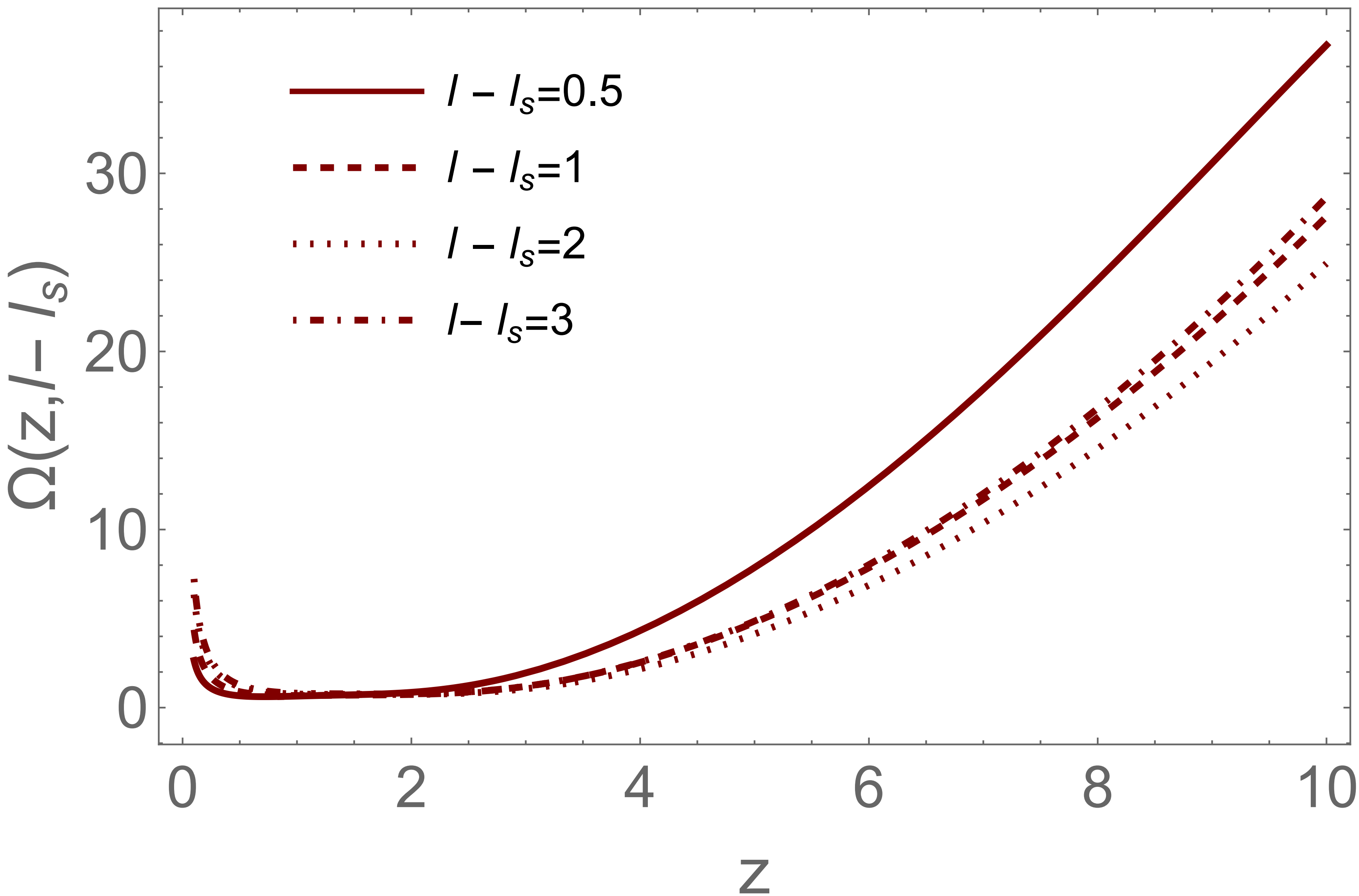
$$\Omega\left(\zeta, l - l_s\right) = \Omega^{(0)}\left(Y, l - l_s\right) + \Omega^{(1)}\left(Y, l - l_s\right) + \Omega^{(2)}\left(Y, l - l_s\right)$$

Results

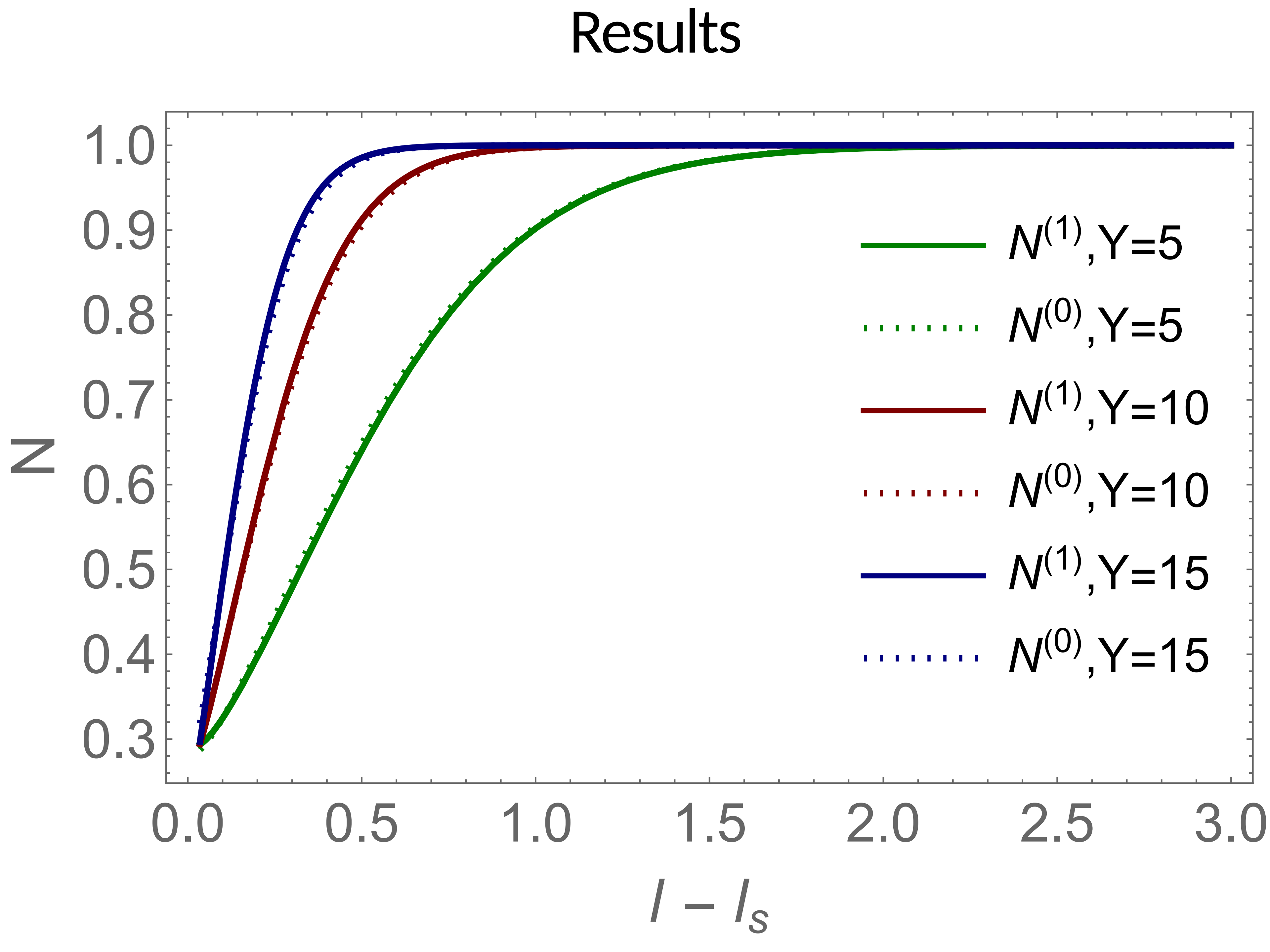


$$\Omega\left(\zeta, l-l_s\right)= \Omega^{(0)}\left(Y, l-l_s\right)+\Omega^{(1)}\left(Y, l-l_s\right)+\Omega^{(2)}\left(Y, l-l_s\right)$$

Results

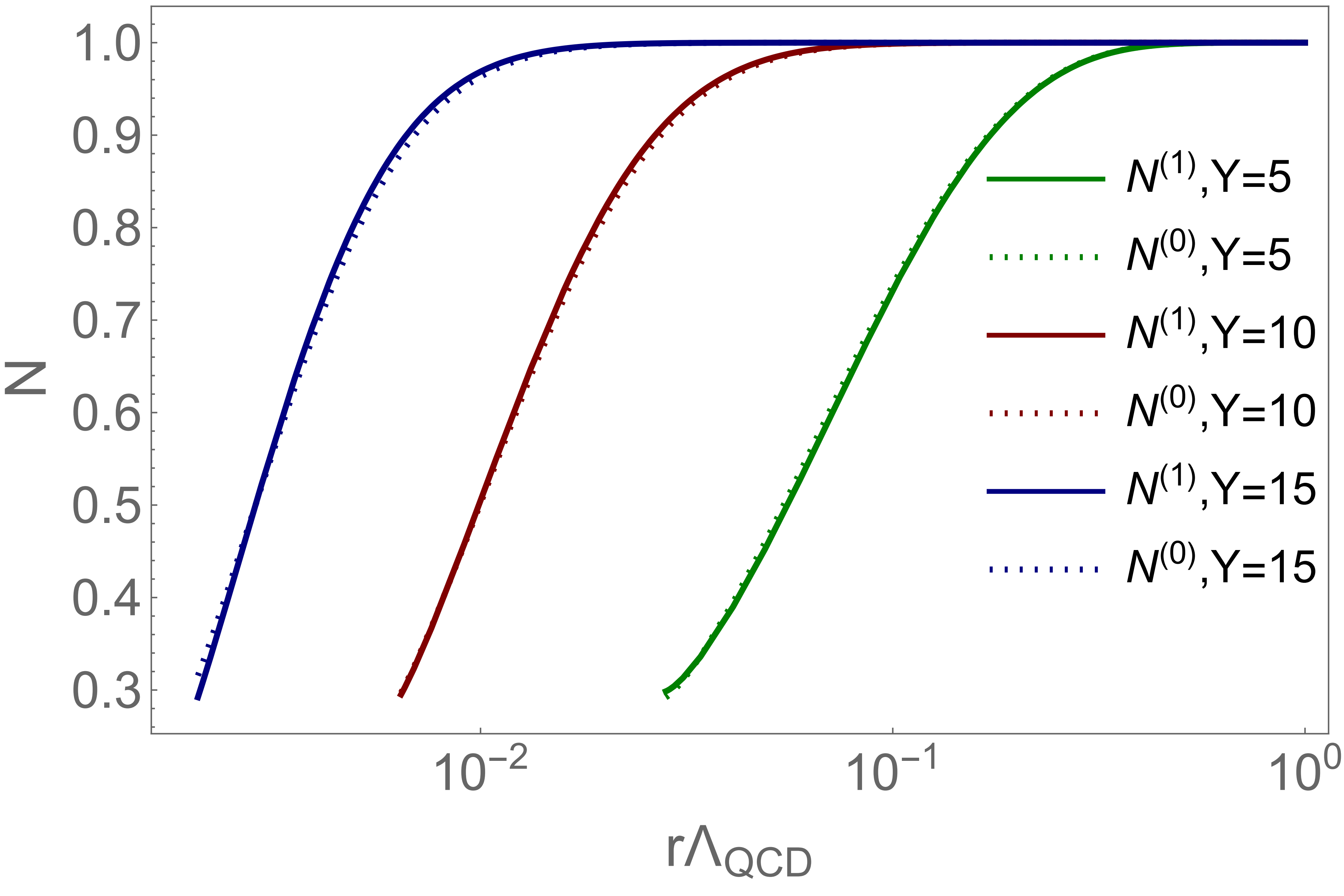


$$\Omega\left(\zeta, l - l_s\right) = \Omega^{(0)}\left(Y, l - l_s\right) + \Omega^{(1)}\left(Y, l - l_s\right) + \Omega^{(2)}\left(Y, l - l_s\right)$$

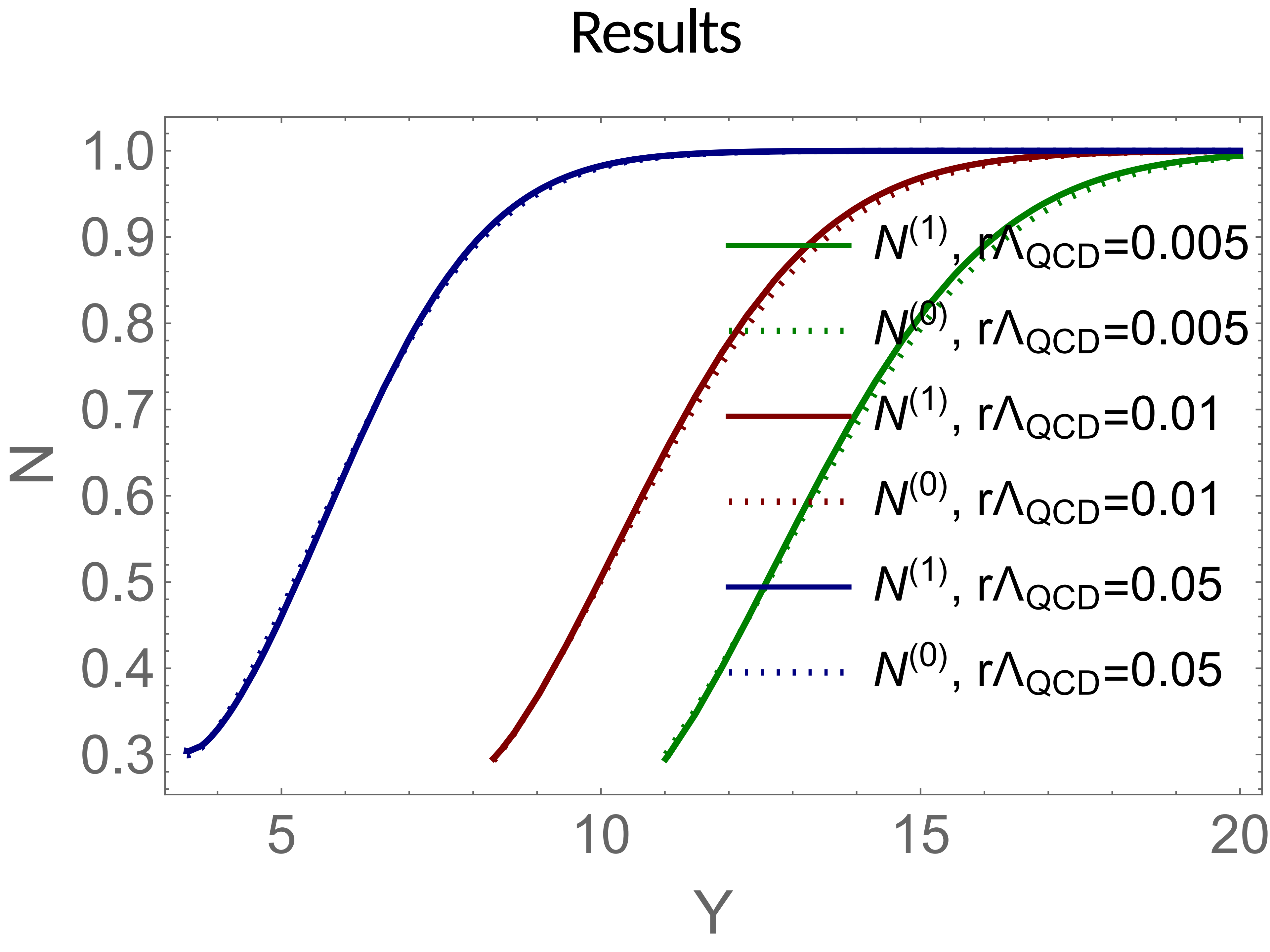


Results
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
N
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$l - l_s$
$N^{(1)}$,Y=5
$N^{(0)}$,Y=5
$N^{(1)}$,Y=10
$N^{(0)}$,Y=10
$N^{(1)}$,Y=15
$N^{(0)}$,Y=15

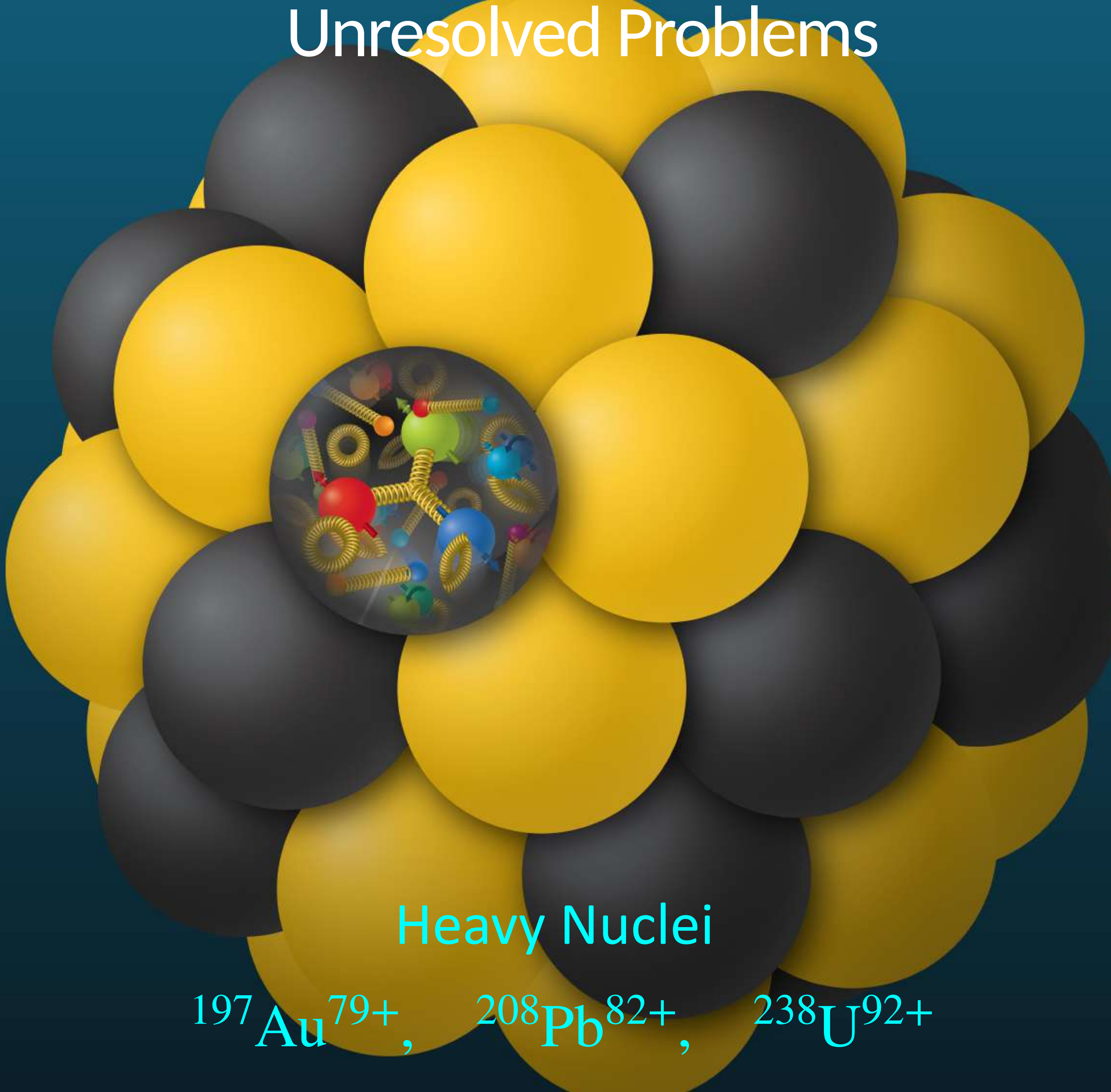
Results



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
N
10^{-2}
10^{-1}
10^{0}
$r\Lambda_{QCD}$
$N^{(1)}$,Y=5
$N^{(0)}$,Y=5
$N^{(1)}$,Y=10
$N^{(0)}$,Y=10
$N^{(1)}$,Y=15
$N^{(0)}$,Y=15


Results
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
N
5
10
15
20
Y
$N^{(1)}$, $r\Lambda_{QCD}$=0.005
$N^{(0)}$, $r\Lambda_{QCD}$=0.005
$N^{(1)}$, $r\Lambda_{QCD}$=0.01
$N^{(0)}$, $r\Lambda_{QCD}$=0.01
$N^{(1)}$, $r\Lambda_{QCD}$=0.05
$N^{(0)}$, $r\Lambda_{QCD}$=0.05

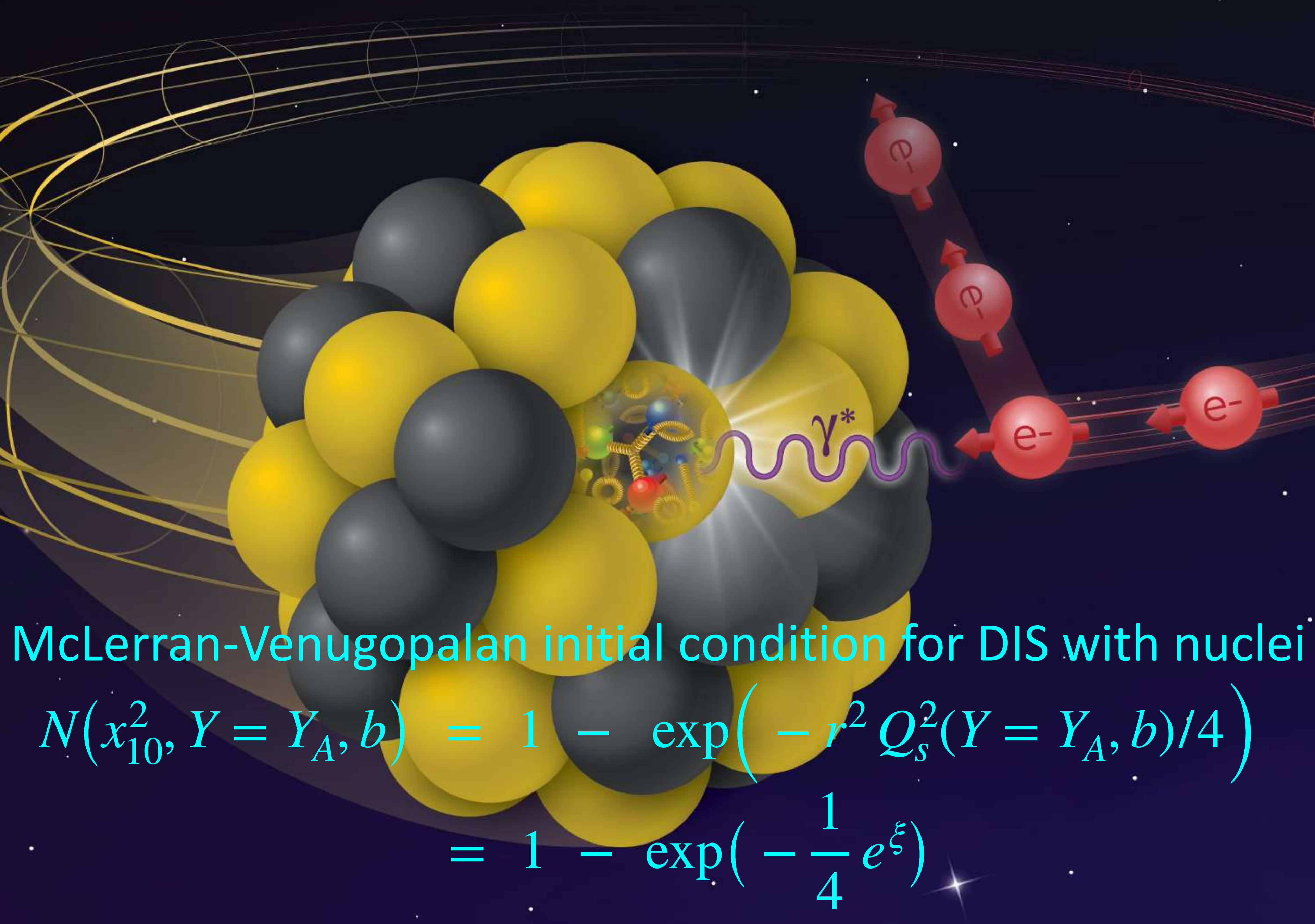
Unresolved Problems



Heavy Nuclei

$^{197}Au^{79+}$, $^{208}Pb^{82+}$, $^{238}U^{92+}$

Unresolved Problems



McLerran-Venugopalan initial condition for DIS with nuclei

$$N\left(x_{10}^2, Y = Y_A, b\right) = 1 - \exp\left(-r^2 Q_s^2(Y = Y_A, b)/4\right)$$
$$= 1 - \exp\left(-\frac{1}{4} e^{\xi}\right)$$

Conclusion

Homotopy approach successfully applied to the rcBK equation.
Next step use MV initial condition.

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Thank you for listening!