A Toolbox for Physicists ... and others:

PDFs, symmetries, scaling, dimensional analysis, and ...

Fred Olness
SMU

Thanks for substantial input
from my friends & colleagues

nCTEQ
nuclear parton distribution functions

xFitter

CTEQ

IITB-CFNS-CTEQ School
Indian Institute of Technology
8-15 February 2026

The Precison

ROCKET SCIENCE

Systems Labs, Sherman Oaks, California • 1957

Technical papers and pieces of chalk in hand, six scientists work out orbit equations for the first satellite launches in the United States. Their calculations include measurements of the earth's rotation as well as mathematical laws of motion, gravity, and centripetal force. The following year, Explorer I was launched from Cape Canaveral in Florida, and made the first major discovery of the space age: the Van Allen Radiation Belt, two zones of high-speed charged particles made from solar wind and cosmic rays. Since then, nearly 7,000 satellites have gone into orbit, covering everything from communications and space navigation to military observations and weather.

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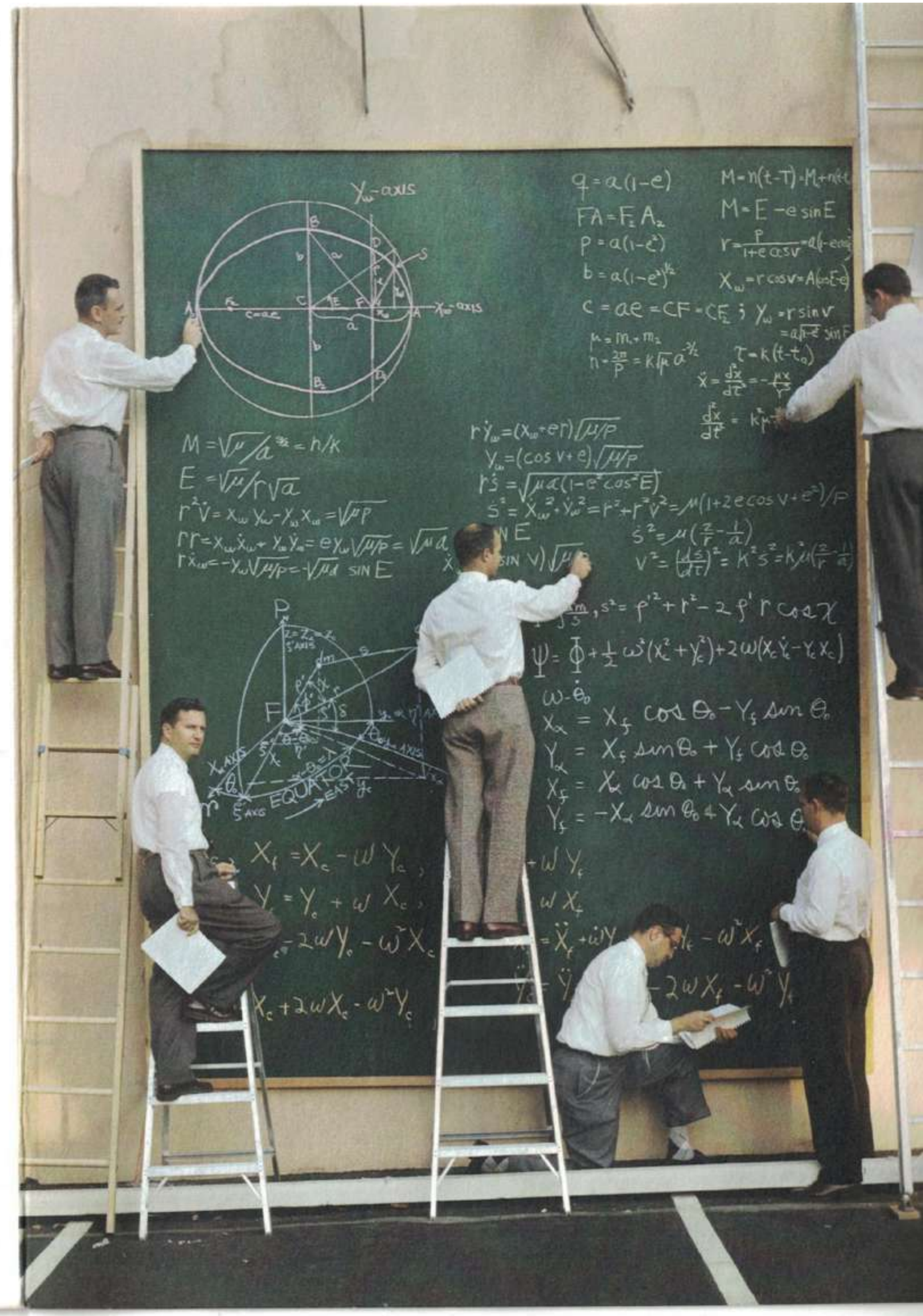
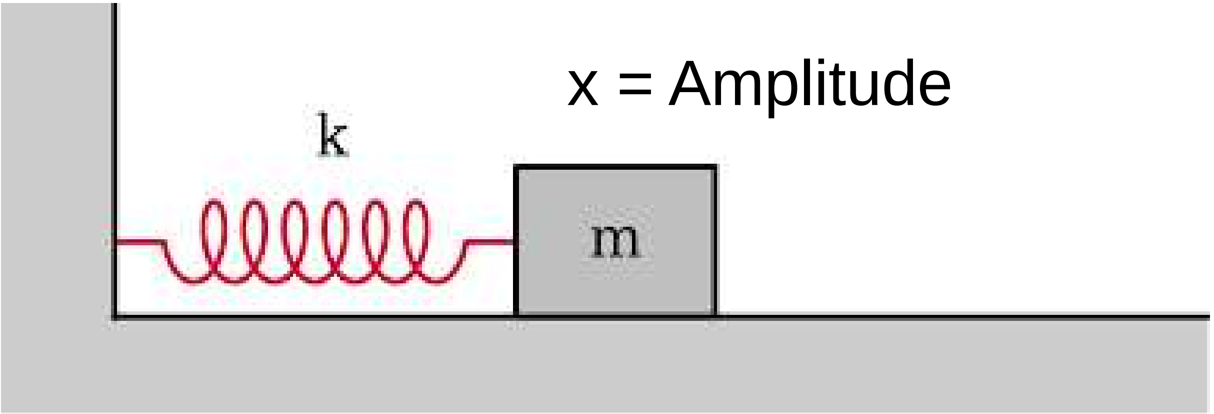


The Relations

Dimensional Analysis

Simple Harmonic Oscillations (SHO)



Hooke's Law:

F = - k x and F = m a

so, dimensionally,

$k = F/x = (m\ a)/x = (Kg)(m/s^2)/(m)$

$k = Kg/s^2$

x = meters

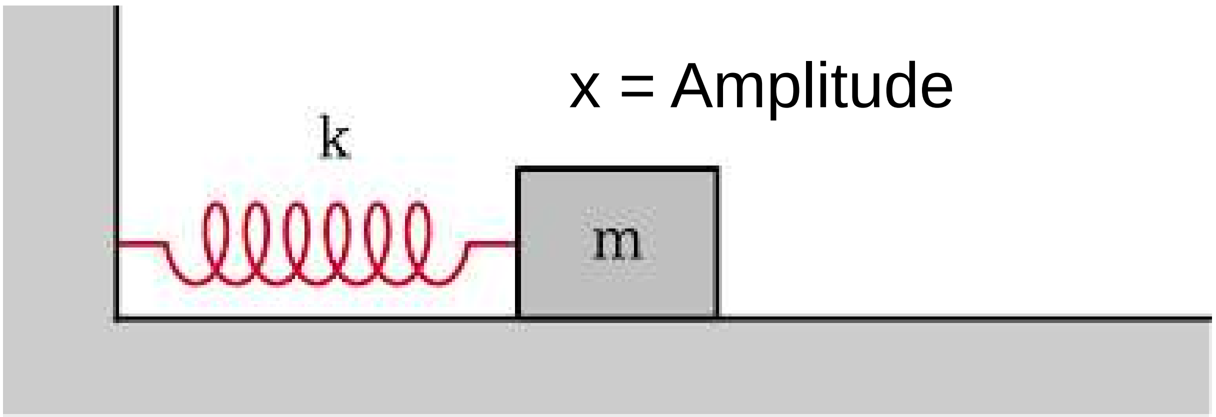
m = kilograms

$k = Kg/sec^2$

Period: (sec)

T=

Simple Harmonic Oscillations (SHO)



x = meters

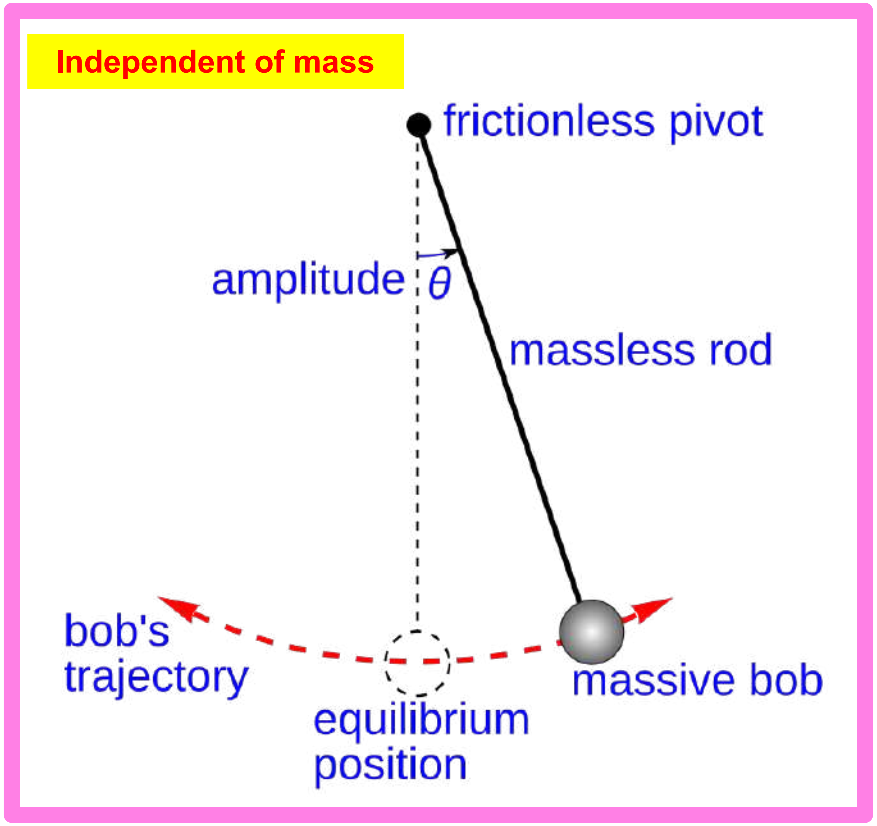
m = kilograms

k = Kg/sec²

Period: (sec)

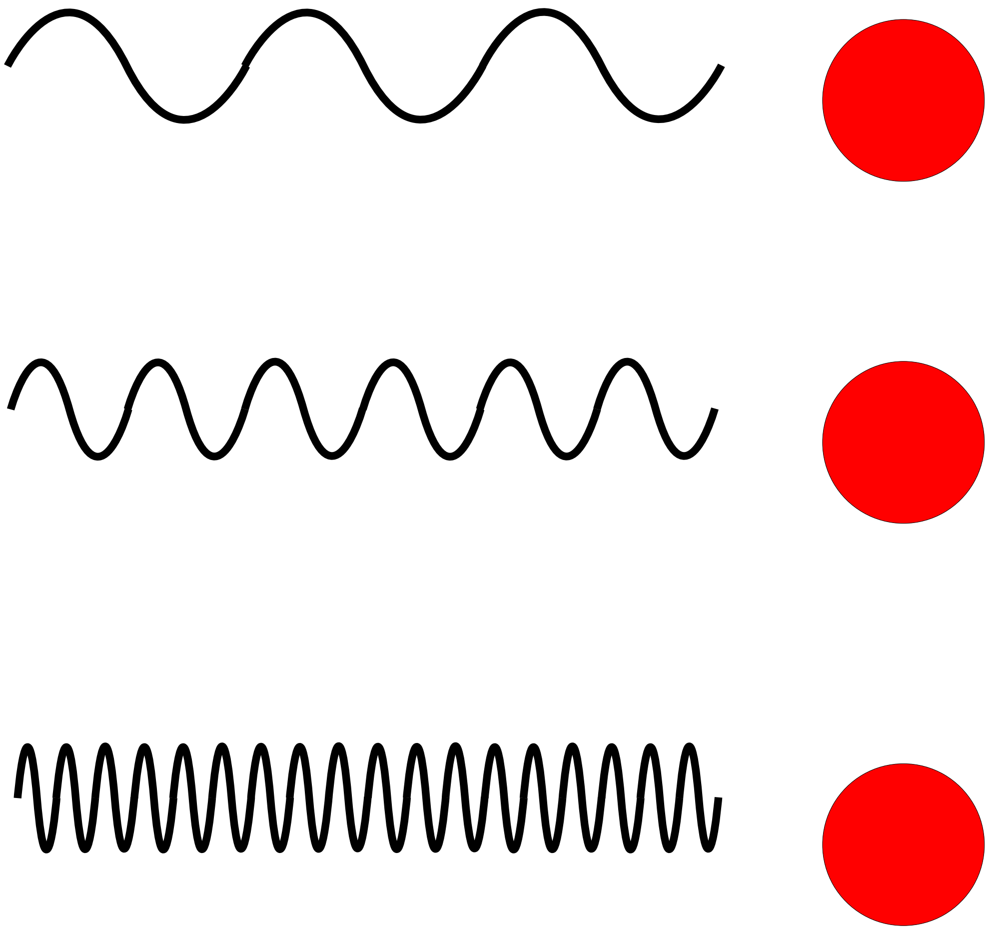
$$T = 2\pi\sqrt{\frac{m}{k}}$$

Independent of amplitude



Scale Invariance:

What if the observable does NOT depend on a specific variable



$$d\sigma \sim \sigma_0 \times 1$$

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$$d\sigma \sim \sigma_0 \times 1$$

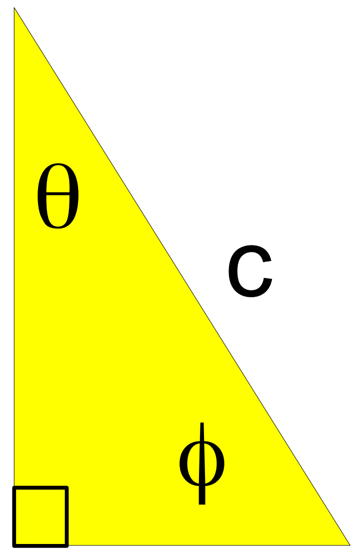
Dimensional considerations

Structure Function

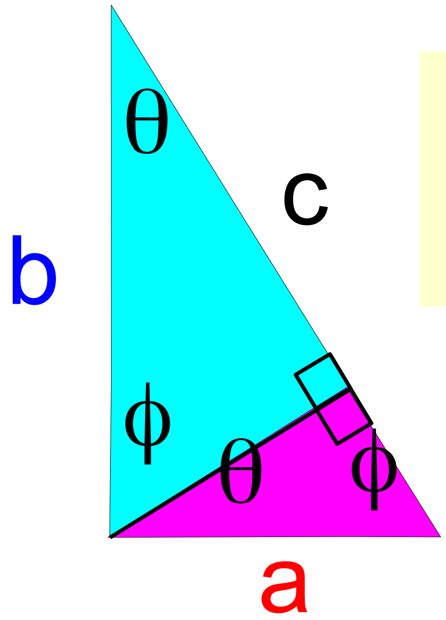
Warm-up: Pythagorean Theorem

GOAL:
Pythagorean Theorem

METHOD:
Dimensional Analysis



$$A_c = c^2 f(\theta, \phi)$$



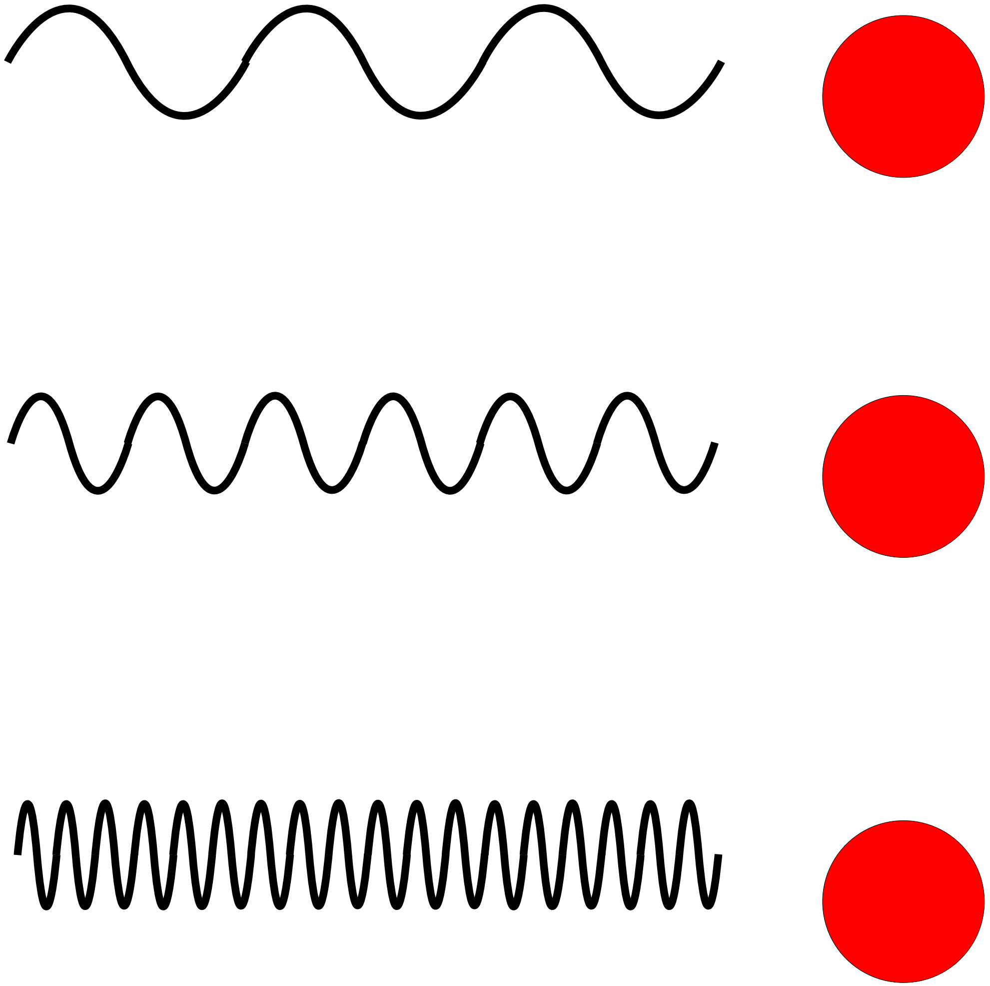
$$A_a + A_b = $$
$$a^2 f(\theta, \phi) + b^2 f(\theta, \phi)$$

$$A_a + A_b = A_c$$

$$a^2 + b^2 = c^2$$

Scaling

What if the structure does not depend on RESOLUTION???



$$d\sigma \sim \sigma_0 \times 1$$

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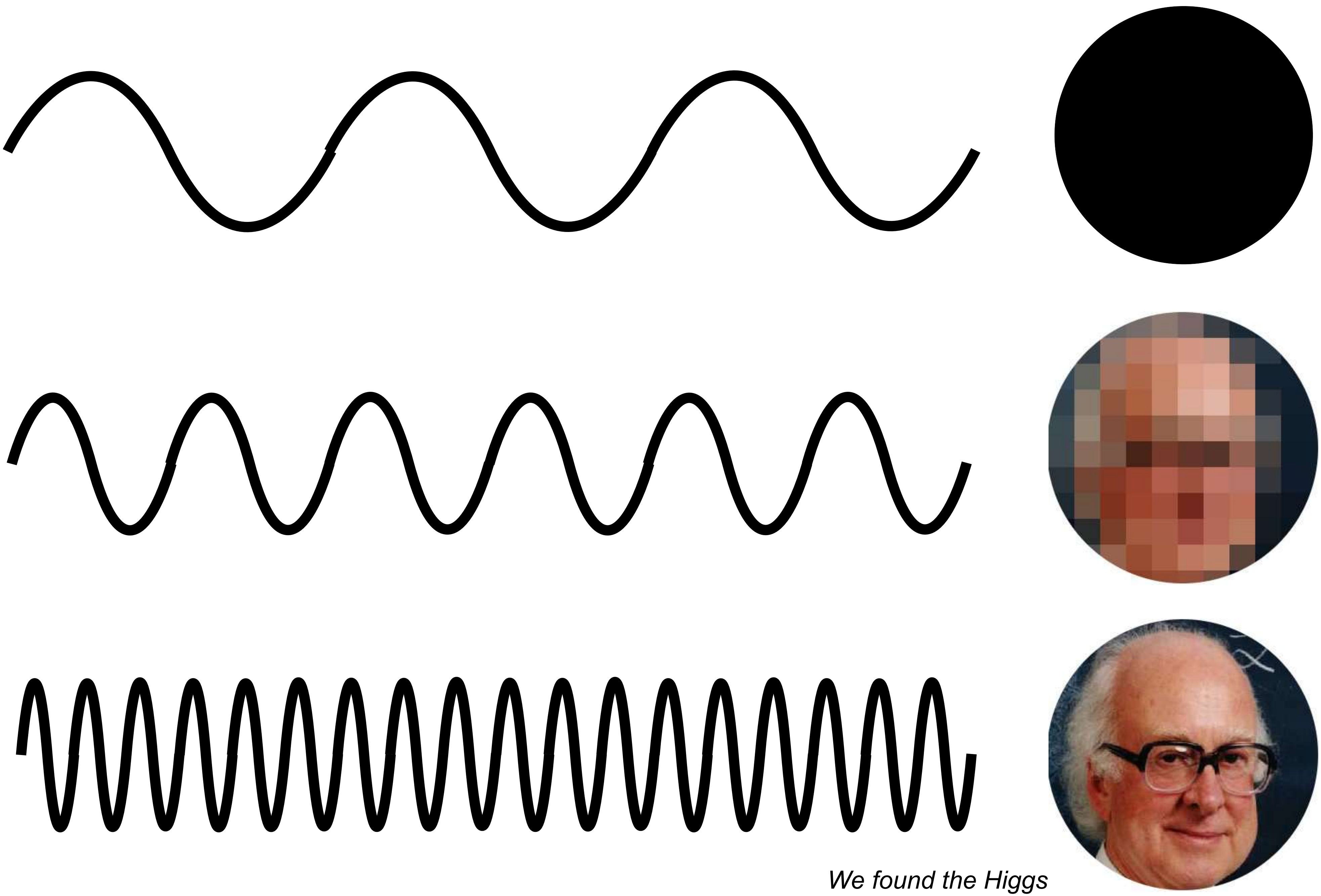
$$d\sigma \sim \sigma_0 \times 1$$

Dimensional considerations

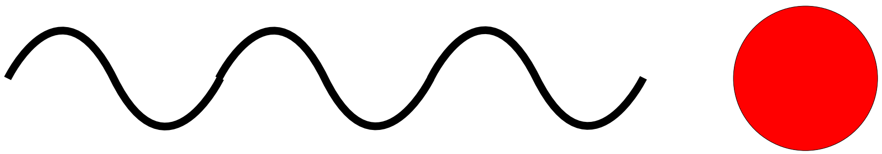
Structure Function

Is this a point like particle ???

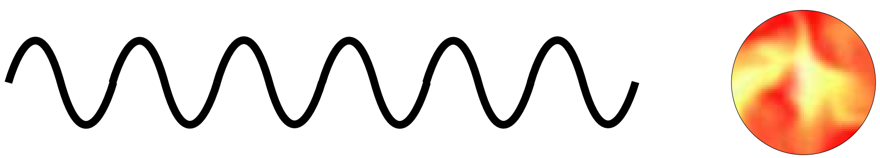


We found the Higgs

Structure of the Proton

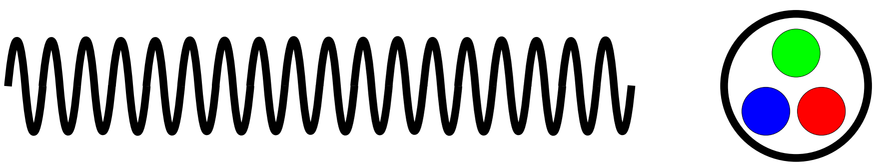


$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times F\left(\frac{Q^2}{\Lambda^2}\right)$$

Λ of order of the proton mass scale

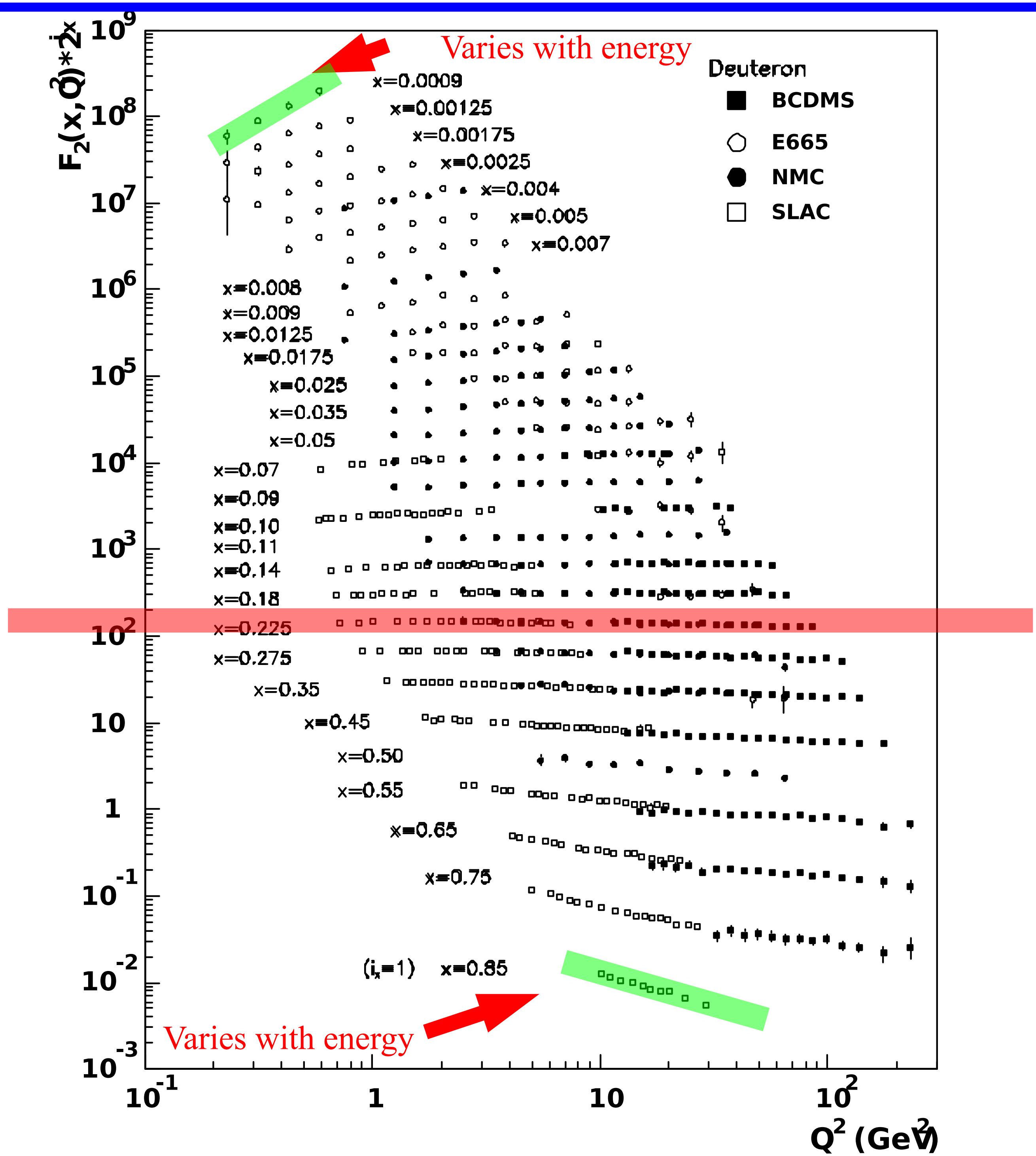


$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times \sum_i e_i^2$$

The Scaling of the Proton Structure Function

Data is (relatively) independent of energy

Scaling Violations observed at extreme x values



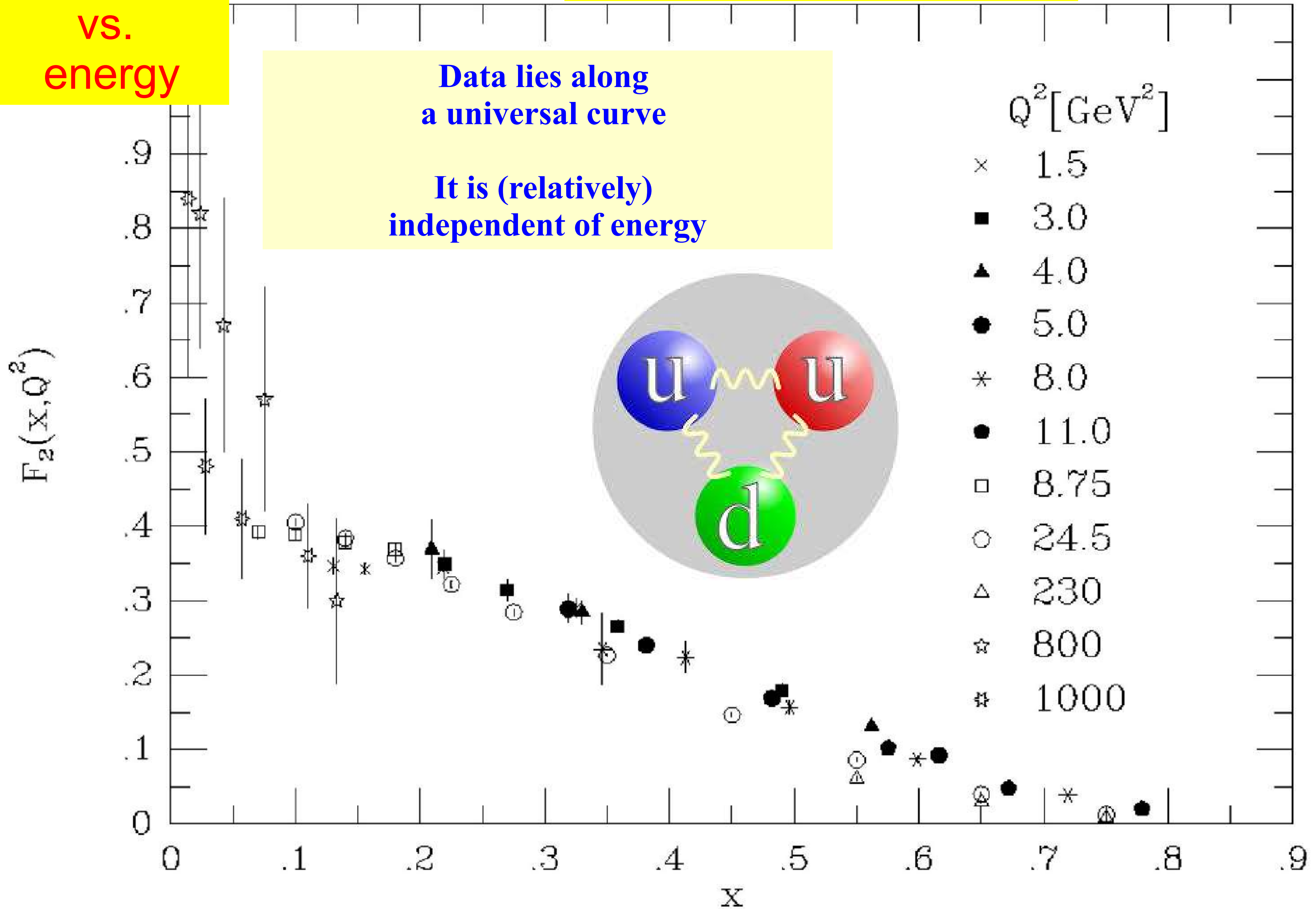
The Scaling of the Proton Structure Function

Structure vs. energy

$$F_2^{\gamma}(x,Q) = x e_q^2 \{q + \bar{q}\}$$

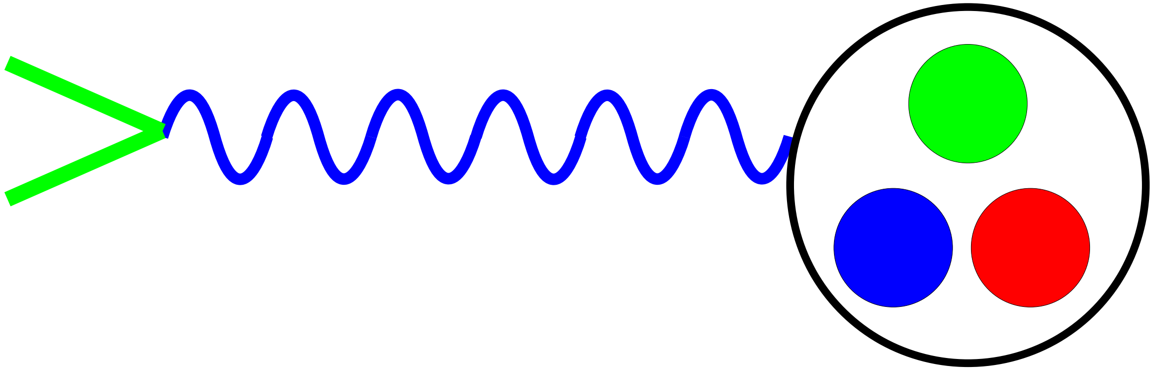
Data lies along a universal curve

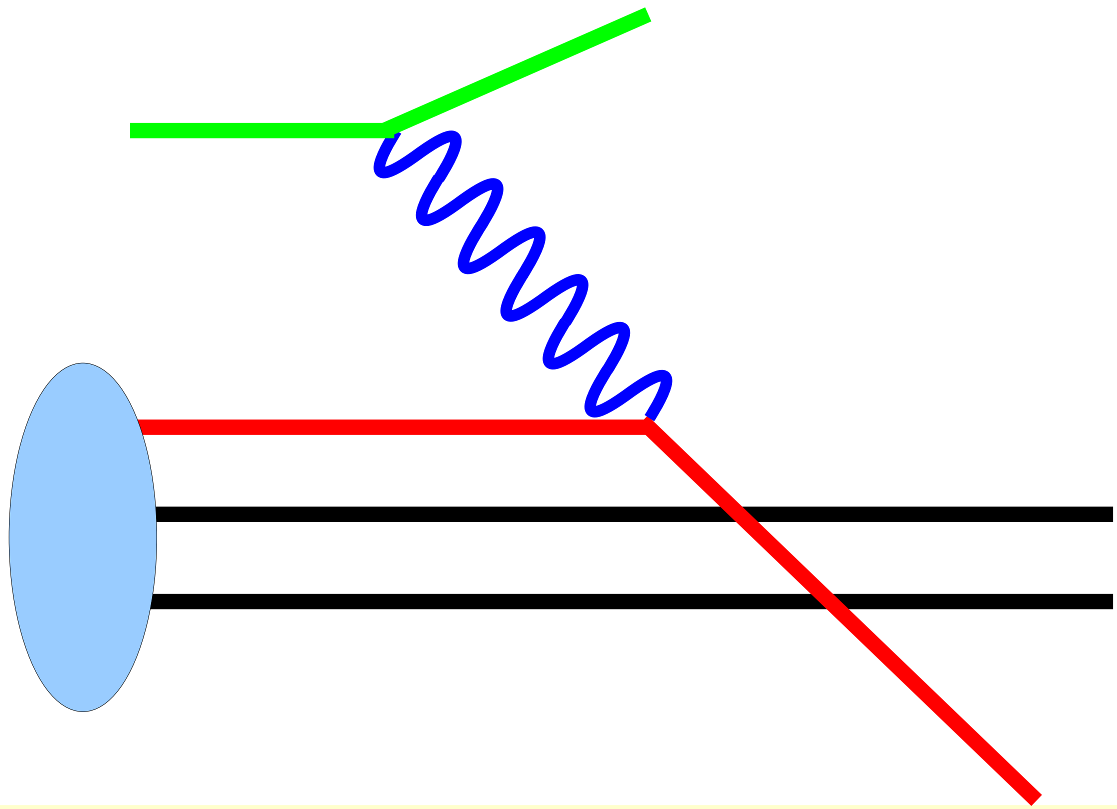
It is (relatively) independent of energy

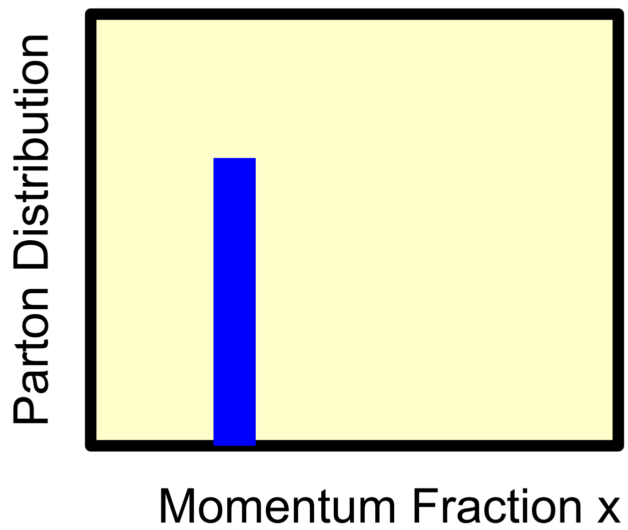


Parton Model

Proton as a bag of free Quarks



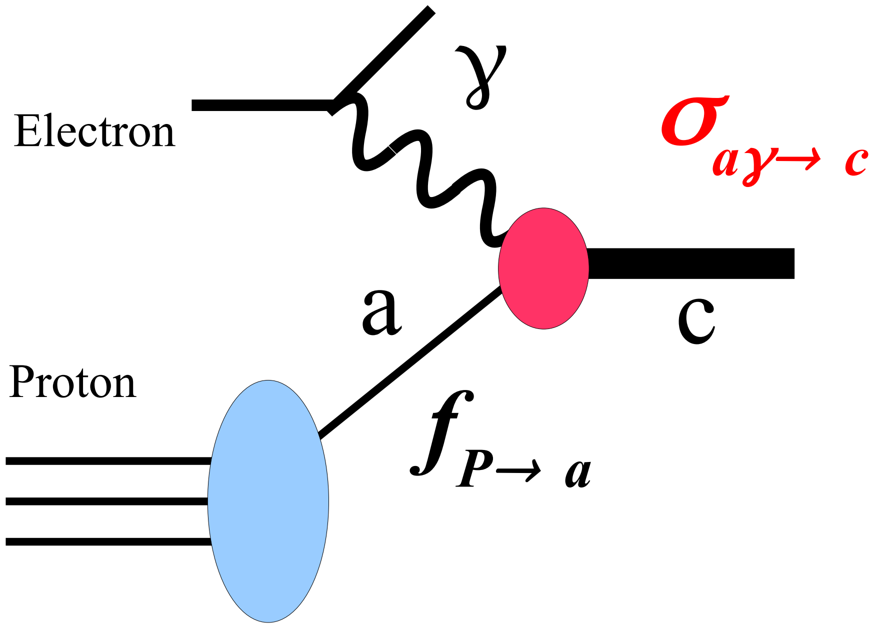




Fred's PDFs

$$f(x,Q) = u(x,Q) + d(x,Q) = 2\ \delta(x - \tfrac{1}{3}) + 1\ \delta(x - \tfrac{1}{3})$$

The Parton Model and Factorization



Parton Distribution Functions (PDFs) $f_{P\to a}$ are the key to calculations involving hadrons!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \hat{\sigma}_{a\gamma\to c}$$

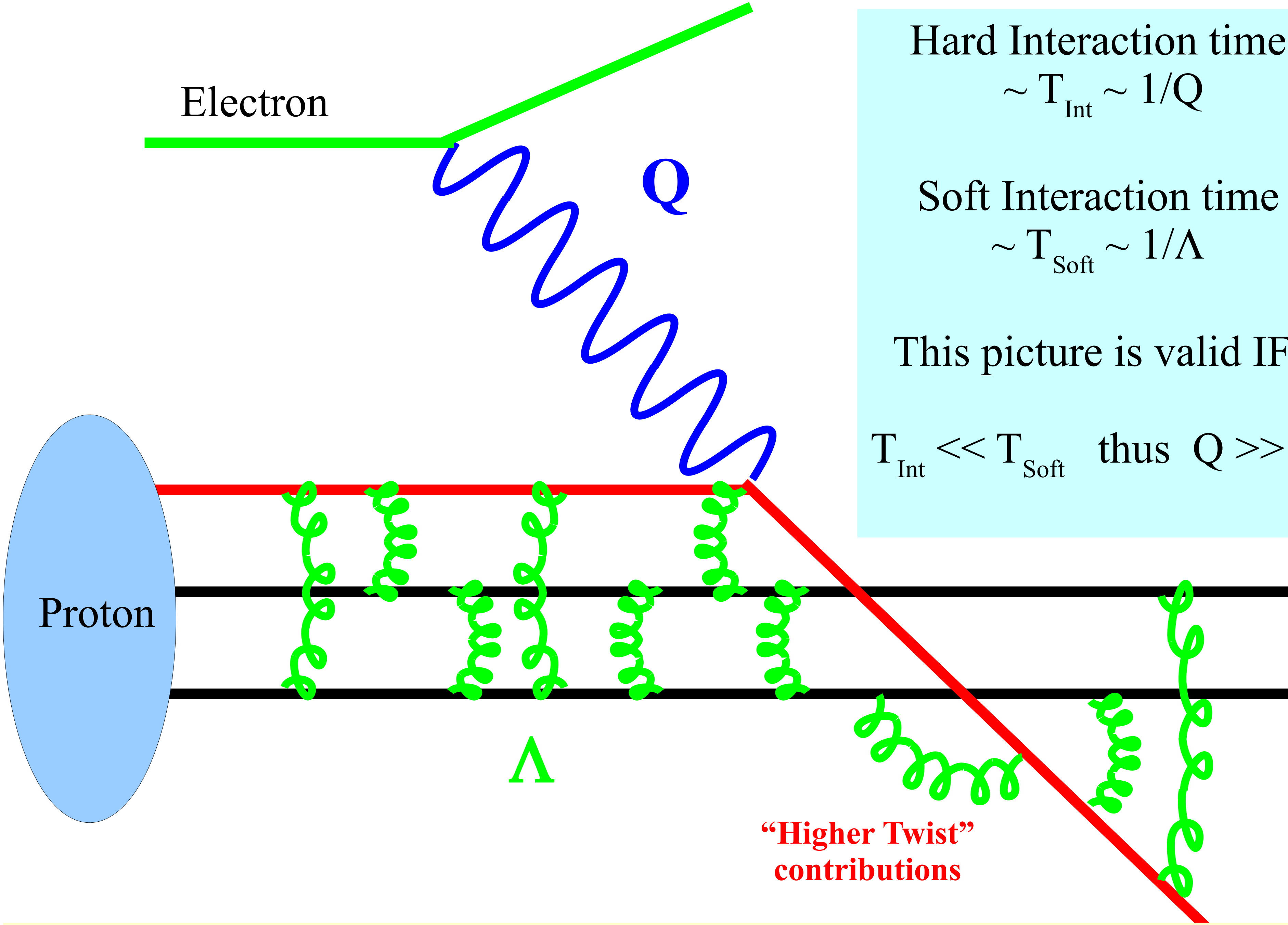
Corrections of order (Λ^2/Q^2)

must extract from experiment

calculable from theoretical model

Cross section is product of independent probabilities!!! *(Homework Assignment)*

Quarks are not quite free



Hard Interaction time

$\sim T_{Int} \sim 1/Q$

Soft Interaction time

$\sim T_{Soft} \sim 1/\Lambda$

This picture is valid IF:

$T_{Int} << T_{Soft}$ thus $Q >> \Lambda$

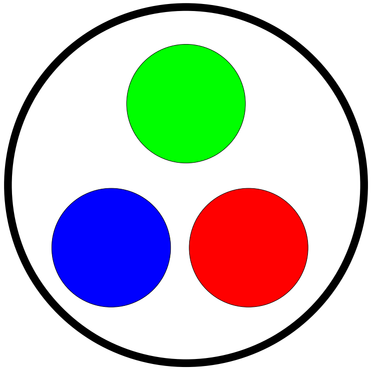
Corrections to this picture (non-factorizable/ higher twist) terms are suppressed by powers of Λ/Q

TOY

PDFs

Proton as a bag of free Quarks: Part 2

$$f(x,Q) = u(x,Q) + d(x,Q) = 2\ \delta(x - \tfrac{1}{3}) + 1\ \delta(x - \tfrac{1}{3})$$



$$u(x,Q) = 2\ \delta(x - \tfrac{1}{3})$$

$$d(x,Q) = 1\ \delta(x - \tfrac{1}{3})$$

Perfect Scaling PDFs
Q independent

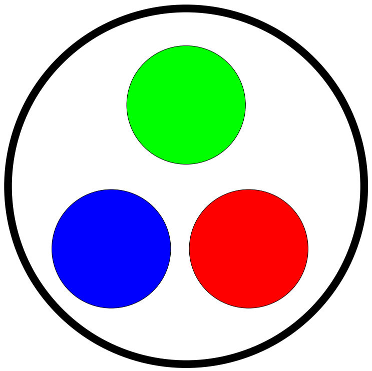
Quark Number Sum Rule

$$\langle q \rangle = \int_0^1 dx\, q(x) \qquad \langle u \rangle = 2 \quad \langle d \rangle = 1 \quad \langle s \rangle = 0$$

Quark Momentum Sum Rule

$$\langle x\, q \rangle = \int_0^1 dx\, x\, q(x) \qquad \langle x\, u \rangle = \tfrac{2}{3} \quad \langle x\, d \rangle = \tfrac{1}{3}$$

Problem #1: The proton does not add up???



$$\begin{aligned} F_+ &= 2\bar{q} \\ F_- &= 2q \\ F_L &= \phi \end{aligned}$$

$$q + \bar{q} = \frac{F_+ + F_-}{2}$$

Momentum Sum Rule

$$\sum_i \langle x\, q_i \rangle = \int_0^1 dx \sum x\,[q_i(x) + \bar{q}_i(x)] = 50\% \neq 100\%$$

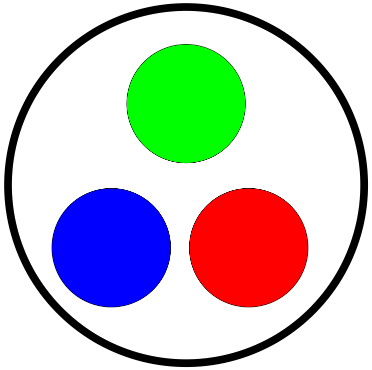
Substitute F

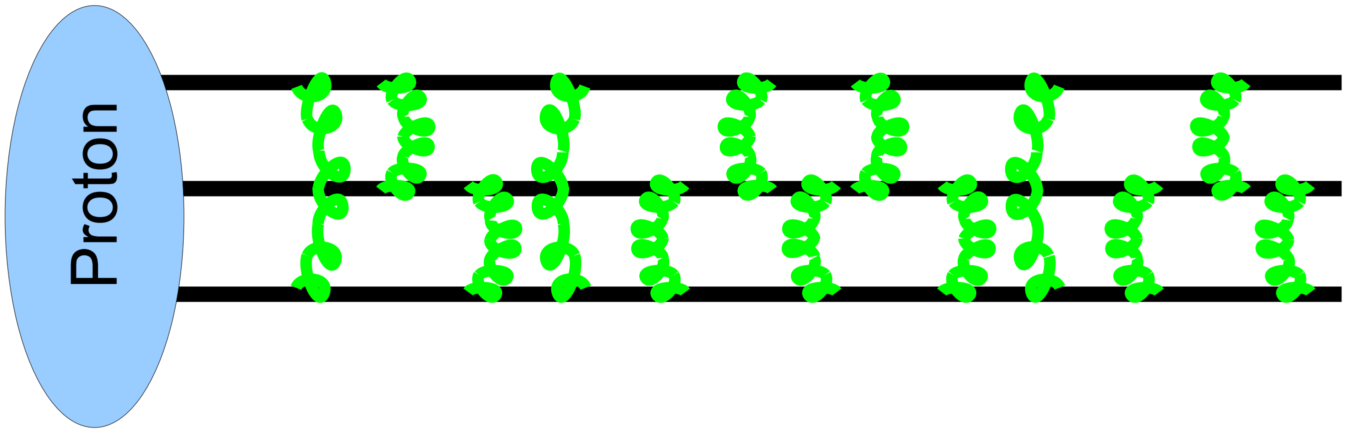
SOLUTION:

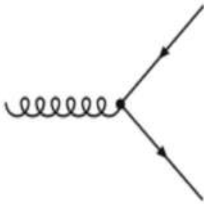
Gluons carry half the momentum,

but don't couple to the photons

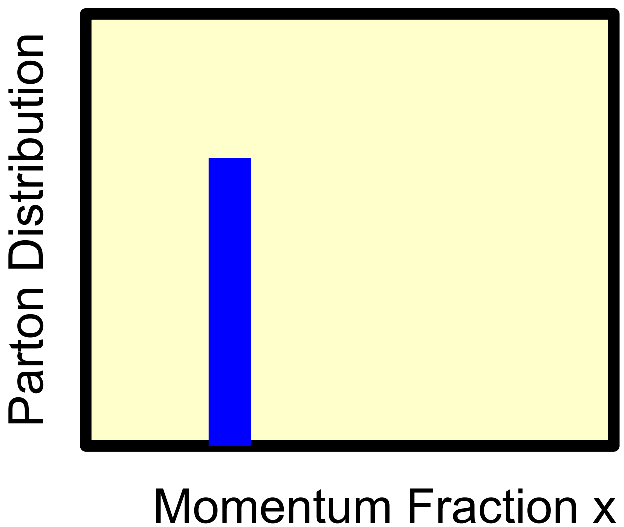
Gluons smear out PDF momentum

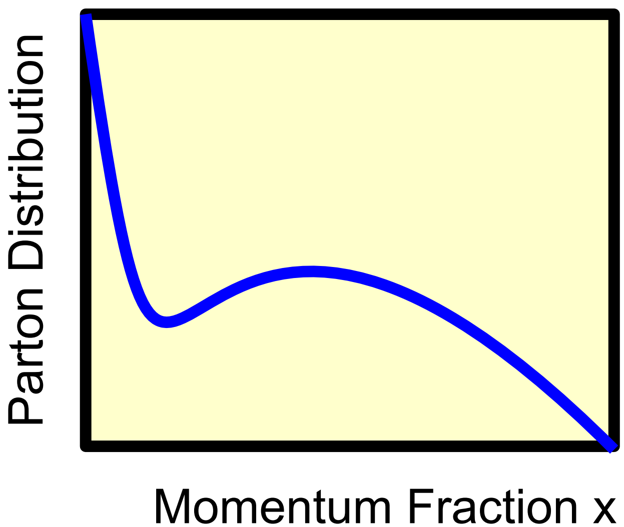






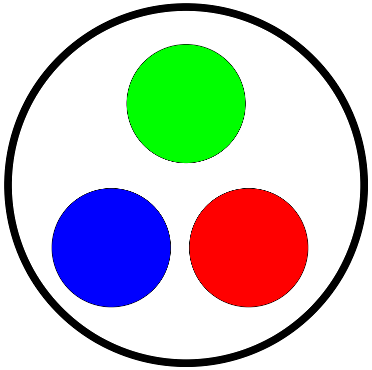
Gluons allow partons to exchange momentum fraction





α_s is large at low Q, so it is easy to emit soft gluons

Problem #2: Infinitely many quarks

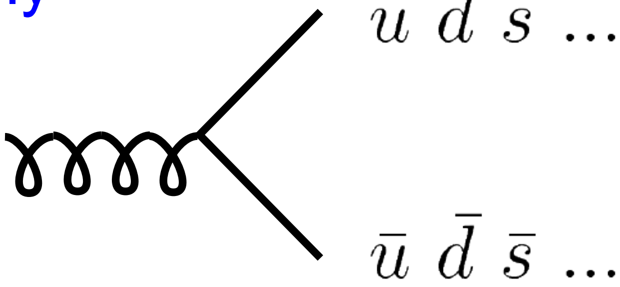


Reconsider the Quark Number Sum Rule

$$\langle u, d \rangle = \infty \qquad \langle q \rangle = \int_0^1 dx\, q(x)$$

Quark Number Sum Rule: More Precisely

$$q(x) \sim 1/x^{1.5}$$



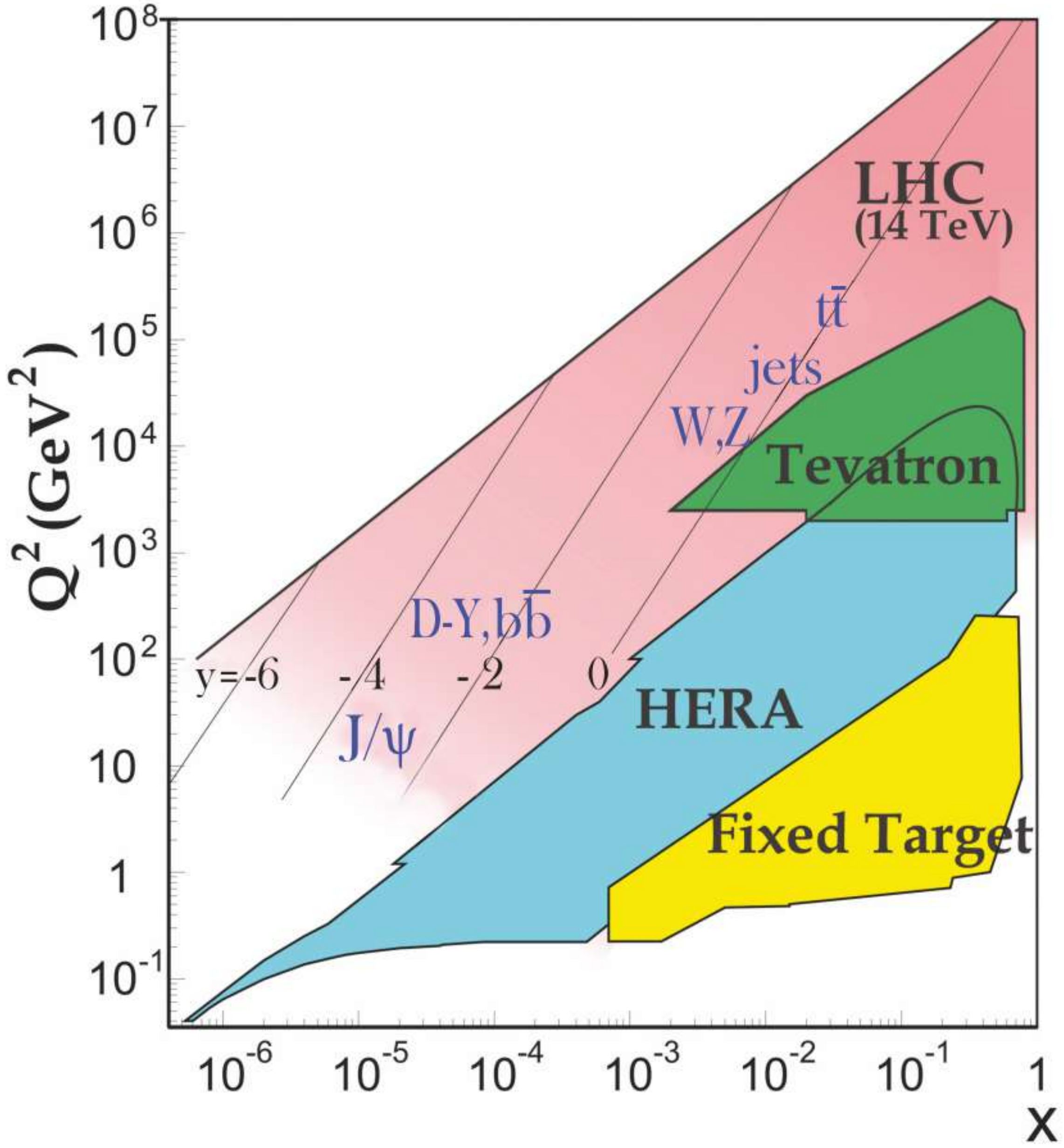
$$\langle u - \bar{u} \rangle = 2 \qquad \langle d - \bar{d} \rangle = 1 \qquad \langle s - \bar{s} \rangle = 0$$

SOLUTION: Infinite number of u quarks in proton, because they can be pair produced: *(We neglect saturation ....)*

PDFs

Where do PDFs come from???? Universality!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \sigma_{a\gamma\to c}$$



Must extract from experiment

Calculable from theoretical model

Note we can combine different experiments.
FACTORIZATION!!!

HOMEWORK

Sum Rules
&
Structure Functions

$$F_2^{ep} = \tfrac{4}{9}x\left[u+\bar{u}+c+\bar{c}\right] + \tfrac{1}{9}x\left[d+\bar{d}+s+\bar{s}\right]$$

$$F_2^{en} = \tfrac{4}{9}x\left[d+\bar{d}+c+\bar{c}\right] + \tfrac{1}{9}x\left[u+\bar{u}+s+\bar{s}\right]$$

$$F_2^{\nu p} = 2x\left[d+s+\bar{u}+\bar{c}\right]$$

$$F_2^{\nu n} = 2x\left[u+s+\bar{d}+\bar{c}\right]$$

$$F_2^{\bar{\nu} p} = 2x\left[u+c+\bar{d}+\bar{s}\right]$$

$$F_2^{\bar{\nu} n} = 2x\left[d+c+\bar{u}+\bar{s}\right]$$

$$F_3^{\nu p} = 2\left[d+s-\bar{u}-\bar{c}\right]$$

$$F_3^{\nu n} = 2\left[u+s-\bar{d}-\bar{c}\right]$$

$$F_3^{\bar{\nu} p} = 2\left[u+c-\bar{d}-\bar{s}\right]$$

$$F_3^{\bar{\nu} n} = 2\left[d+c-\bar{u}-\bar{s}\right]$$

Verify:

i.e., Check for typos ...

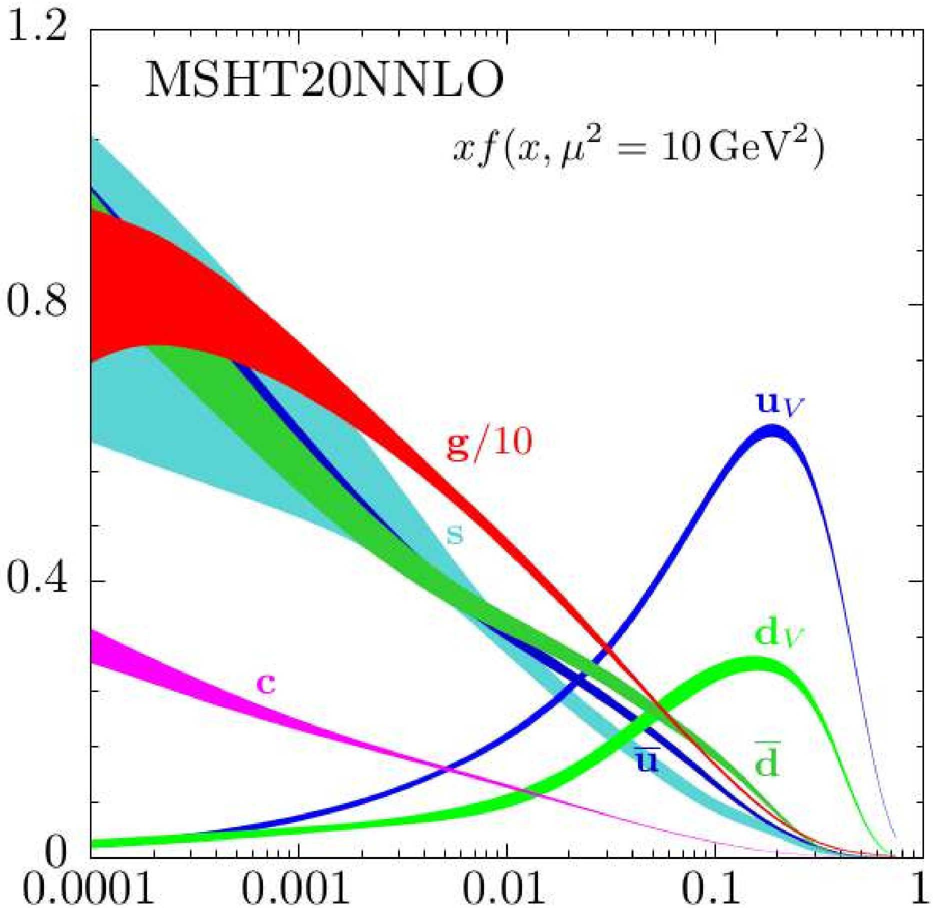
We use these different observables to dis-entangle the flavor structure of the PDFs

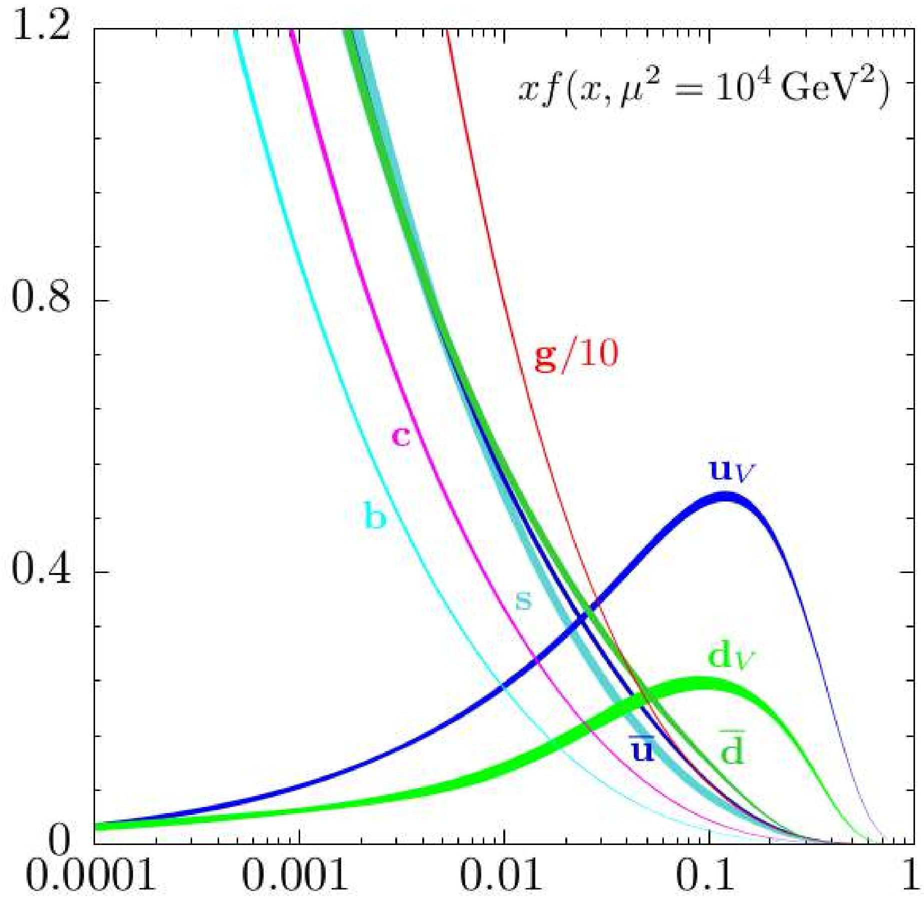
In the limit

$$\theta_{Cabibbo} = 0$$

$$m_c = 0$$

Sample PDFs: The rich structure of the proton





Shown for different μ scales

Scaling violations are essential feature of PDFs

What about the μ scale

$$\sigma_{p\gamma\to c}(x,Q) = f_{p\to a}(x,Q,\mu) \otimes \hat{\sigma}_{a\gamma\to c}(x,Q,\mu)$$

Dimensional Regularization

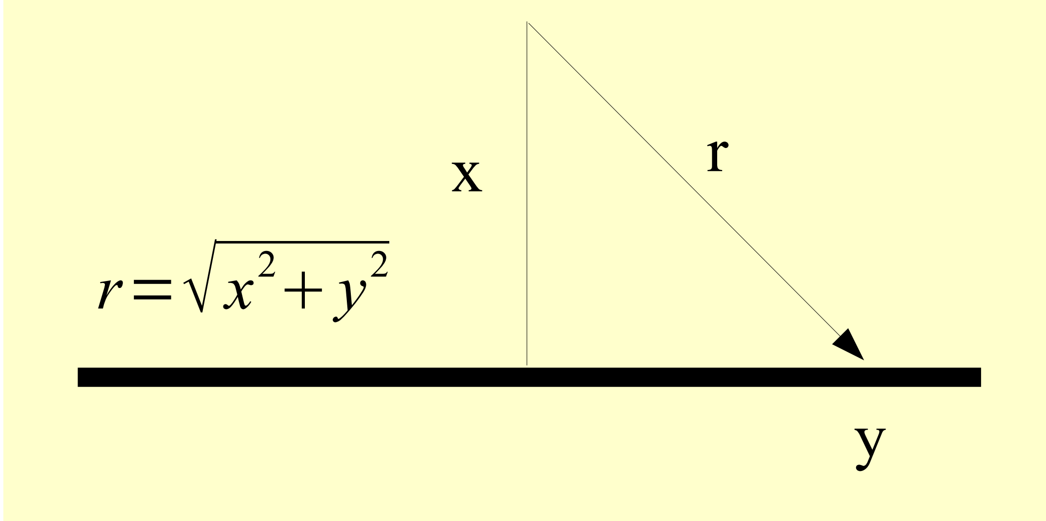
meets

Freshman E&M

M. Hans, Am.J.Phys. 51 (8) August (1983). p.694
C. Kaufman, Am.J.Phys. 37 (5), May (1969) p.560
B. Delamotte, Am.J.Phys. 72 (2) February (2004) p.170

Regularization, Renormalization, and Dimensional Analysis:
Dimensional Regularization meets Freshman E&M.
Olness & Scalise, arXiv:0812.3578 [hep-ph]

Infinite Line of Charge



$$r = \sqrt{x^2 + y^2}$$

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r} \qquad \lambda = \frac{Q}{y}$$

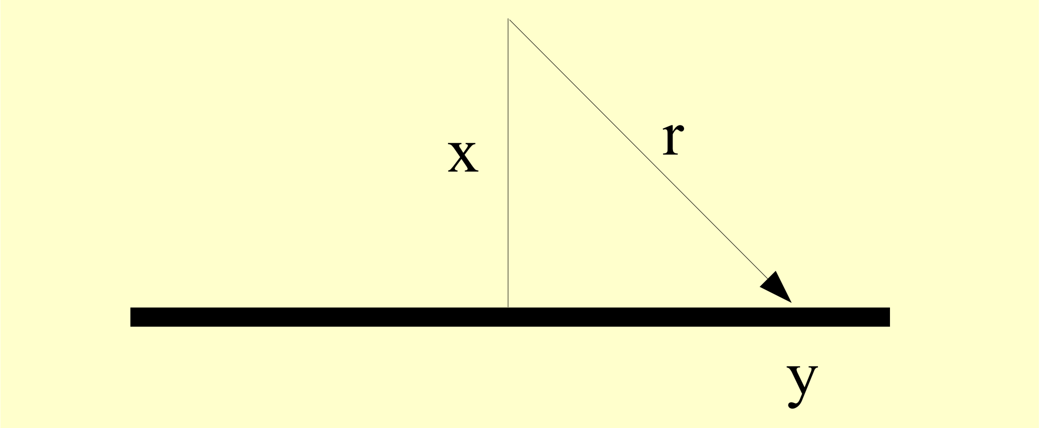
$$V = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} dy \; \frac{1}{\sqrt{x^2+y^2}} = \infty$$

potential

dimensionless f(x)

Note: ∞ can be very useful

Scale Invariance



$$V(k\,x) =$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \; \frac{1}{\sqrt{(kx)^2 + y^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} d\left(\frac{y}{k}\right) \; \frac{1}{\sqrt{x^2 + (y/k)^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dz \; \frac{1}{\sqrt{x^2 + z^2}}$$
$$= V(x)$$

Note: $\infty + c = \infty$

$\therefore \quad \infty - \infty = c$

How do we distinguish this from

$\infty - \infty = c + 17$

need to regulate

$$V(k\,x) = V(x)$$

Naively Implies:
V(k x) – V(x) = 0

Regularization Method #1: Cutoff Method

$$V=\frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{+L} dy \frac{1}{\sqrt{x^2+y^2}}$$

$$V=\frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{+L+\sqrt{L^2+x^2}}{-L+\sqrt{L^2+x^2}}\right]$$

V(x) depends on artificial regulator L

We cannot remove the regulator L

Argument of Log is dimensionless

All physical quantities are independent of the regulator:

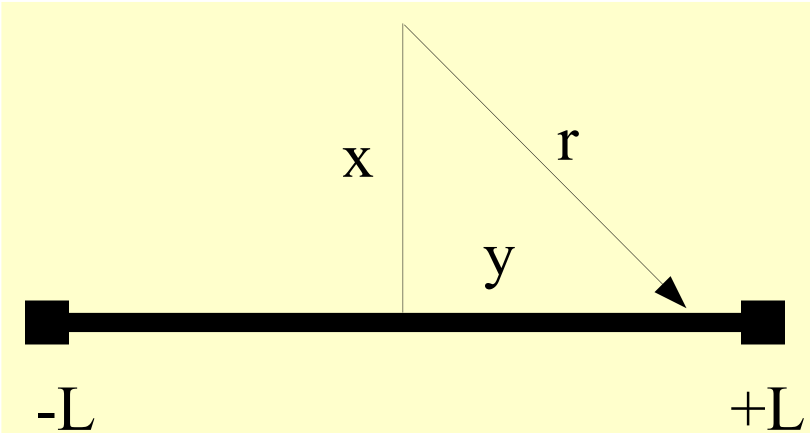
Electric Field $$E(x)=\frac{-dV}{dx}=\frac{\lambda}{2\pi\epsilon_0 \; x} \frac{L}{\sqrt{L^2+x^2}} \to \frac{\lambda}{2\pi\epsilon_0 \; x}$$

Energy $$\delta V=V(x_1)-V(x_2) \underset{L\to\infty}{\to} \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{x_2^2}{x_1^2}\right]$$

Problem solved at the expense of an extra scale L

AND we have a broken symmetry: translation invariance

Broken Translational Symmetry



Shift: $y \rightarrow y' = y - c$

$y = [+L+c, -L+c]$

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-L+c}^{+L+c} dy \, \frac{1}{\sqrt{x^2+y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{+(L+c)+\sqrt{(L+c)^2+x^2}}{-(L-c)+\sqrt{(L-c)^2+x^2}}\right]$$

V(r) depends on "y" coordinate!!!

In QFT, gauge symmetries are important. E.g., Ward identies

Compute in n-dimensions

$$dy \rightarrow d^n y = \frac{d\Omega_n}{2} \; y^{n-1} dy$$

$$\Omega_n = \int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

$$\Omega_{1,2,3,4} = \{2, 2\pi, 4\pi, 2\pi^2\}$$

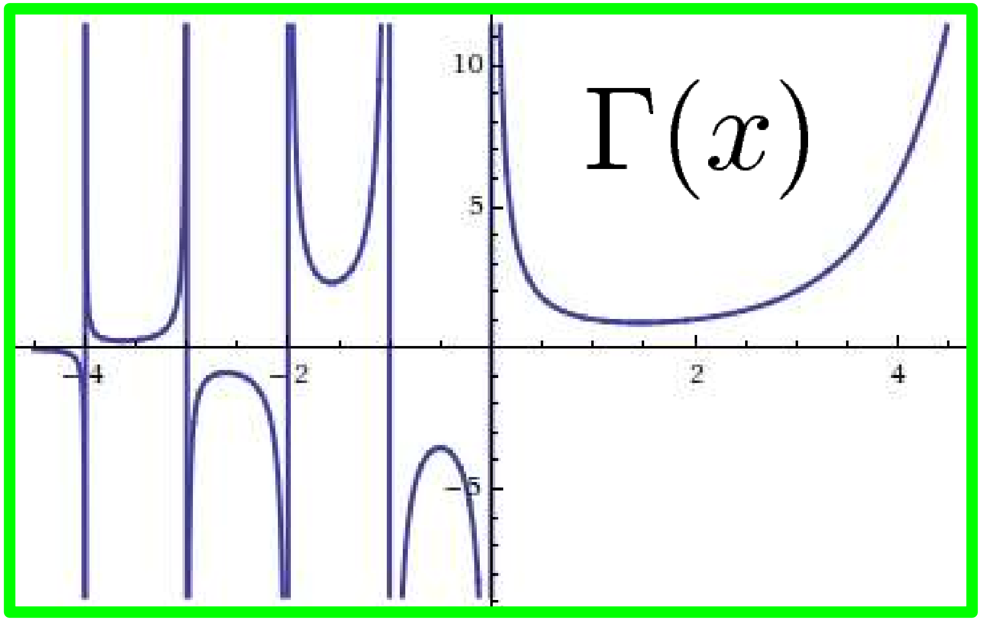
$$V = \frac{\lambda}{4\pi\epsilon_0} \int_0^{+\infty} d\Omega_n \; \frac{y^{n-1}}{\mu^{n-1}} \; \frac{dy}{\sqrt{x^2+y^2}}$$

Each term is individually dimensionless

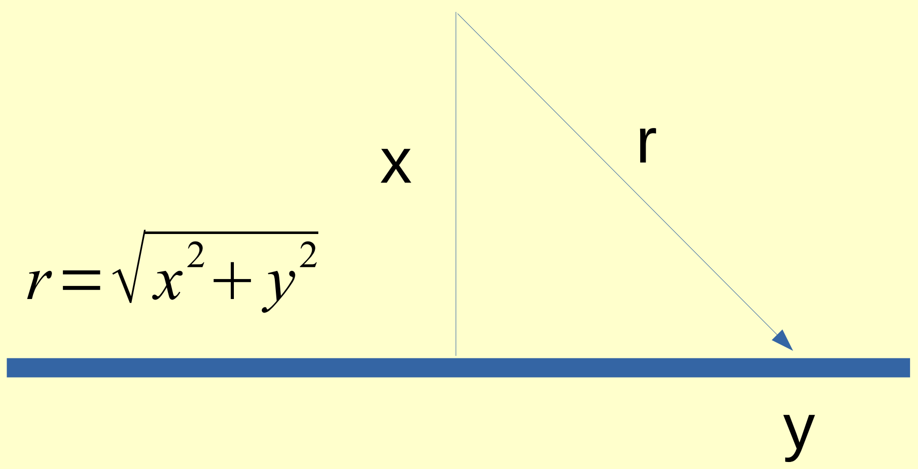
New scale μ

$$n = 1 - 2\epsilon$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}} \frac{\Gamma[\epsilon]}{\pi^{\epsilon}} \right)$$



Why do we need an extra scale μ ???



$$r=\sqrt{x^2+y^2}$$

$$dV=\frac{1}{4\pi\epsilon_0}\,\frac{dQ}{r} \qquad \lambda=\frac{Q}{y}$$

$$V=\frac{\lambda}{4\pi\epsilon_0}\left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}}\,\frac{\Gamma[\epsilon]}{\pi^{\epsilon}}\right)$$

$$V=\underbrace{\frac{\lambda}{4\pi\epsilon_0}}_{\text{potential}}\times\underbrace{f\left(\frac{x}{\mu}\right)}_{\text{dimensionless } f(x/\mu)}$$

potential

dimensionless
f(x/μ)

Dimensional Regularization

$$V = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}} \frac{\Gamma[\epsilon]}{\pi^{\epsilon}} \right)$$

All physical quantities are independent of the regulators:

Electric Field
$$E(x) = \frac{-dV}{dx} = \frac{\lambda}{4\pi\epsilon_0} \left[\frac{2\epsilon\mu^{2\epsilon}\Gamma[\epsilon]}{\pi^{\epsilon} x^{1+2\epsilon}} \right] \underset{\epsilon\to 0}{\to} \frac{\lambda}{2\pi\epsilon_0} \frac{1}{x}$$

Energy
$$\delta V = V(x_1) - V(x_2) \underset{\epsilon\to 0}{\to} \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{x_2^2}{x_1^2} \right]$$

Problem solved at the expense of an extra scale μ **AND** regulator ε

Translation invariance is preserved!!!

Dimensional Regularization respects symmetries

Connection to QFT

$$V = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}} \frac{\Gamma[\epsilon]}{\pi^{\epsilon}} \right)$$

Expand in Taylor series in ε

The was the potential from our "Toy" calculation:

$$V \to \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{1\ \pi}\right] + \ln\left[\frac{\mu^2}{x^2}\right] \right]$$

This is a partial result from
a <u>real</u> NLO Drell-Yan Calculation:
Cf., B. Potter

$$\frac{D(\epsilon)}{\epsilon} = \left(\frac{4\pi\mu^2}{Q^2}\right) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \to \left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{4\pi}\right] + \ln\left[\frac{\mu^2}{Q^2}\right] \right]$$

Renormalization

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$

Original

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$

MS

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\ln\left[\frac{\mu^2}{x^2}\right]\right]$$

MS-Bar

Physical quantities are independent of renormalization scheme!

$$V_{\overline{MS}}(x_1)-V_{\overline{MS}}(x_2)=\delta V=V_{MS}(x_1)-V_{MS}(x_2)$$

But only if performed consistently:

$$V_{\overline{MS}}(x_1)-V_{MS}(x_2)\neq\delta V\neq V_{MS}(x_1)-V_{\overline{MS}}(x_2)$$

Regulator provides unique definition of V, f, ω

Cutoff regulator L:
simple, but does NOT respect symmetries

Dimensional regulator ε:
respects symmetries: translation, Lorentz, Gauge invariance
introduces new scale μ

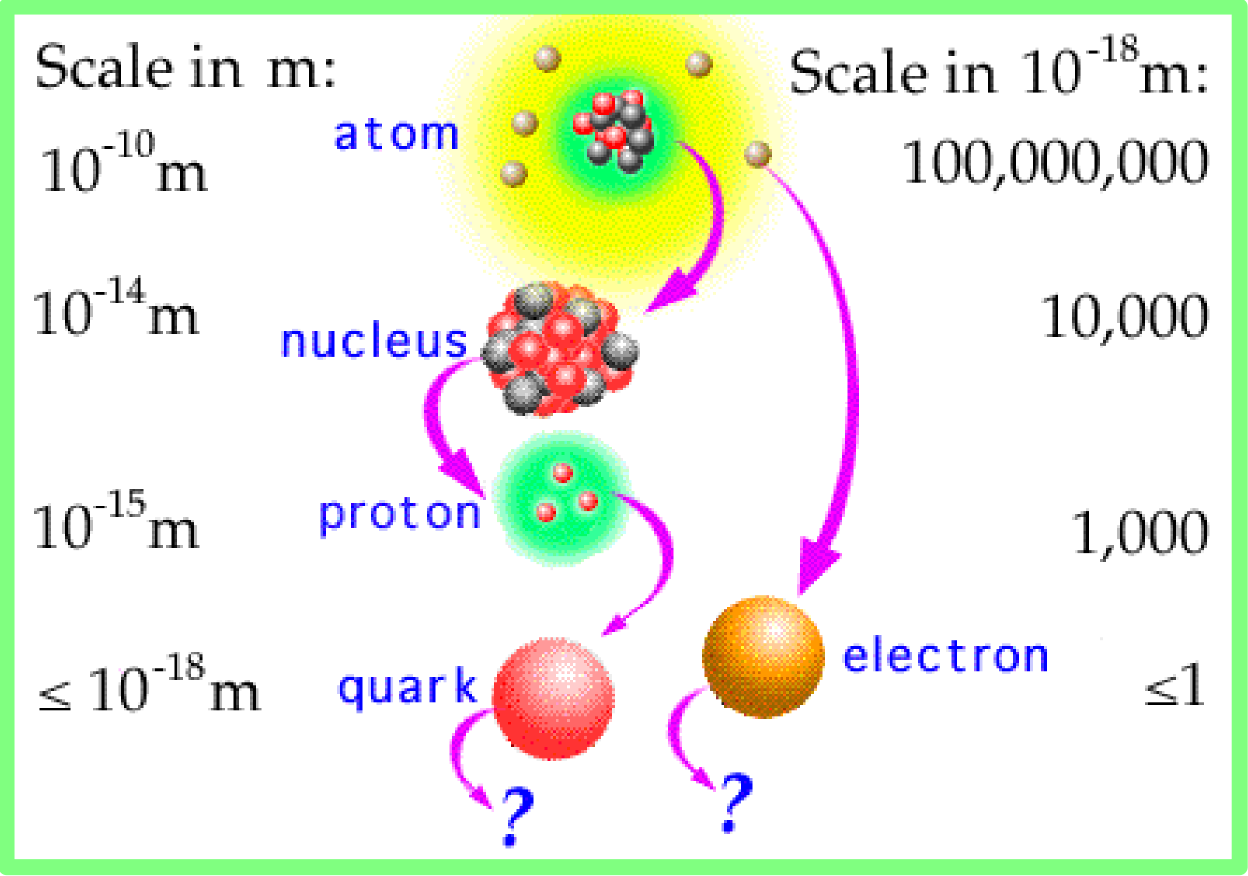
All physical quantities (E, dV, σ) are independent of the regulator
AND the new scale μ
Renormalization group equation: $d\sigma/d\mu=0$

We can define renormalized quantities (V,f,ω)
Renormalized (V,f,ω) are scheme dependent and arbitrary
Physical quantities (E,dV, σ) are unique and scheme independent
if we apply the scheme consistently

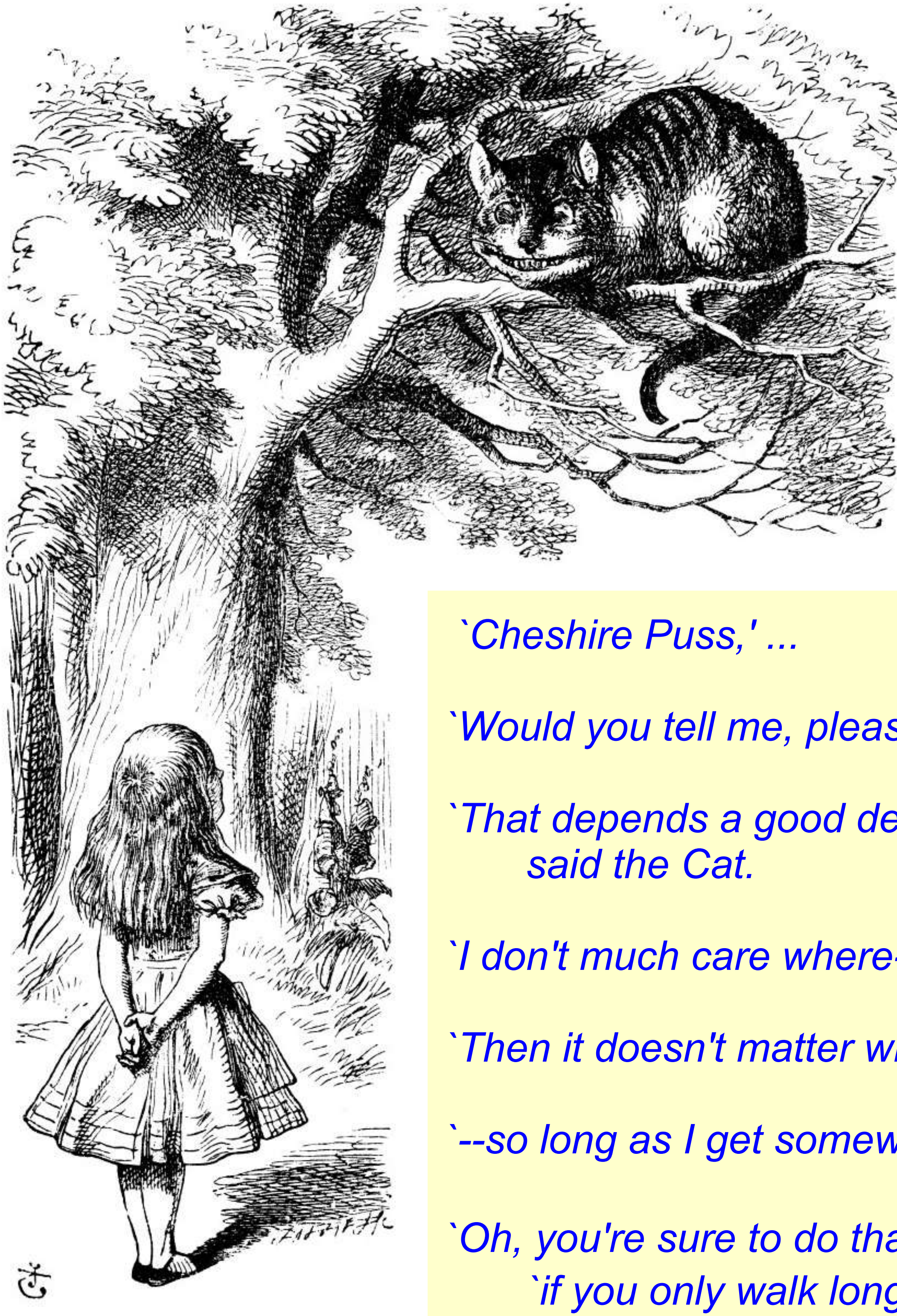
What about the μ scale

$$\sigma_{p\gamma\to c}(x,Q) = f_{p\to a}(x,Q,\mu) \otimes \hat{\sigma}_{a\gamma\to c}(x,Q,\mu)$$

When we do our calculations, where does the mysterious μ does renormalization scale come from???



What is inside the proton/nucleon???
The answer depends on how closely you look.



The answer is dependent upon the question

... an old preprint by Charles Dodgson

`Cheshire Puss,' ...

`Would you tell me, please, which way I ought to go from here?'

`That depends a good deal on where you want to get to,'
said the Cat.

`I don't much care where--' said Alice.

`Then it doesn't matter which way you go,' said the Cat.

`--so long as I get somewhere,' Alice added as an explanation.

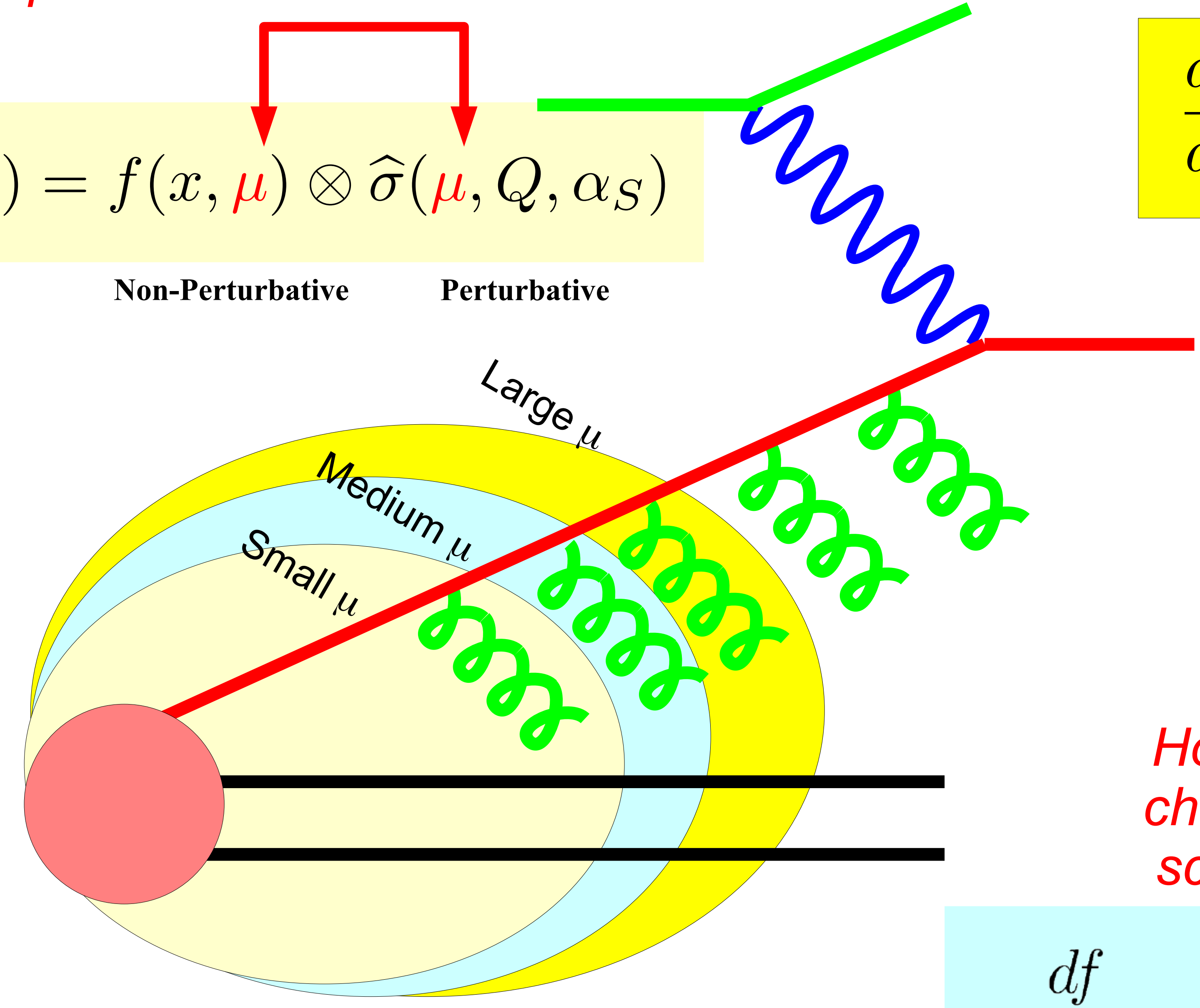
`Oh, you're sure to do that,' said the Cat,
`if you only walk long enough.'

μ dependence must balance

$$\sigma(Q,x) = f(x,\mu) \otimes \widehat{\sigma}(\mu, Q, \alpha_S)$$

Observable **Non-Perturbative** **Perturbative**

$$\frac{d\sigma}{d\mu} = 0$$



How does f change with scale μ???

$$\frac{df}{d\ \ln[\mu]} = ???$$

Renormalization Group Equation

Parton Model

$$\sigma = f(\mu) \otimes \omega(\mu)$$

$$\omega \equiv \hat{\sigma}$$

Not physical!
Poor notation

Renormalization Group Equation

$$\frac{d\sigma}{d\mu} = 0 = \frac{d\tilde{f}}{d\mu}\,\tilde{\omega} + \tilde{f}\,\frac{d\tilde{\omega}}{d\mu}$$

Take Mellin Transform

Separation of variables

$$\frac{1}{\tilde{f}}\,\frac{d\tilde{f}}{d\ln[\mu]} = -\gamma = -\frac{1}{\tilde{\omega}}\,\frac{d\tilde{\omega}}{d\ln[\mu]}$$

DGLAP

$$\frac{d\tilde{f}}{d\ln[\mu]} = -\tilde{f}\,\gamma \qquad \frac{df}{d\ln[\mu]} = P \otimes f$$

DGLAP Equation

$$\tilde{f} \sim \mu^{-\gamma}$$

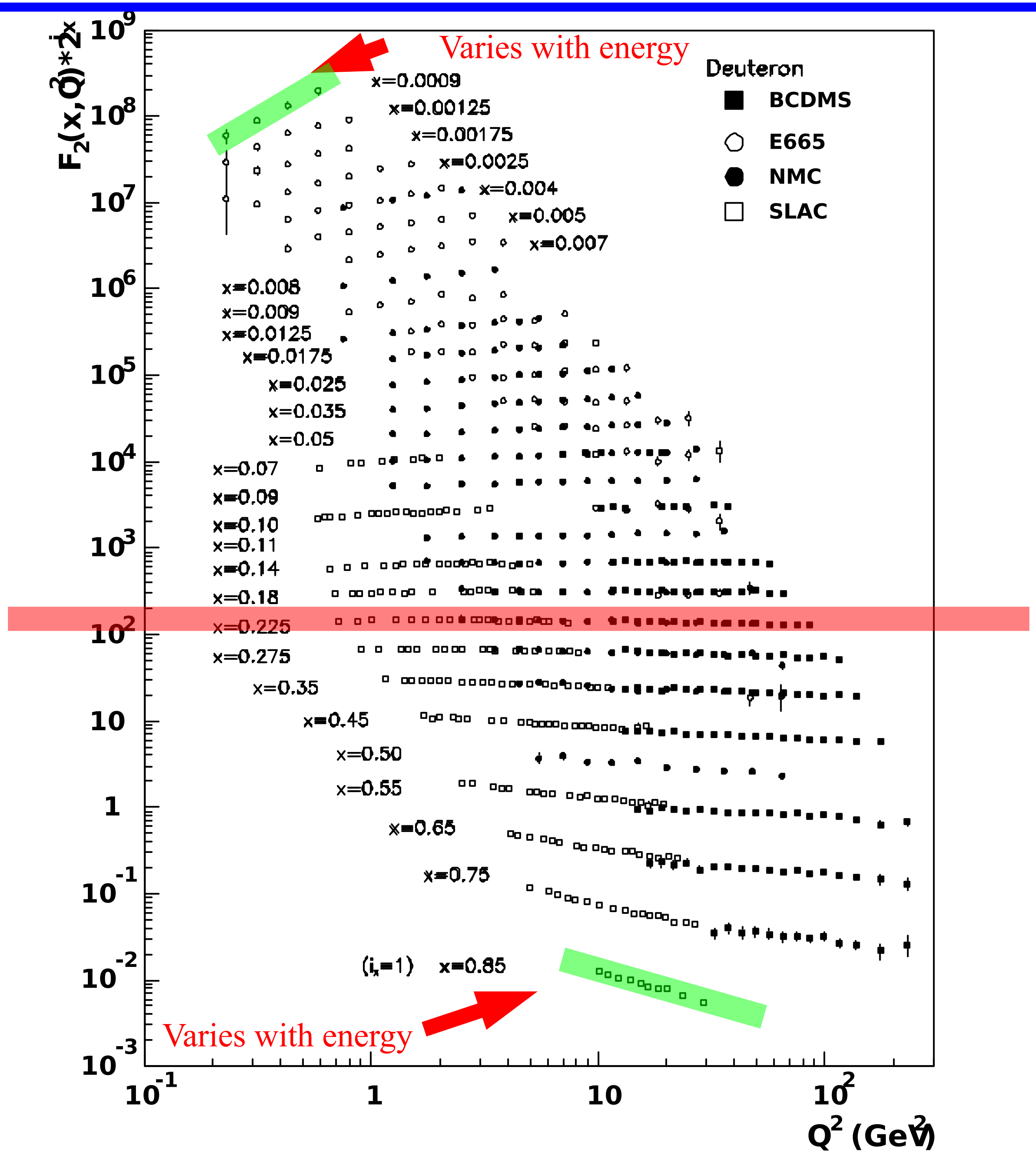
Anomalous Dimension

If "*f*" scaled, γ would vanish

It is the dimension of the mass scaling

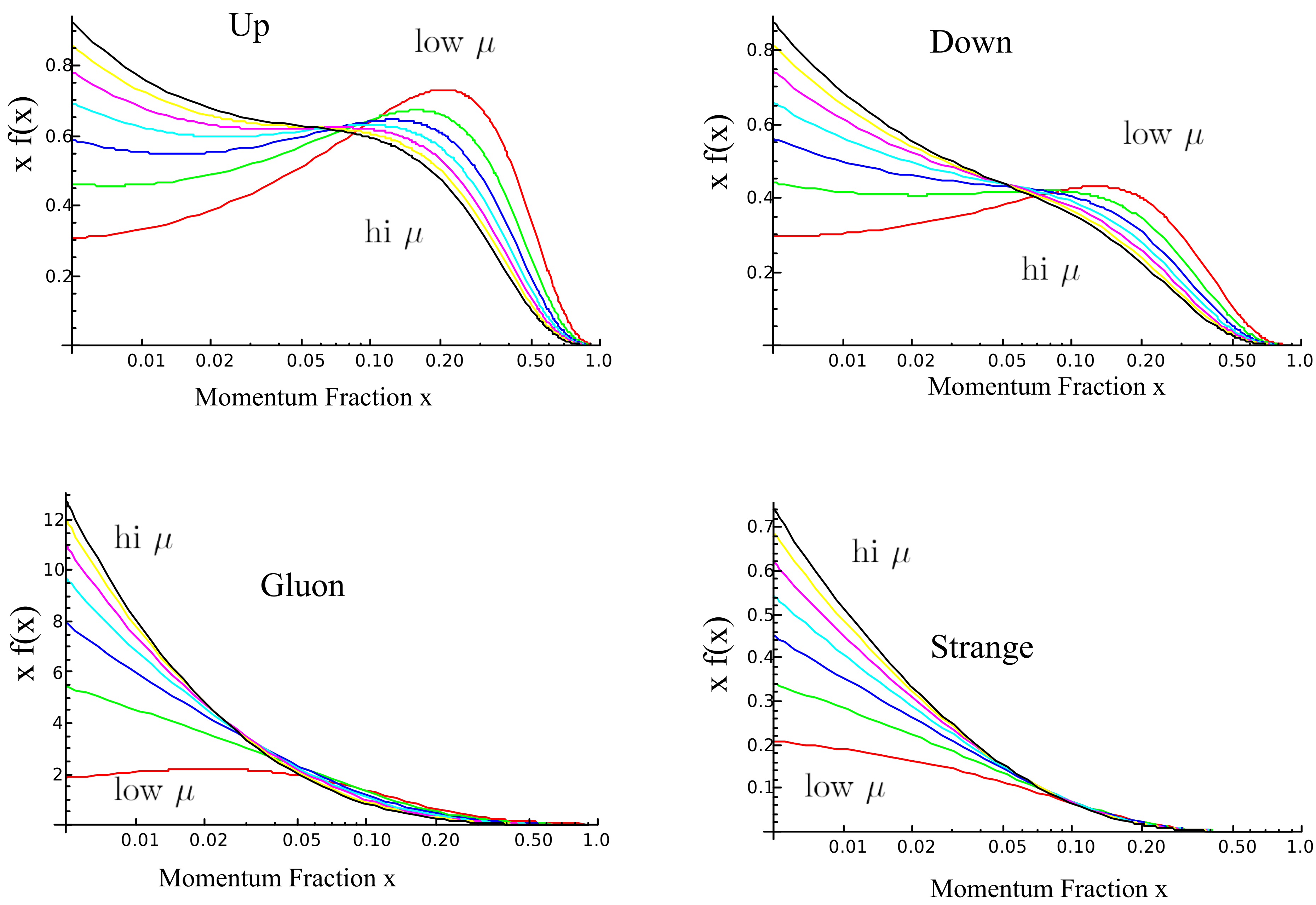
The Scaling of the Proton Structure Function

Data is (relatively) independent of energy

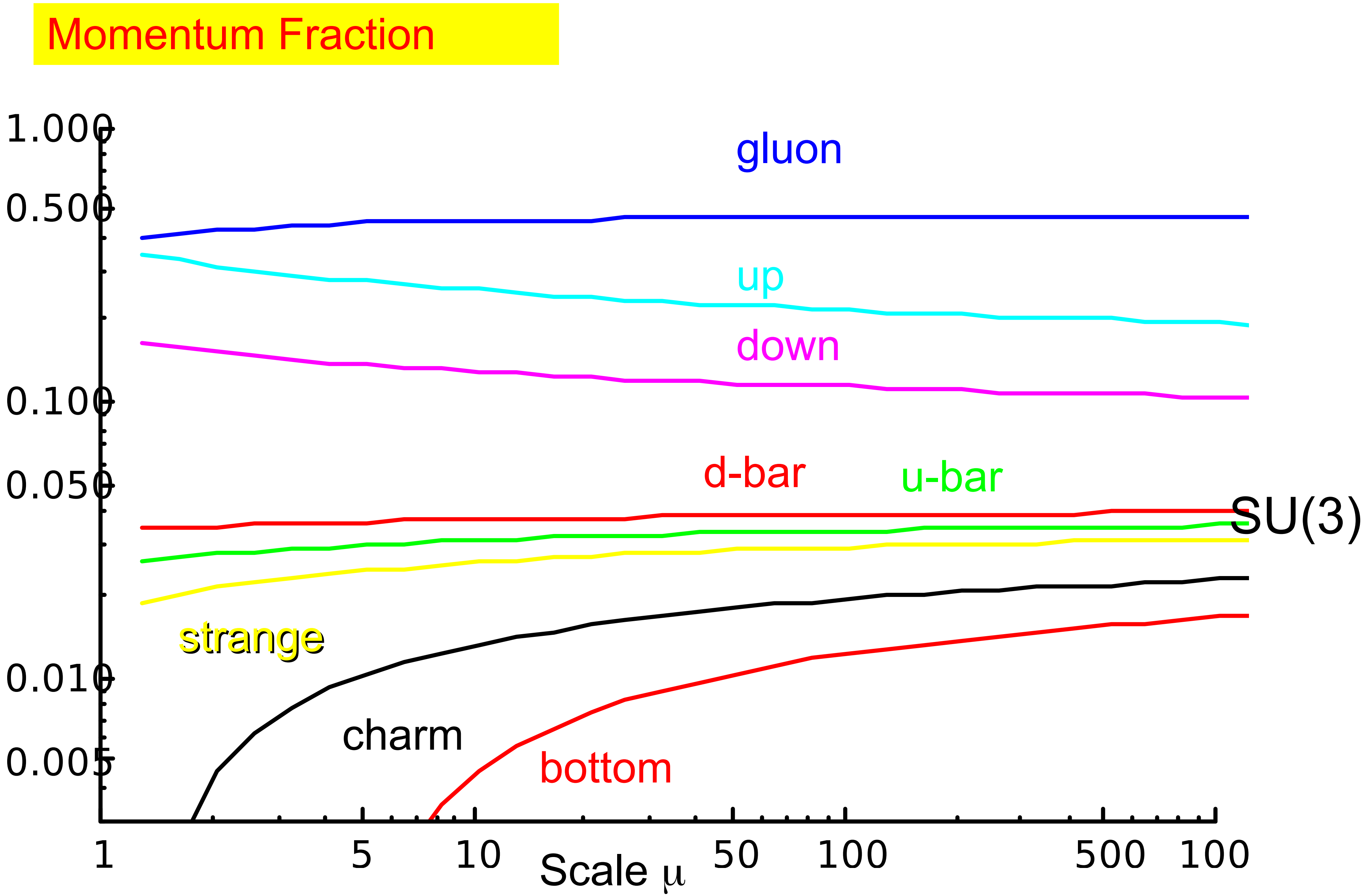


Scaling Violations observed at extreme x values

Evolution of the PDFs



PDF Momentum Fractions vs. scale μ



Scaling violations are essential feature of PDFs

Homework: Mellin Transform

$$\widetilde{f}(n) = \int_0^1 dx\, x^{n-1}\, f(x)$$

$$\sigma = f \otimes \omega$$

$$f(x) = \frac{1}{2\pi i} \int_C dn\, x^{-n}\, \widetilde{f}(n)$$

$$\widetilde{\sigma} = \widetilde{f}\, \widetilde{\omega}$$

C is parallel to the imaginary axis, and to the right of all singularities

1) Take the Mellin transform of $f(x) = \sum_{m=1}^{\infty} a_m x^m$, and verify the inverse transform of $\widetilde{f}$ regenerates $f(x)$

2) Take the Mellin transform of $\sigma = f \otimes \omega$ to demonstrate that the Mellin transform separates a convolution yields $\widetilde{\sigma} = \widetilde{f}\, \widetilde{\omega}$.

A useful reference:
Courant, Richard and Hilbert, David. Methods of Mathematical Physics, Vol. 1. New York: Wiley, 1989. 561 p.

More

Scaling

$$A_c = c^2 f(\theta, \phi)$$

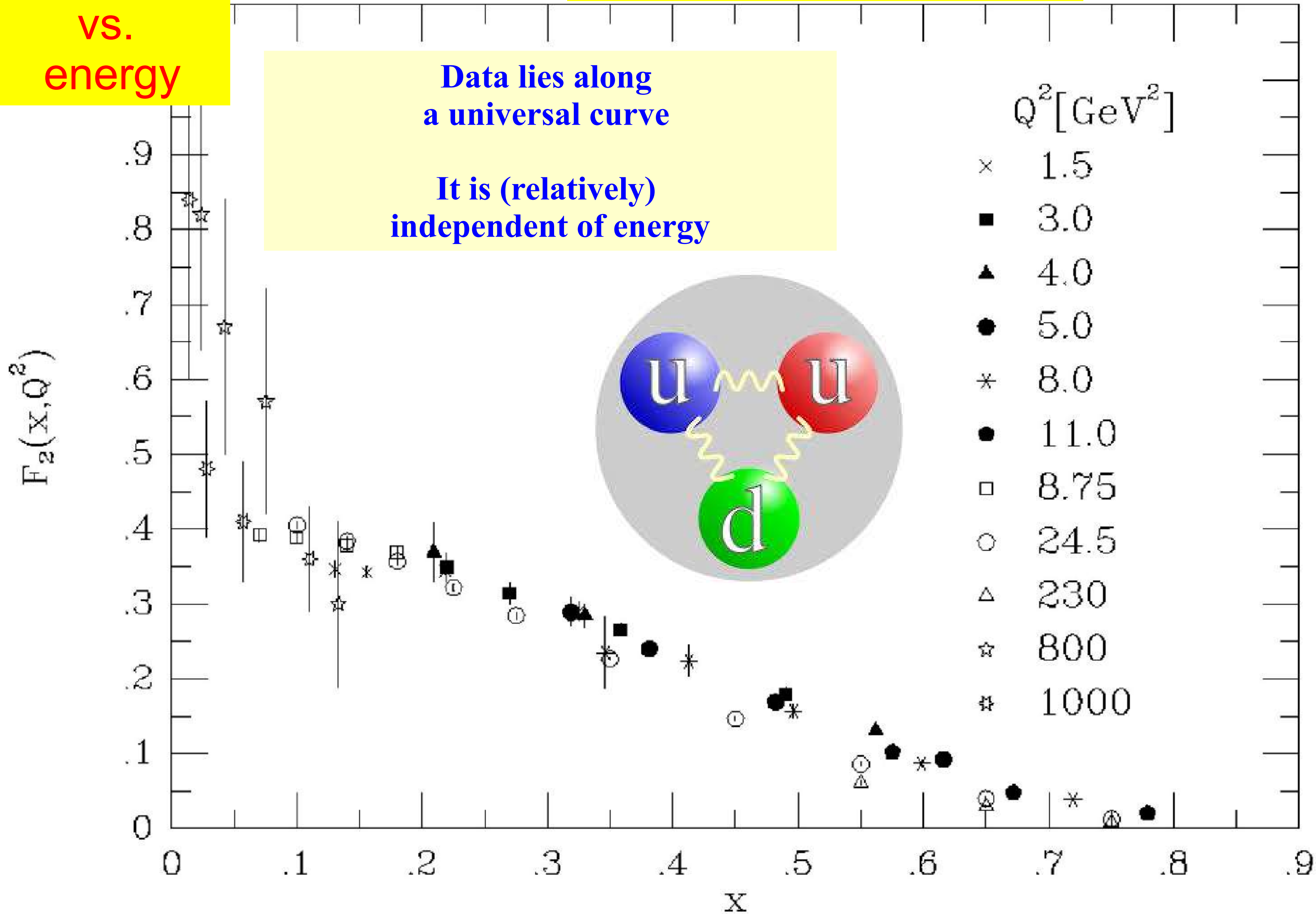
The Scaling of the Proton Structure Function

Structure vs. energy

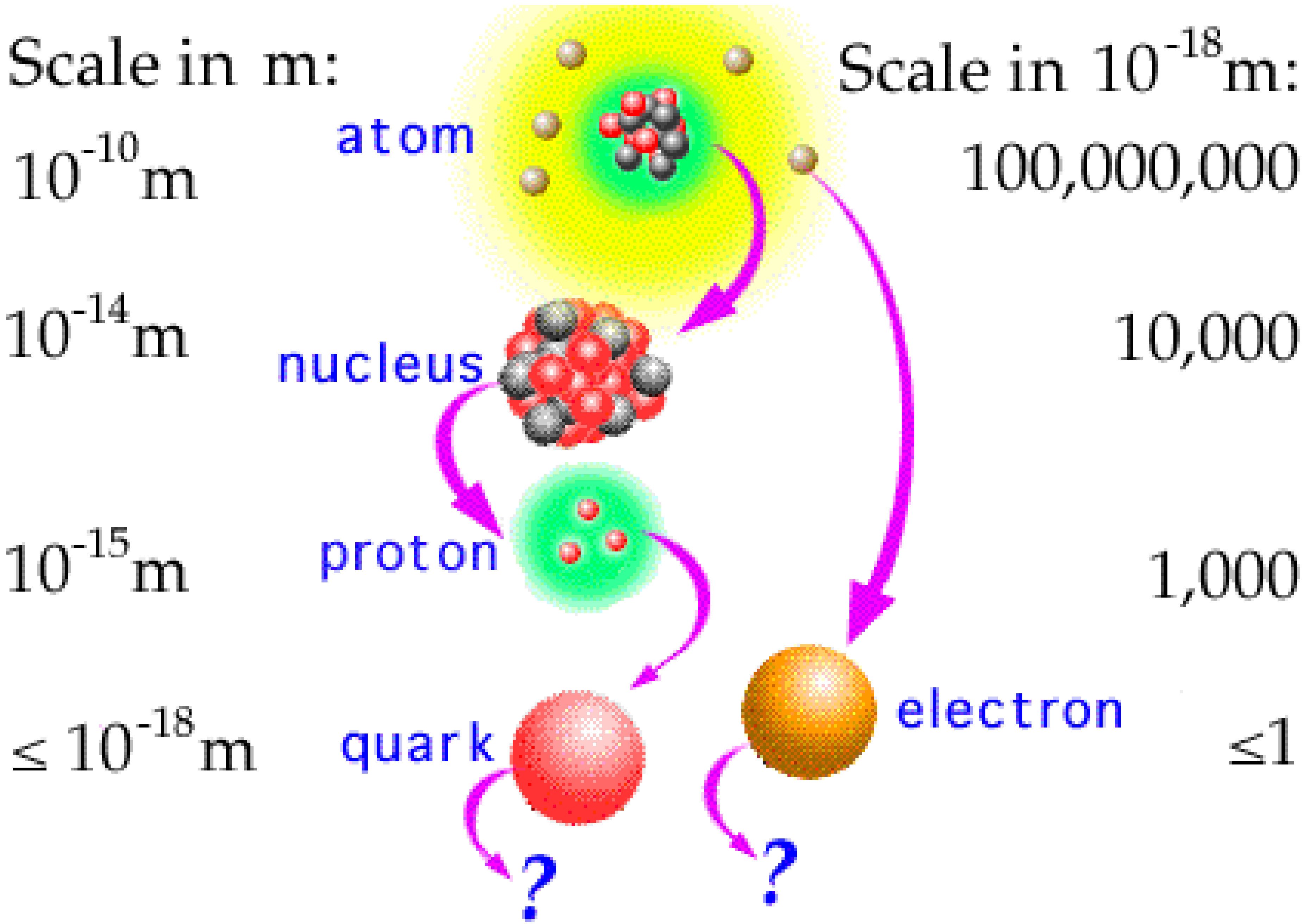
$$F_2^{\gamma}(x,Q) = xe_q^2\{q+\bar{q}\}$$

Data lies along a universal curve

It is (relatively) independent of energy



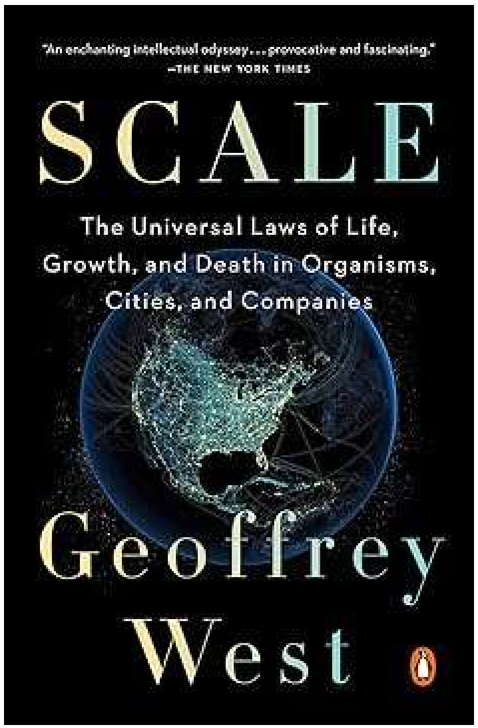
Small distance ~ High Energy

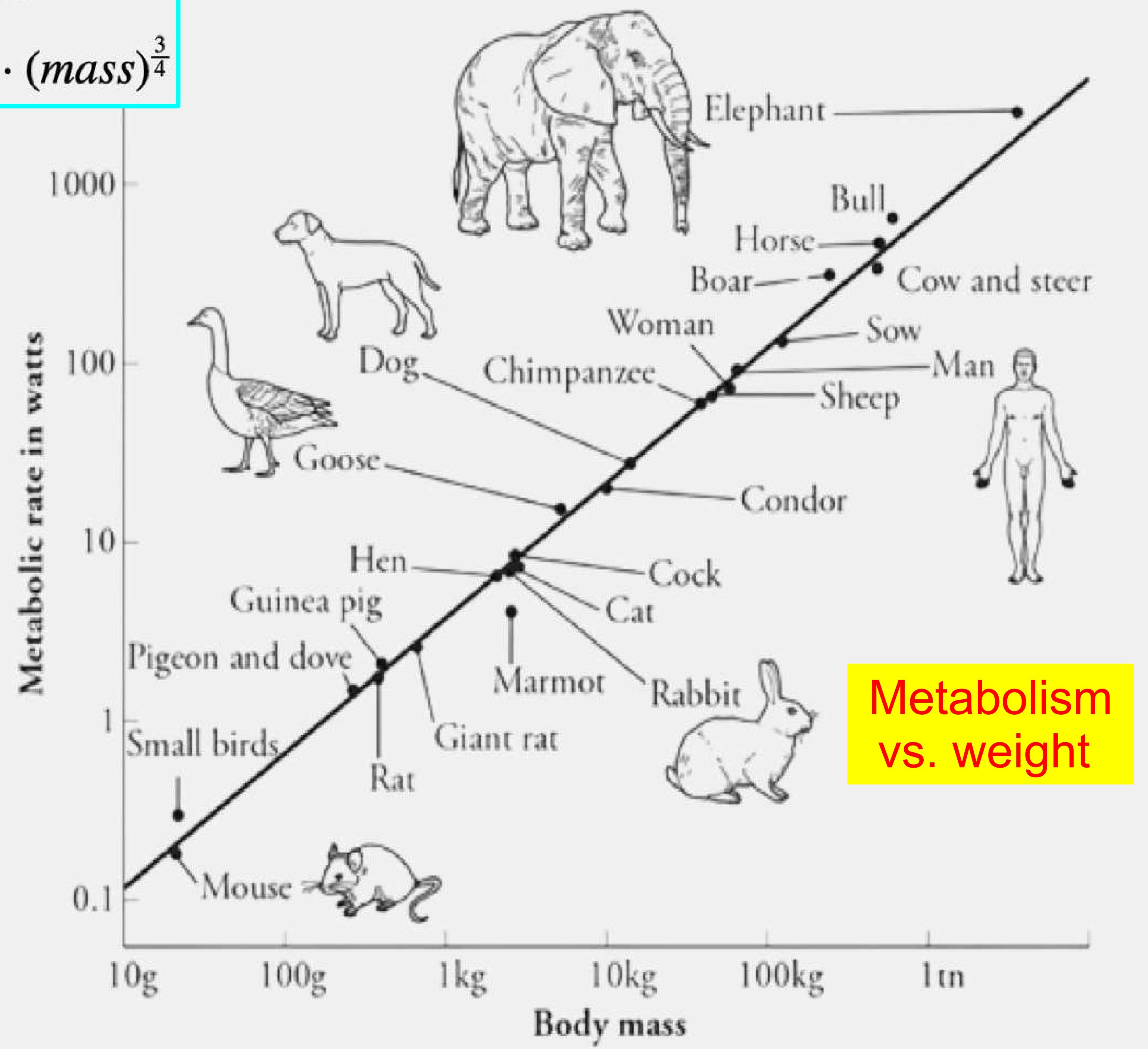


Going to smaller scale, we get to simpler, more fundamental objects

Kleiber's law

$$metabolic\ rate \approx k \cdot (mass)^{\frac{3}{4}}$$





Metabolism vs. weight

Scale: The Universal Laws of Growth, Innovation, Sustainability, and the Pace of Life in Organisms, Cities, Economies, and Companies. By Geoffrey West

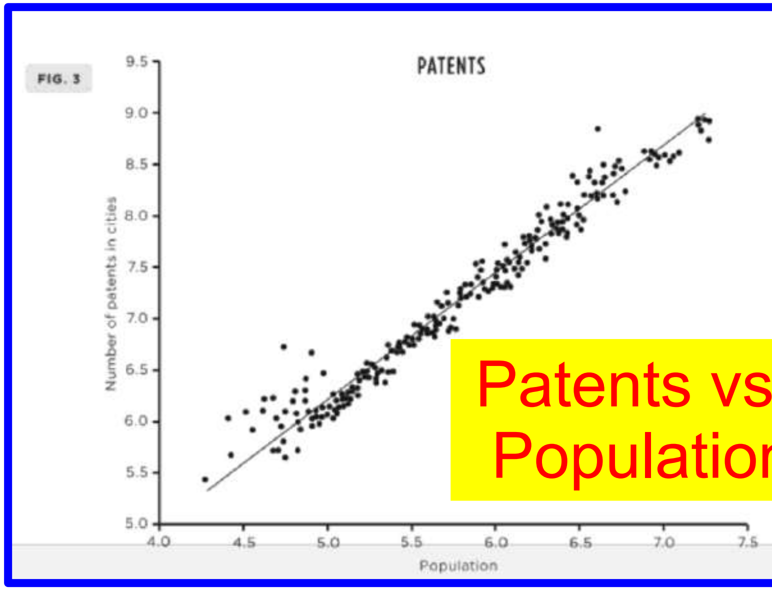
The Reynolds number is defined as:[6]

$$\mathrm{Re} = \frac{uL}{\nu} = \frac{\rho u L}{\mu}$$

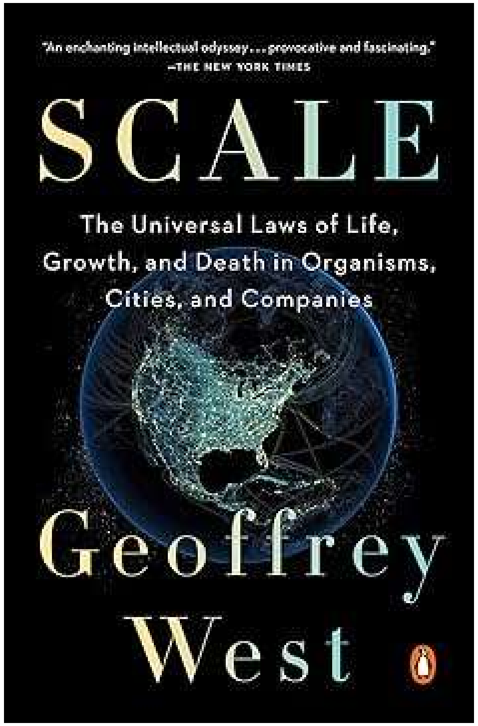
where:

- ρ is the density of the fluid (SI units: kg/m^3)
- u is the flow speed (m/s)
- L is a characteristic length (m)
- μ is the dynamic viscosity of the fluid (Pa·s or N·s/m^2 or kg/(m·s))
- ν is the kinematic viscosity of the fluid (m^2/s).

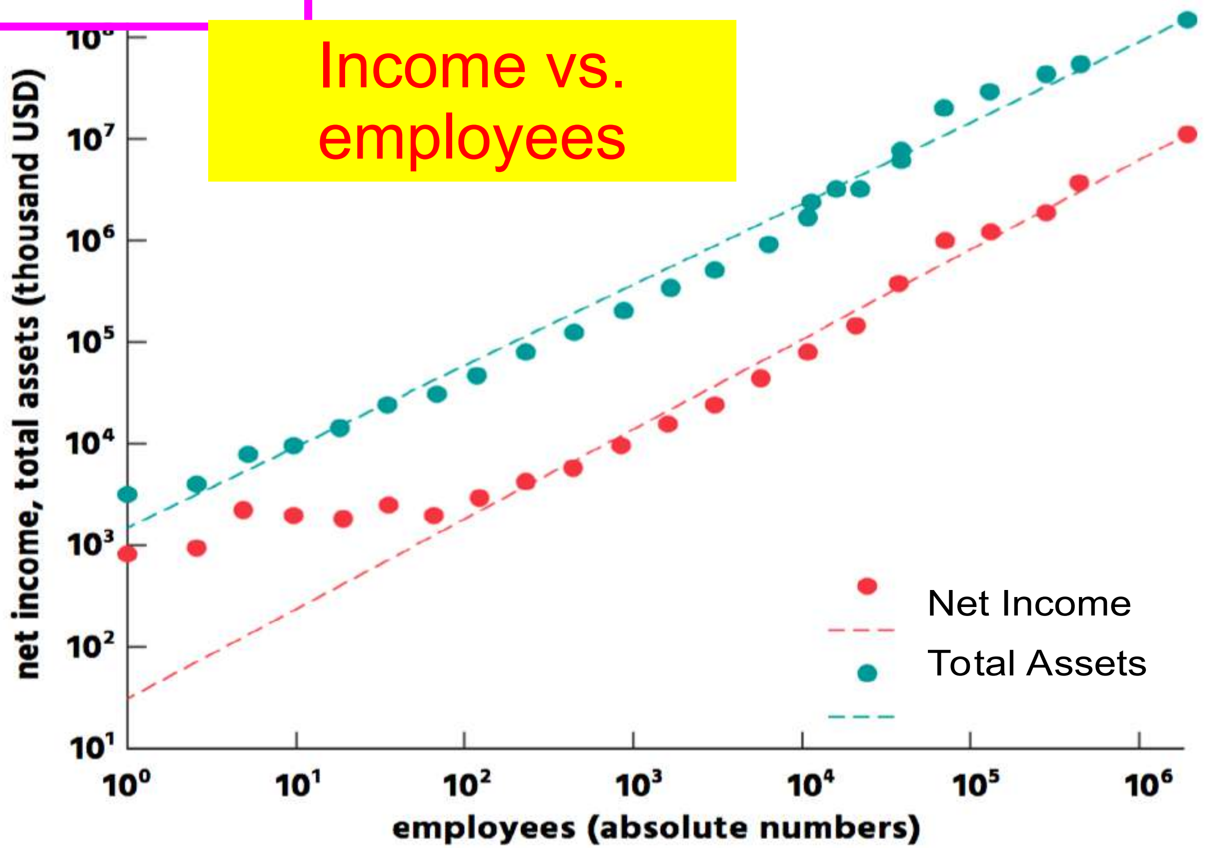
Fluid Flow

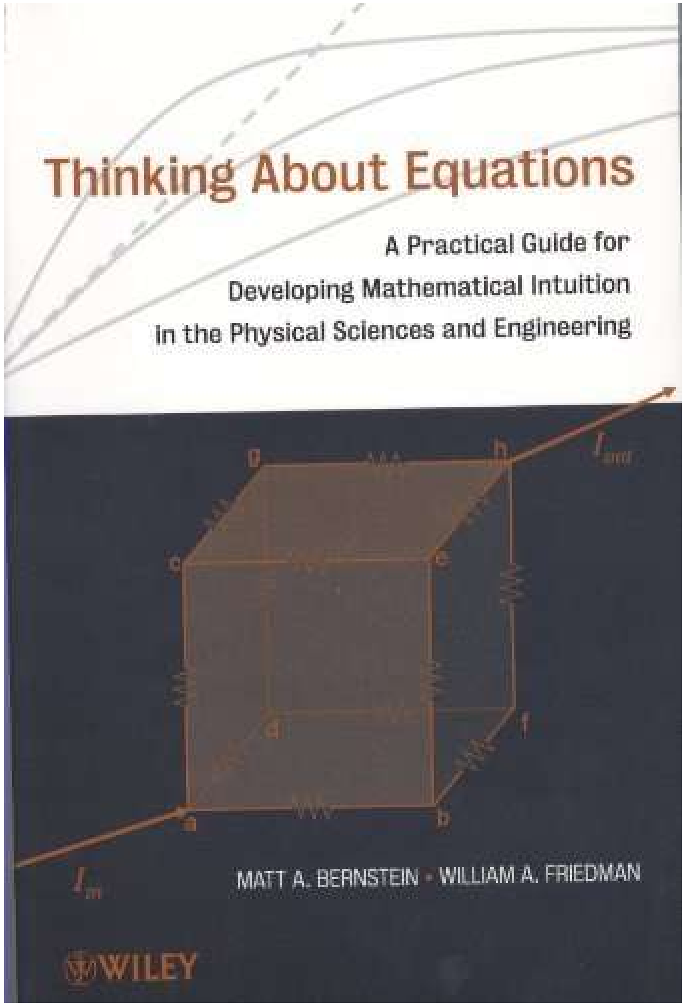


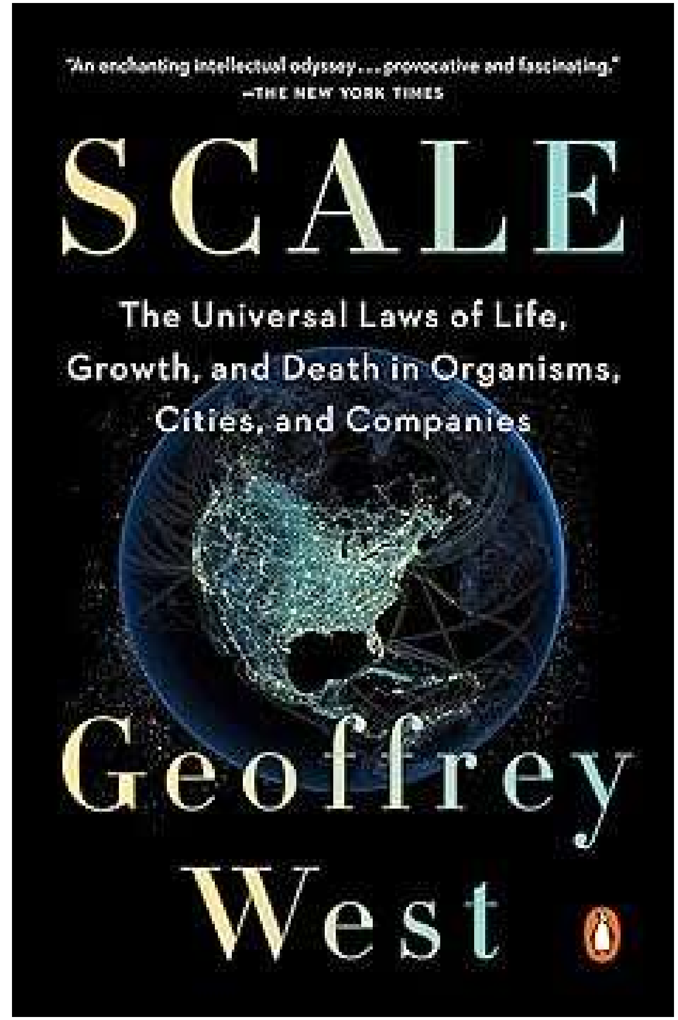
Patents vs. Population



Income vs. employees







Thinking About Equations

Physicist Richard Feynman attributes his success at solving problems, in part, to the "Different Box of Tools" he acquired in his early studies.

SCALE

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Matt Bernstein has been shaping the field of MRI for over 30 years, first as a researcher at GE Medical Systems, and then as a clinical medical physicist at Mayo Clinic. During this time, he has authored over 130 research articles, 250 abstracts, as well as two books including the widely-read Handbook of MRI Pulse Sequences that can be found on the desks of most MRI engineers around the world.

arXiv:0812.357

Regularization, Renormalization, and Dimensional Analysis:

*Dimensional Regularization Meets Freshman E&M**

Fredrick Olness & Randall Scalise
Department of Physics, Southern Methodist University, Dallas, TX 75275-0175, U.S.A.

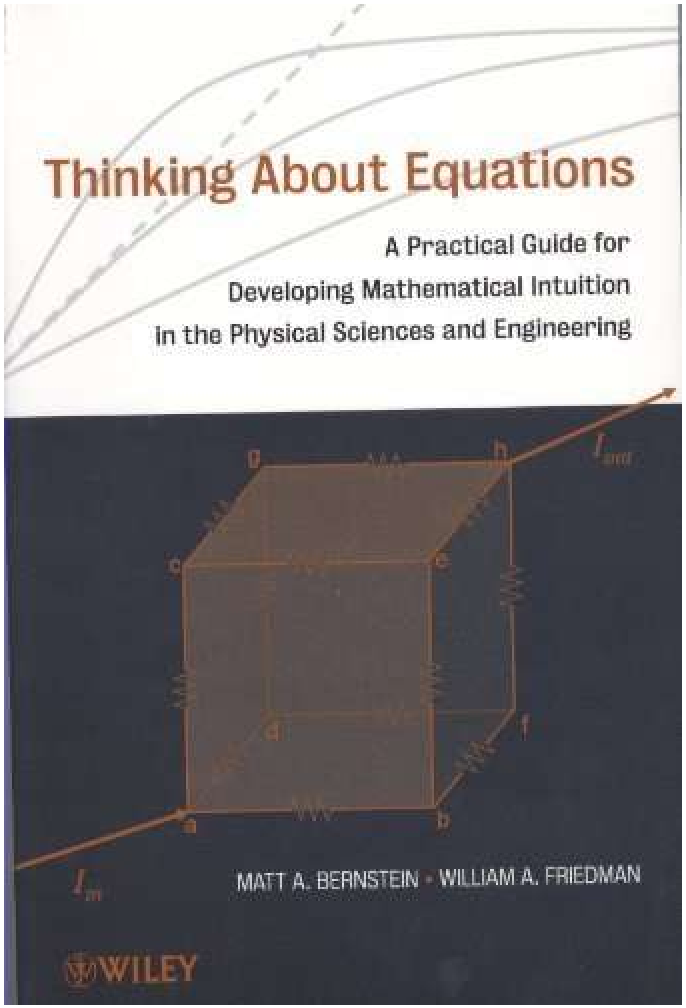
(Dated: August 22, 2017)

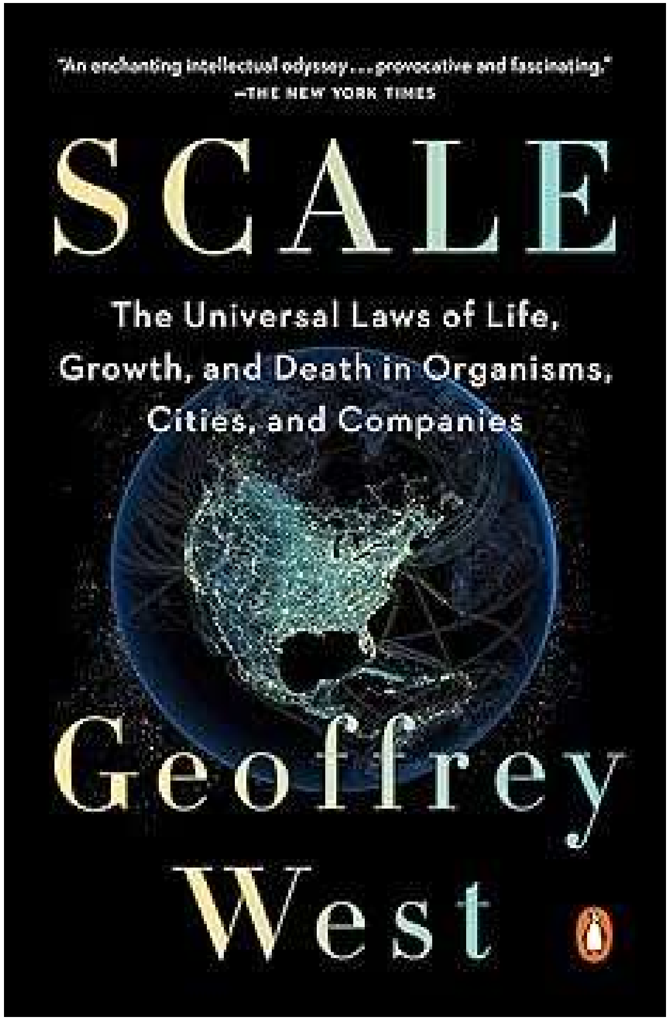
We illustrate the dimensional regularization (DR) technique using a simple problem from elementary electrostatics. This example illustrates the virtues of DR without the complications of a full quantum field theory calculation. We contrast the DR approach with the cutoff regularization approach, and demonstrate that DR preserves the translational symmetry. We then introduce a Minimal Subtraction (MS) and a Modified Minimal Subtraction ($\overline{MS}$) scheme to renormalize the result. Finally, we consider dimensional transmutation as encountered in the case of compact extra-dimensions.

PACS numbers:
11.30.-j Symmetry and conservation laws
11.10.Kk Field theories in dimensions other than four
11.15.-q Gauge field theories
11.10.Gh Renormalization

Keywords: Renormalization, Dimensional Regularization, Regularization, Gauge Symmetries

Regularization, Renormalization, and Dimensional Analysis:





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