Modern Methods in QCD

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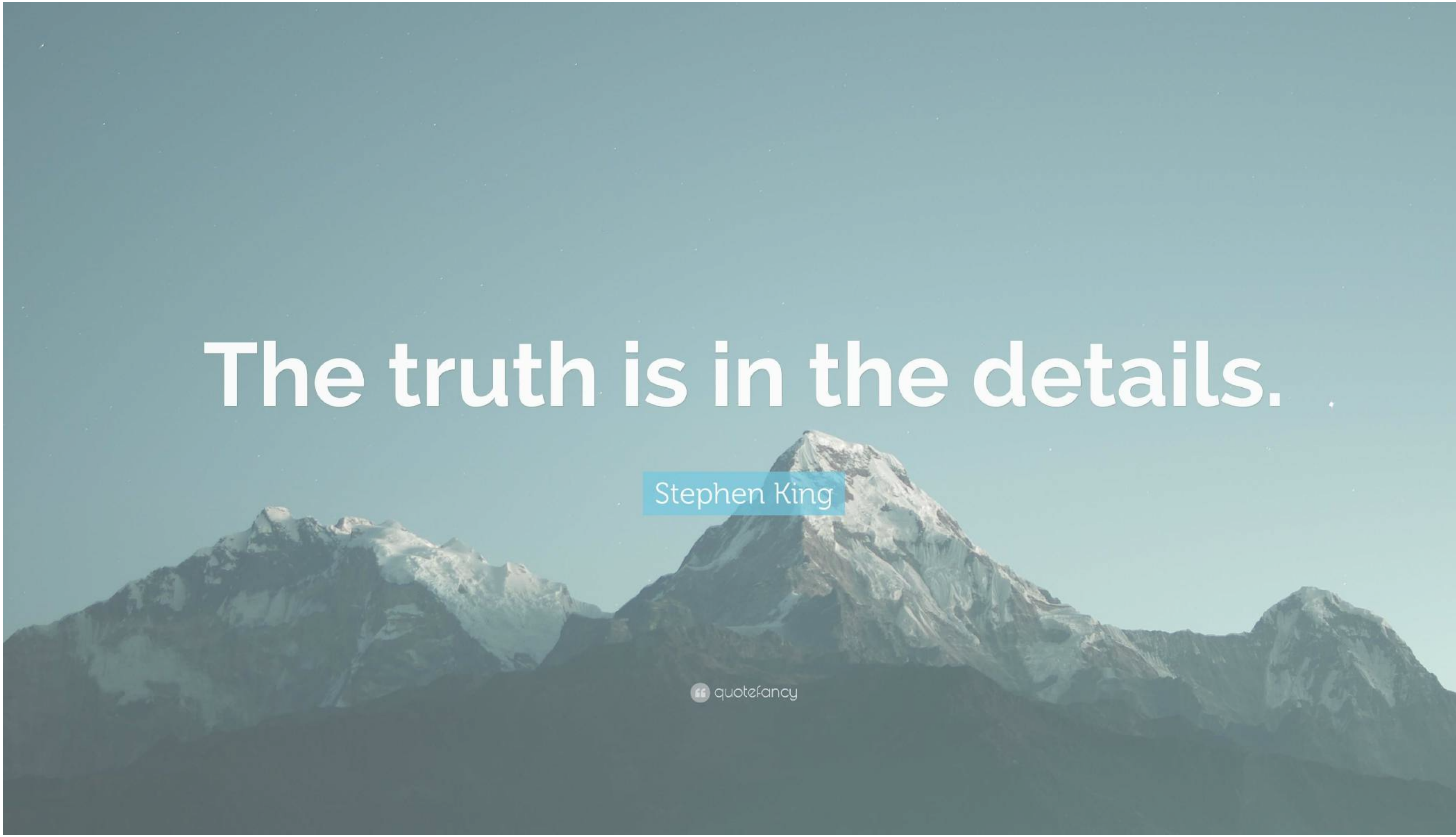

The truth is in the details.
Stephen King
quotefancy

Plan

- Why Precision Calculation (PC)
- Impact of PCs on discoveries
- Part-1

  Methods for Multi-leg processes
- Part-2

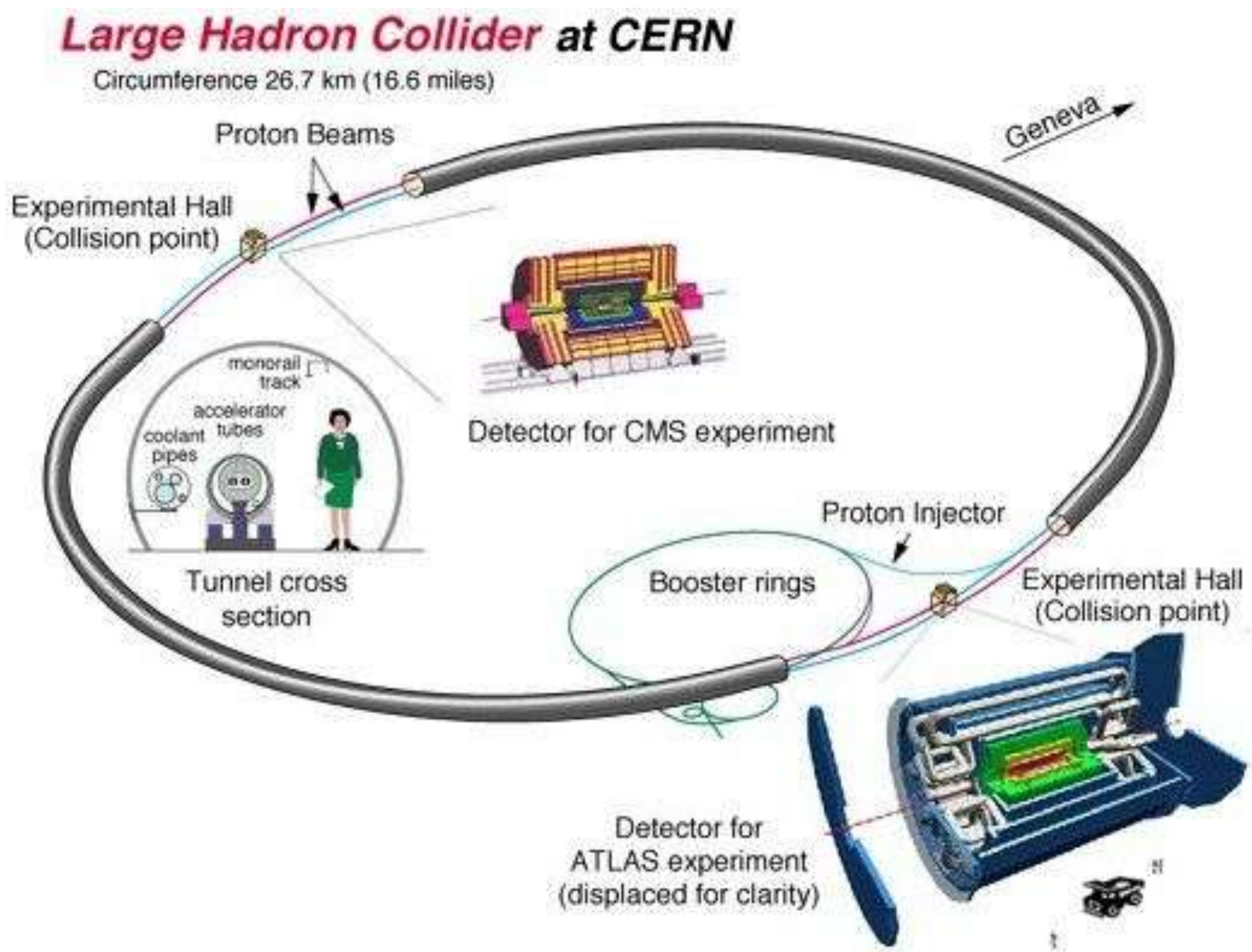
  Methods for Multi-loops processes
- Infrared physics

Precision Physics

Large Hadron Collider

- Excellent Discovery Reach

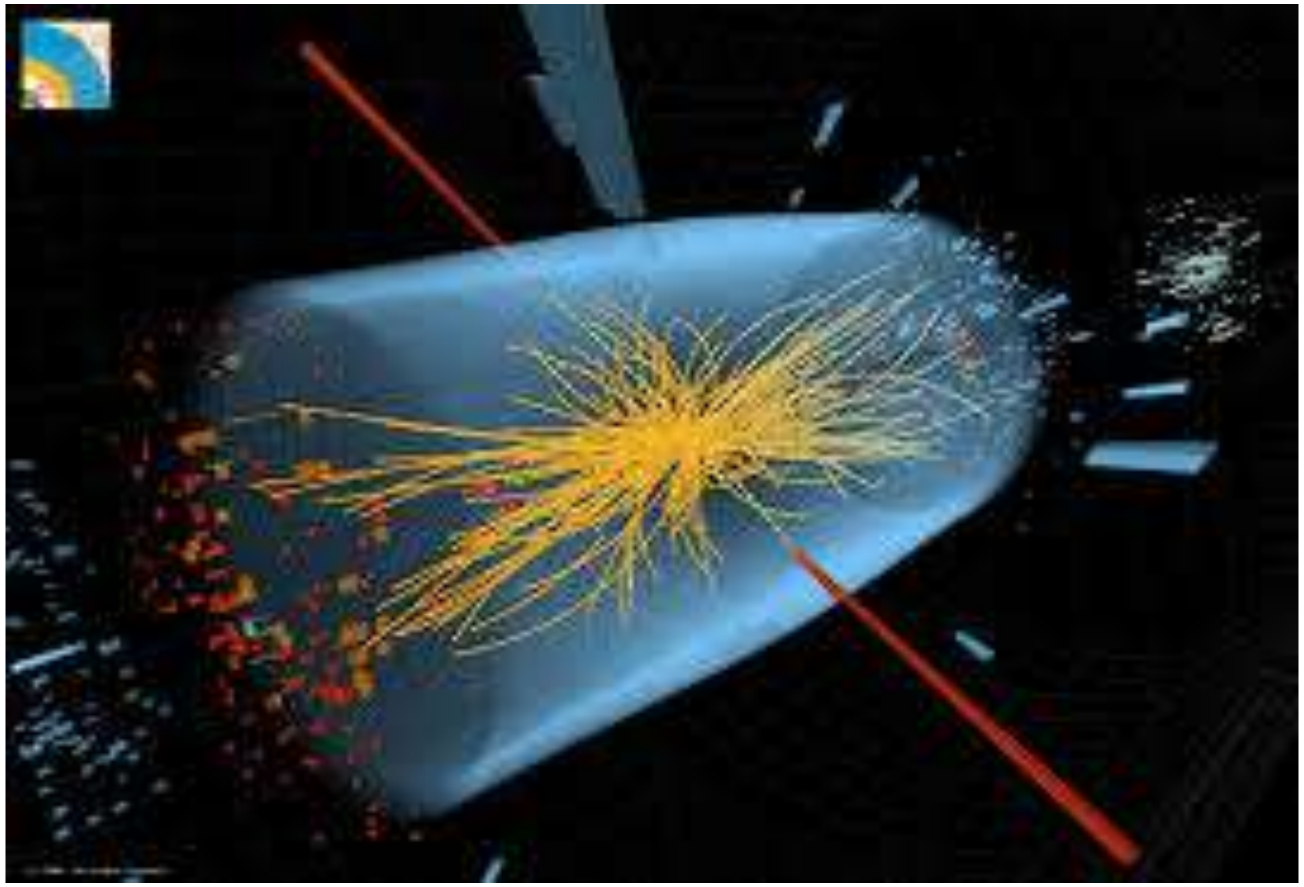
- Higgs
- Supersymmetry
- Extra-Dimensions
- Anything else



Large Hadron Collider

- Large amount of events

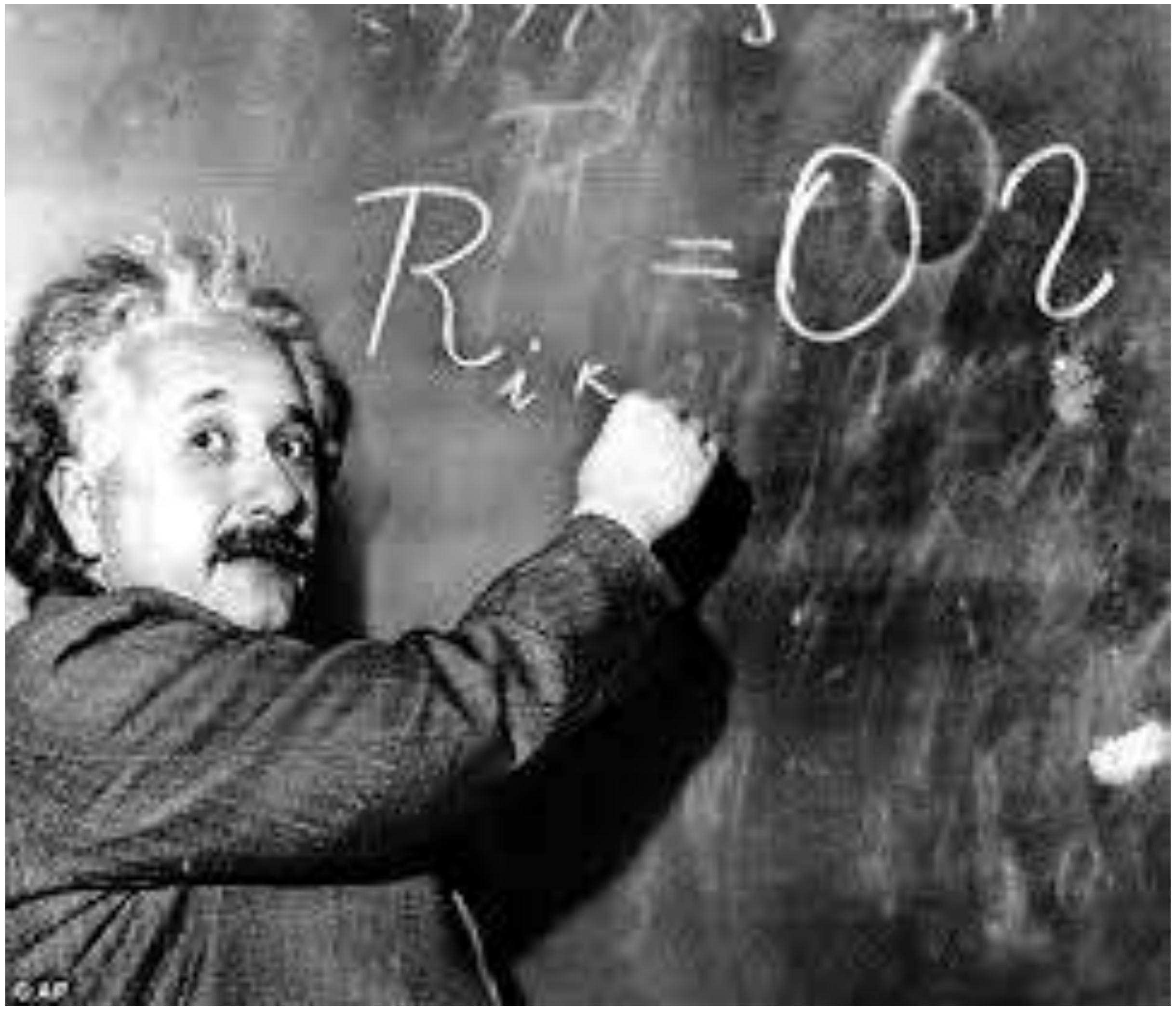
- $W \rightarrow e\nu$: 10^8 events
- $Z \rightarrow e^+e^-$: 10^7 events
- $t\bar{t}$ production 10^7 events
- Higgs production 10^5 events



Standard Model

Testing the known

- Theories
- Quantum Chromodynamics
- Electroweak Theory (SM)
- Theory of Gravity

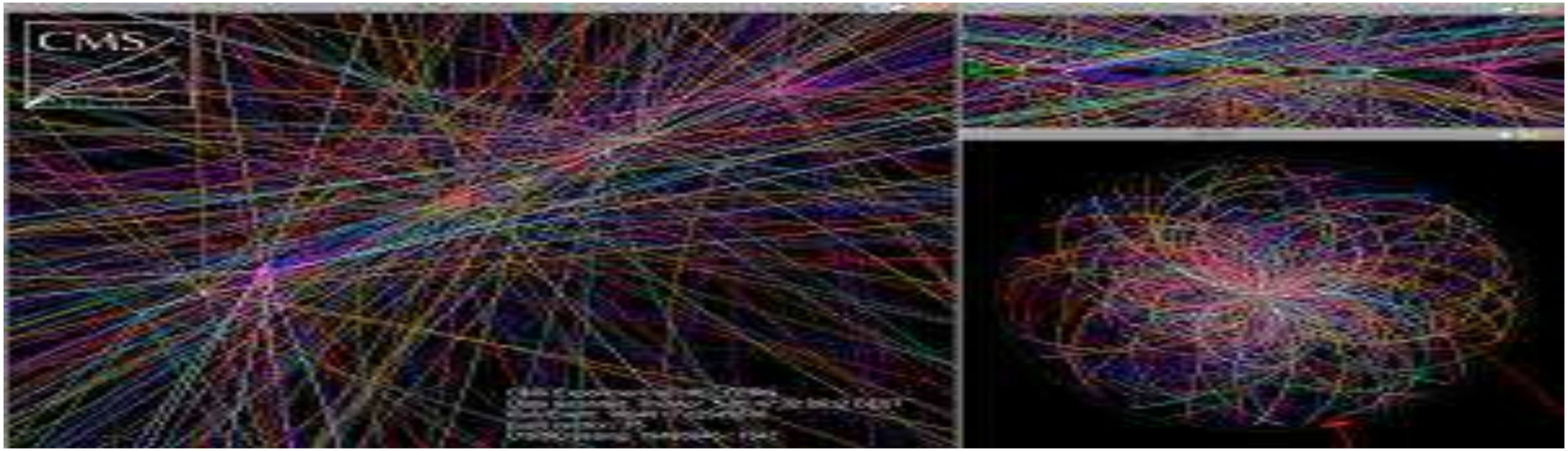


Constraining the parameters of New theories

Large Hadron Collider

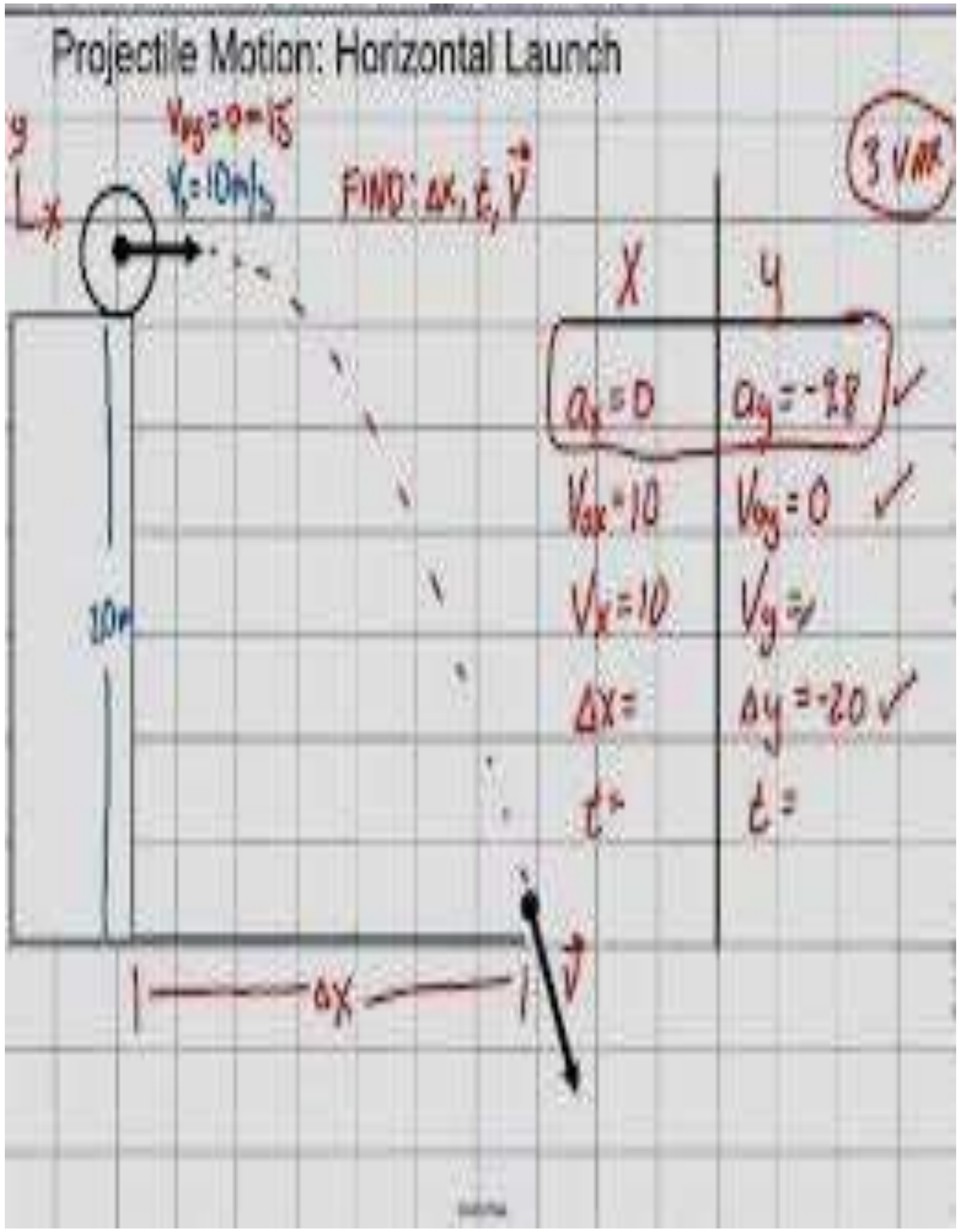
Challenges

- Large background
- Large number of $\gamma, l^{\pm}, Z, W^{\pm}$
- Jets
- Large number of $t\bar{t}, b\bar{b}$



- Issues to be tackled

- Kinematics
- Normalisation
- Renormalisation and Factorisation Scales
- Parton distribution functions
- Phase space boundary effects, resummation



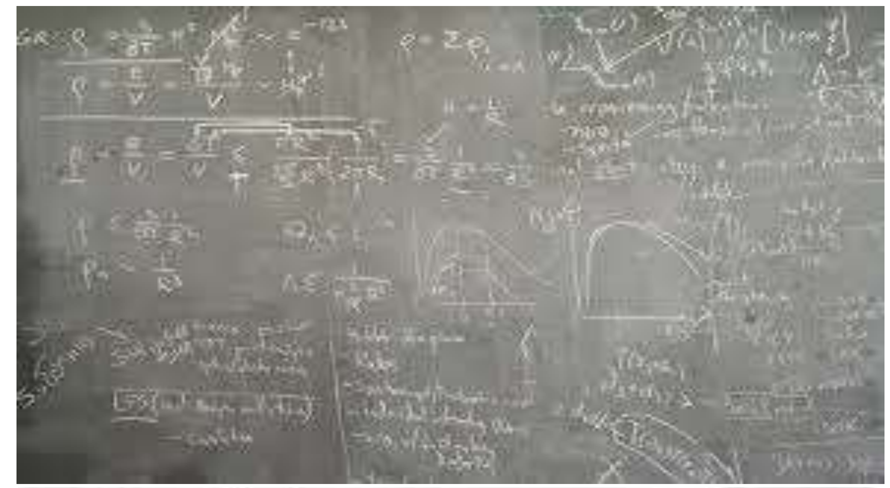
Why Precision ?

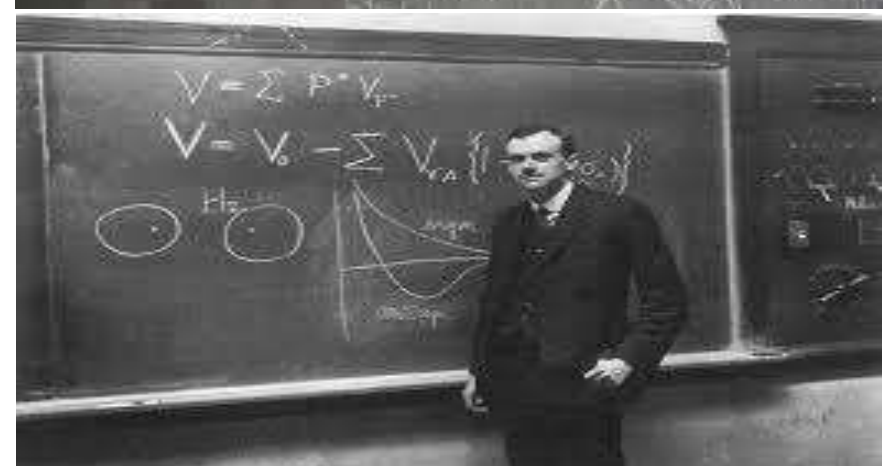
Precision
Units Ltd

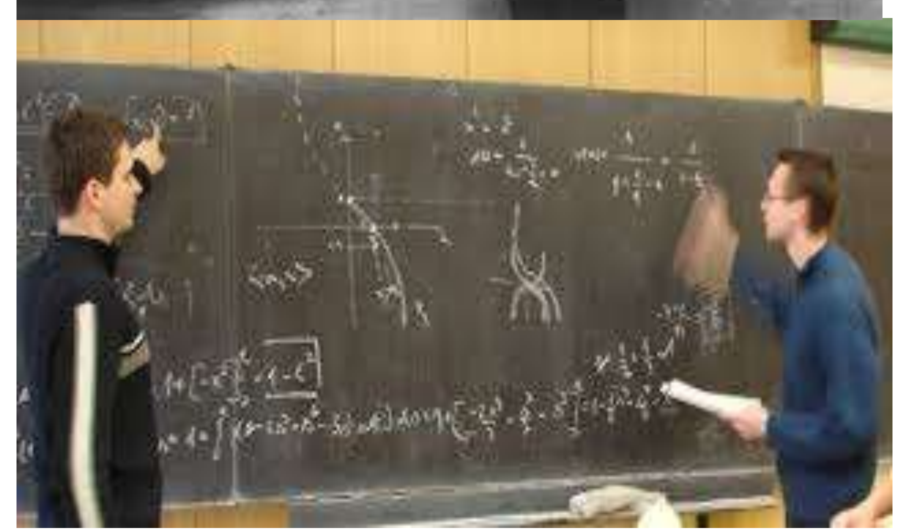
The truth is in the Details

Precision Physics

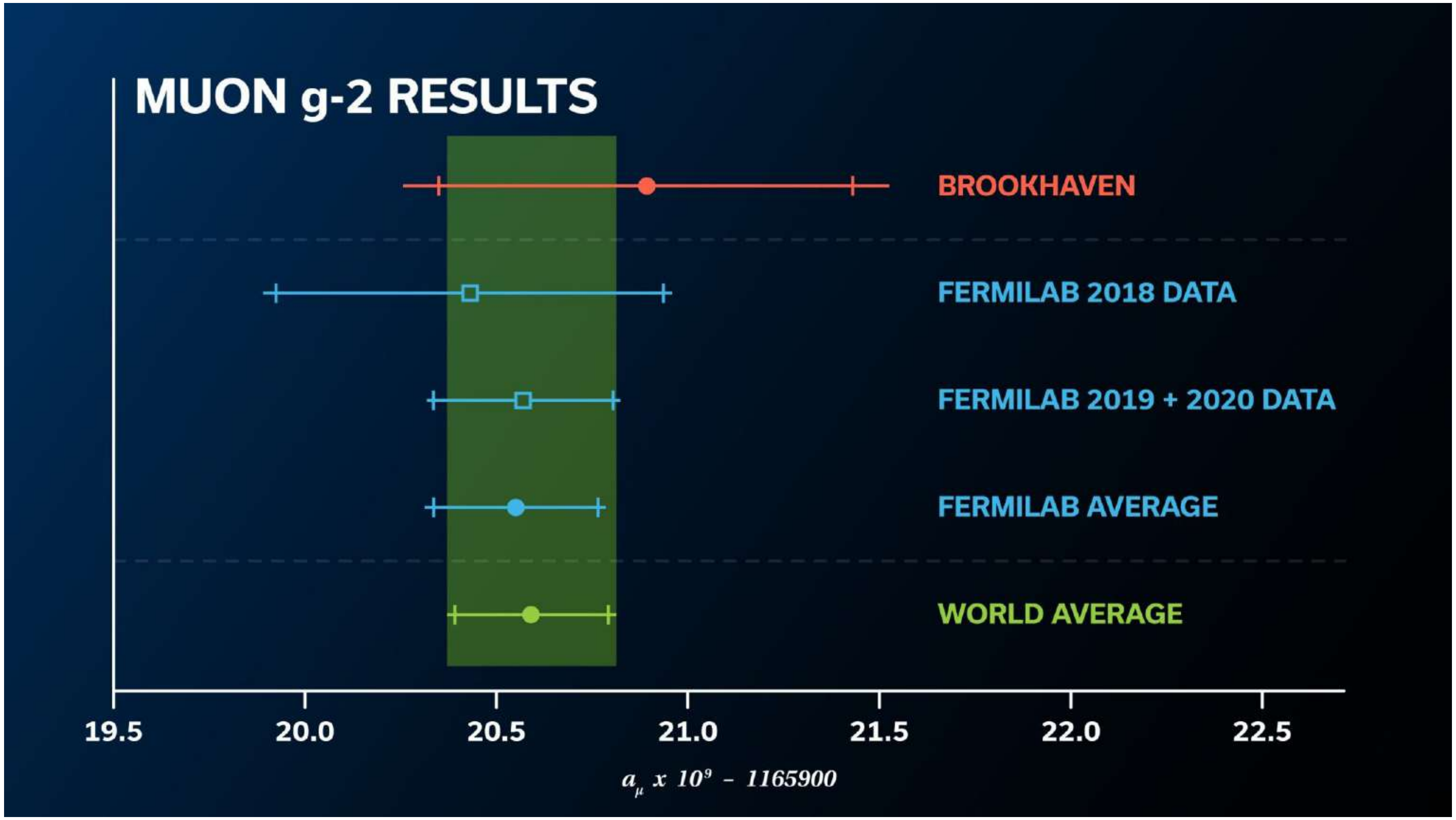
Experiment ⟷ Theory







Anomalous magnetic moment of muon



$$a_\mu = 116\ 592\ 059\ (22) \times 10^{-11} (0.19ppm)$$

$$a_\mu = \frac{g-2}{2}$$

Anomalous Magnetic moment of muon

$a_\mu^{QED} = (1/2)(\alpha/\pi)$ Schwinger 1948

$+\ 0.765857426\ (16)\ (\alpha/\pi)^2$

Sommerfield; Petermann; Suura&Wichmann '57; Elend '66; MP '04

$+\ 24.05050988\ (28)\ (\alpha/\pi)^3$

Remiddi, Barbieri, Laporta … ; Czarnecki, Skrzypek '99; MP '04;
Friot, Greynat & de Rafael '05, Ananthanarayan, Friot, Ghosh 2020

$+\ 130.8780\ (60)\ (\alpha/\pi)^4$

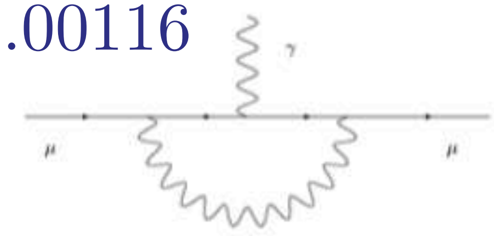
Kinoshita & Lindquist '81, … , Kinoshita & Nio '04, '05;
Aoyama, Hayakawa,Kinoshita & Nio, 2007, Kinoshita et al. 2012 & 2015;
Steinhauser et al. 2013, 2015 & 2016 (all electron & τ loops, analytic);
Laporta, PLB 2017 (mass independent term) COMPLETED²!

$+\ 750.86\ (88)\ (\alpha/\pi)^5$ COMPLETED!

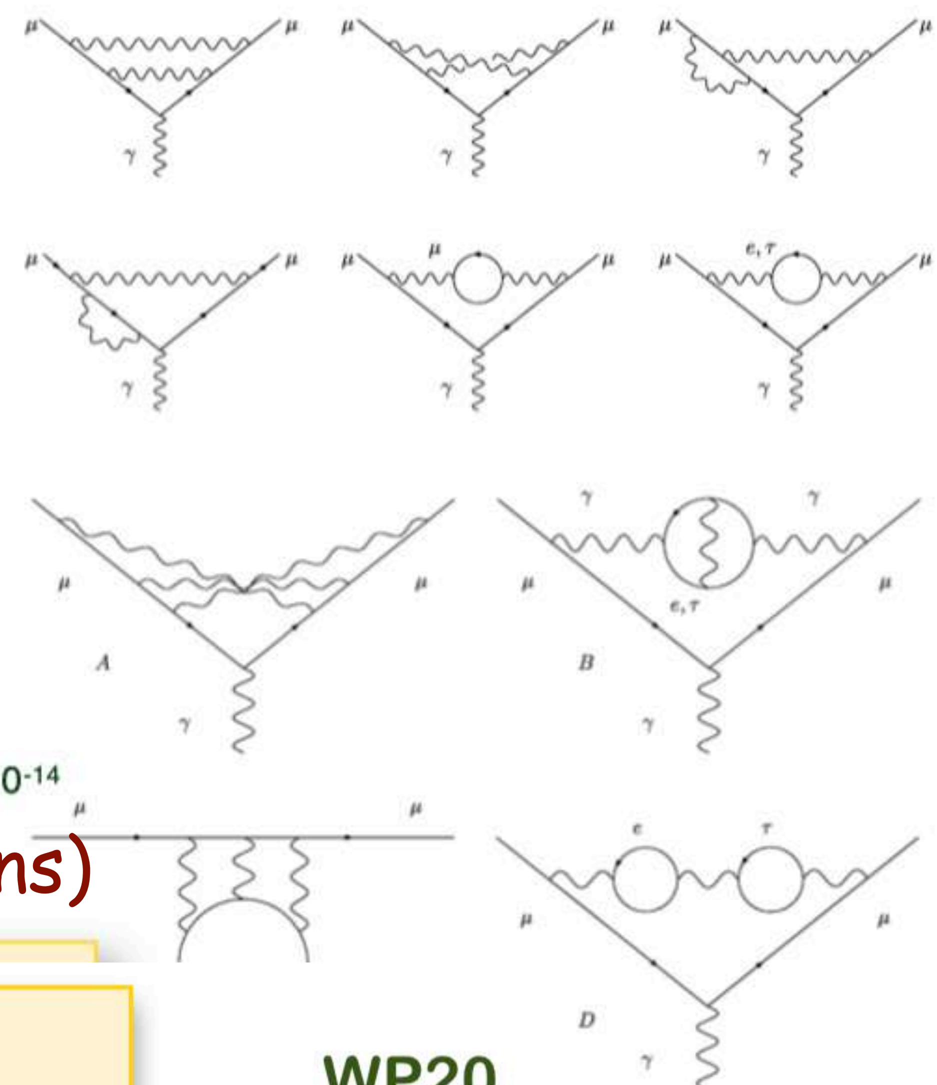
Kinoshita et al. '90, Yelkhovsky, Milstein, Starshenko, Laporta,…
Aoyama, Hayakawa, Kinoshita, Nio 2012, 2015, 2017 & 2019.
Volkov 1909.08015: $A_1^{(10)}$[no lept loops] at variance, but negligible $\delta a_\mu \sim 6\times10^{-14}$

Adding up, we get:(hadronic contributions)

$a_\mu = 0.00116$



talk by Passera



$a_\mu^{SM} = 116591810\ (43) \times 10^{-11}$

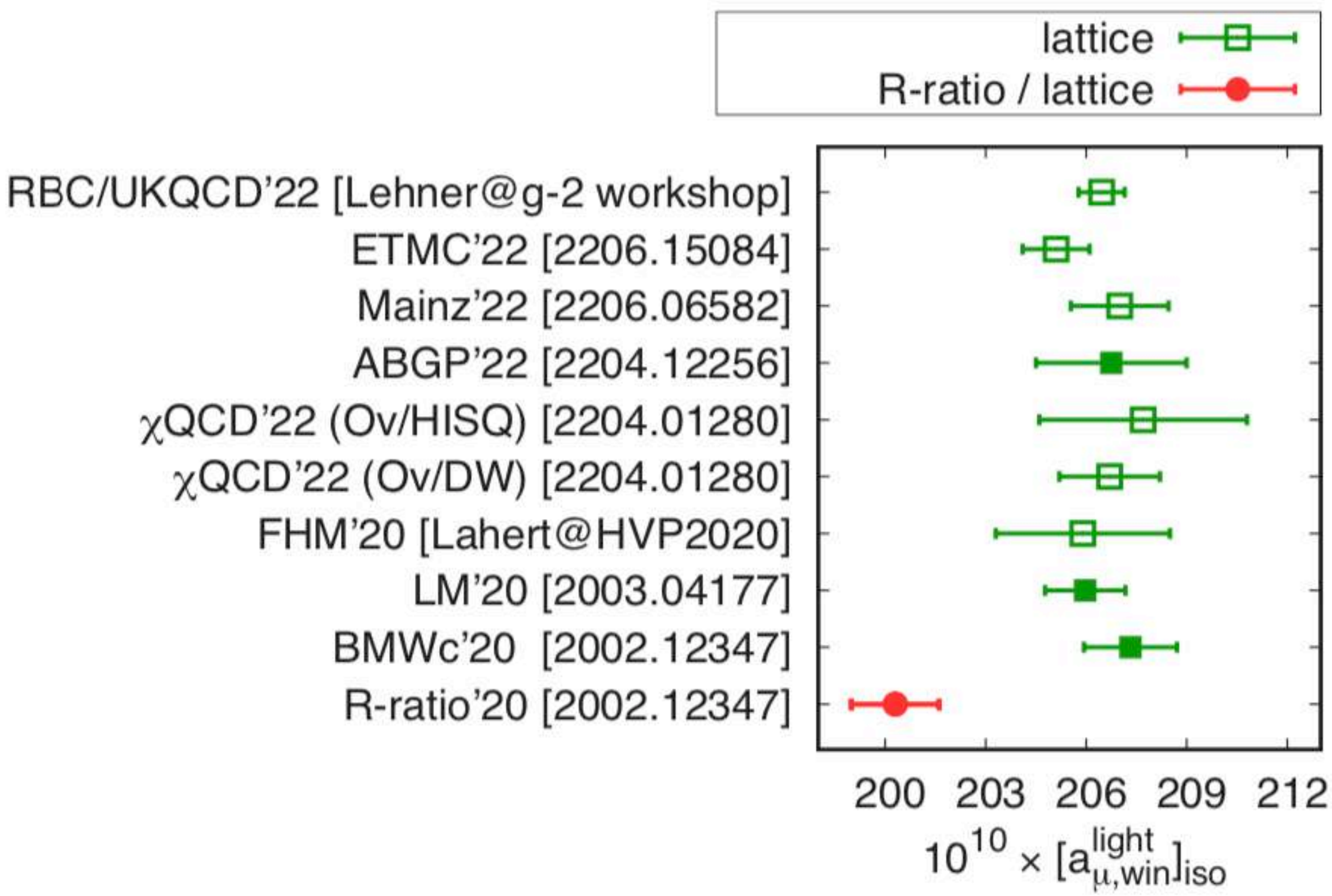
WP20

$$a_\mu = 116\ 592\ 059\ (22) \times 10^{-11} (0.19 ppm)$$

The discrepancy is more than 4 sigma

Hadronic contribution from Lattice

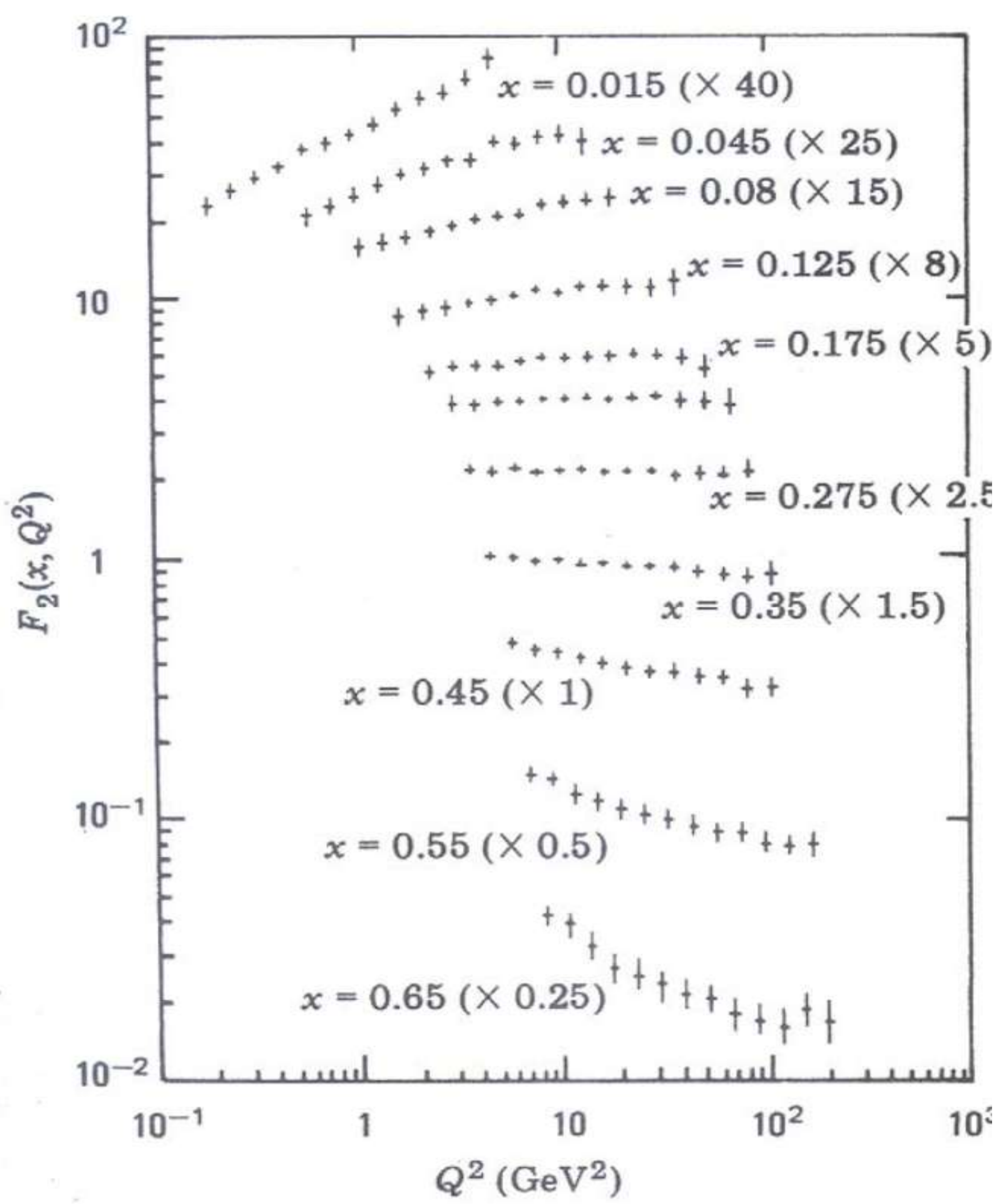
- Status now:



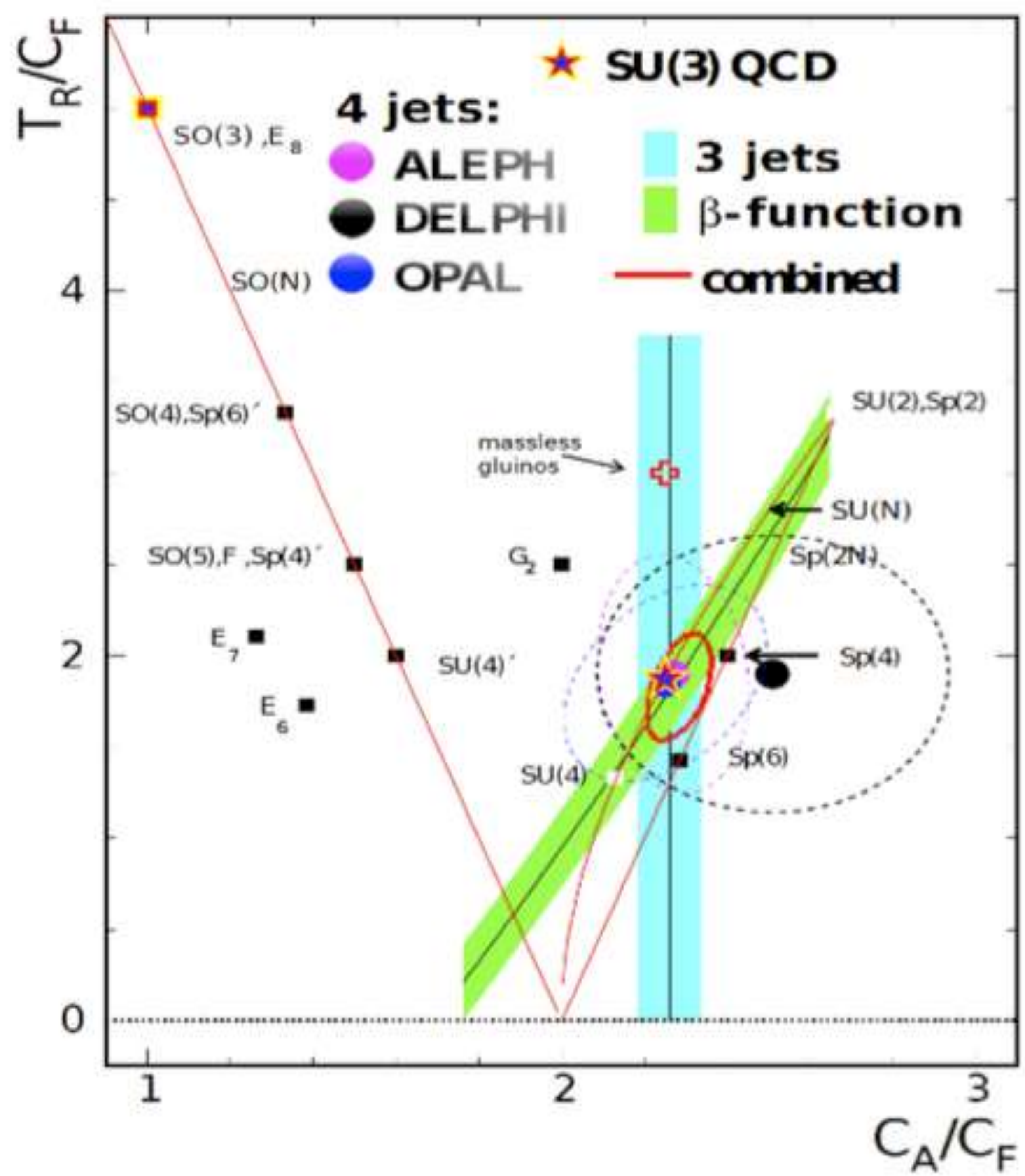
With lattice results, the discrepancy is less than 2 sigma

Tests of Quantum Chromodynamics

QCD RGE prediction for DIS



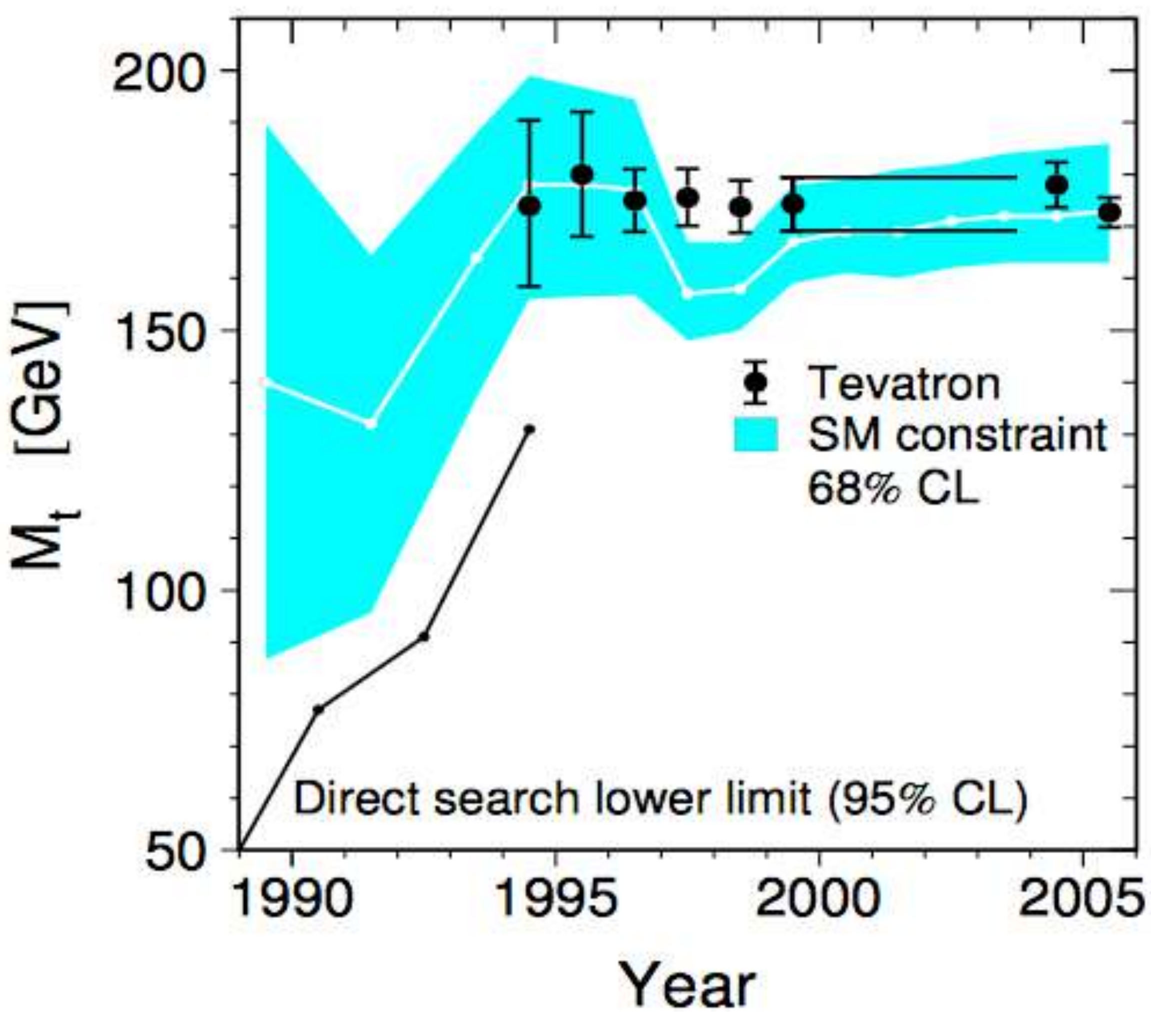
QCD Jets at LEP

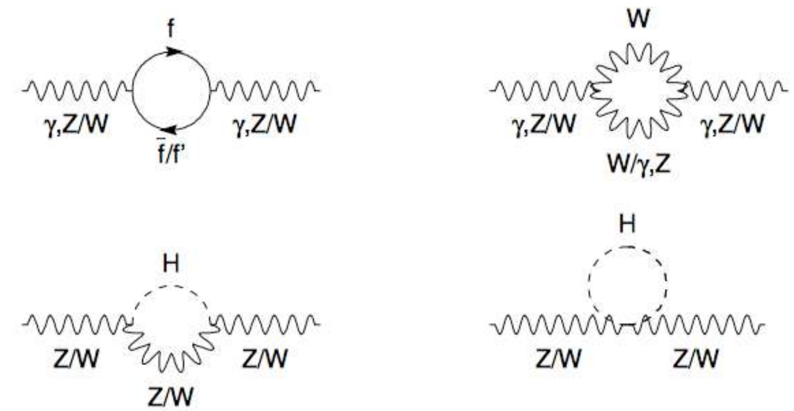


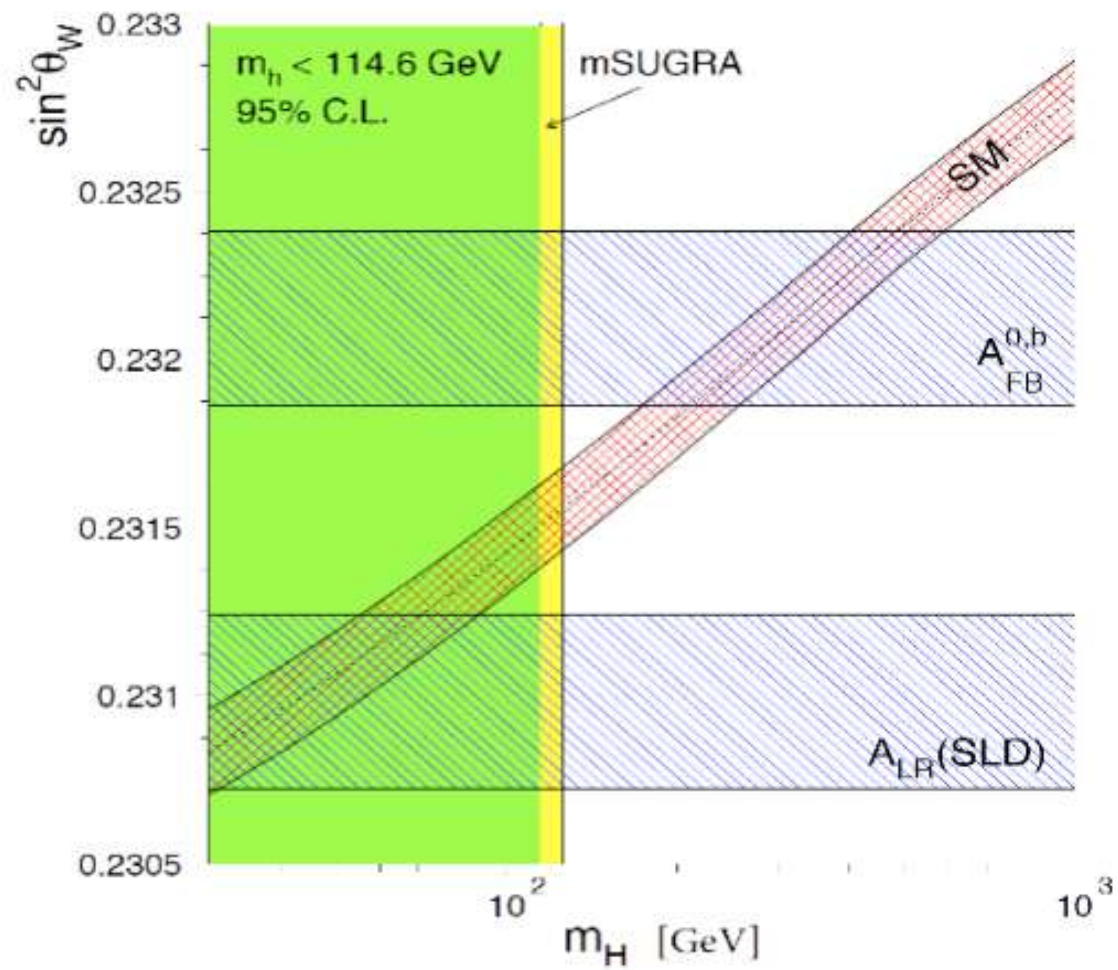
Hits from LEP for Top and Higgs

EW Radiative Corrections

$M_Z, M_H, m_t, \alpha_s(M_Z), \alpha(M_Z)$



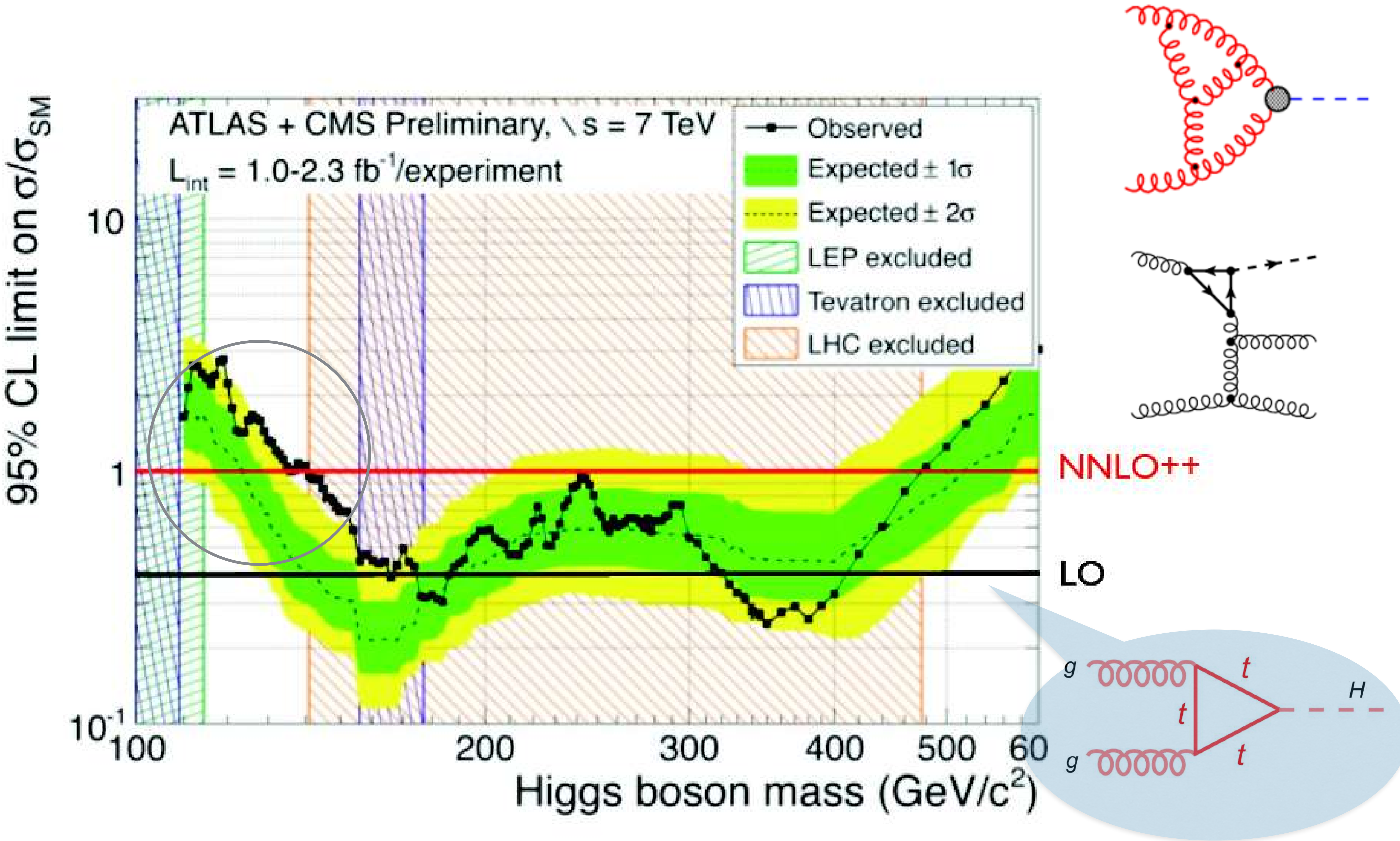




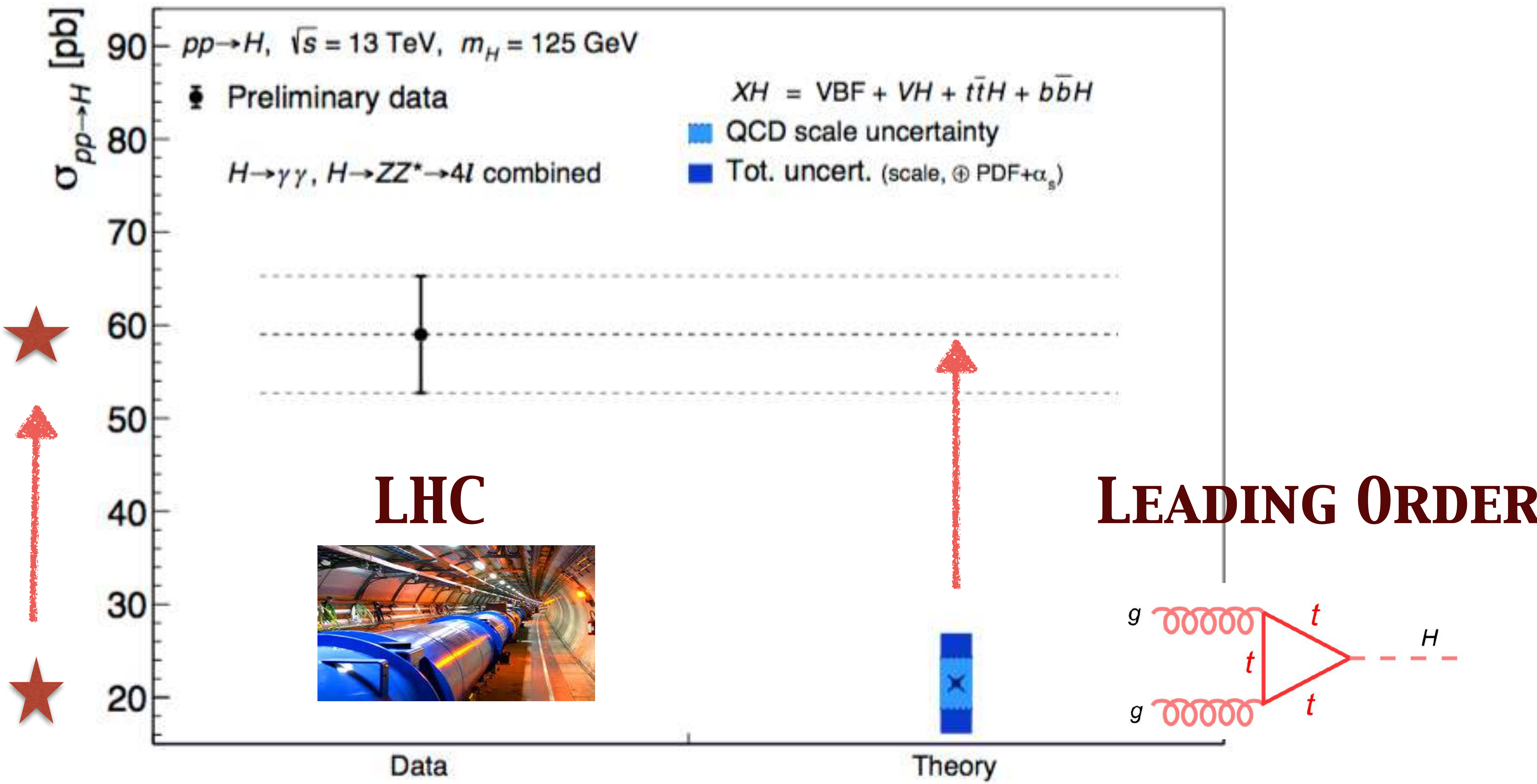
$$m_t = 178.5 \pm 3.9\ GeV$$

$$m_H = 129^{+74}_{-49}\ GeV$$

Exclusion Plot for Higgs mass



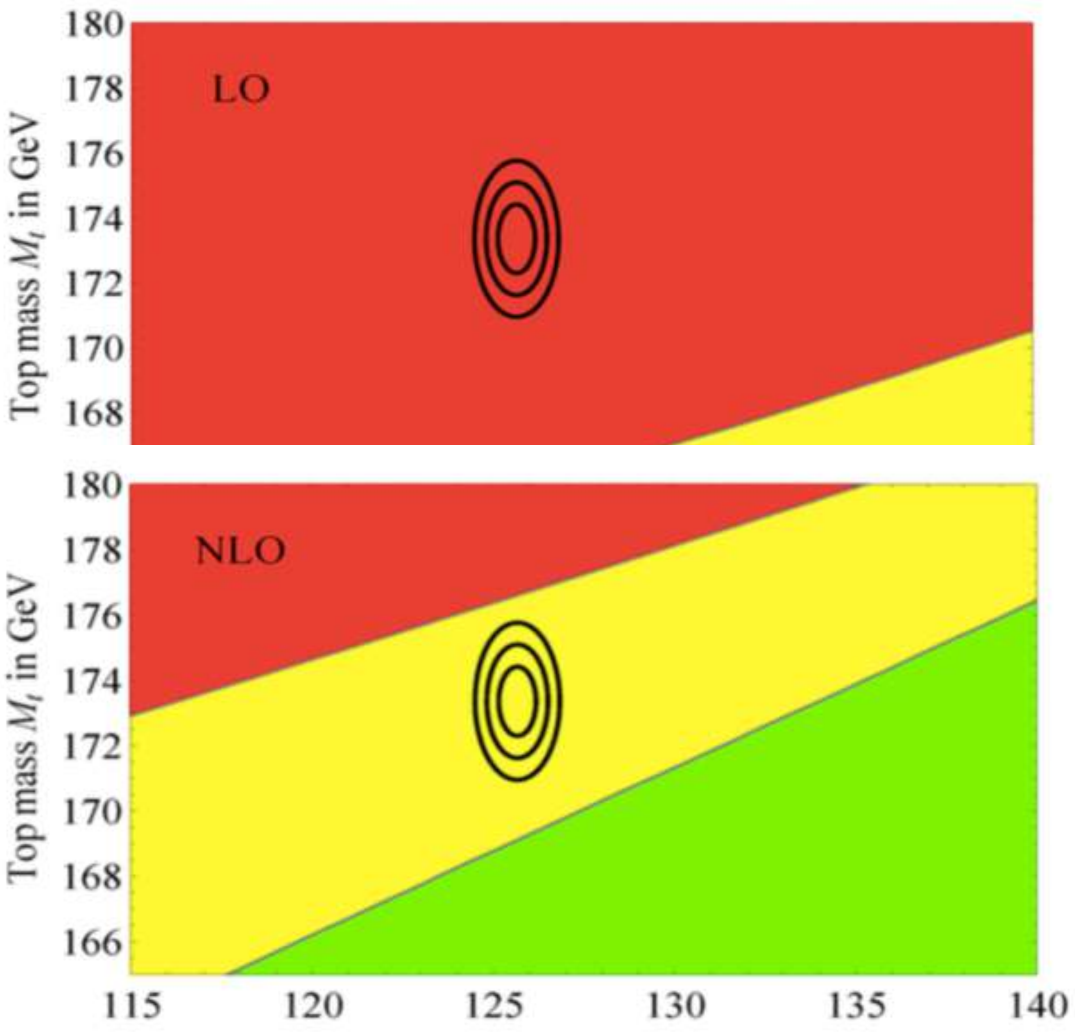
Leading order is often Crude in QCD

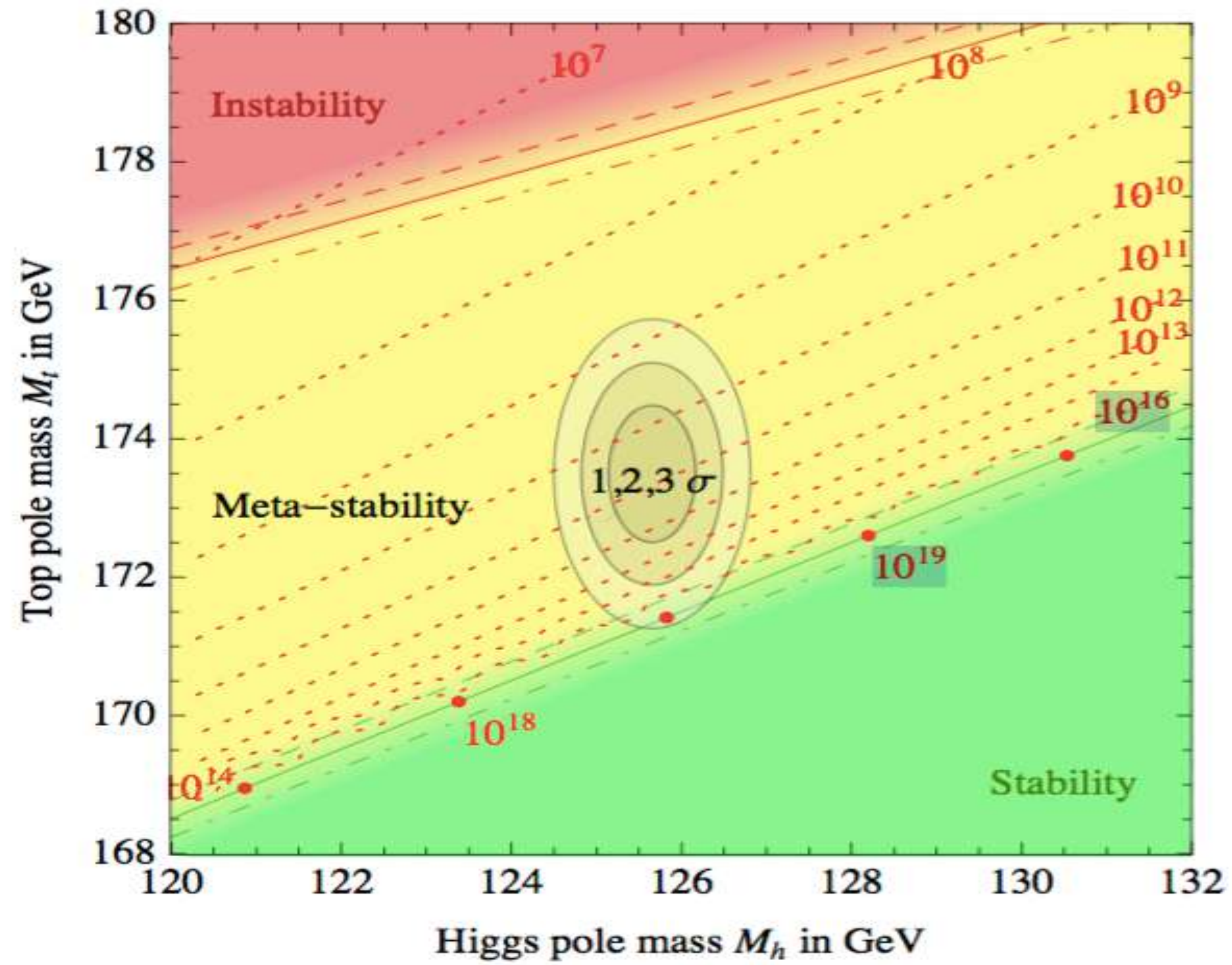


Stability of our Vacuum

NNLO Electroweak Correction :

$$M_h[GeV] > 129.6 + 2.0\,[M_t(GeV) - 173.35] - 0.5\left[\frac{\alpha_s(M_z) - 0.1184}{0.0007}\right] \pm 0.3\,.$$





Fate of the universe depends on the mass of top

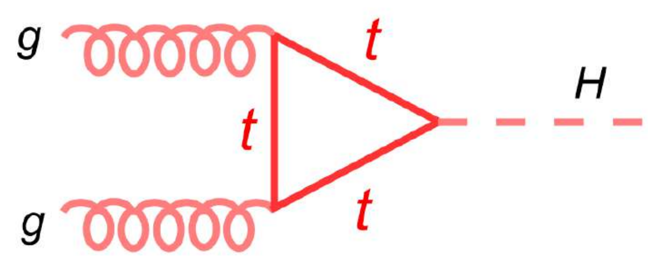
Parton Model in QCD

Hadronic Cross section:

$$\sigma^A(\tau, Q^2) = \tau \sum_{ab} \int_\tau^1 \frac{dx_1}{x_1} \int_{\frac{\tau}{x_1}}^1 \frac{dx_2}{x_2} f_a(x_1, \mu_F^2) f_b(x_2, \mu_F^2) \frac{1}{z} \hat{\sigma}_{ab} \left(\frac{\tau}{x_1 x_2}, \frac{\mu_F^2}{Q^2} \right) + \mathcal{O}\left(\frac{1}{Q^2}\right)$$

Partonic distribution functions (non-perturbative)

Partonic cross section (perturbative)



Unphysical scales!

μ_F - Factorisation Scale

μ_R - Renormalisation Scale

$$\hat{\sigma}_{ab}\left(\frac{\mu_F^2}{Q^2}\right) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2) \hat{\sigma}_{ab}^{(i)} \left(\frac{\mu_F^2}{Q^2}, \frac{\mu_R^2}{Q^2}\right)$$

$$a_s = \frac{g_s^2}{16\pi^2}$$

Renormalisation Group μ_F

RG invariance:

$$\mu_F^2 \frac{d}{d\mu_F^2} \left[\sigma^A(\tau, Q^2))\right] = 0$$

Altarelli-Parisi Evolution Equation:

$$\mu_F^2 \frac{d}{d\mu_F^2} f_a(\tau, \mu_F^2) = \frac{1}{2} P_{aa'}(\tau, \mu_F^2) \otimes f_{a'}(\tau, \mu_F^2)$$

Prediction: RGE for the Partonic Cross section:

$$\mu_F^2 \frac{d}{d\mu_F^2} \left[\hat{\sigma}_{ab}\left(\frac{\mu_F^2}{Q^2}\right)\right] = S\left(\frac{\mu_F^2}{Q^2}\right)$$

All the logarithms $\log\left(\frac{\mu_F^2}{Q^2}\right)$ predictable from previous order!

Renormalisation Group Eqn. μ_R

Partonic cross section:
(perturbative)

$$\hat{\sigma}_{ab}\left(\frac{\mu_F^2}{Q^2}\right) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2)\hat{\sigma}_{ab}^{(i)}\left(\frac{\mu_F^2}{Q^2}, \frac{\mu_R^2}{Q^2}\right)$$

RG invariance:

$$\mu_R^2 \frac{d}{d\mu_R^2}\left[\sigma_{ab}^{(i)}\left(\frac{\mu_F^2}{Q^2}\right)\right] = 0$$

RGE for coupling constant

$$\begin{aligned} \mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) &= \beta\left(a_s(\mu_R^2)\right) \\ &= -\beta_0\, a_s^2(\mu_R^2) - \beta_1\, a_s^3(\mu_R^2) - \beta_2\, a_s^4(\mu_R^2) - \cdots \end{aligned}$$

Prediction: RGE for $\sigma_{ab}^{(i)}$

$$\mu_R^2 \frac{d}{d\mu_R^2}\left[\hat{\sigma}_{ab}^{(i)}\left(\frac{\mu_F^2}{Q^2}, \frac{\mu_R^2}{Q^2}\right)\right] = R^{(i)}\left(\frac{\mu_F^2}{Q^2}, \frac{\mu_R^2}{Q^2}\right)$$

All the logarithms $\log\left(\frac{\mu_R^2}{Q^2}\right)$ predictable from previous order!

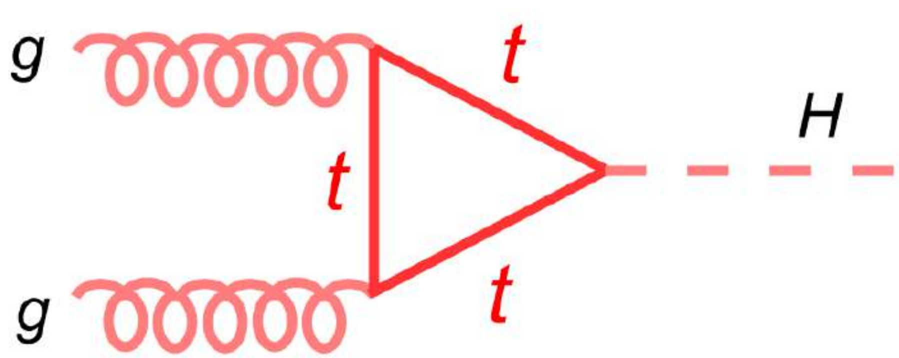
LO is often a crude estimate

$$2S\sigma^H(x, m_H) = \int_x^1 \frac{dz}{z} \Phi_{gg}^{(0)}(z, \mu_F) 2\hat{s}\hat{\sigma}_{gg}^{(0)} \left(\frac{x}{z}, m_H^2, \mu_R \right) + \cdots$$

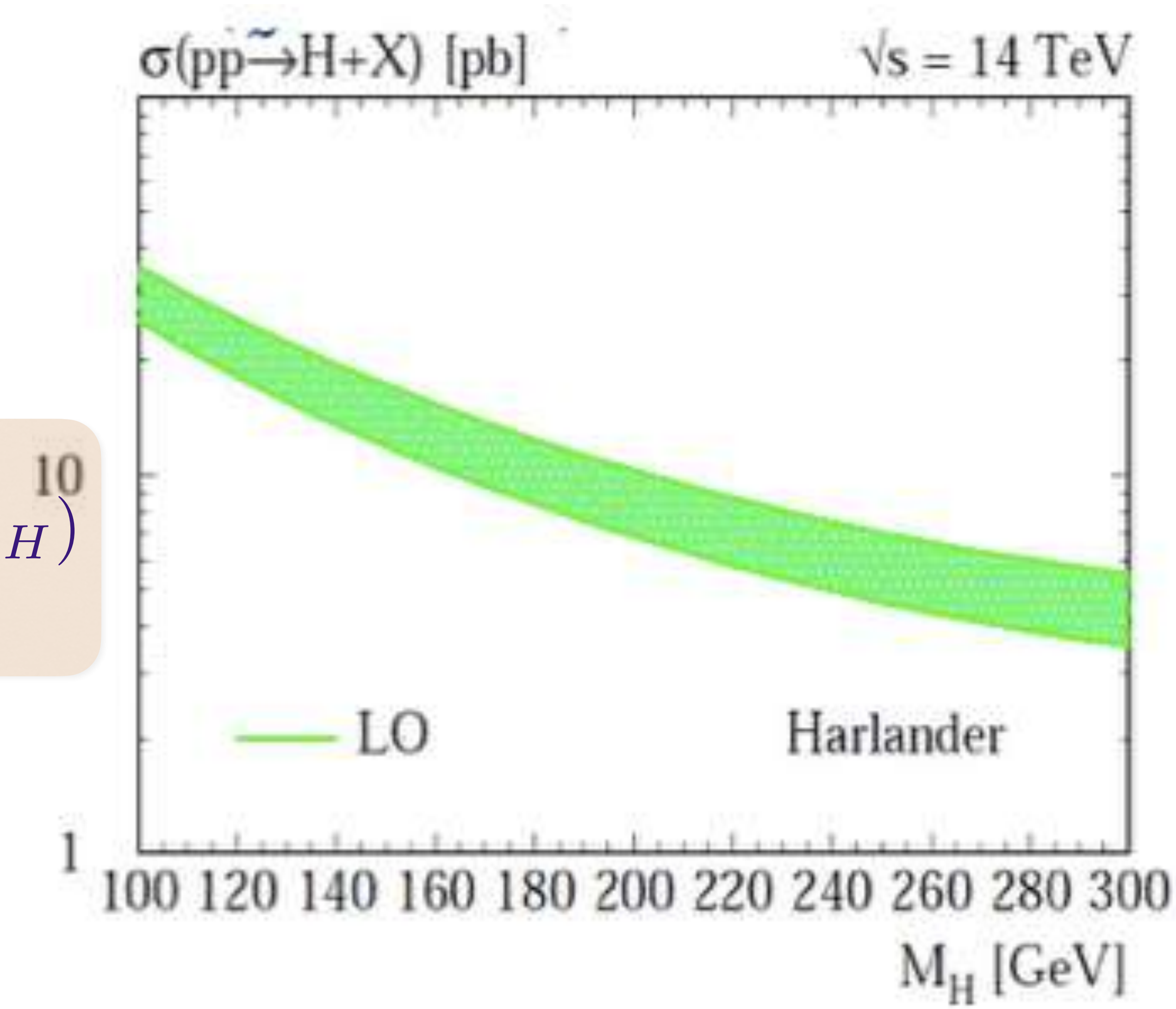
Partonic Flux:

$$\Phi_{ab}(y, \mu_F^2) = \int_y^1 \frac{dx}{x} f_a(x, \mu_F^2) f_b\left(\frac{y}{x}, \mu_F^2\right)$$

LO cross section



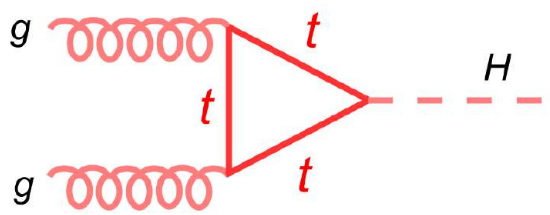
$$2\hat{s}\hat{\sigma}_{gg}^{(0)} \left(\frac{x}{z}, m_H^2, \mu_R \right) = \alpha_s^2(\mu_R) G_F F(m_t, m_H)$$

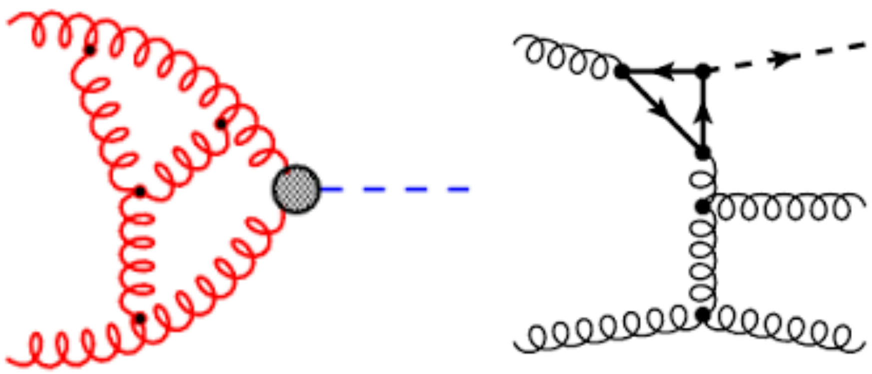


Most Unreliable Result!

True Result for Higgs

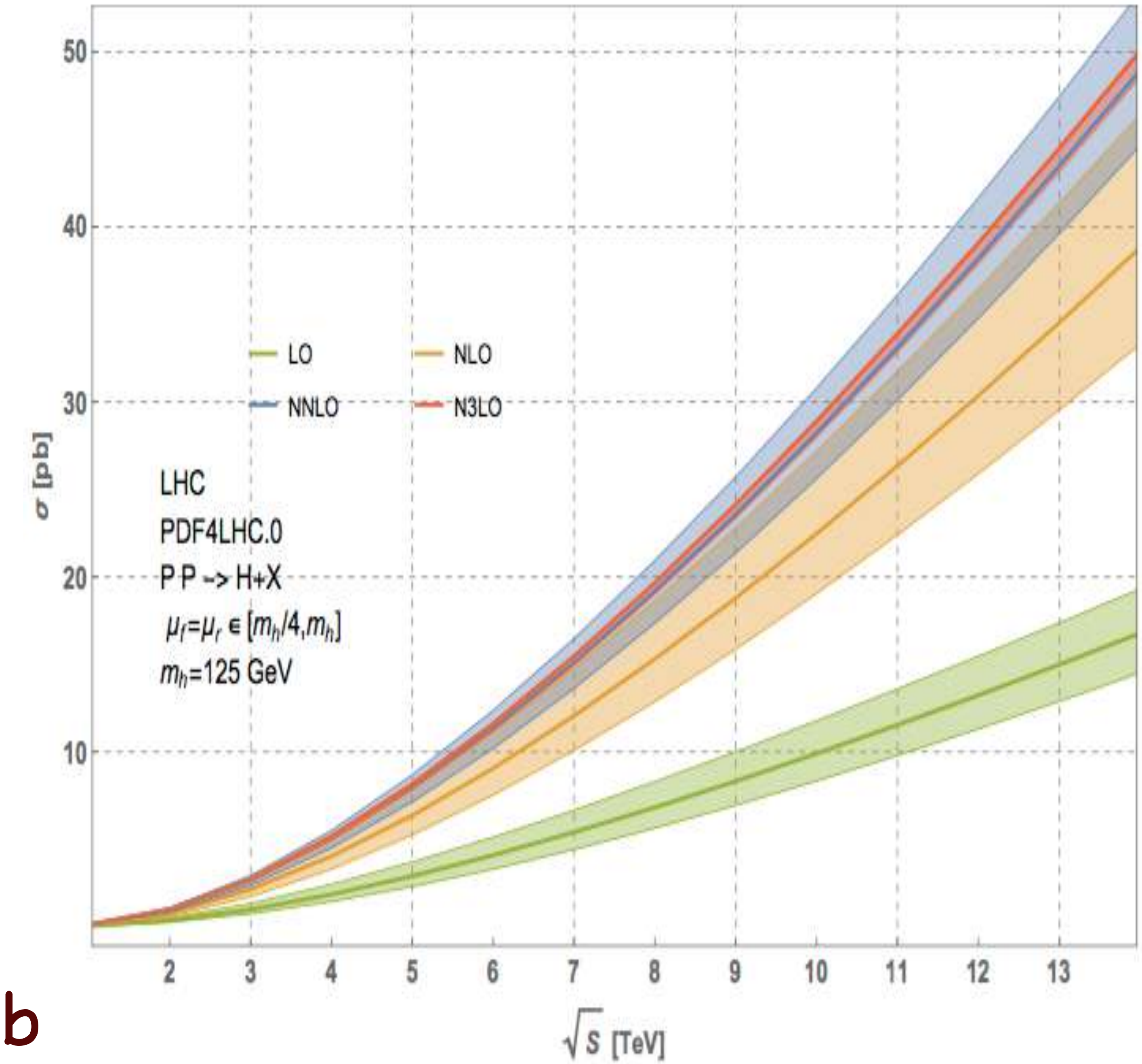
$$\hat{\sigma}(z) = \hat{\sigma}^{LO}(z) + \alpha_S \hat{\sigma}^{NLO}(z) + \alpha_S^2 \hat{\sigma}^{NNLO}(z) + \alpha_S^3 \hat{\sigma}^{N3LO}(z) + \mathcal{O}(\alpha_S^4)$$





LO	$15.05 \pm 14.8\%$
NLO	$38.2 \pm 16.6\%$
NNLO	$45.1 \pm 8.8\%$
N3LO	$45.2 \pm 1.9\%$

pb



Anastasiou, Duhr, Dulat and Mistleberger

Inclusive Prediction and The errors

Anastasiou, Duhr, Dulat and Mistleberger

Precision Prediction

$$\sigma_{PP\to H+X}(\mu_R,\mu_F) = \tau \sum_{i,j} \int_\tau^1 \frac{dz}{z} \int_{\frac{\tau}{z}}^1 \frac{dx_1}{x_1} f_i(x_1,\mu_F) f_j\left(\frac{\tau}{x_1 z},\mu_F\right) \frac{1}{z}\widehat{\sigma}_{ij}(\mu_F,\mu_R).$$

$$\begin{aligned}\hat{\sigma}_{ij} &= \mathrm{R_{LO}}C^2\left[\sigma_{ij}^{\text{LO, EFT}} + \sigma_{ij}^{\text{NLO, EFT}} + \sigma_{ij}^{\text{NNLO, EFT}} + \sigma_{ij}^{\text{N}^3\text{LO, EFT}}\right] \\ &+ \delta\sigma_{ij}^{\text{LO, (t,b,c)}} + \delta\sigma_{ij}^{\text{NLO, (t,b,c)}} + \delta\sigma_{ij}^{\text{NNLO, (t)}} + \mathrm{R_{LO}}C^2\,\delta\sigma_{ij}^{\text{Res}}\,.\end{aligned}$$

Theoretical Errors

$$\delta(\text{scale}) = \begin{matrix}\sigma^{\max}_{PP\to H+X} - \sigma_{PP\to H+X}\left(\frac{m_h}{2},\frac{m_h}{2}\right) \\ \sigma_{PP\to H+X}\left(\frac{m_h}{2},\frac{m_h}{2}\right) - \sigma^{\min}_{PP\to H+X}\end{matrix}\,.$$

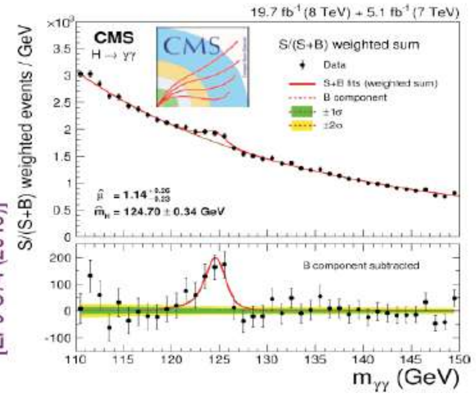
$$\begin{aligned}\sigma^{\max}_{PP\to H+X} &= \max_{\mu\in[m_h/4,m_h]} \sigma_{PP\to H+X}(\mu,\mu)\,. \\ \sigma^{\min}_{PP\to H+X} &= \min_{\mu\in[m_h/4,m_h]} \sigma_{PP\to H+X}(\mu,\mu)\,.\end{aligned}$$

$$\delta(\text{PDF-TH}) = \pm\frac{1}{2}\left|\sigma^{\text{(2), EFT, NNLO}}_{PP\to H+X} - \sigma^{\text{(2), EFT, NLO}}_{PP\to H+X}\right|\,.$$

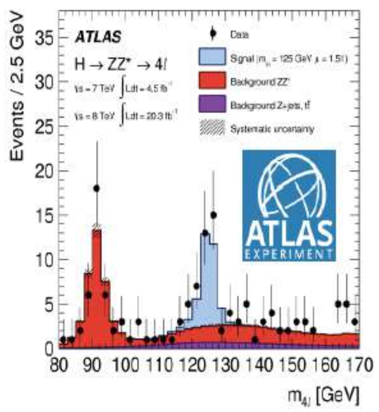
$$\delta(t,b,c)^{\overline{\text{MS}}} = \pm\left|\frac{\delta\sigma^{t,\text{ NLO}} - \delta\sigma^{t,b,c,\text{ NLO}}}{\delta\sigma^{t,\text{ NLO}}}\right| \times \left(\mathrm{R_{LO}}\delta\sigma^{EFT,\text{ NNLO}} + \delta\sigma^{1/m_t^2,\text{ NNLO}}\right)\,.$$

$$\delta(\alpha_S) = \frac{1}{2}\left|\sigma_{PP\to H+X}(\alpha_S(m_Z)=0.1195) - \sigma_{PP\to H+X}(\alpha_S(m_Z)=0.1165)\right|\,.$$

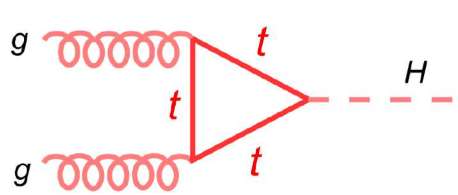
Theory Vs Experiment

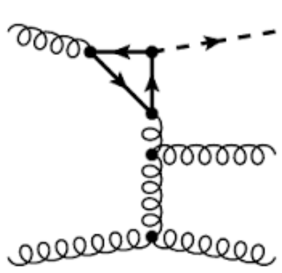


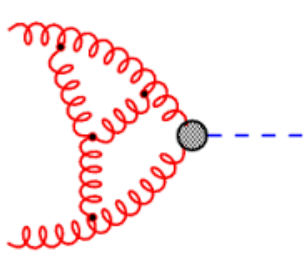
[EPJ C74 (2015)]



[PRD 91 (2015)]







Significance of excess:

$\gamma\gamma$: 5.6 σ (5.1 exp.)

ZZ: 6.6 σ (5.5 exp.)

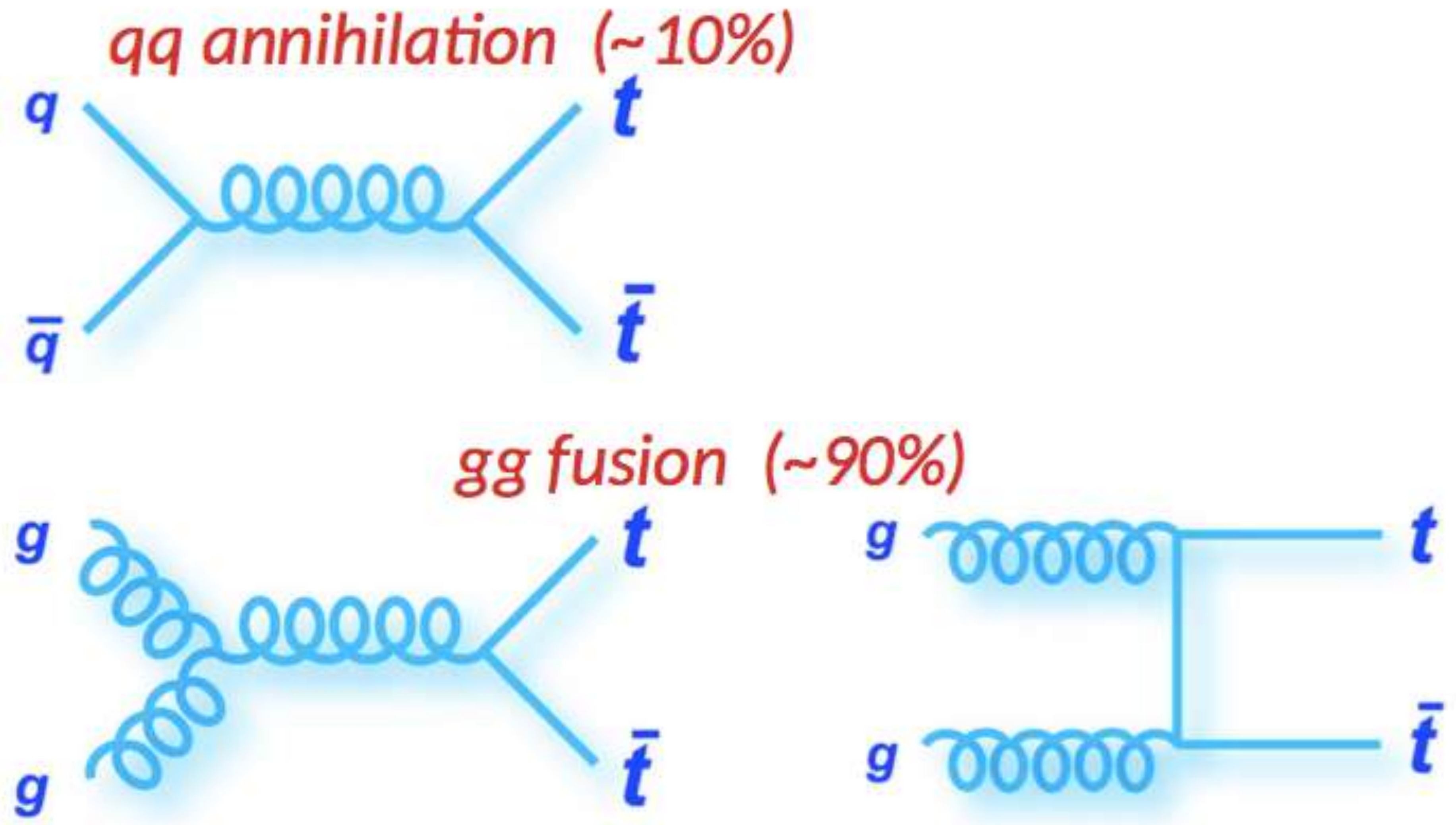
Signal strength $\mu = \sigma_{obs} / \sigma_{SM}$

$$\mu = 1.12^{+0.25}_{-0.23}$$

$$\mu = 1.51^{+0.39}_{-0.34}$$

Agreement with SM Higgs Boson

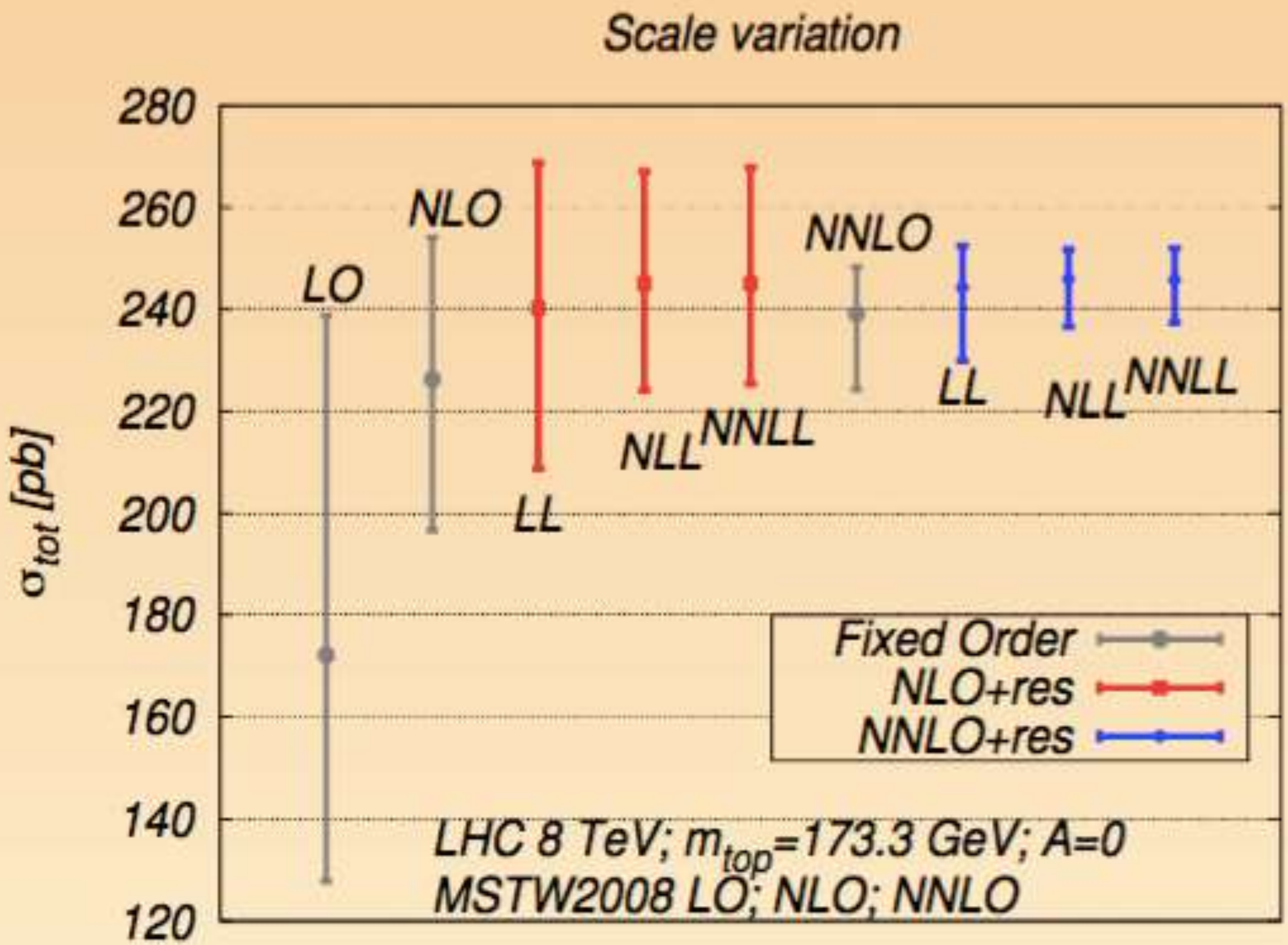
Top production



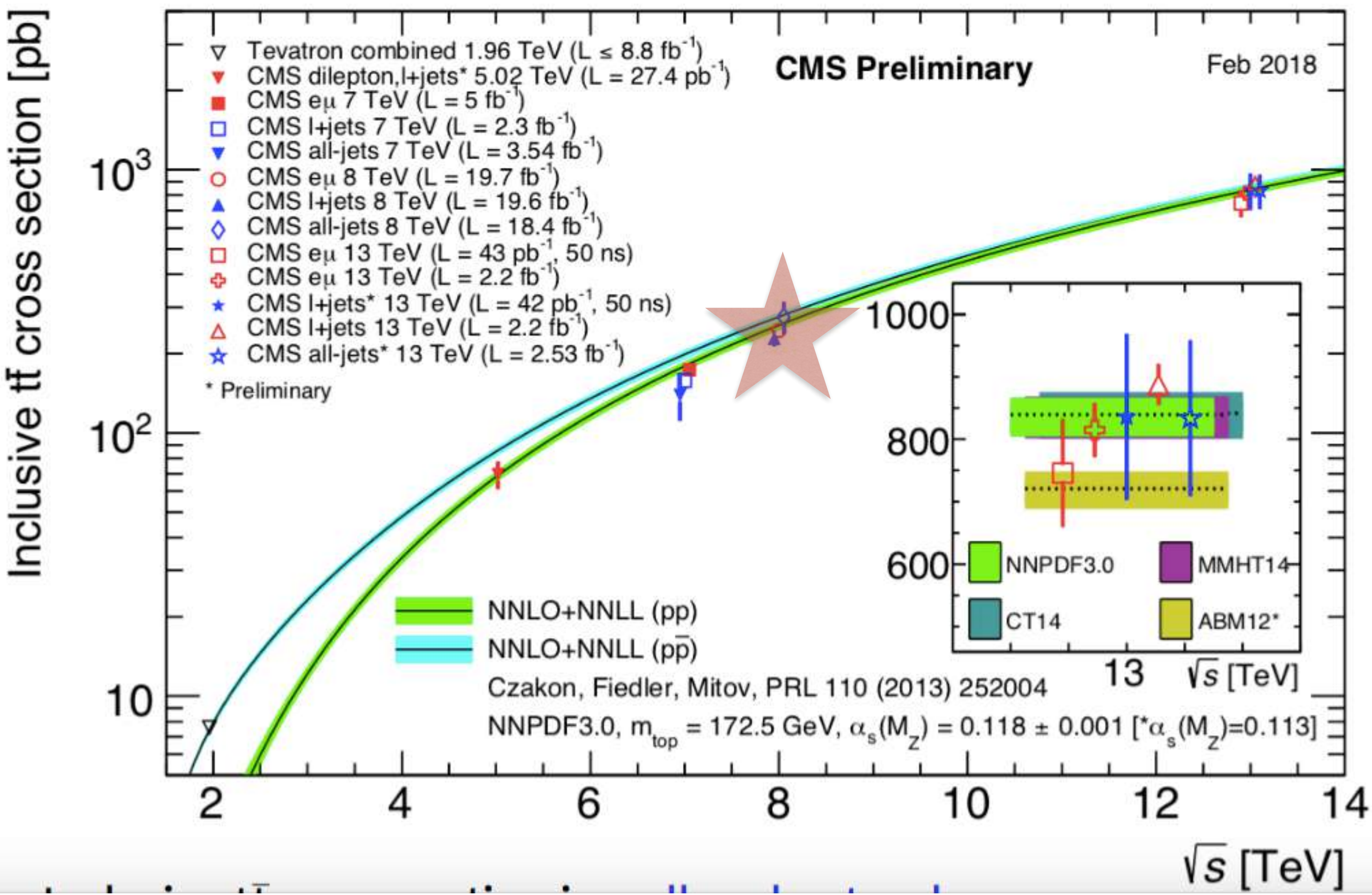
Large theory uncertainty

$\alpha_s(\mu_R^2) \quad f_g(x, \mu_F^2)$

Theory Prediction



Theory Vs Experiment



Parton Model in QCD

Hadronic Cross section:

$$\sigma^A(\tau, m_A^2) = \sigma^{A,(0)}(\mu_R^2) \sum_{a,b=q,\bar{q},g} \int_\tau^1 dy\, \Phi_{ab}(y,\mu_F^2) \Delta_{ab}^A\left(\frac{\tau}{y}, m_A^2, \mu_R^2, \mu_F^2\right)$$

Partonic cross section:

Partonic Flux:

$$\Phi_{ab}(y,\mu_F^2) = \int_y^1 \frac{dx}{x} f_a(x,\mu_F^2) f_b\left(\frac{y}{x},\mu_F^2\right),$$

Precision Measurements

Precise theory

Discover/Test Physics

Inputs that can affect

- UV Renormalisation Scale, Strong coupling

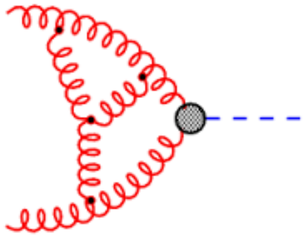
$$\alpha_s(\mu_R)$$

- Factorisation Scale and Parton Distribution Functions

$$f_a(x, \mu_F)$$

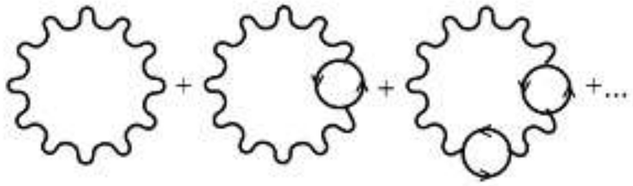
- Missing Higher Order corrections

- Stability of the perturbation theory

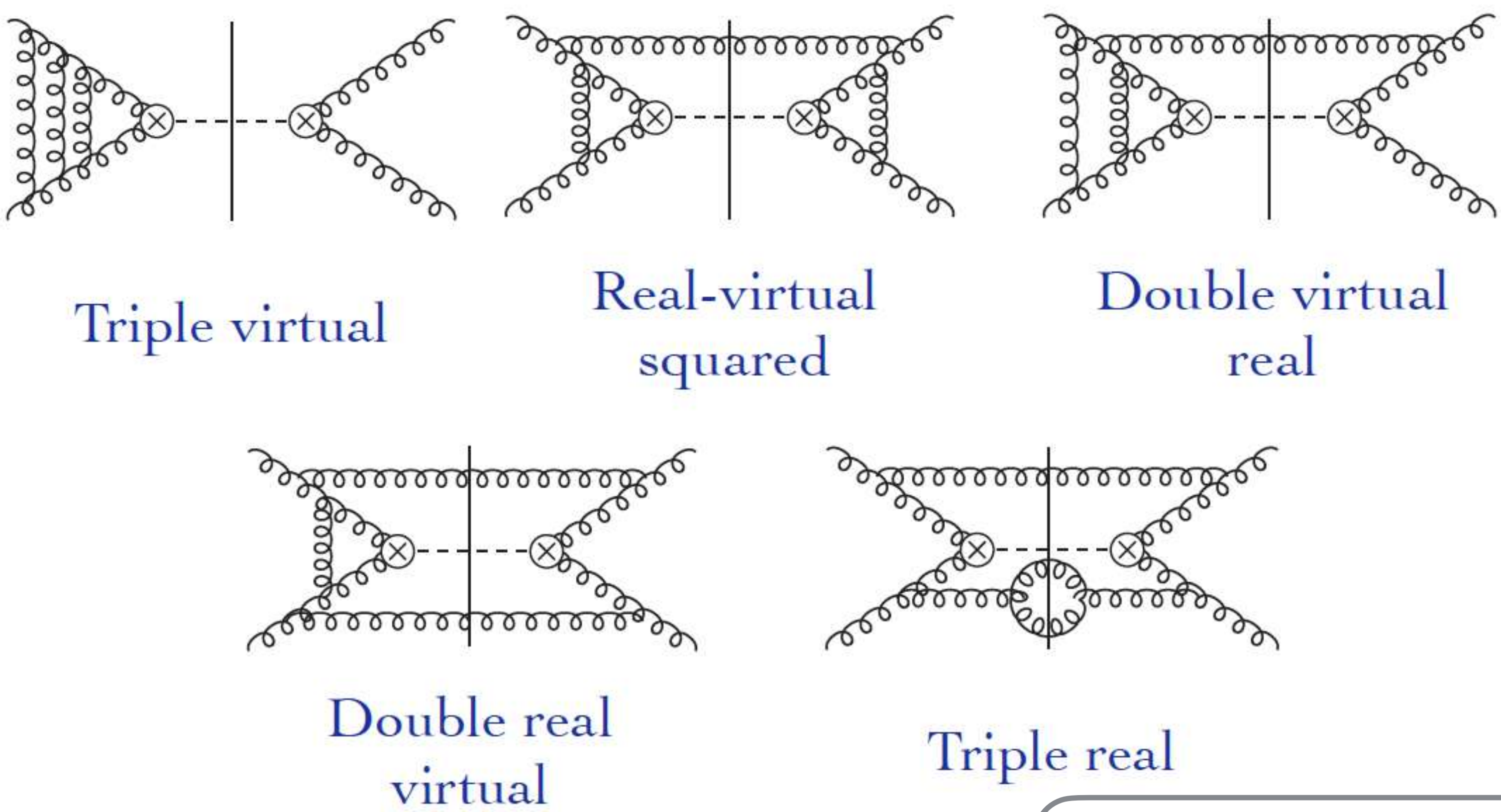


- Resummation Methods

- Hadronisation models



Higgs production at 3rd order



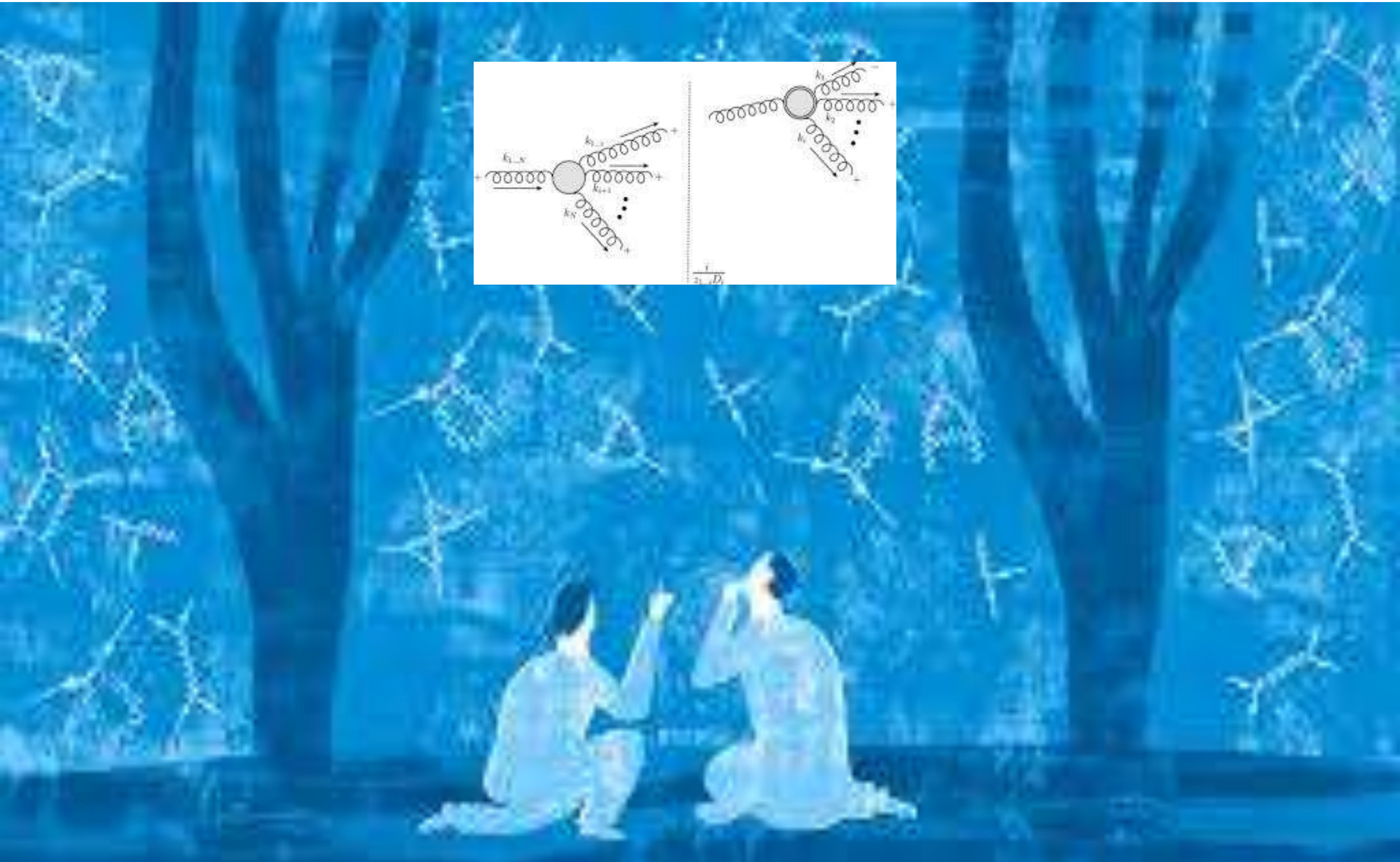
100 000 diagrams

	Integrals
NNLO	50 000
N3LO	517 531 178

$$\int \frac{d^dk_1}{(4\pi)^d} \int \frac{d^dk_2}{(4\pi)^d} \int \frac{d^dk_3}{(4\pi)^d} \frac{1}{\Pi_{i=1}^{k} D_i(k_l, p_m)}$$

$$D_i(k_l, p_m) = \Big(\sum_{n_1}^{l} k_{n_1} + \sum_{n_2}^{m} p_{n_2}\Big)^2 - m^2$$

Real Emission Processes



No. of diagrams

$$g + g \to n\ g$$

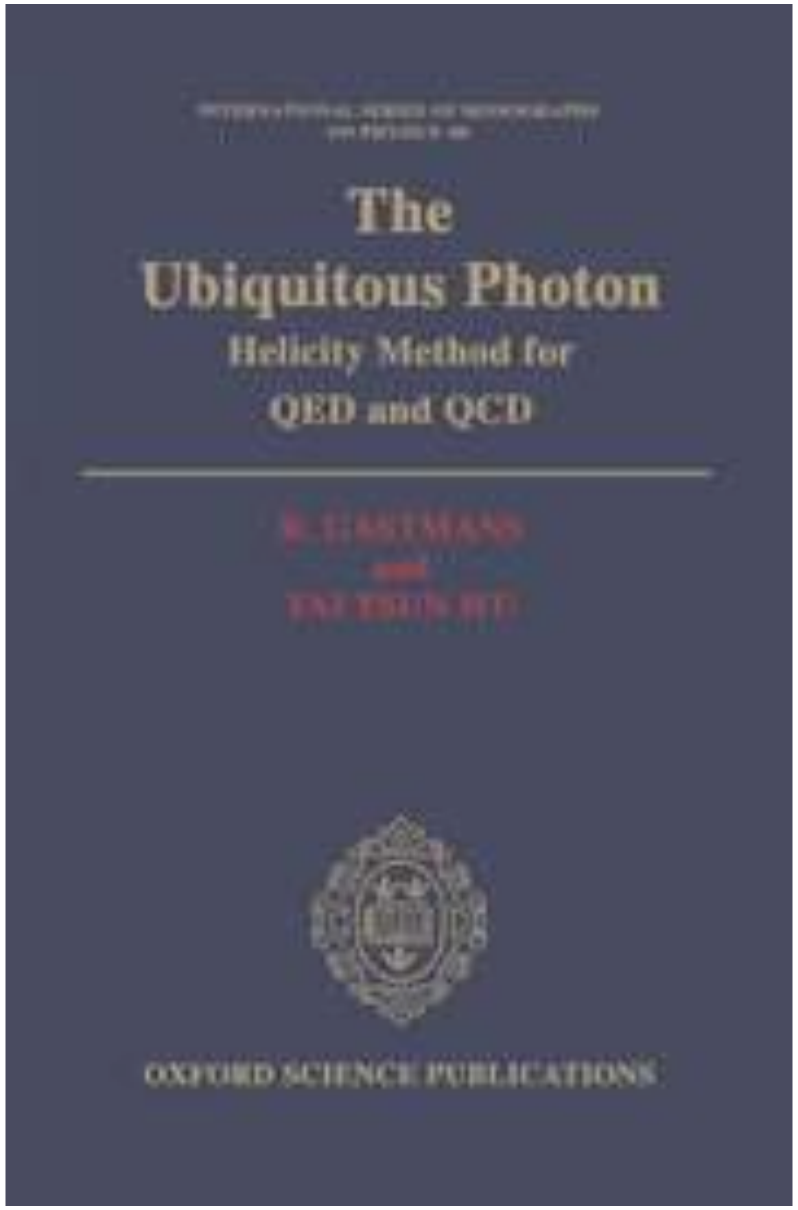
For Jet / background to BSM

$$g + g \to g + g$$
$$g + g \to g + g + g$$
$$g + g \to g + g + g + g$$

n	no. of diagrams
2	4
3	25
4	220
5	2485
6	34300
7	559405
8	10525900

Helicity Amplitudes

- Helicity Amplitude -
  Convenient way
- Along fermion line Helicity is conserved
- Clever choice of photon/gluon polarization - Gauge invariance



- Different Helicity amplitudes do not interfere

Notations

Weyl Spinors

$$u_\pm(k) = \tfrac{1}{2}(1 \pm \gamma_5)u(k)$$ Particle

$$v_\mp(k) = \tfrac{1}{2}(1 \pm \gamma_5)v(k)$$ Anti-Particle

$$|i^\pm\rangle \equiv |k_i^\pm\rangle \equiv u_\pm(k_i) = v_\mp(k_i)$$

$$\langle i^\pm| \equiv \langle k_i^\pm| \equiv \overline{u_\pm(k_i)} = \overline{v_\mp(k_i)}.$$

$$|i\rangle = |i^+\rangle$$

$$|i] = |i^-\rangle$$

Dot Products

$$\overline{u_-(k_i)}u_+(k_j) = \langle i^-|j^+\rangle = \langle i\,j\rangle$$

$$\overline{u_+(k_i)}u_-(k_j) = \langle i^+|j^-\rangle = [i\,j]$$

Properties

$$i\gamma^2 u_\pm = u_\mp^*, \qquad i\gamma^2 \,|i\rangle = |i]^*, \quad i\gamma^2 |i] = |i\rangle^*$$

- **Antisymmetric:**

$$\begin{aligned}
\langle ij\rangle &= u_-^\dagger(p_i)\gamma^0 u_+(k_j) = u_-^\dagger(p_i)\gamma^0 u_+(k_j)^T \\
&= u_+^T(p_j)\gamma^0 u_-^*(k_i) = u_-^\dagger(p_j)(i\gamma^2)^\dagger\gamma^0 i\gamma^2 u_+(k_i) \\
&= -u_-^\dagger(p_j)\gamma^2\gamma^0\gamma^2 u_+(k_i) = -u_-^\dagger(p_j)\gamma^0 u_+(k_i) \\
&= -\langle ji\rangle\,.
\end{aligned}$$

Identity

- **Gordon identity**

$$
\begin{aligned}
[i|\gamma^\mu|j\rangle &= u_+^\dagger(p_i)\gamma^0\gamma^\mu u_+(p_j) \\
&= (u_+^\dagger(p_i)\gamma^0\gamma^\mu u_+(p_j))^T \\
&= (u_-^T(p_i)(i\gamma^2)^\dagger\gamma^0\gamma^\mu(i\gamma^2)u_-^*(p_j))^T \\
&= u_-^\dagger(p_j)(-\gamma^2\gamma^0\gamma^\mu\gamma^2)^T u_-(p_i) \\
&= u^\dagger(k_j)\gamma^0\gamma^\mu u_-(k_i) \\
&= \langle j|\gamma^\mu|i].
\end{aligned}
$$

$$[i|\gamma^\mu|i\rangle = \langle i|\gamma^\mu|i] = 2p_i^\mu.$$

Useful Identities

- **Schouten Identity**

$$|i\rangle = c_1 |j\rangle + c_2 |k\rangle$$

★ Multiply

$$\langle k| \longrightarrow c_2 = \frac{\langle ji\rangle}{\langle jk\rangle}$$

$$\langle j| \longrightarrow c_1 = \frac{\langle ki\rangle}{\langle kj\rangle}$$

$$|i\rangle = \frac{\langle ki\rangle}{\langle kj\rangle} |j\rangle - \frac{\langle ji\rangle}{\langle kj\rangle} |k\rangle$$

$$|i\rangle \langle jk\rangle + |j\rangle \langle ki\rangle + |k\rangle \langle ij\rangle = 0$$

Useful Identities

- $\langle pq \rangle = -\langle qp \rangle, \quad [pq] = -[qp]$
- $\langle pp \rangle = 0 = [pp], \quad \langle pq] = 0 = [pq\rangle$
- $\langle pq \rangle = [qp]^*$
- $\langle pq \rangle [qp] = |\langle pq \rangle|^2 = |[qp]|^2 = 2\, p \cdot q = (p+q)^2 \equiv s_{pq}$
- $\langle p\gamma^\mu q] = [q\gamma^\mu p\rangle$
- $\gamma_\mu [p\gamma^\mu q\rangle = 2|p]\langle q| + |q\rangle [p|$
- Gordon Identity $[p\gamma^\mu p\rangle = 2p^\mu$
- Fierz identity $\langle p\gamma^\mu q][r\gamma_\mu s\rangle = 2\langle ps\rangle [rq]$
- Schouten identity $\langle pq\rangle\langle rs\rangle + \langle pr\rangle\langle sq\rangle + \langle ps\rangle\langle qr\rangle = 0$

Polarisation Vectors

$$\epsilon^+(p,q) = \frac{\langle q|\gamma_\mu p]}{\sqrt{2}\,\langle qp\rangle}$$

$$\epsilon^-(p,q) = -\frac{[q|\gamma_\mu|p\rangle}{\sqrt{2}[qp]}$$

$$p_\mu \epsilon^\mu = 0.$$

$$(\varepsilon_\mu^+)^* = \varepsilon_\mu^-$$

★ Gauge Choice

$$\epsilon_\mu^-(\tilde{q}) - \epsilon_\mu^-(q) = \frac{[\tilde{q}\gamma^\mu|p\rangle[qp] - [q\gamma^\mu|p\rangle[\tilde{q}p]}{\sqrt{2}[qp][\tilde{q}p]}$$

$$= \frac{\sqrt{2}[\tilde{q}q]}{[qp][\tilde{q}p]}p^\mu.$$

★ Polarisation Sum

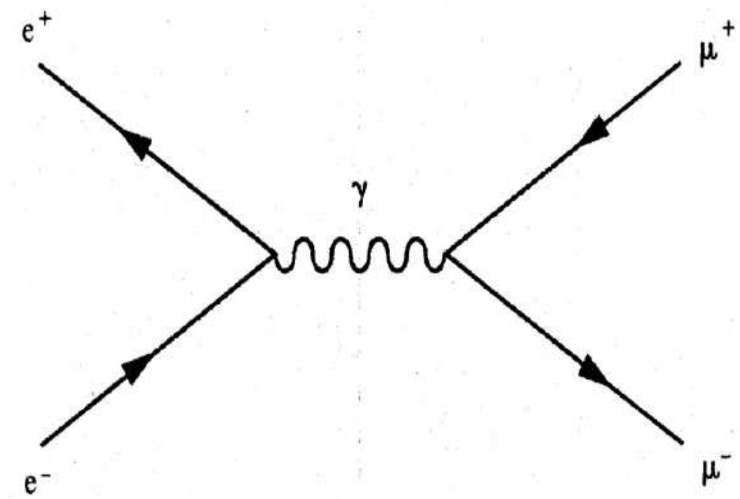
$$\sum_\lambda \epsilon_\mu^\lambda(p,q)(\epsilon_\nu^\lambda(p,q))^* = \frac{q_\mu p_\nu + q_\nu p_\mu}{p.q} - \eta_{\mu\nu}$$

$$\epsilon_i^\pm(q).q = 0$$

$$\epsilon_i^\pm(q).\epsilon_j^\pm(q) = 0$$

$$\epsilon_i^\pm(p_j).\epsilon_j^\mp(q) = 0$$

Using Helicity Amplitude Method



$$i\mathcal{M} = (-ie)^2 \, \frac{-i}{q^2} \, \overline{U}_L(3)\gamma^\mu U_L(4)\,\overline{U}_L(2)\gamma_\mu U_L(1)$$

$$= \frac{ie^2}{q^2}\langle 3\gamma^\mu 4]\langle 2\gamma_\mu 1]$$

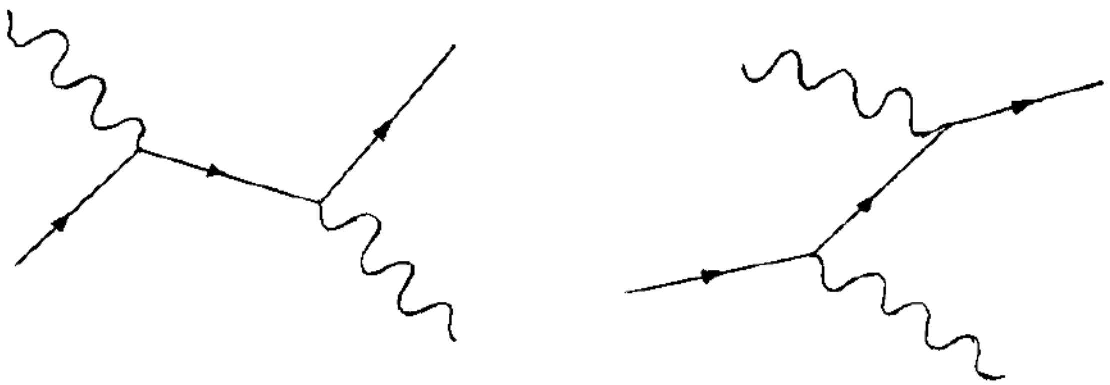
$$= \frac{2ie^2}{q^2}\langle 32\rangle[14] \ ,$$

$$i\mathcal{M} = 2ie^2 \frac{\langle 32\rangle[14]\langle 32\rangle}{\langle 12\rangle[21]\langle 32\rangle} \ .$$

$$[21]\langle 32\rangle = [12]\langle 23\rangle = [1\ \not{2}\ 3\rangle = [1(-\ \not{1} -\ \not{3} -\ \not{4})3\rangle$$

$$i\mathcal{M} = 2ie^2 \frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle} \ .$$

Compton Scattering



$$\epsilon^\mu(3) = \frac{1}{\sqrt{2}}\frac{\langle 2\gamma^\mu 3]}{\langle 23\rangle}\,, \qquad \epsilon^\mu(4) = \frac{1}{\sqrt{2}}\frac{\langle 2\gamma^\mu 4]}{\langle 24\rangle}\,.$$

$$\epsilon^\mu(3) = \frac{1}{\sqrt{2}}\frac{\langle 2\gamma^\mu 3]}{\langle 23\rangle}\,, \qquad \epsilon^\mu(4) = -\frac{1}{\sqrt{2}}\frac{[1\gamma^\mu 4\rangle}{[14]}\,.$$

$$i\mathcal{M} = (-ie)^2\langle 2\left\{\gamma\cdot\epsilon(4)\frac{i(2+4)}{s_{24}}\gamma\cdot\epsilon(3) + \gamma\cdot\epsilon(3)\frac{i(2+3)}{s_{23}}\gamma\cdot\epsilon(4)\right\}1]$$

$$\begin{aligned} i\mathcal{M} &= \frac{-ie^2}{s_{24}}\frac{2\cdot 2}{(-2)\langle 23\rangle[14]}\langle 24\rangle[1(2+4)2\rangle[31] \\ &= \frac{2ie^2}{s_{13}\langle 23\rangle[14]}\langle 24\rangle[14]\langle 42\rangle[31] \\ &= \frac{2ie^2}{\langle 13\rangle[31]\langle 23\rangle[14]}\langle 24\rangle[14]\langle 42\rangle[31] \\ &= 2ie^2\frac{(\langle 24\rangle)^2}{\langle 23\rangle\langle 31\rangle} \end{aligned}$$

SU(N) colour factors

$SU(N)$ Special Unitary Group

Element of the group - $g = e^{i\theta^a(x) T^a}$

Generators of the group - T^a $\quad a = 1, \cdots, N^2 - 1$

Lie Algebra - $\left[T^a, T^b\right] = i f^{abc} T^c,$ $\quad \mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}.$

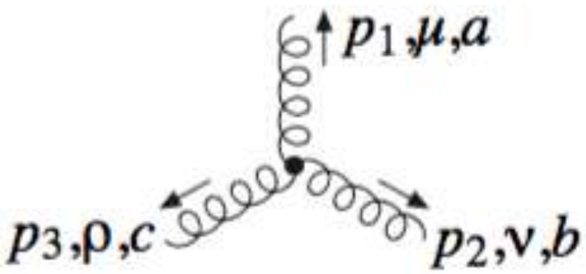
f^{abc} - Structure constants

Trace Properties -

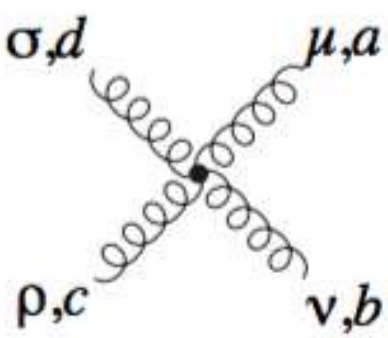
$$\begin{aligned}
\mathrm{Tr}(T^a X)\,\mathrm{Tr}(T^a Y) &= \frac{1}{2}\left[\mathrm{Tr}(XY) - \frac{1}{N}\mathrm{Tr}(X)\,\mathrm{Tr}(Y)\right], \\
\mathrm{Tr}(T^a X T^a Y) &= \frac{1}{2}\left[\mathrm{Tr}(X)\,\mathrm{Tr}(Y) - \frac{1}{N}\mathrm{Tr}(XY)\right].
\end{aligned}$$

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$$

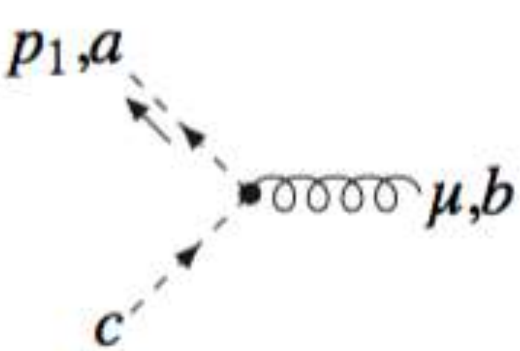
★ Feynman Rules:



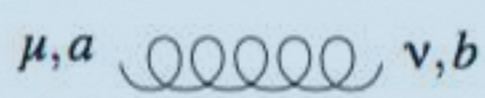
$$= g\left(if^{abc}\right) i\left[g^{\mu\nu}\left(p_1^{\rho} - p_2^{\rho}\right) + g^{\nu\rho}\left(p_2^{\mu} - p_3^{\mu}\right) + g^{\rho\mu}\left(p_3^{\nu} - p_1^{\nu}\right)\right],$$



$$= ig^2\Big[\left(if^{abe}\right)\left(if^{ecd}\right)\left(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}\right) + \left(if^{bce}\right)\left(if^{ead}\right)\left(g^{\nu\mu}g^{\rho\sigma} - g^{\rho\mu}g^{\nu\sigma}\right) + \left(if^{cae}\right)\left(if^{ebd}\right)\left(g^{\rho\nu}g^{\mu\sigma} - g^{\mu\nu}g^{\rho\sigma}\right)\Big],$$



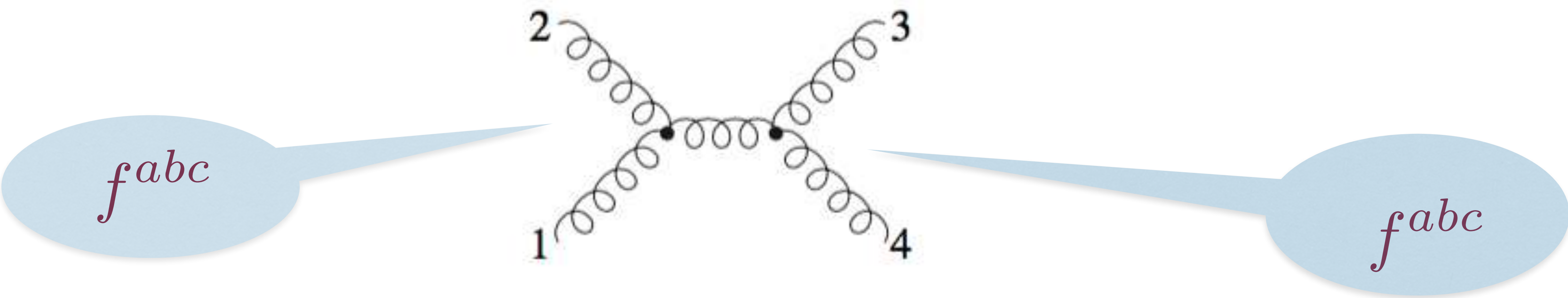
$$= ig\left(if^{abc}\right)p_1^{\mu}.$$



$$= \frac{-ig^{\mu\nu}}{p^2}\delta^{ab},$$

$$= \frac{i}{p^2}\delta^{ab}.$$

SU(N) color algebra

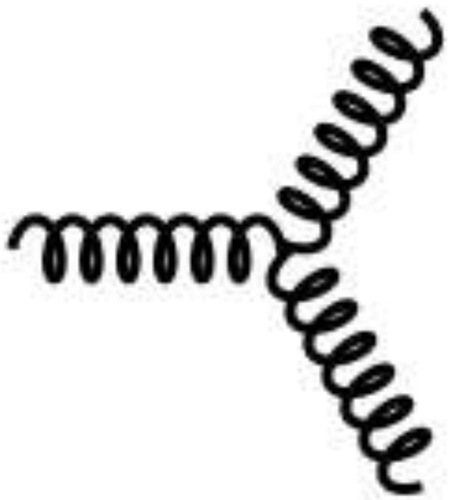


$$[T^a, T^b] = i f^{abc} T^c \quad \longrightarrow \quad i f^{abc} = 2\left[\mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^b T^a T^c\right)\right]$$

$$\begin{aligned} i f^{a_1 a_2 b} i f^{b a_3 a_4} &= 4\left[\mathrm{Tr}\left(T^{a_1} T^{a_2} T^b\right) - \mathrm{Tr}\left(T^{a_2} T^{a_1} T^b\right)\right]\left[\mathrm{Tr}\left(T^{a_3} T^{a_4} T^b\right) - \mathrm{Tr}\left(T^{a_4} T^{a_3} T^b\right)\right] \\ &= 2\,\mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) - 2\,\mathrm{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) - 2\,\mathrm{Tr}(T^{a_2} T^{a_1} T^{a_3} T^{a_4}) \\ &\quad + 2\,\mathrm{Tr}(T^{a_2} T^{a_1} T^{a_4} T^{a_3}) \end{aligned}$$

Three gluon vertex

$$V^3(1,2,3) = (-ig) f^{a_1 a_2 a_3} \left[\eta_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + \eta_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + \eta_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2}\right]$$

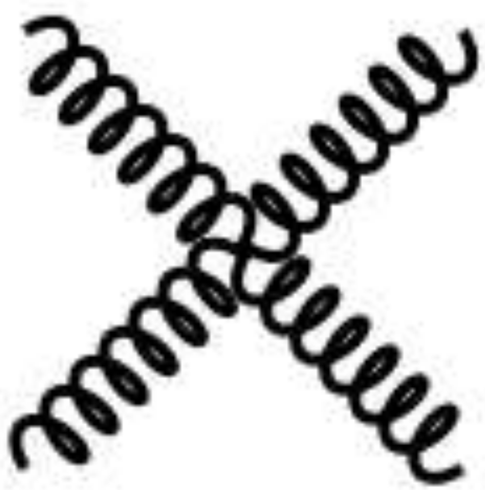


$$\begin{aligned}V^3(1,2,3) =&(-ig)\left(\frac{-i}{\sqrt{2}}\right)\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3} - T^{a_2}T^{a_1}T^{a_3})\\ &\left[\eta_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + \eta_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + \eta_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2}\right]\\ =&(-ig)\left(\frac{-i}{\sqrt{2}}\right)\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\\ &\left(\eta_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + \eta_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + \eta_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2}\right)\\ &+(-ig)\left(\frac{-i}{\sqrt{2}}\right)\mathrm{tr}(T^{a_2}T^{a_1}T^{a_3})\\ &\left(\eta_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + \eta_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + \eta_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2}\right)\end{aligned}$$

Cyclic -1

Cyclic -2

Four Gluon Vertex

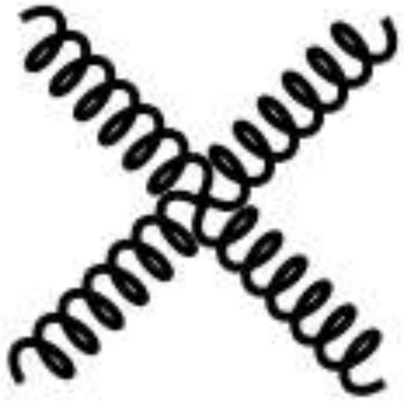


$$V^4(1,2,3,4) = (-g)^2 \Big[f^{a_1a_2d} f^{a_3a_4d} \big(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}\big)$$
$$+ f^{a_1a_3d} f^{a_2a_4d} \big(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}\big)$$
$$+ f^{a_1a_4d} f^{a_2a_3d} \big(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \eta_{\mu_1\mu_3}\eta_{\mu_3\mu_4}\big) \Big].$$

$$f^{a_1a_2d} f^{a_3a_4d} = -2\mathrm{tr}\Big(T^{a_1}T^{a_2}T^{d} - T^{a_1}T^{a_2}T^{d}\Big)$$
$$\times \mathrm{tr}\Big(T^{a_3}T^{a_4}T^{d} - T^{a_4}T^{a_3}T^{d}\Big)$$
$$= -2\Big[\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - \mathrm{tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})$$
$$- \mathrm{tr}(T^{a_2}T^{a_1}T^{a_3}T^{a_4}) + \mathrm{tr}(T^{a_2}T^{a_1}T^{a_4}T^{a_3})\Big]$$

Cyclic Permutations

Four gluon vertex



Six Cyclic Permutations

$$
\begin{aligned}
V^4(1,2,3,4) = {} & g^2 \mathrm{tr} T^{1234}\left(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{1}{2}\left(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}\right)\right) \\
& + g^2 \mathrm{tr} T^{1243}\left(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \frac{1}{2}\left(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}\right)\right) \\
& + g^2 \mathrm{tr} T^{1342}\left(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \frac{1}{2}\left(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}\right)\right) \\
& + g^2 \mathrm{tr} T^{1432}\left(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{1}{2}\left(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}\right)\right) \\
& + g^2 \mathrm{tr} T^{1324}\left(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \frac{1}{2}\left(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}\right)\right) \\
& + g^2 \mathrm{tr} T^{1423}\left(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} - \frac{1}{2}\left(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}\right)\right)
\end{aligned}
$$

QCD improved Parton Model

$$\mathcal{A}_n^{(0)}(g_1, g_2, \ldots, g_n) = g^{n-2} \sum_{\sigma \in S_n/Z_n} 2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}\right) A_n^{(0)}\left(g_{\sigma(1)}, \ldots, g_{\sigma(n)}\right)$$

Partial Amplitude

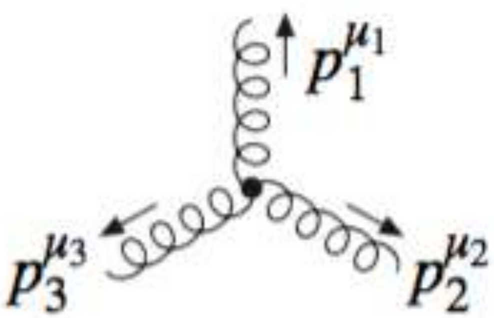
Partial Amplitudes
- No color information
- Gauge Invariant

n	4	5	6	7	8	9	10
unordered	4	25	220	2485	34300	559405	10525900
cyclic ordered	3	10	38	154	654	2871	12925

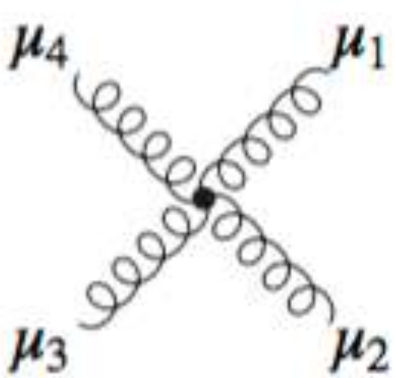
Cyclic Ordered Feynman Rules

No Color Factors !!!

$$= \frac{-ig^{\mu\nu}}{p^2},$$



$$= i\left[g^{\mu_1\mu_2}\left(p_1^{\mu_3} - p_2^{\mu_3}\right) + g^{\mu_2\mu_3}\left(p_2^{\mu_1} - p_3^{\mu_1}\right) + g^{\mu_3\mu_1}\left(p_3^{\mu_2} - p_1^{\mu_2}\right)\right]$$



$$= i\left[2g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right].$$

n-gluon Amplitude

- Choosing same reference momentum for all the n gluons implies

$$A_n(1^{\pm} \cdots n^{\pm}) = 0$$

as there is always atleast one $\epsilon_i^{\pm}(q).\epsilon_j^{\pm}(q)$ term in the amplitude.

- The n gluon amplitude with one flipped helicity state gives

$$A_n(1^{\mp}, 2^{\pm}, \cdots n^{\pm}) = 0, \qquad n > 3.$$

because $\epsilon_i^{\pm}(q).\epsilon_j^{\pm}(q) = 0$ for $i, j \in 2, \cdots, n$ and $\epsilon_i^{\pm}(q).\epsilon_1^{\mp}(q) = 0$.

for $n = 3$ we have $p_i.p_j = 0$ from momentum conservation

singular denominator

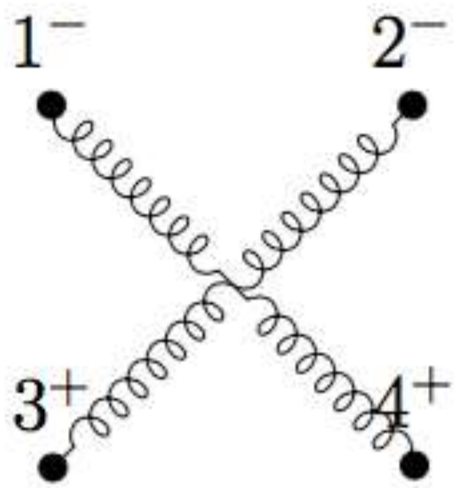
gg -> gg Amplitude

$A(1^-, 2^-, 3^+, 4^+)$

$$\epsilon^+(p,q) = \frac{\langle q|\gamma_\mu p]}{\sqrt{2}\langle qp\rangle}$$

$$\epsilon^-(p,q) = -\frac{[q|\gamma_\mu|p\rangle}{\sqrt{2}[qp]}$$

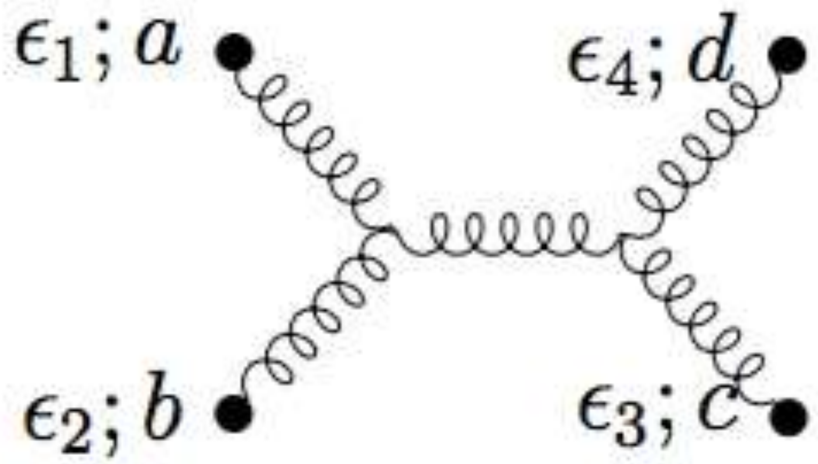
for ϵ_1 and ϵ_2 as $q = p_4$ and for ϵ_3 and ϵ_4 as p_1



$\longrightarrow$ 0

$$2(\epsilon_2.\epsilon_4)(\epsilon_1.\epsilon_3) - (\epsilon_2.\epsilon_3)(\epsilon_1.\epsilon_4) - (\epsilon_2.\epsilon_1)(\epsilon_3.\epsilon_4) = 0$$

gg -> gg Amplitude

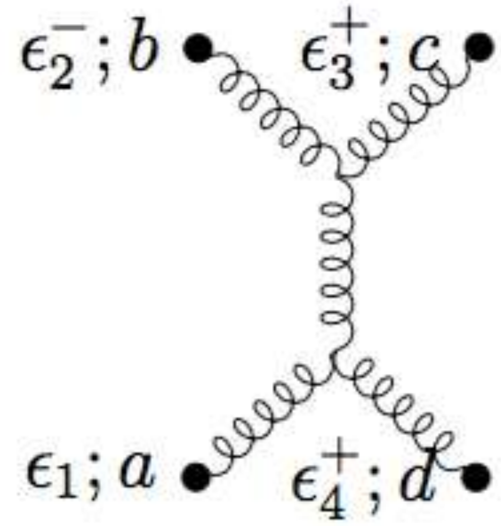
$iA_s =$ 

$$A_s = \frac{g^2}{s} f^{abe} f^{cde} [(\epsilon_1.\epsilon_2)(p_1 - p_2)^\mu + 2\epsilon_2^\mu(p_2.\epsilon_1) - 2\epsilon_1^\mu(p_1.\epsilon_2)]$$
$$\times [(\epsilon_3.\epsilon_4)(p_3 - p_4)^\mu + 2\epsilon_4^\mu(p_4.\epsilon_3) - 2\epsilon_3^\mu(p_3.\epsilon_4)]$$
$$= \frac{4g^2}{s} f^{abe} f^{cde} (\epsilon_2^-.\epsilon_3^+)(p_2.\epsilon_1^-)(p_3.\epsilon_4^+)$$

$$= -2g^2 f^{abe} f^{cde} \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 34\rangle \langle 23\rangle \langle 41\rangle}.$$

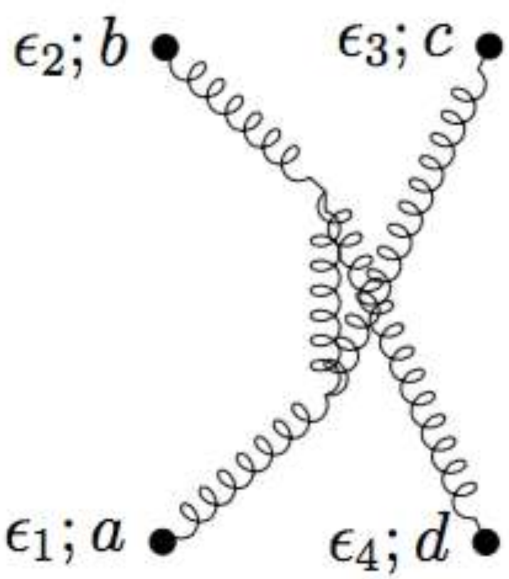
gg -> gg Amplitude

$iA_t =$



$= 0$

$iA_u =$



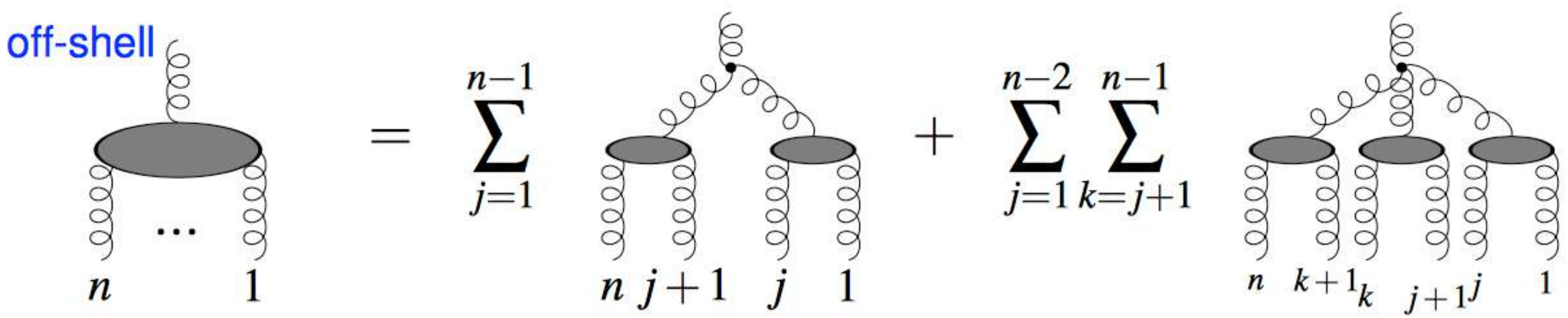
$$A_u = \quad -2g^2 f^{ace} f^{bde} \frac{\langle 21 \rangle^4}{\langle 14 \rangle \langle 42 \rangle \langle 23 \rangle \langle 31 \rangle}.$$

$$A(-,-;+,+) = -2g^2 \left[f^{abe} f^{cde} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 34 \rangle \langle 23 \rangle \langle 41 \rangle} + f^{ace} f^{bde} \frac{\langle 21 \rangle^4}{\langle 14 \rangle \langle 42 \rangle \langle 23 \rangle \langle 31 \rangle} \right]$$

Berends-Giele Recursion

- Off-Shell currents

- No Feynman diagrams

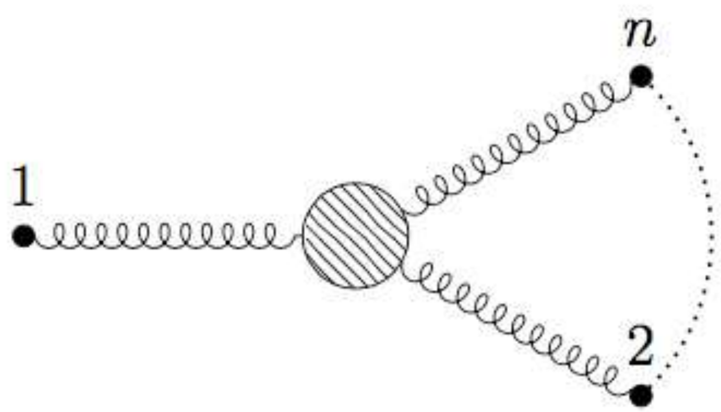


BCFW relation

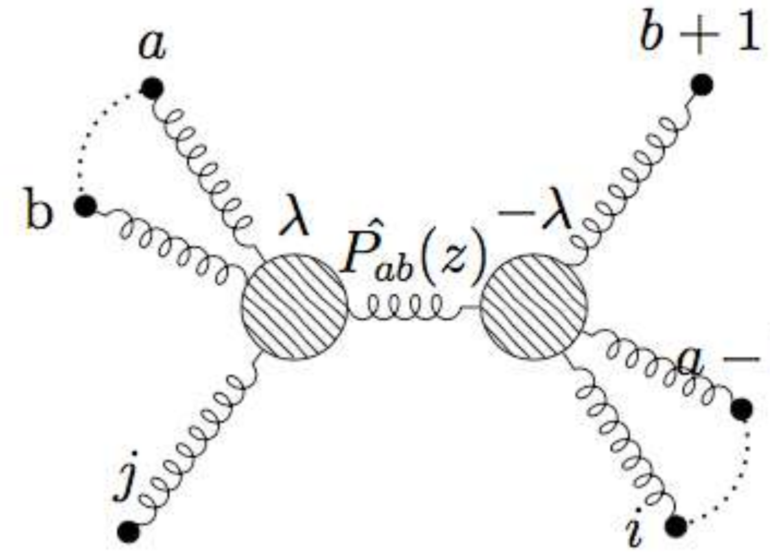
$$0 = \frac{1}{2\pi i}\oint_C \mathrm{d}z \frac{A(z)}{z} = A(0) + \sum_{poles(z_\alpha \neq 0)} \mathrm{Res}\left(\frac{A(z)}{z}, z_\alpha\right)$$

$$\hat{P}_{ab}(z) = \sum_{k=a}^{b} |k\rangle\, [k| - z\, |i\rangle\, [j|$$

$$A(1, \cdots, n) \sim \sum_{\lambda} A_L(a, \cdots, b, -\hat{P}_{ab}^{\lambda}) \frac{1}{\hat{P}_{ab}^2} A_R(\hat{P}_{ab}^{-\lambda}, b+1, \cdots, a-1)$$



$\sim \sum_{\lambda}$



$, \hat{P}_{ab}^2 \to 0.$

BCFW relation

BCFW on n - gluon amplitude

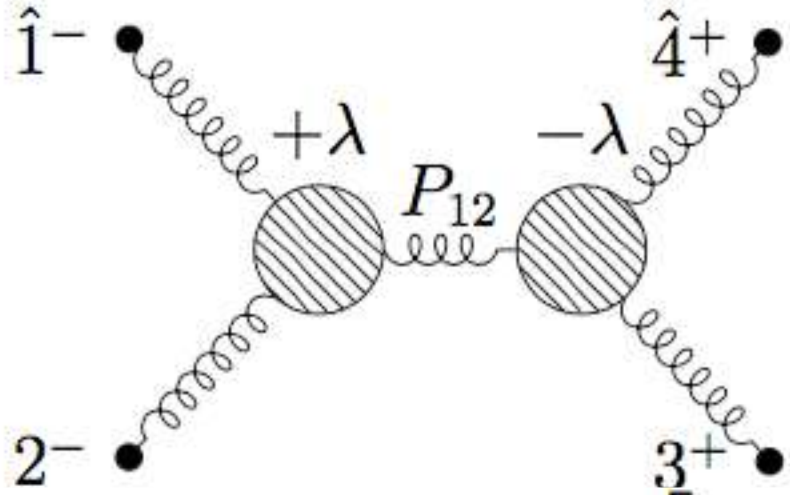
$$A(0) = \frac{-1}{z_\alpha} \sum_\lambda \operatorname*{Res}_{z \to z_\alpha} \left(A_L(z) \frac{1}{(P_a + \cdots + P_b)^2 - z \sum_k \langle ik \rangle [kj]} A_R(z) \right)$$

$$= \sum_\lambda A_L(a, \cdots, b, -\hat{P}_{ab}^{\lambda}) \frac{1}{(P_a + \cdots + P_b)^2} A_R(a-1, \cdots, b+1, \hat{P}_{ab}^{-\lambda})$$

RHS contains n-1 gluons

4-gluon MHV Amplitude

$$A_4(1^-,2^-,3^+,4^+) =$$



$$A_4(1^-,2^-,3^+,4^+) = \left[A_3(\hat{1}^-,2^-,-\hat{P}_{12}^+)\frac{1}{P_{12}^2}A_3(\hat{4}^+,3^+,\hat{P}_{12}^-) \right]_{z_{12}}$$

$$= \left(\frac{\langle \hat{1}2 \rangle^3}{\langle 2\hat{P}_{12}\rangle \langle \hat{P}_{12}\hat{1}\rangle} \right) \frac{1}{\langle 12 \rangle [21]} \left(\frac{-[43]^3}{[3\hat{P}_{12}][\hat{P}_{12}4]} \right)$$

$$= \frac{-\langle 12 \rangle^3 [43]^3}{\langle 2\hat{P}_{12}\rangle [\hat{P}_{12}4] \langle \hat{1}\hat{P}_{12}\rangle [\hat{P}_{12}3] \langle 12 \rangle [21]}$$

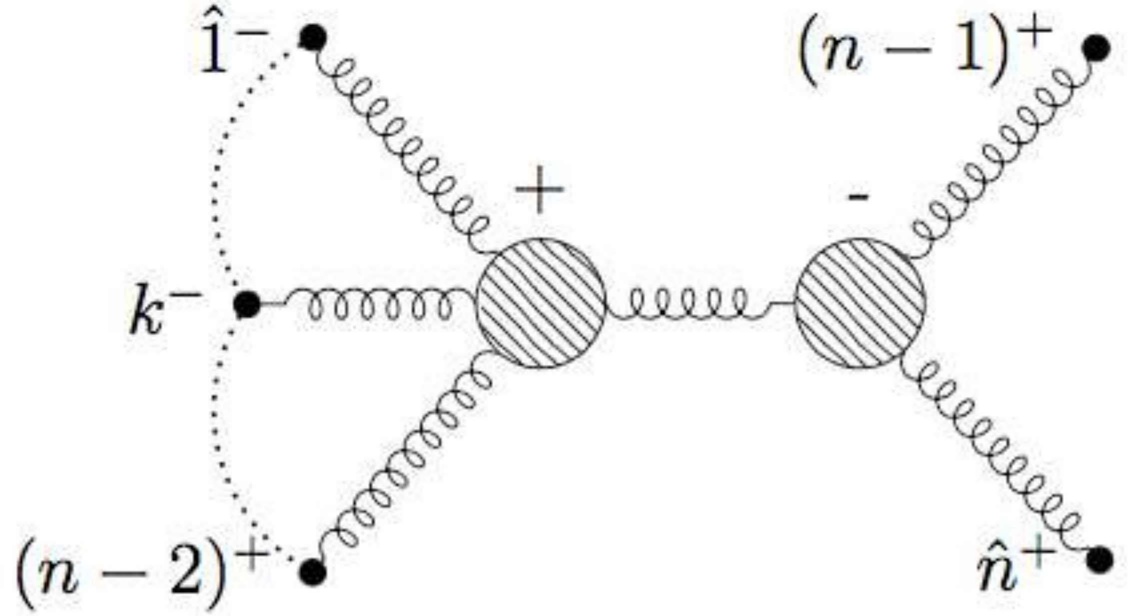
$$A_1(1234) = f^{abe}f^{cde}A(1234) + f^{bae}f^{cde}A(2134) + f^{abe}f^{dce}A(1243) + f^{bae}f^{cde}A(2143)$$

$$= f^{abe}f^{cde}\left(\frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} - \frac{\langle 12 \rangle^4}{\langle 21 \rangle \langle 13 \rangle \langle 34 \rangle \langle 42 \rangle} \right)$$

Park-Taylor Amplitude

MHV n - gluon Amplitudes

$$A_n(1^-, 2^+, \cdots, k^-, \cdots, n^+) =$$



$$A_n(1^-, 2^+, \cdots, k^-, \cdots, n^+)$$

$$= \frac{\langle 1k \rangle^4}{\langle 12 \rangle \cdots \langle n-3|n-2 \rangle \langle n-2|n-1 \rangle \langle n-1|n \rangle \langle n1 \rangle}.$$

Twister space

Momenta in bi-spinor $p^\mu \to \lambda_a \lambda_{\dot{a}}$

Scaling

$$\lambda_a \to z\lambda_a$$

$$\lambda_{\dot{a}} \to \frac{1}{z}\lambda_{\dot{a}}$$

Transform

$$\lambda_{\dot{a}} \to i\frac{\partial}{\partial\lambda^{\dot{a}}}$$

$$-i\frac{\partial}{\partial\lambda^{\dot{a}}} \to \lambda_{\dot{a}}$$

Fourier Transform

$$f(\overline{\lambda}^{\dot{a}}) = \int \frac{d^2p}{(2\pi)^2}\ exp(i\overline{\lambda}^{\dot{a}}\lambda_{\dot{a}})\ \ f(\lambda_{\dot{a}})$$

Weinzierl's comparison

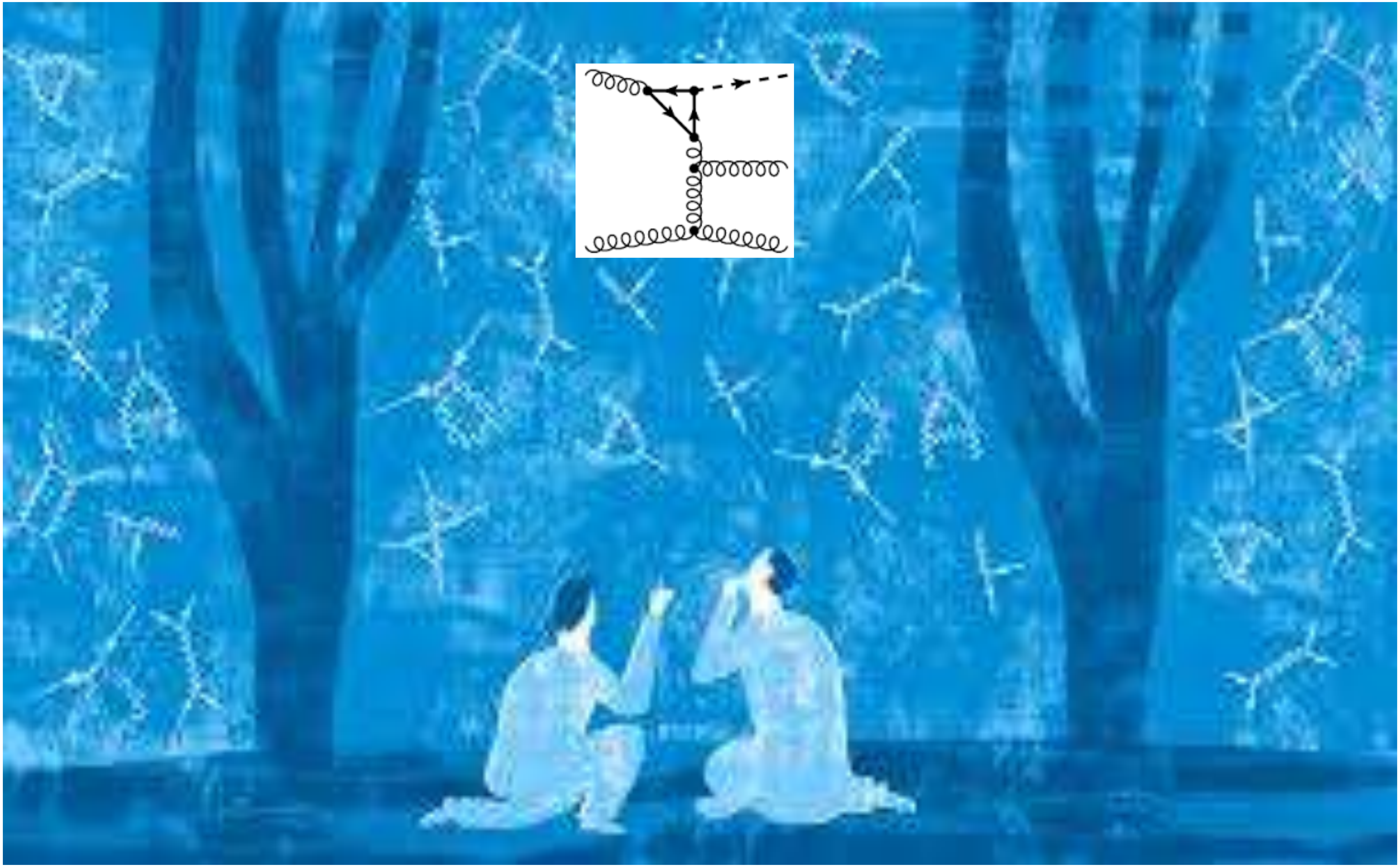
Compare algorithms based on different methods for the numerical computation of the Born gluon amplitude:

n	4	5	6	7	8	9	10	11	12
Berends-Giele	0.00011	0.00043	0.0015	0.005	0.016	0.047	0.13	0.37	1
Scalar	0.00014	0.00083	0.0033	0.011	0.033	0.097	0.26	0.7	1.8
MHV	0.00001	0.00053	0.0056	0.073	0.62	3.67	29	217	—
BCF	0.00002	0.00007	0.0004	0.003	0.017	0.083	0.47	2.5	14.5

CPU time in seconds for the computation of the n gluon amplitude on a standard PC (Pentium IV with 2 GHz), summed over all helicities.

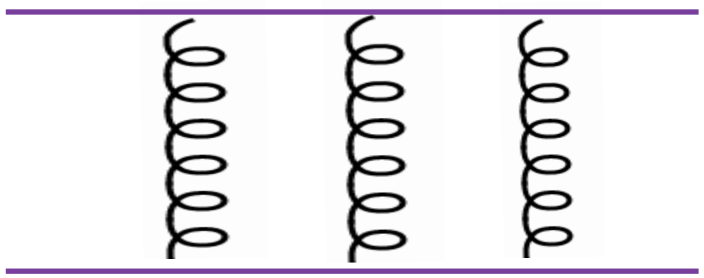
All methods give identical results within an accuracy of 10^{-12}.

NLO

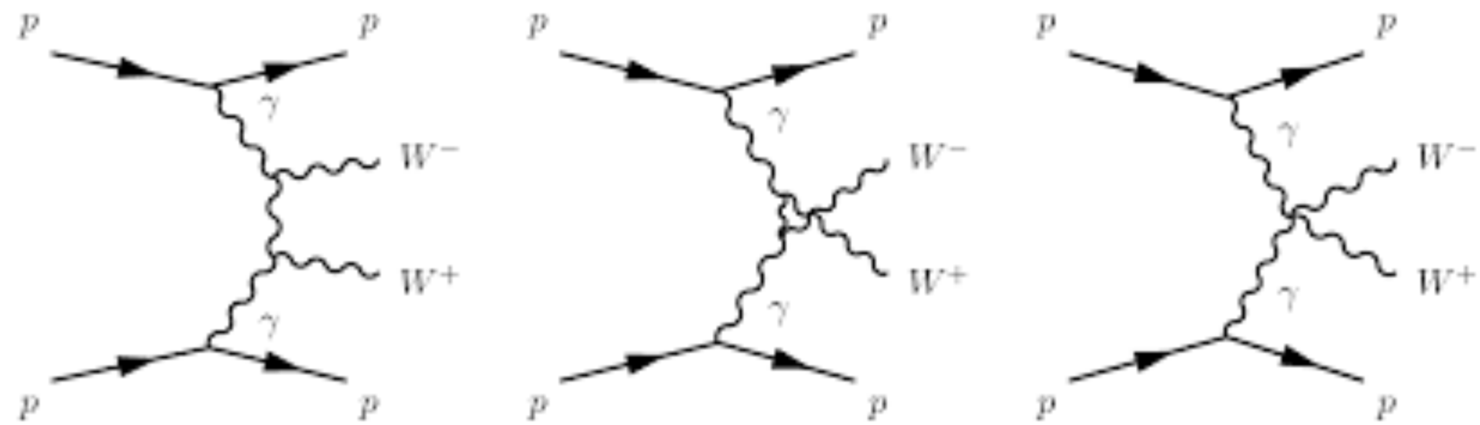


Beyond LO

Loop integral

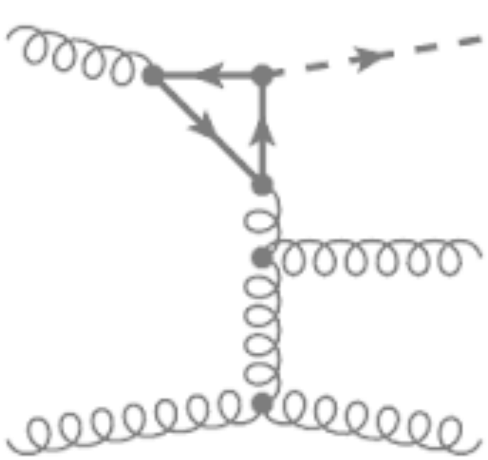


$$\mathbf{I} = \prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n} \frac{N(\{k_l\},\{p_m\})}{D_1^{a_1} D_2^{a_2} \cdots D_{N_d}^{a_{N_d}}}$$



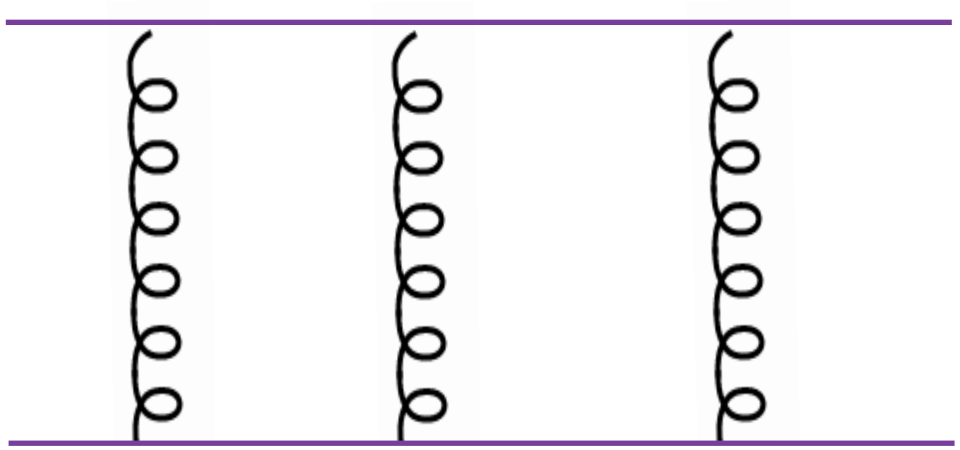
Phase space integrals

$$d\Phi_N = \prod_{i=1}^{N} \left(\int \frac{d^n p_i}{(2\pi)^n} \delta(p_i^2 - m_i^2) \right) (2\pi)^n \delta^n (q - \sum_i p_i)$$



Mixed L-V integrals

Loop Integrals



Loop integral

$$\prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n}$$

$$\mathbf{I} = \prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n} \frac{N(\{k_l\}, \{p_m\})}{D_1^{a_1} D_2^{a_2} \cdots D_{N_d}^{a_{N_d}}}$$

Propagator: $D_i = q_i^2 + i\epsilon$ $\quad q_i = \sum_j k_l + \sum_l p_j$

Numerator:

$$N(\{k_l\}, \{p_m\}) = \prod_{i=1}^{N_k} \prod_{j=i}^{N_k} (k_i \cdot k_j)^{\delta_{ij2}} \times \prod_{i=1}^{N_p} \prod_{j=1}^{N_k} (p_i \cdot k_j)^{\delta_{ij1}}$$

$\gamma_i \geq 0, \; \delta_{ijl} \geq 0$

Loop Integrals

External momenta	p_i	$i = 1, \cdots N_e$
Propagator momenta	q_i	$i = 1, \cdots N_d$
Loop momenta	k_i	$i = 1, \cdots N_k$

Momentum Conservation $\sum_{i=1}^{N_e} p_i = 0$

Number of Scalar Product

$$N_{sp} = N_p N_k + \frac{N_k(N_k+1)}{2}$$

$$N_p = N_e - 1$$

Loop Integrals

Numerator: Reducible

Irreducible

Reducible if

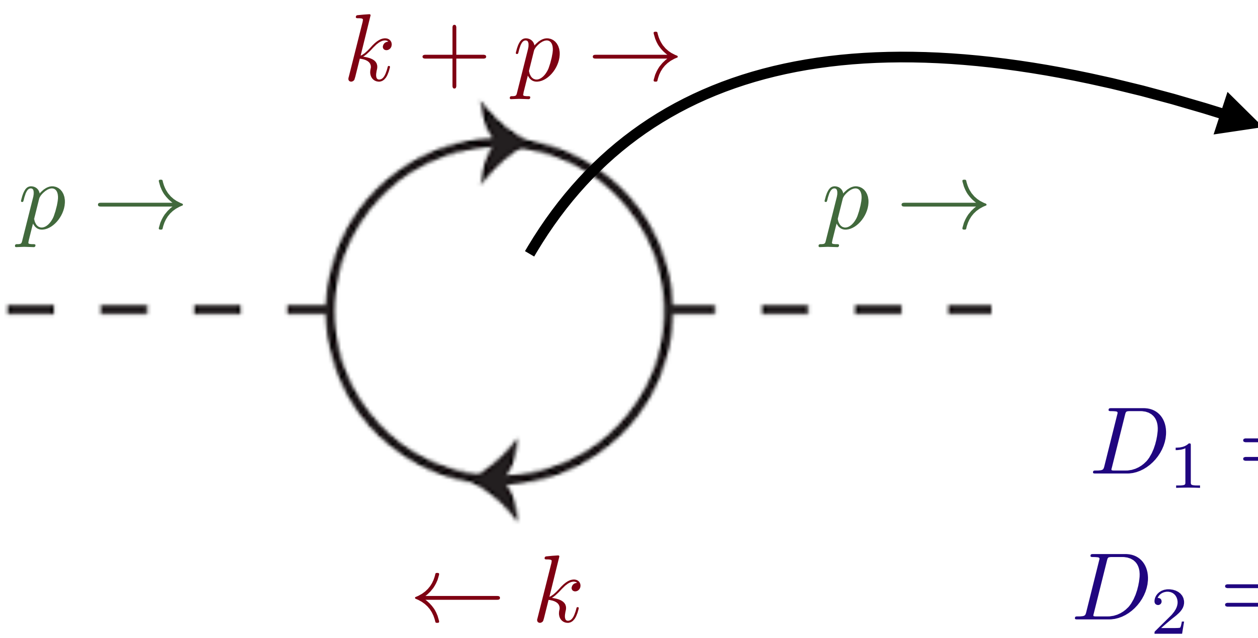
$(p_i \cdot k_j)^{a_k}$

$(k_i \cdot k_j)^{b_k}$

} are expressible in terms of Denominators

$$\frac{(p \cdot k)_j}{D_j} = \frac{1}{C_j} \left(1 - \frac{D_j - C_j \, (p \cdot k)_j}{D_j} \right) \, , \qquad j = 1, \ldots, N_d \, ,$$

Loop Integrals



$$\int \frac{d^n k}{(2\pi)^n} \frac{(k \cdot p)^a}{D_1 D_2}$$

$$D_1 = k^2 + i\epsilon$$

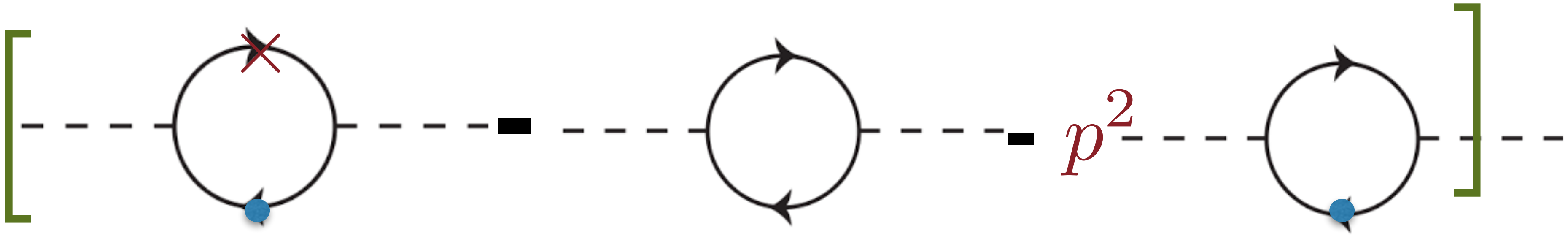
$$D_2 = (k+p)^2 + i\epsilon, \qquad p^2 < 0$$

$$\frac{k \cdot p}{D_1} = \frac{1}{2}\left(\frac{D_2}{D_1} - 1 - \frac{p^2}{D_1}\right)$$

$$\frac{k \cdot p}{D_2} = \frac{1}{2}\left(1 - \frac{D_1}{D_2} - \frac{p^2}{D_2}\right)$$

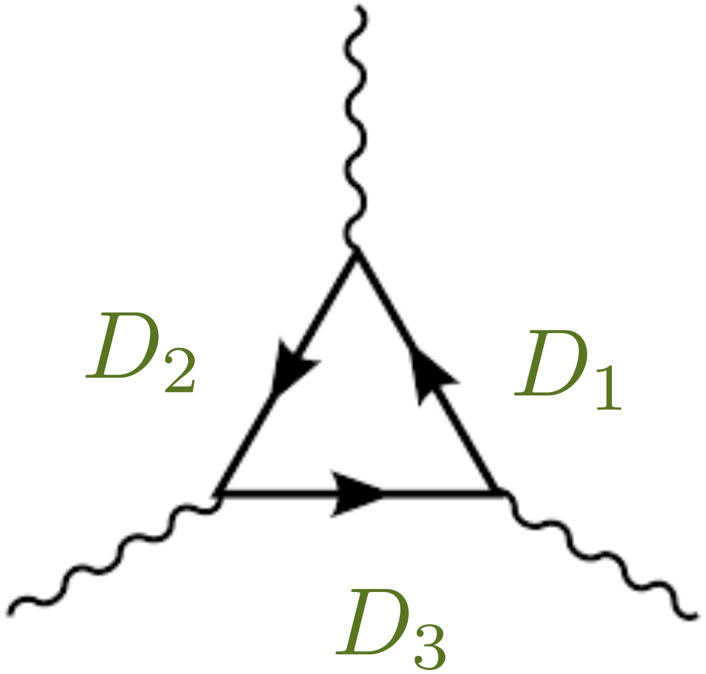
Reducible

$$\int \frac{d^n k}{(2\pi)^n} \frac{k \cdot p}{D_1 D_2} = \frac{1}{2}\Big[$$



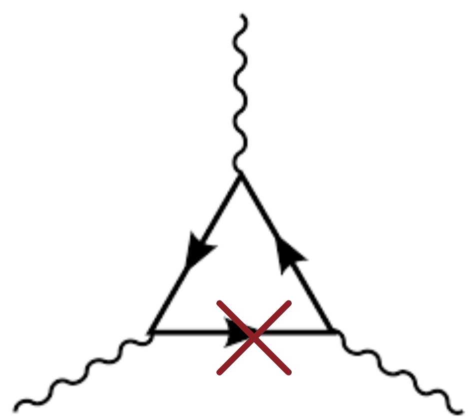
$\Big]$

Reducible integrals



$$D_1 = k^2 - m^2$$
$$D_2 = (k + p_1)^2 - m^2$$
$$D_3 = (k + p_1 + p_2)^2 - m^2$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{k \cdot p_2}{D_1 D_2 D_3} = \frac{1}{2} \Bigg[$$



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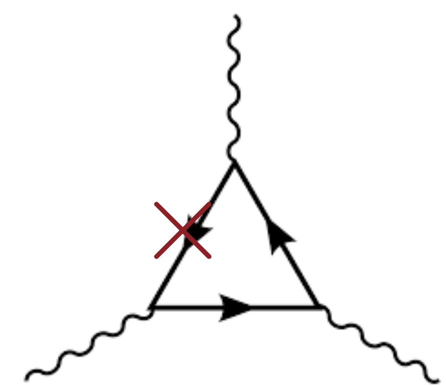
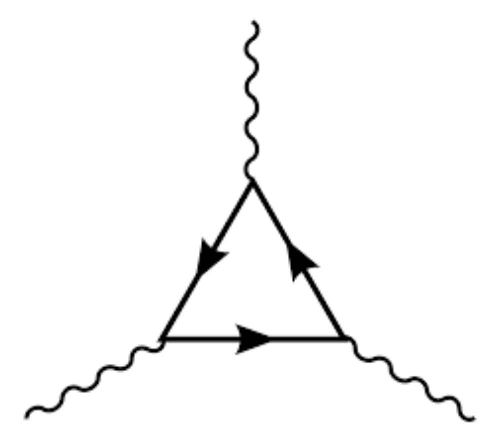


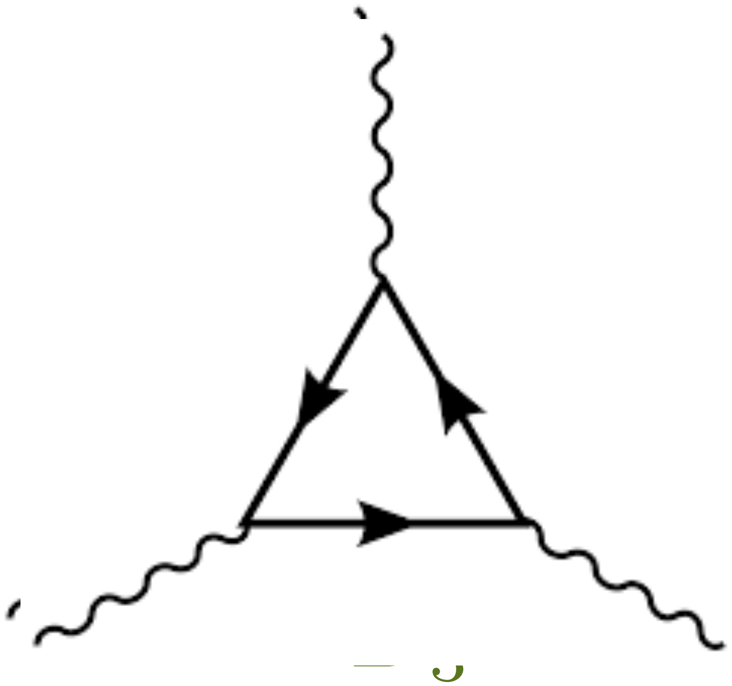
$$k \cdot p_2 = \frac{1}{2} (D_3 - D_2 - s)$$

- s



$\Bigg]$

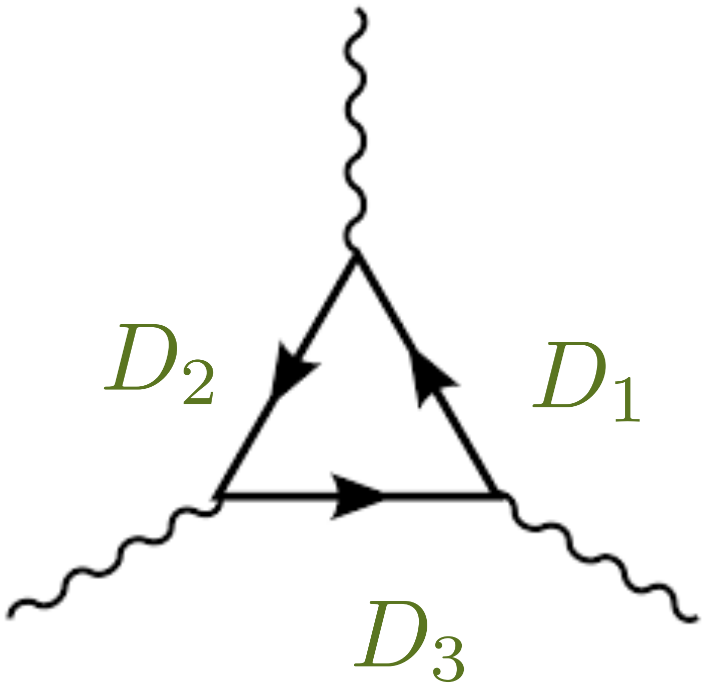
Tensorial Reduction



$$I_{\mu_1\cdots\mu_m} = \int \frac{d^n k}{(2\pi)^n} \frac{k_{\mu_1}\cdots k_{\mu_m}}{D_1 D_2 D_3}$$

$$\begin{aligned}
I_{\mu_1\cdots\mu_m} = A_1 \;\; & g_{\mu_1\mu_2}\cdots g_{\mu_{m-1}\mu_m} \\
+A_2 \;\; & \{p_{1\mu_1}p_{2\mu_2}\}\, g_{\mu_3\mu_4}\cdots g_{\mu_{m-1}\mu_m} \\
+\cdots & \\
+A_\alpha \;\; & \{p_{1\mu_1}p_{2\mu_2}p_{2\mu_3}p_{3\mu_4}\}\cdots g_{\mu_{m-1}\mu_m} \\
+\cdots & \\
+A_\beta \;\; & p_{1\mu_1}p_{2\mu_2}\cdots p_{1\mu_{m-1}}p_{m\mu_m}
\end{aligned}$$

Tensorial Reduction



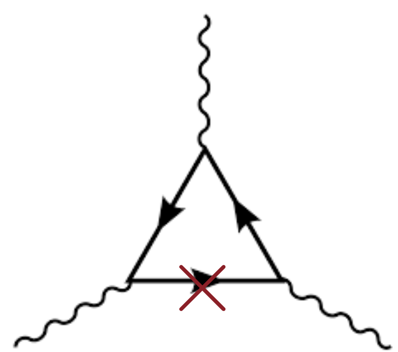
$$I_\mu = \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{D_1 D_2 D_3}$$

$$I_\mu = A_1 \; p_{1\mu} + A_2 \; p_{2\mu}$$

$$I_1 = p_1 \cdot I = A_2 \; p_1 \cdot p_2$$

$$I_2 = p_2 \cdot I = A_1 \; p_2 \cdot p_1$$

$$A_1 = \frac{1}{2p_1 \cdot p_2}$$ [



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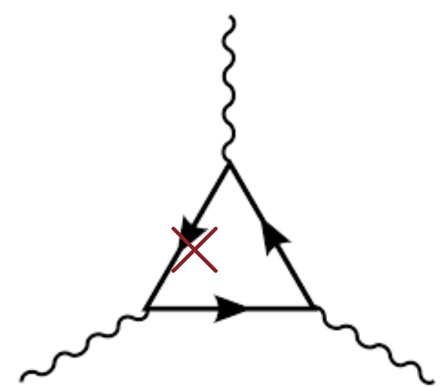

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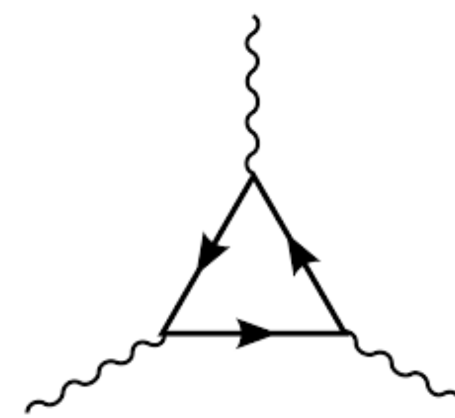


]

Feynman and Schwinger Parametrisations

$$A(k) = \frac{N(k, p_i)}{D_1 \cdots D_m}, \quad D_i = (k + p_i)^2 - m_i^2$$

Feynman:

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{\Gamma(n_1 + n_2 \cdots + n_k)}{\Gamma(n_1)\Gamma(n_2) \cdots \Gamma(n_k)}$$

$$\int \frac{\delta(x_1 + x_2 \cdots + x_l - 1) x_1^{n_1 - 1} x_2^{n_2 - 1} \cdots x_k^{n_k - 1} dx_1\, dx_2 \cdots dx_k}{[a_1 x_1 + a_2 x_2 \cdots + a_k x_k]^{n_1 + n_2 \cdots + n_k}}$$

Schwinger:

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{1}{\Gamma(n_1)\Gamma(n_2) \cdots \Gamma(n_k)}$$

$$\int e^{-a_1 \alpha_1 - a_2 \alpha_2 \cdots - a_k \alpha_k} \alpha_1^{n_1 - 1} \alpha_2^{n_2 - 1} \cdots \alpha_k^{n_k - 1} d\alpha_1\, d\alpha_2 \cdots d\alpha_k\, .$$

Ossola-Papadopoulos-Pittau (OPP method)

Integrand : $\dfrac{N(k,p_i)}{D_1 D_2 D_3 D_4}$

$$D_i = (k + \sum_j c_j p_j)^2 - m_i^2$$

$$N(k,p_i) = \sum_{i_1<i_2<i_3<i_4}^{m} \left[d(i_1 i_2 i_3 i_4) + \tilde{d}(k, i_1 i_2 i_3 i_4)\right] \prod_{i \neq i_1 i_2 i_3 i_4}^{m} D_i$$

$$+ \sum_{i_1<i_2<i_3}^{m} \left[c(i_1 i_2 i_3) + \tilde{c}(k, i_1 i_2 i_3)\right] \prod_{i \neq i_1 i_2 i_3}^{m} D_i$$

$$+ \sum_{i_1<i_2}^{m} \left[b(i_1 i_2) + \tilde{b}(k, i_1 i_2)\right] \prod_{i \neq i_1 i_2}^{m} D_i$$

$$+ \sum_{i_1}^{m} \left[b(i_1 i_2) + \tilde{b}(k, i_1)\right] \prod_{i \neq i_1}^{m} D_i + P(q) \prod_{i=1}^{m} D_i$$

Best suited for Numerical Methods

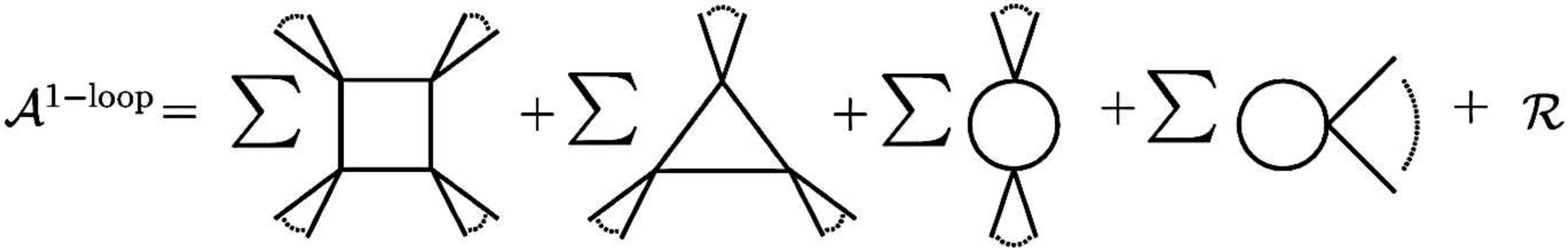
Ossola-Papadopoulos-Pittau (OPP method)

Integrand :

$$A(k) = \frac{N(k, p_i)}{D_1 \cdots D_m}, \quad D_i = (k + p_i)^2 - m_i^2$$

$$N(k, p_i) = \sum_{i_1 < i_2 < i_3 < i_4}^{m} \left[d(i_1 i_2 i_3 i_4) + \tilde{d}(k, i_1 i_2 i_3 i_4) \right] \prod_{i \neq i_1 i_2 i_3 i_4}^{m} D_i$$

$$+ \sum^{m} \left[c(i_1 i_2 i_3) + \tilde{c}(k, i_1 i_2 i_3) \right] \prod^{m} D_i$$



$$+ \sum_{i_1}^{m} \left[b(i_1 i_2) + \tilde{b}(k, i_1) \right] \prod_{i \neq i_1}^{m} D_i + P(q) \prod_{i=1}^{m} D_i$$

Best suited for Numerical Methods

NLO QCD - Tool Kits

Analytical Tools

Faster generation of Feynman diagram

QGRAF

Symbolic Manupulation:

FORM,Mathematica

On-Shell Methods

BCFW

Recursion techniques

BG

Merging NLO with Showers

MC@NLO
POWEG
SHERPA
VINCIA
GENeVa
aMC@NLO
KRKMC

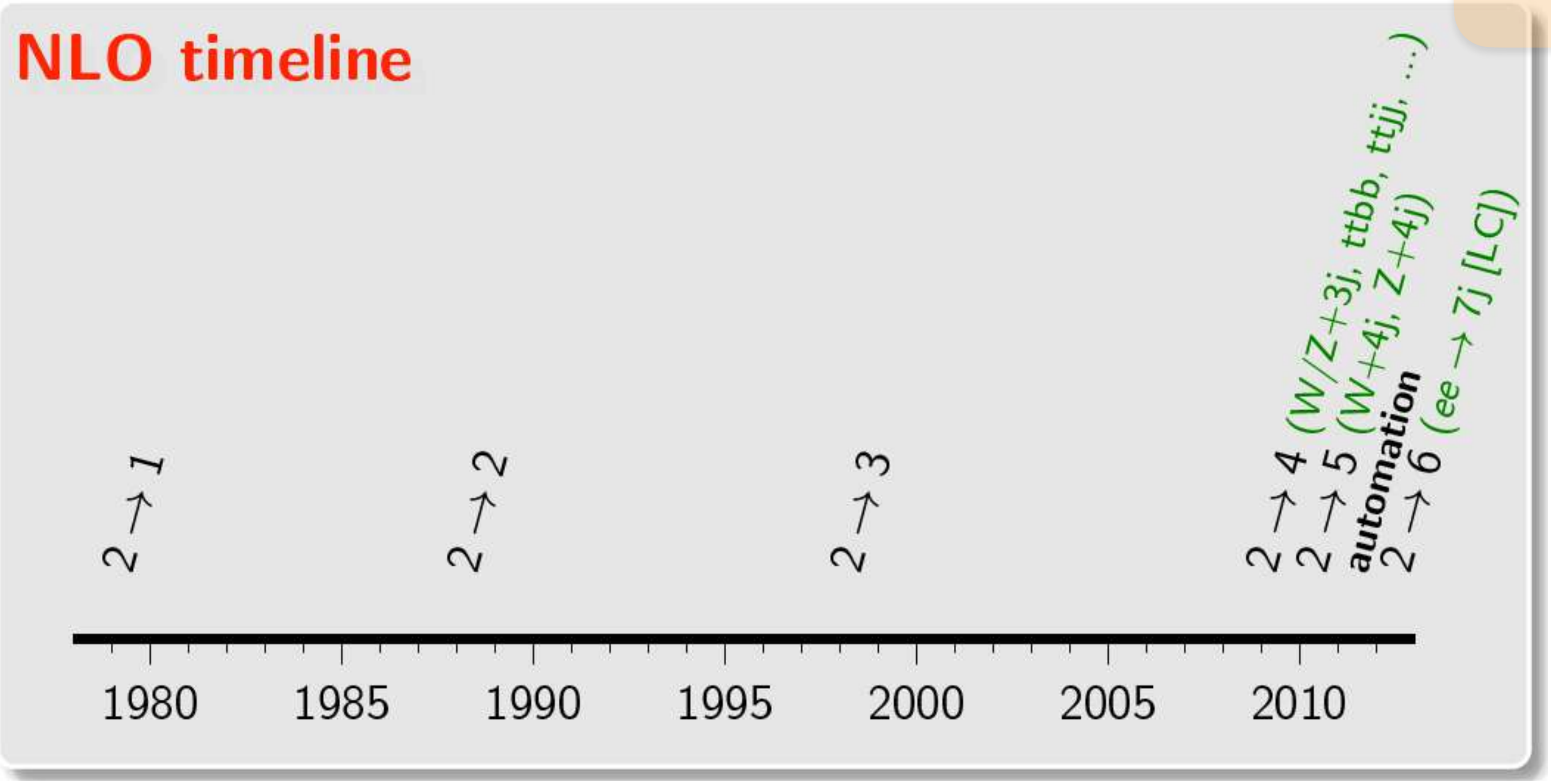
Semi-Numerical Methods

Madgraph
Helac-NLO
CutTools
BlackHat
Rocket
SAMURAI
MADLoop
GoSam

NLO revolution

1979: NLO Drell-Yan [Altarelli, Ellis & Martinelli]
1991: NLO $gg \to$ Higgs [Dawson; Djouadi, Spira & Zerwas]

1987: NLO high-p_t photoproduction [Aurenche et al]
1988: NLO $b\bar{b}$, $t\bar{t}$ [Nason et al]
1988: NLO dijets [Aversa et al]
1993: Vj [JETRAD, Giele, Glover & Kosower]

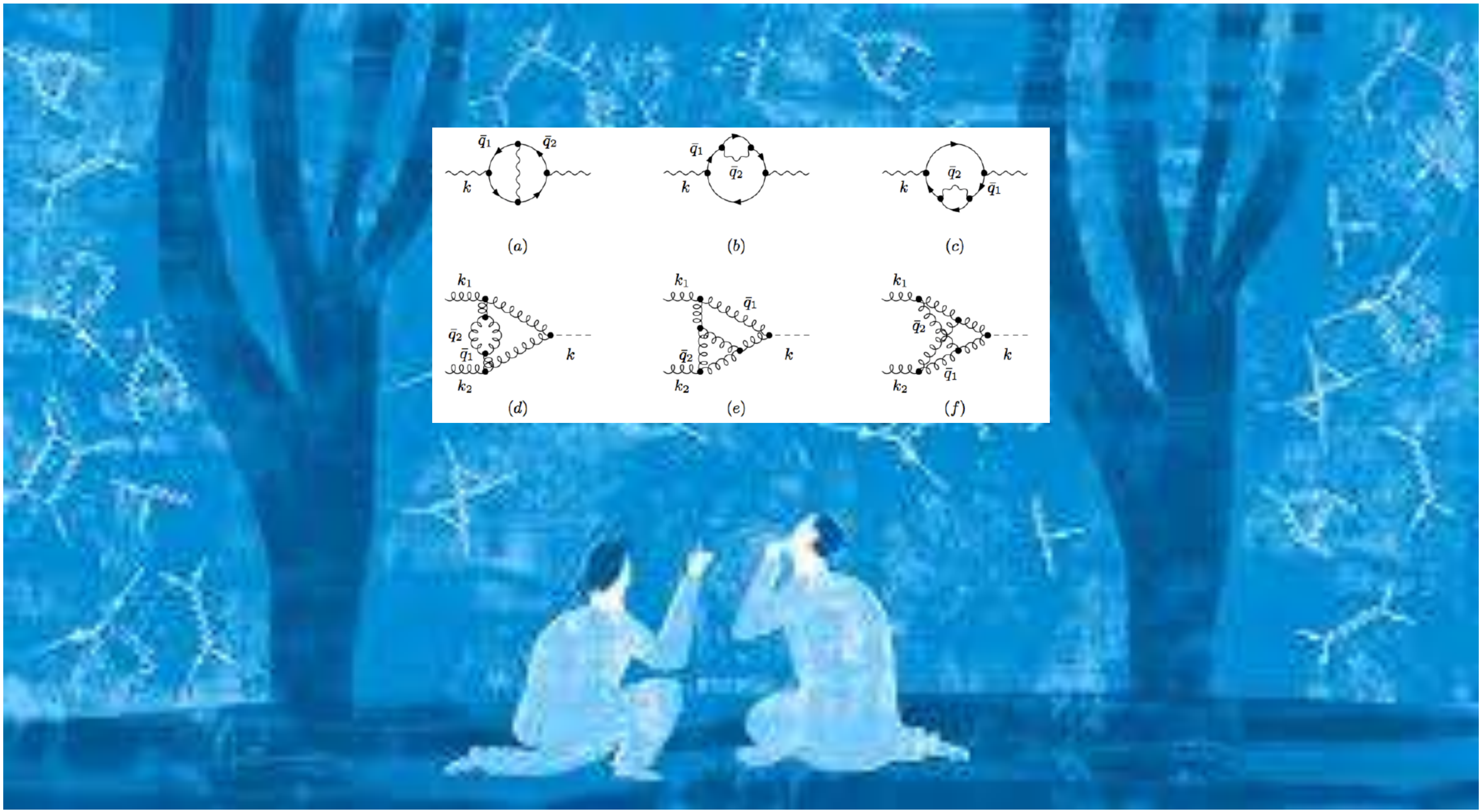


Golem, HELAC
BlackHat

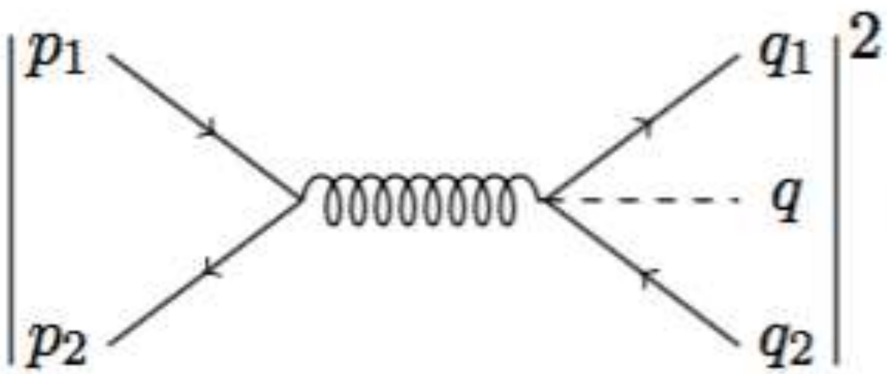
1998: NLO $Wb\bar{b}$ [MCFM: Ellis & Veseli]
2000: NLO $Zb\bar{b}$ [MCFM: Campbell & Ellis]
2001: NLO 3j [NLOJet++: Nagy]
...
2007: NLO $t\bar{t}j$ [Dittmaier, Uwer & Weinzierl '07]
...

2009: NLO W+3j [Rocket: Ellis, Melnikov & Zanderighi]
2009: NLO W+3j [BlackHat+Sherpa: Berger et al]
2009: NLO $t\bar{t}b\bar{b}$ [Bredenstein et al]
2009: NLO $t\bar{t}b\bar{b}$ [HELAC-NLO: Bevilacqua et al]
2009: NLO $q\bar{q} \to b\bar{b}b\bar{b}$ [Golem: Binoth et al]
2010: NLO $t\bar{t}jj$ [HELAC-NLO: Bevilacqua et al]
2010: NLO Z+3j [BlackHat+Sherpa: Berger et al]
...

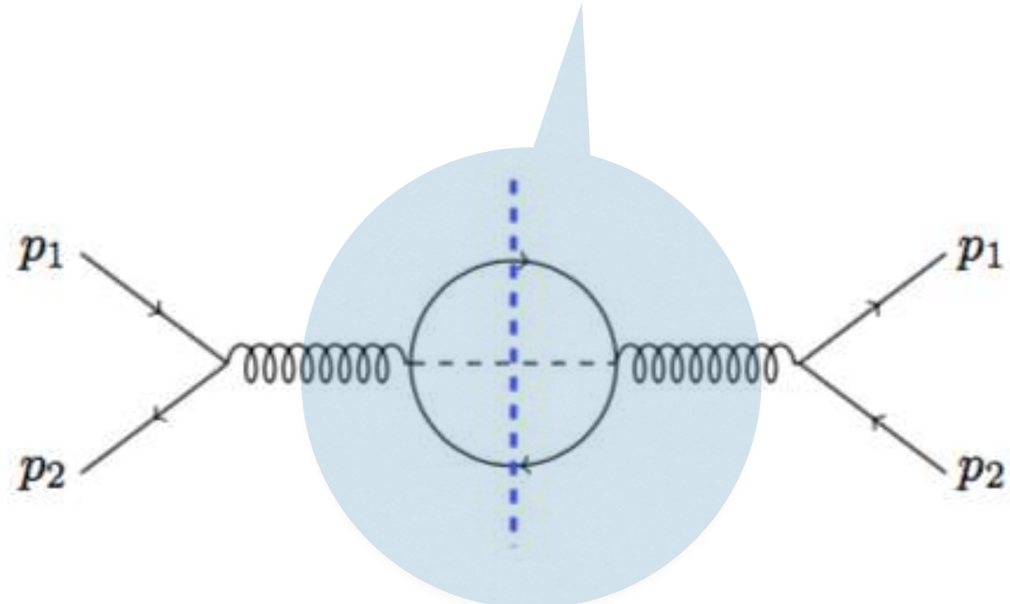
NNLO



Phase Space Integrals



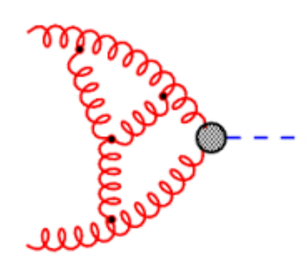
$$\propto \int \frac{d^n q_1}{(2\pi)^{n-1}} \frac{d^n q_2}{(2\pi)^{n-1}} \delta_+(q_1^2)\delta_+(q_2^2)\delta_+(q^2 - m_h^2)[\cdots]$$



Reverse Unitarity

$$\delta_+(q^2 - m^2) \sim \frac{1}{q^2 - m^2 + i\varepsilon} - \frac{1}{q^2 - m^2 - i\varepsilon}$$

→ **Loop Integrals**



Integration By Parts (IBP) — First Revolution

$D_i = q_i^2 + i\epsilon$ $\qquad q_i = \sum_j k_l + \sum_l p_j$

$$\prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n} \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_{N_d}^{a_{N_d}}}$$

n-Dimensional Gauss theorem

$$\int \prod_{i=1}^{N_k} \frac{d^n k_i}{(2\pi)^n} \frac{\partial}{\partial k_{j,\mu}} \left(\frac{k_{l,\mu}}{D_1^{a_1} D_2^{a_2} \cdots D_{N_d}^{a_{N_e}}} \right) = 0 \qquad j, l = 1, \cdots N_k$$

$$\int \prod_{i=1}^{N_k} \frac{d^n k_i}{(2\pi)^n} \frac{\partial}{\partial k_{j,\mu}} \left(\frac{p_{l,\mu}}{D_1^{a_1} D_2^{a_2} \cdots D_{N_d}^{a_{N_e}}} \right) = 0 \qquad l = 1, \cdots N_e - 1$$

Integration By Parts (IBP) identities

Integration By Parts (IBP)

Number of IBP identities = $N_k(N_k + N_e - 1)$

IBP identities:

$$\sum_i C_i(s_{ij}, n)\ I_i(\{p_j\}, n, a_1, \cdots a_{N_e}) = 0$$

Integrals:

$$i = 1, \cdots N_I$$
$$s_{ij} = (p_i + p_j)^2$$

Solving IBP identities -> **Master Integrals:**

$$I_i(\{p_j\}, a_1, \cdots a_{N_e})$$

$$i = 1, \cdots N_{MI}$$

Good News $N_{MI} << N_I$

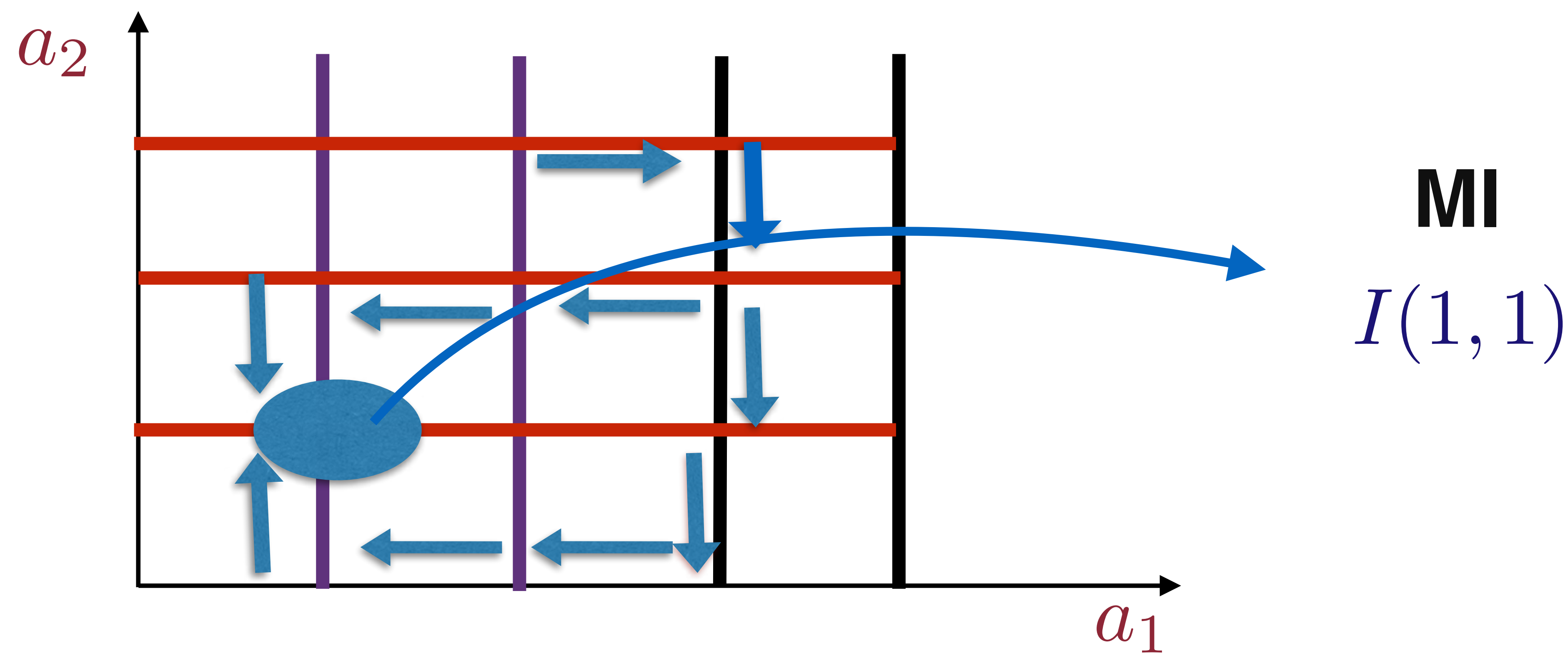
Integration By Parts (IBP)

$$I(a_1, a_2) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2)^{a_1}((k+p)^2)^{a_2}}$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{\partial}{\partial k_\mu} \left(\frac{v_\mu}{(k^2)^{a_1}((k+p)^2)^{a_2}} \right) = 0$$

$$v = k, p$$

$$I(a_1, a_2) = \frac{a_1 + a_2 - n - 1}{p^2(a_2 - 1)} I(a_1, a_2 - 1) + \frac{1}{p^2} I(a_1 - 1, a_2)$$



Lorentz Invariant Identities

Integrals are Lorentz scalars

$$\omega^{\mu\nu} = -\omega^{\nu\mu}$$

$$p_i^\mu \to p_i^\mu + \delta p_i^\mu = p_i^\mu + \omega^{\mu\nu} p_\nu ,$$

$$I(p_i + \delta p_i) = I(p_i) + \omega^{\mu\nu} \sum_j p_{j,\nu} \frac{\partial}{\partial p_j^\mu} I(p_i) = I(p_i)$$

Anti-symmetry

$$\omega^{\mu\nu} = -\omega^{\nu\mu}$$

$$\sum_i p_{i,[\mu} \frac{\partial}{\partial p_i^{\nu]}} I(p_i) = 0$$

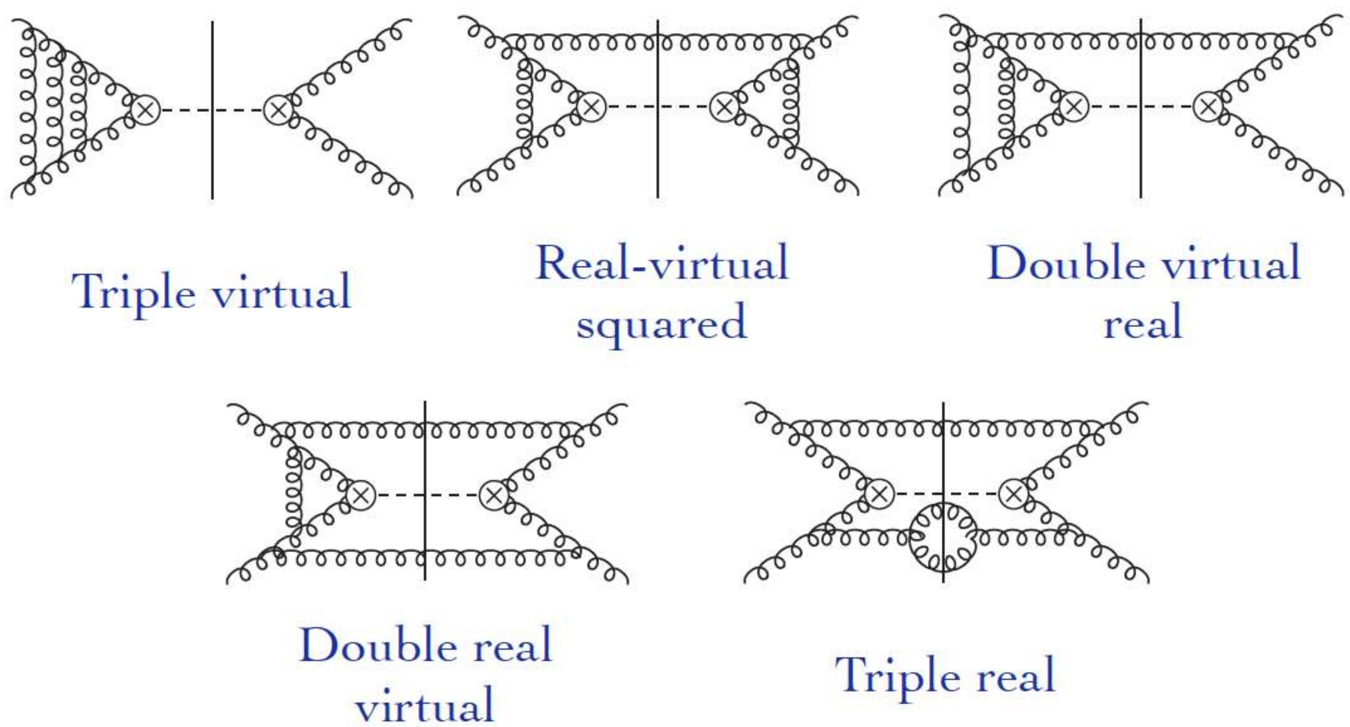
Anti-symmetry of $p_j^{[\mu} p_k^{\nu]}$

$$p_j^{[\mu} p_k^{\nu]} \sum_i p_{i,[\mu} \frac{\partial}{\partial p_i^{\nu]}} I(p_i) = 0$$

Higgs production to N^3LO in QCD at the LHC

$Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger$

100 000 diagrams



	Integrals	Master Integrals
NNLO	50 000	27
N3LO	517 531 178	1028

Integration By Parts

$$\int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_3}{(2\pi)^d} \frac{\partial}{\partial k_i} \cdot \left(v_j \frac{1}{\prod_l D_l^{n_i}} \right) = 0$$

Lorentz Invariance

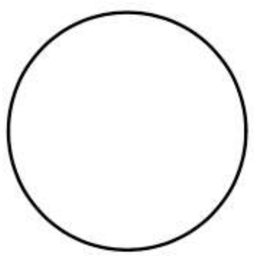
$$p_i^\mu p_j^\nu \left(\sum_k p_{k[\nu} \frac{\partial}{\partial p_k^{\mu]}} \right) J(\vec{n}) = 0.$$

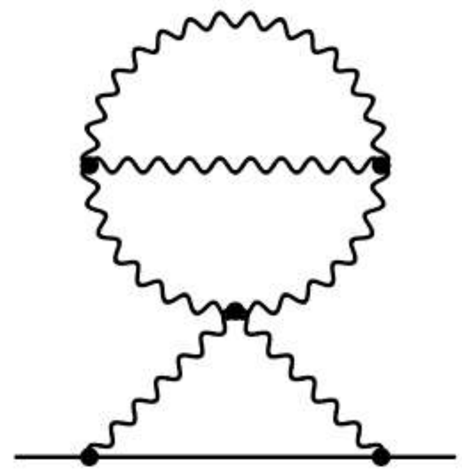
↓

Master Integrals

Symmetry relations

Scaless integrals are zero





$= 0$

$$\int \frac{d^d k}{(-k^2 - i0)^n} = 0\,.$$

Under Translation

$$k_i \to \sum_i c_{il} k_l + \sum_j d_{ij} p_j$$

$$\prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n}$$

Jacobian is invariant

Exchange of external momenta

$$p_i \leftrightarrow p_j$$

Master Integrals

Start with set of scalar integrals $D_i = q_i^2 + i\epsilon$ $\quad q_i = \sum_j k_l + \sum_l p_j$

$$\left\{ \prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n} \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_{N_d}^{a_{N_d}}} \right\}$$

IBP and LI identities give Master Integrals

$$I_{\vec{b}}(s_{ij}, n) = \prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n} \frac{1}{D_1^{b_1} D_2^{b_2} \cdots D_{N_f}^{b_{N_f}}}$$

$$s_{ij} = (p_i + p_j)^2$$

$$\vec{b} = (b_1, b_2, \cdots, b_{N_f})$$

Solving Master Integrals - Second Revolution

Consider a Master Integral

$$D_i = q_i^2 + i\epsilon \qquad q_i = \sum_j k_l + \sum_l p_j$$

$$I(s_{ij}, n) = \prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n} \frac{1}{D_1 D_2 \cdots D_{N_f}}$$

Define $s_{12} = s$ $\qquad s = (p_1 + p_2)^2$

Differential w.r.t s

$$s \frac{\partial}{\partial s} I(s_{ij}, n) = \prod_{i=1}^{N_k} \int \frac{d^n k_i}{(2\pi)^n} \boxed{s \frac{\partial}{\partial s}} \left(\frac{1}{D_1 D_2 \cdots D_{N_f}} \right)$$

Generate System of Linear Differential Equations for Sij

Knowing Boundary integrals give results for MIs

Master Integrals

$$s\frac{\partial}{\partial s}\left(\frac{1}{D_j}\right)$$

$$D_i = q_i^2 + i\epsilon \qquad q_i = \sum_j k_l + \sum_l p_j$$

Expand

$$s\frac{\partial}{\partial s} = \sum_{ij} a_{ij}^{(s)} p_i^\mu \frac{\partial}{\partial p_j^\mu}$$

a_{ij} functions of $s_{ij} = (p_i + p_j)^2$

Determine a_{ij} using

$$s\frac{\partial}{\partial s} p_i^2 = 0 \qquad i = 1, \cdots N_e - 1$$

$$s\frac{\partial}{\partial s} s = s \qquad s\frac{\partial}{\partial s} s_{ij} = 0 \qquad s_{ij} \neq s$$

Master Integrals

$$s\frac{\partial}{\partial s} = \sum_{ij} a_{ij}^{(s)} p_i^{\mu} \frac{\partial}{\partial p_j^{\mu}}$$: $$\frac{1}{(k+\sum_i a_i p_i)^2}$$

$$\frac{\partial}{\partial p_{j,\mu}} \frac{1}{(k+\sum_i a_i p_i)^2} = -\frac{2a_j (k+\sum_i a_i p_i)^{\mu}}{[(k+\sum_i a_i p_i)^2]^2}$$

$$k\cdot p_i = \sum_j d_{ij} D_j + \sum_j h_{ij} s_{ij}$$

$$s\frac{\partial}{\partial s}\frac{1}{D_i} = \sum_j b_{ij}\left(\frac{1}{D_j}\right)^{d_j}$$

Master Integrals

$$s\frac{\partial}{\partial s}I(s_{ij},n) = \sum_{l=1}^{N_J} h_l^{(s)}(s_{ij},n)\ J_l(s_{ij},n)$$

- RHS - NOT master integrals

Using IBP and LI identities

$$s\frac{\partial}{\partial s}I(s_{ij},n) = \sum_{l=1}^{N_{MI}} C_l^{(s)}(s_{ij},n)\ I_l(s_{ij},n)$$

I_l - Master Integrals

$C_l^{(s)}(s_{ij},n)$ - Rational functions

Master Integrals

Generalization with set of MIs

$$\vec{I} = \Big(I_1, I_2, \cdot, \cdot, \cdot, I_N\Big)$$

$\{I_i(\vec{x})\}$ depend on Scaling variables

$$\vec{x} = \Big(x_1, x_2, \cdot, \cdot, \cdot, x_M\Big)$$

$$x_i = f_i\left(\frac{s_{ij}}{Q^2}\right)$$

Differential equation:

$$d\vec{I} = \sum_{i=1}^{M} \mathbf{A}_i dx_i\ \vec{I}$$

$$\frac{\partial}{\partial x_i}\begin{bmatrix} I_1 \\ \cdot \\ I_N \end{bmatrix} = \begin{bmatrix} \mathbf{A_{11}} & \cdots & \mathbf{A_{1N}} \\ \cdot & \cdots & \cdot \\ \mathbf{A_{N1}} & \cdots & \mathbf{A_{NN}} \end{bmatrix} \begin{bmatrix} I_1 \\ \cdot \\ I_N \end{bmatrix}$$

$$i = 1, 2, \cdot, \cdot, \cdot M$$

Master Integrals

Consider Diff equation:

$$s\frac{\partial}{\partial s}I(s,n) = A(s,n)I(s,n)$$

Expand around n = 4

$$I(s,n) = I^{(0)}(s) + (n-4)I^{(1)}(s) + \mathcal{O}((n-4)^2)$$

$$A(s,n) = A^{(0)}(s) + (n-4)A^{(1)}(s) + \mathcal{O}((n-4)^2)$$

$$s\frac{\partial}{\partial s}I^{(0)}(s) = A^{(0)}(s)I^{(0)}(s) \qquad \text{——(1)}$$

Solution

$$I^{(0)}(s) = I^{(0)}(s_0)e^{\int_{s_0}^{s}\frac{d\lambda}{\lambda}A^{(0)}(\lambda)}$$

$$s\frac{\partial}{\partial s}I^{(1)}(s) = A^{(0)}(s)I^{(1)}(s) + A^{(1)}(s)I^{(0)}(s) \qquad \text{——(2)}$$

Master Integrals

Consider Diff equation:

$$d\vec{I} = \sum_{i=1}^{M} \mathbf{A}_i dx_i \; \vec{I}$$

with

$$\begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1N} \\ \cdot & \cdots & \cdot \\ \mathbf{A}_{N1} & \cdots & \mathbf{A}_{NN} \end{bmatrix}$$

Under U Transformation

$$\vec{I} = \mathbf{U} \cdot \vec{\overline{I}}$$

$$d\vec{\overline{I}} = \sum_{i=1}^{M} \overline{\mathbf{A}} \; dx_i \; \vec{\overline{I}}$$

where

$$\overline{\mathbf{A}} = U^{-1}\mathbf{A}U - U^{-1}dU$$

$$= \begin{bmatrix} \overline{\mathbf{A}}_{11} & \overline{\mathbf{A}}_{12} & \cdots & \overline{\mathbf{A}}_{1N} \\ 0 & \overline{\mathbf{A}}_{22} & \cdots & \overline{\mathbf{A}}_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \overline{\mathbf{A}}_{NN} \end{bmatrix}$$

- Triangular Matrix

Canonical/Henn's Basis

Consider Diff equation:

$$d\vec{I}(\vec{x},n) = \sum_i \mathbf{A}_i(\vec{x},n)dx_i \ \vec{I}(\vec{x},n)$$

Choose U Transformation such that

$$U^{-1}\mathbf{A}(\vec{x},n)U - U^{-1}dU = (n-4)\ \overline{\mathbf{A}}(\vec{x})$$

Diff equation contains 'n' independent A

$$d\vec{\overline{I}}(\vec{x},n) = (n-4)\sum_i \overline{\mathbf{A}}_i(\vec{x})dx_i \ \vec{\overline{I}}(\vec{x},n)$$

Solution

$$\vec{\overline{I}}(\vec{x},n) = \vec{\overline{I}}(\vec{x}_0,n)\ \mathbf{P}\exp\left((n-4)\int\frac{d\lambda}{\lambda}\overline{\mathbf{A}}(\lambda)\right)$$

$\mathbf{P}$ - Path Ordered exponential

Canonical/Henn's Basis

Start with Henn's Diff equation:

$$s\frac{\partial}{\partial s}\overline{I}(s,n) = (n-4)\overline{A}(s)\ \overline{I}(s,n)$$

If $\overline{A}$ contains poses at s_i $\qquad \overline{A}(s) = \sum_i \frac{\tilde{A}_i(s_i)}{s-s_i}$

$$\overline{I}(s,n) = \overline{I}^{(0)}(s_0) + (n-4)\sum_i \tilde{A}_i(s_i)\log\left(\frac{s-s_i}{s_0-s_i}\right)$$

$$+(n-4)^2\sum_i \tilde{\tilde{A}}(s_i)\mathcal{L}_i(s_0,s_i) + \cdots$$

Polylogarithms - Uniform transcendental terms

Iterated Integrals

$$\int^{s_m} \frac{ds_{m-1}}{s_{m-1}} \cdots \int^{s_3} \frac{ds_2}{s_2} \int^{s_2} \frac{ds_1}{s_1} \frac{1}{s_1 - s_i}$$

★ $$\int^{s} \frac{ds_1}{s_1} \frac{1}{s_1 - s_i} \rightarrow \log(s - s_i)$$

★ $$\int^{s} \frac{ds_1}{s_1} \log(s_1 - s_i) \rightarrow \log^2(s - s_i) \quad \text{or} \quad \mathcal{L}_2(s - s_i)$$

• • •

★ $$\int^{s} \frac{ds_1}{s_1} \mathcal{L}_m(s_1 - s_i) \rightarrow \mathcal{L}_{m+1}(s - s_i)$$

- PolyLog of weight "m" have transcendental weight "m"

Harmonic Polylogarithms (HPL)

One dimensional HPLs

For $w > 1$, $H(m, \vec{m}_w; y)$ is defined by

$$H(m, \vec{m}_w; y) \equiv \int_0^y dx\ f(m, x)\ H(\vec{m}_w; x), \qquad m \in 0, \pm 1\,.$$

Alphabets:

$$f(1; y) \equiv \frac{1}{1-y}, \qquad f(0; y) \equiv \frac{1}{y}, \qquad f(-1; y) \equiv \frac{1}{1+y}\,.$$

$$H(1, y) \equiv -\ln(1-y), \qquad H(0, y) \equiv \ln y, \qquad H(-1, y) \equiv \ln(1+y)\,.$$

Two dimensional HPLs

$$f(2; y) \equiv f(1-z; y) \equiv \frac{1}{1-y-z}, \qquad f(3; y) \equiv f(z; y) \equiv \frac{1}{y+z}$$

$$H(2, y) \equiv -\ln\left(1 - \frac{y}{1-z}\right), \qquad H(3, y) \equiv \ln\left(\frac{y+z}{z}\right).$$

2-Dimensional HPLs

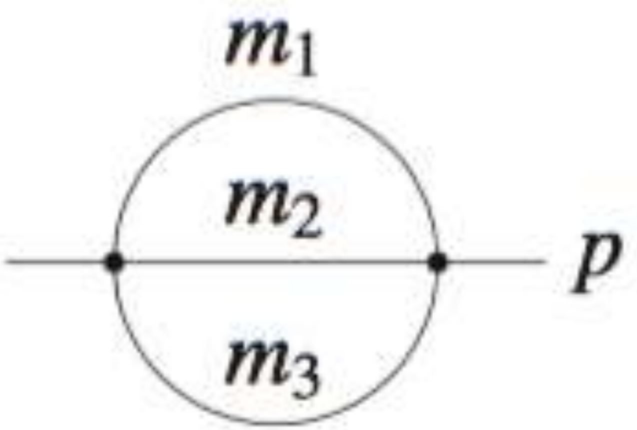
Integration by Parts Identities

$$
\begin{aligned}
H(\vec{m}_w; y) \equiv H(m_1, m_2, ..., m_w; y) &= H(m_1, y) H(m_2, ..., m_w; y) \\
&- H(m_2, m_1, y) H(m_3, ..., m_w; y) \\
&+ ... + (-1)^{w+1} H(m_w, ..., m_2, m_1; y) .
\end{aligned}
$$

Shuffle Algebra

$$
H(\vec{m}_{w_1}; y) H(\vec{m}_{w_2}; y) = \sum_{\vec{m}_w = \vec{m}_{w_1} \uplus \vec{m}_{w_2}} H(\vec{m}_w; y).
$$

Elliptic Integrals



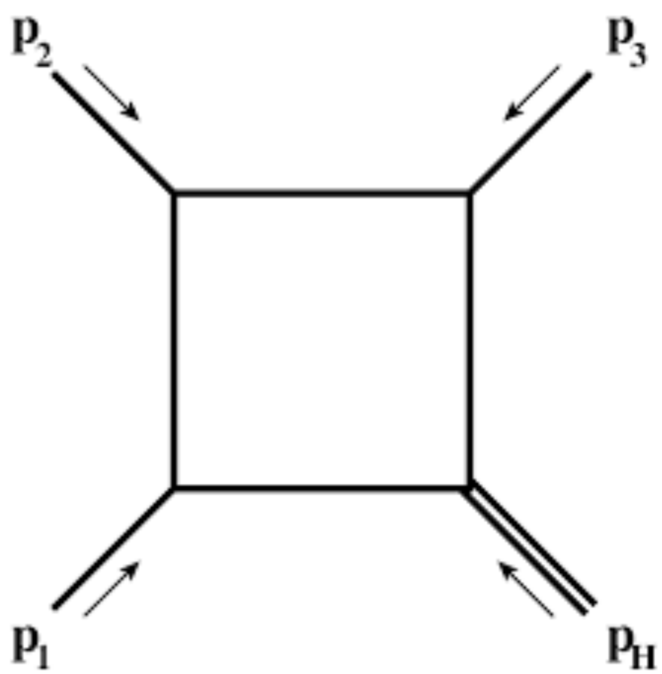
$$\int_{x_j \geq 0} d^3x \; \delta(1 - \sum x_j) \; \frac{1}{\mathcal{D}(\vec{x}, m_i)}$$

$$\mathcal{D}(\vec{x}, m_i) = -x_1 x_2 x_3 t + (x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^3) \times (x_1 x_2 + x_2 x_3 + x_3 x_1), \qquad t = p^2$$

$$\mathrm{E}_{2;0}(x;y;q) = \frac{1}{i}\left[\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2\left(x^{-1}\right) + \mathrm{ELi}_{2;0}(x;y;q) - \mathrm{ELi}_{2;0}\left(x^{-1};y^{-1};q\right)\right]$$

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}. \qquad \mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk}.$$

IBP for box



$$I(a_1, a_2, a_3, a_4) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4}}$$

$$s_{ij} = (p_i + p_j)^2 \qquad p_i^2 = 0, \qquad i = 1, 2, 3$$

$$\begin{aligned} D_1 &= k^2 \\ D_2 &= (k + p_1)^2 \\ D_3 &= (k + p_1 + p_2)^2 \\ D_4 &= (k + p_1 + p_2 + p_3)^2 \end{aligned}$$

IBP equation

$$\int \frac{d^n k}{(2\pi)^n} \frac{\partial}{\partial k_\mu} \left(\frac{v_\mu}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4}} \right) = 0$$

$$\begin{aligned} v &= k \\ v &= k + p_1 \\ v &= k + p_1 + p_2 \\ v &= k + p_1 + p_2 + p_3 \end{aligned}$$

IBP for box

IBP identity:

$$-\mathbf{1}^-\left(a_2\mathbf{2}^+ + a_3\mathbf{3}^+ + a_4\mathbf{4}^+\right) + s a_3\mathbf{3}^+ + q^2 a_4\mathbf{4}^+ + \nu_{11234} = 0$$

+

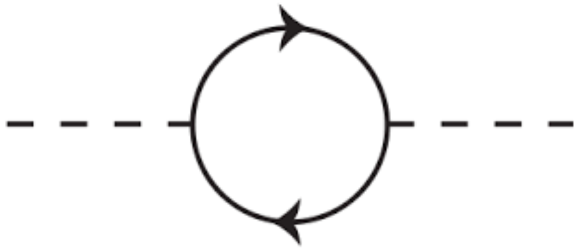
3 more identities

$$\mathbf{j}^{\pm} I(a_1, a_j, a_3, a_4) = I(a_1, a_j \pm 1, a_3, a_4)$$

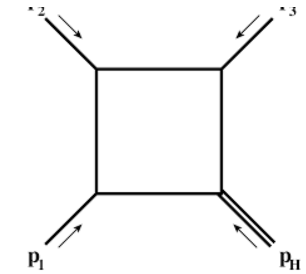
$$\nu_{jj234} = n - a_j - a_j - a_2 - a_3 - a_4$$

Master Integrals

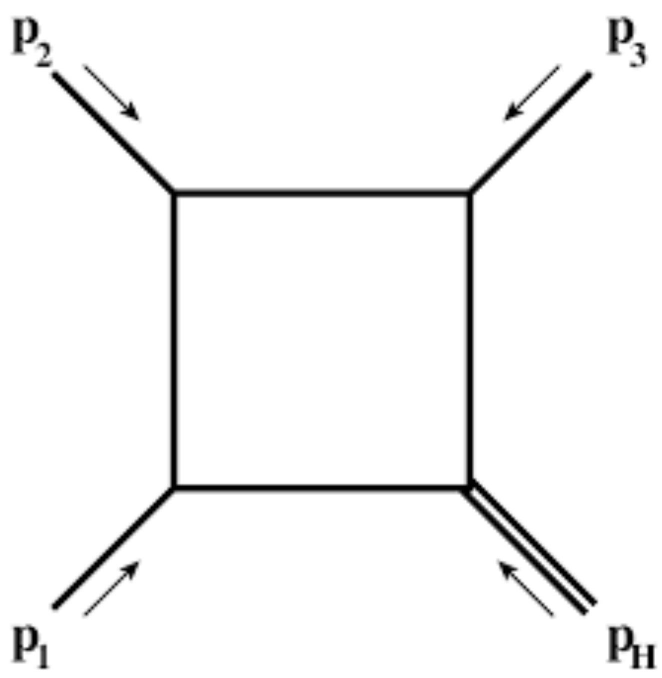
$$I(0,1,0,1),\quad I(1,0,0,1),\quad I(1,0,1,0),$$



$$I(1,1,1,1)$$



Differential equations for box



$$I(a_1,a_2,a_3,a_4) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4}}$$

$$s_{ij} = (p_i + p_j)^2 \qquad p_i^2 = 0, \qquad i = 1,2,3$$

$$\begin{aligned} D_1 &= k^2 \\ D_2 &= (k+p_1)^2 \\ D_3 &= (k+p_1+p_2)^2 \\ D_4 &= (k+p_1+p_2+p_3)^2 \end{aligned}$$

Differential operator

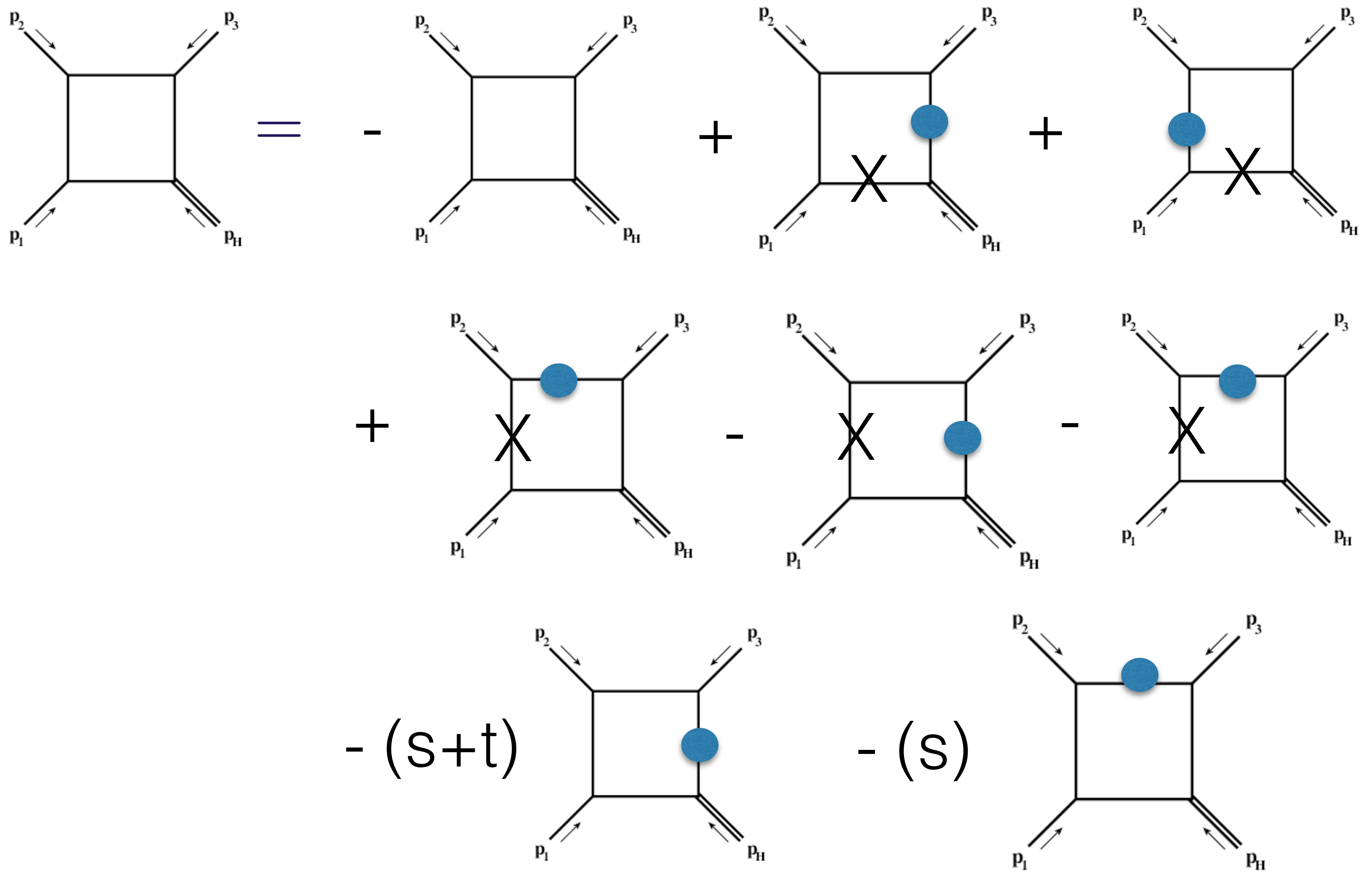
$$\frac{\partial}{\partial s_{12}} = \frac{1}{2 s_{23}} \left[b_{11}\ p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + b_{22}\ p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + b_{33}\ p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} \right]$$

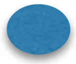
$$\begin{aligned} b_{11} &= s_{23} + s_{12} \\ b_{22} &= s_{23} - s_{12} \\ b_{33} &= -(s_{23} + s_{12}) \end{aligned}$$

Differential equations for box

Differential operator

$$p_{1,\mu}\frac{\partial}{\partial p_{1,\mu}}$$

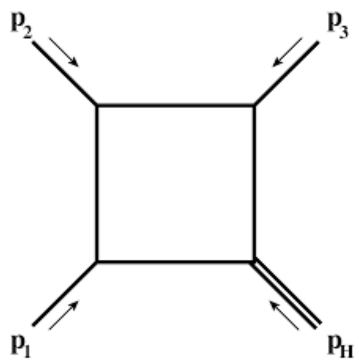


 Increase the power of propagator

X Decrease the power of propagator

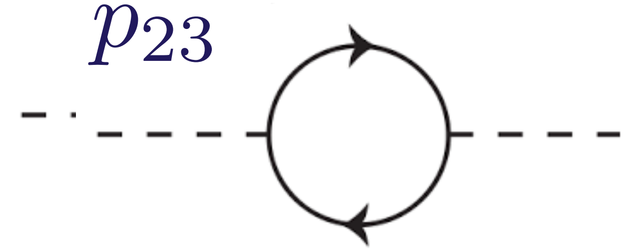
Differential equations for box

$$s_{12}\frac{\partial}{\partial s_{12}}$$



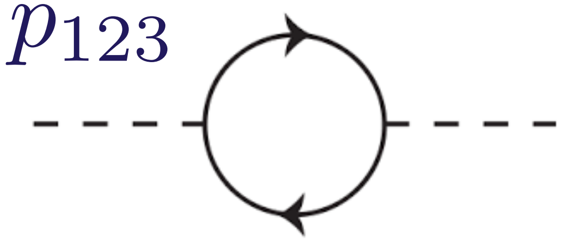
$$= \frac{2(n-3)}{s_{23}(s_{12}+s_{13})}$$

p_{23}



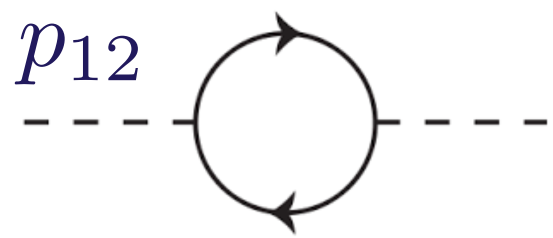
$$+ \frac{2(n-3)(s_{12}-s_{23})}{s_{23}(s_{12}+s_{13})(s_{13}+s_{23})}$$

p_{123}

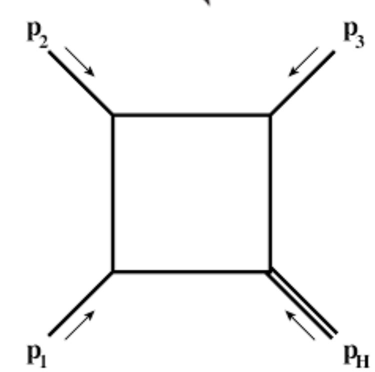


$$+ \frac{2(3-n)}{s_{23}(s_{23}+s_{13})}$$

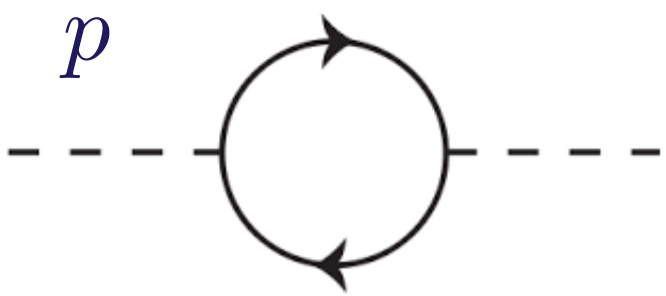
p_{12}



$$+ \frac{(6-n)(s_{12}-s_{23})}{2s_{23}}$$



Differential equations for box



$$= \frac{(4\pi)^{\frac{4-d}{2}}}{16\pi^2}\,\frac{\Gamma\left(3-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma\left(d-3\right)}\left(\frac{-2i}{(d-4)(d-3)}\right)\left(-p^2\right)^{\frac{d-4}{2}}$$

$$x\frac{\partial}{\partial x}f(x) = g(x)f(x) + h(x)$$

Solution to homogeneous part:

$$f_H(x) = f_H(x_0)\exp\left(\int_{x_0}^{x}\frac{d\lambda}{\lambda}g(\lambda)\right)$$

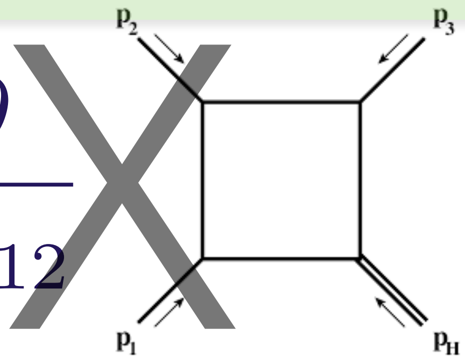
Solution:

$$f(x) = f_H(x)\int\frac{dx}{x}\left(\frac{h(x)}{f_H(x)}\right) + C$$

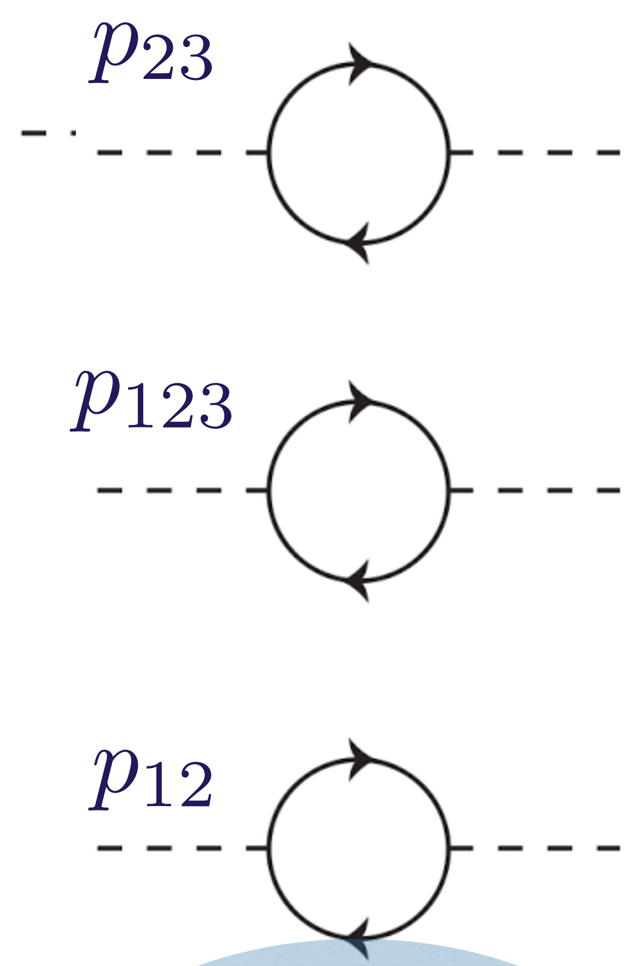
Differential equations for box

Boundary Condition s12 = 0

$$s_{12}\frac{\partial}{\partial s_{12}}$$

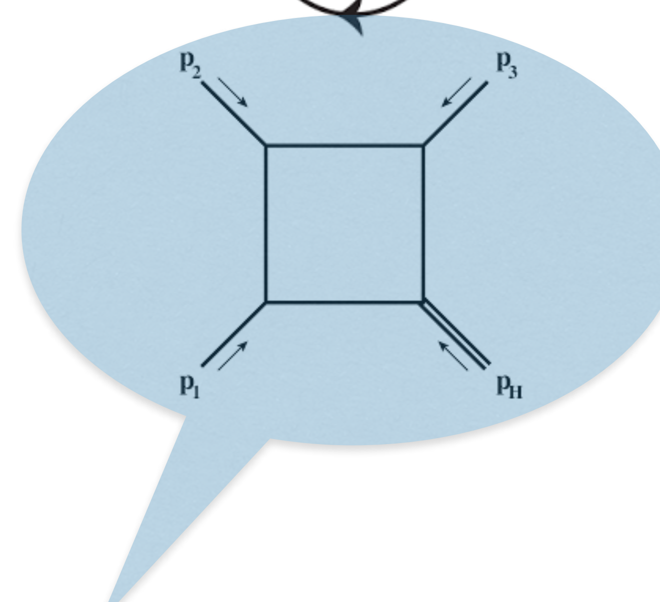


$$= \frac{2(n-3)}{s_{23}(s_{12}+s_{13})} + \frac{2(n-3)(s_{12}-s_{23})}{s_{23}(s_{12}+s_{13})(s_{13}+s_{23})} + \frac{2(3-n)}{s_{23}(s_{23}+s_{13})} + \frac{(6-n)(s_{12}-s_{23})}{2s_{23}}$$



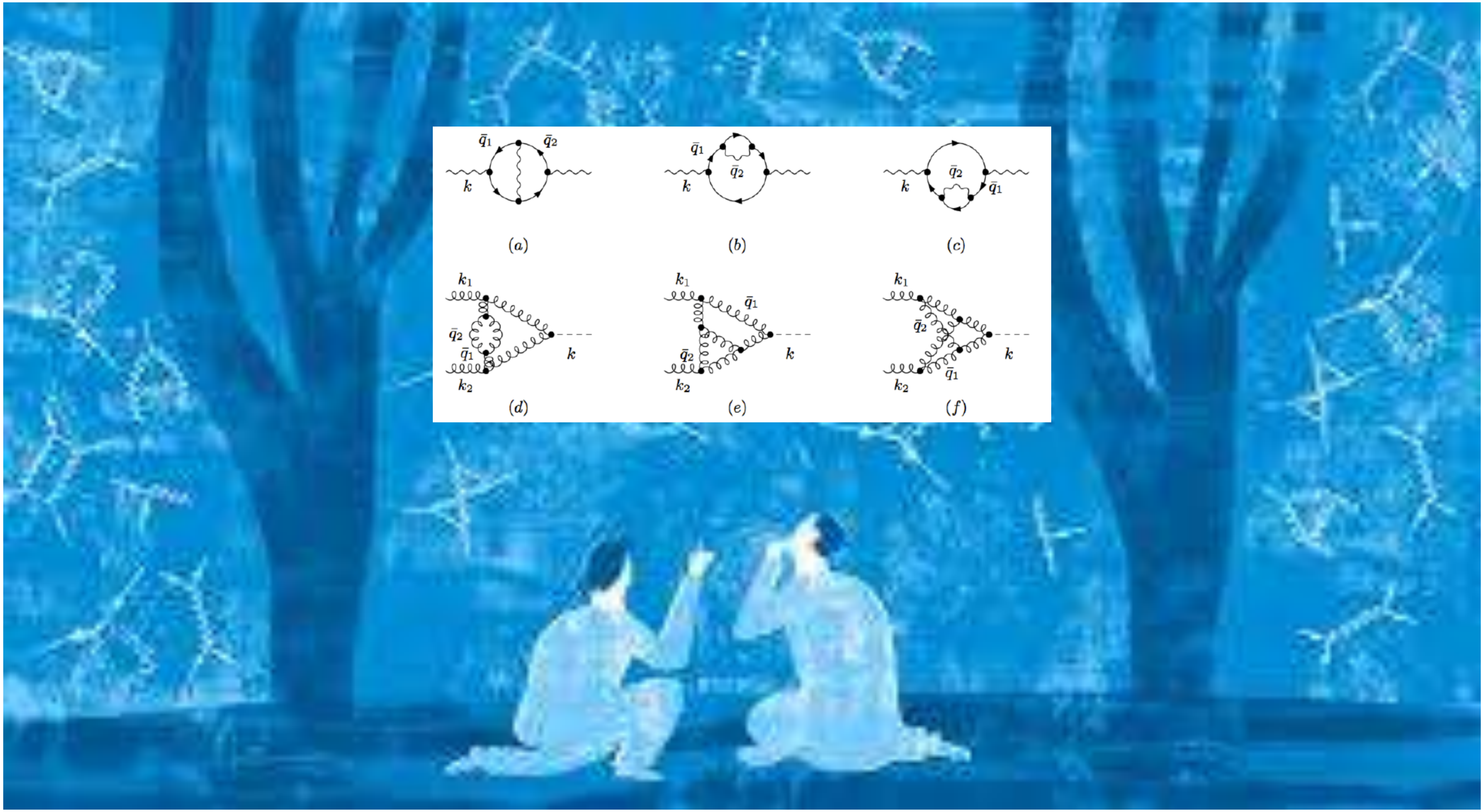
0

S12 =0



S12 =0

NNLO



NNLO subtraction Methods

Radja Boughezal

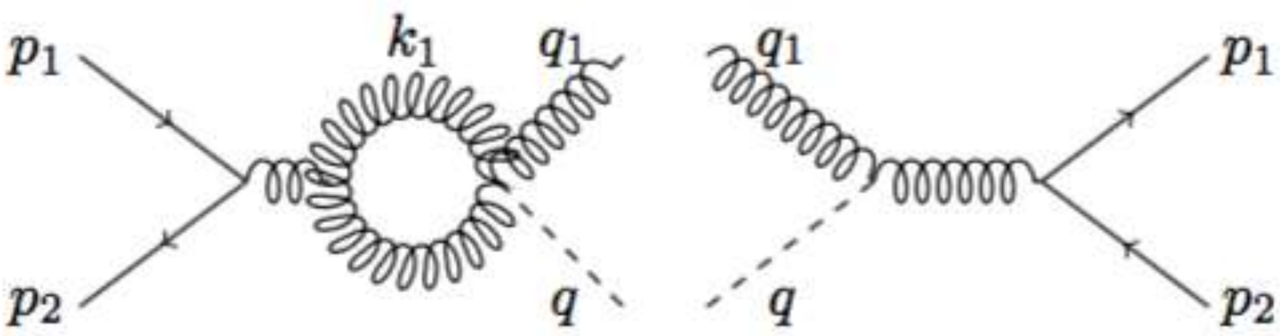
Local subtraction schemes:

- Sector decomposition (Anastasiou, Melnikov, Petriello, 2003)
  - $pp \to H, pp \to V$ including decays

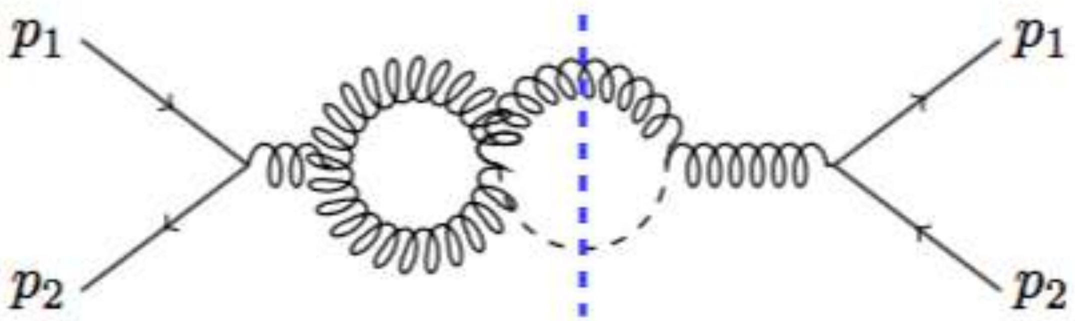
  (Anastasiou, Melnikov, Petriello, 2003-2004)
- Sector-improved subtraction schemes (Czakon, 2010; R.B., Melnikov, Petriello, 2
  - $pp \to t\bar{t}$ (Czakon, Fiedler, Mitov, 2013)
  - $pp \to H + j$ (R.B., Caola, Melnikov, Petriello, Schulze, 2013-2015)
- Antenna subtraction (Gehrmann-De Ridder, Gehrmann, Glover, 2005)
  - $ee \to 3j$ (Gehrmann-De Ridder, Gehrmann, Glover, Heinrich, 2007; Weinzierl, 20
  - $pp \to jj$ partial (Gehrmann-de Ridder, Gehrmann, Glover, Pires, 2013)
  - $pp \to H + j$ gg-only (Chen, Gehrmann, Glover, Jaquier, 2014)
  - $pp \to t\bar{t}$ partial (Abelof, Gehrmann-de Ridder, Maierhofer, Majer, Pozzorini, 20
- 'Colorful NNLO' (Del Duca, Somogyi, Trocsanyi 2005)
  - $H \to b\bar{b}$ (Del Duca, Duhr, Somogyi, Tramontano, Trocsanyi 2015)

NNLO corrections:

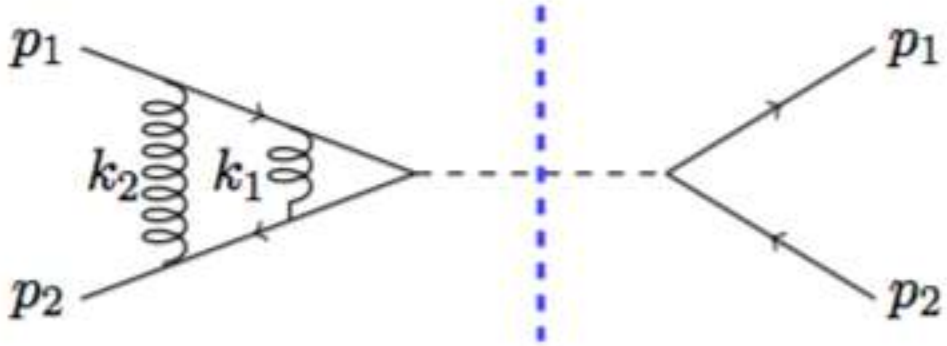
Virtual - Real



$$\propto \int \frac{d^n q_1}{(2\pi)^{n-1}} \frac{d^n k_1}{(2\pi)^n} \delta_+(q_1^2)\delta_+(q^2 - m_h^2)[\cdots]$$



Pure Virtual



$$\delta_+(q^2 - m^2) \sim \frac{1}{q^2 - m^2 + i\varepsilon} - \frac{1}{q^2 - m^2 - i\varepsilon}$$

$$\propto \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \delta_+(q^2 - m_h^2)[\cdots]$$

NNLO corrections:

Integration by Parts

$$\int \frac{d^dk_1}{(2\pi)^d} \cdots \int \frac{d^dk_L}{(2\pi)^d} \frac{\partial}{\partial k_i} \cdot \left(v_j \frac{1}{\prod_l \mathcal{D}_l^{n_l}} \right) = 0$$

IBP identities

$$\sum_i a_i J(b_{i,1} + n_1, ..., b_{i,N} + n_N) = 0$$

where

$$J(\vec{m}) = J(m_1, \cdots, m_N) = \int \frac{d^dk_1}{(2\pi)^d} \cdots \frac{d^dk_L}{(2\pi)^d} \frac{1}{\prod_l \mathcal{D}_l^{m_l}}$$

Master Phase Space Integrals

Infrared divergences

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

Quantum Field Theories with massless particles encounter two kinds of divergences:

Soft :

On-shell amplitudes in gauge theories contain Soft divergences due to massless gauge bosons.

Collinear :

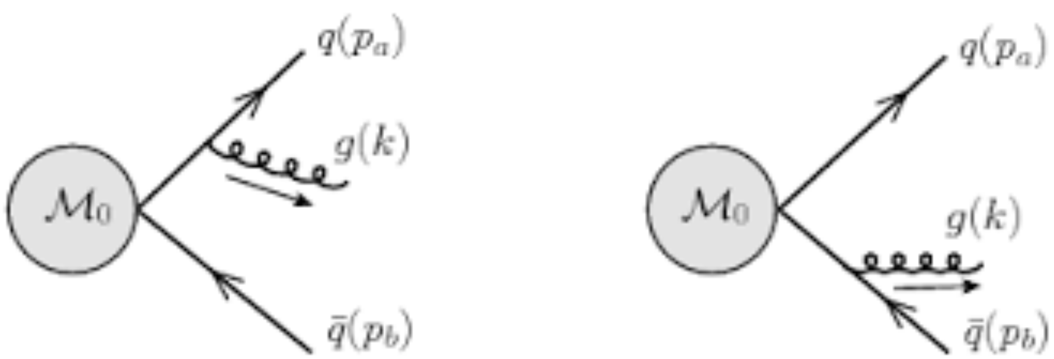
If the matter fields in the theory are light (mass of the particles are negligible compared to hard scale of the process), there will be mass singularities, called Collinear divergences

Infrared divergences

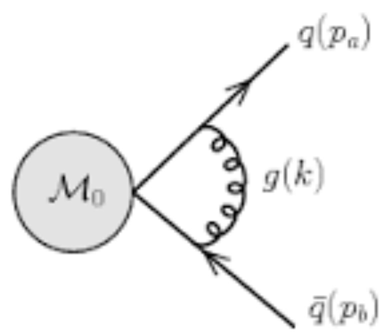
[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

In the Limit $k \to p \;\; (p_a \text{ or } \mathrm{p_b})$ $\quad m_a, m_b << Q$

Real emission



Virtual



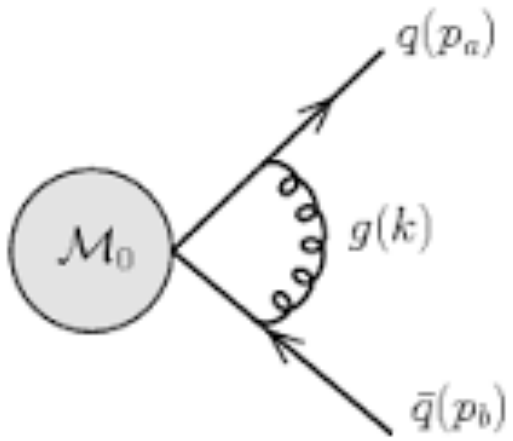
$$\frac{1}{(p+k)^2} = \frac{1}{2p^0 k^0 \, (1 - \cos\theta)}$$

$k^0 \to 0$	Soft divergence
$\cos\theta \to 0$	Collinear divergence

Sudakov Form Factor

One loop on-shell form factor

$$(p-k)^2 = 0,$$
$$p_a^2 = p_b^2 = m^2 << q^2$$



Soft

$k \to 0$

Collinear

$p_a || k \ or \ p_b || k$

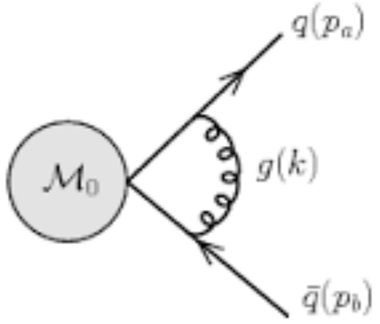
$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2((p_a+k)^2 - m^2)((p_b-k)^2 - m^2)} \to \infty$$

Ill defined

Virtual effect

One loop on-shell form factor

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2((p_a+k)^2-m^2)((p_b-k)^2-m^2)} \to \infty$$



$$p_a^2 = p_b^2 = m^2 << q^2$$

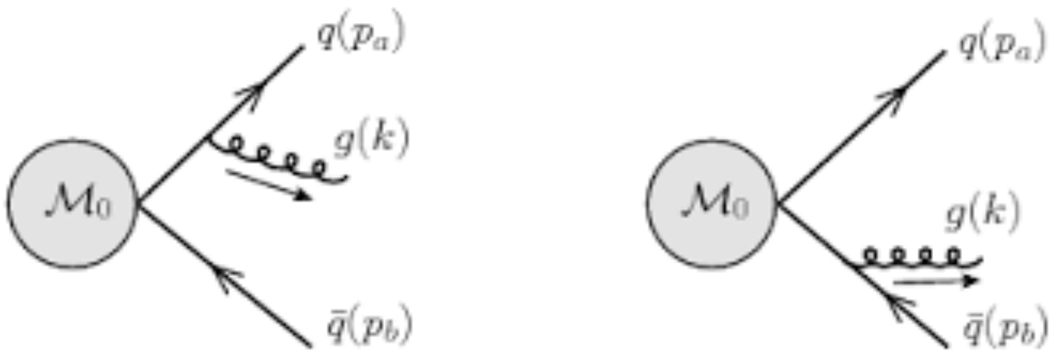
Summing to all orders in g^2

$$1 - g^2\left(\infty\right) + \frac{1}{2!}g^4\left(\infty\right) - \frac{1}{3!}g^6\left(\infty\right) + \cdots = exp(-g^2\infty)$$

Probability to happen this is ZERO

Real emission

Real photon emission:



$$\int \frac{d^4k}{(2\pi)^4} \frac{\delta^+(k^2)}{((p_a+k)^2-m^2)((p_b-k)^2-m^2)} \to \infty$$

Summing multiple emissions

$$p_a^2 = p_b^2 = m^2 << q^2$$

$$1 + g^2\infty + g^4\infty + \cdots$$

Probability grows uncontrollably

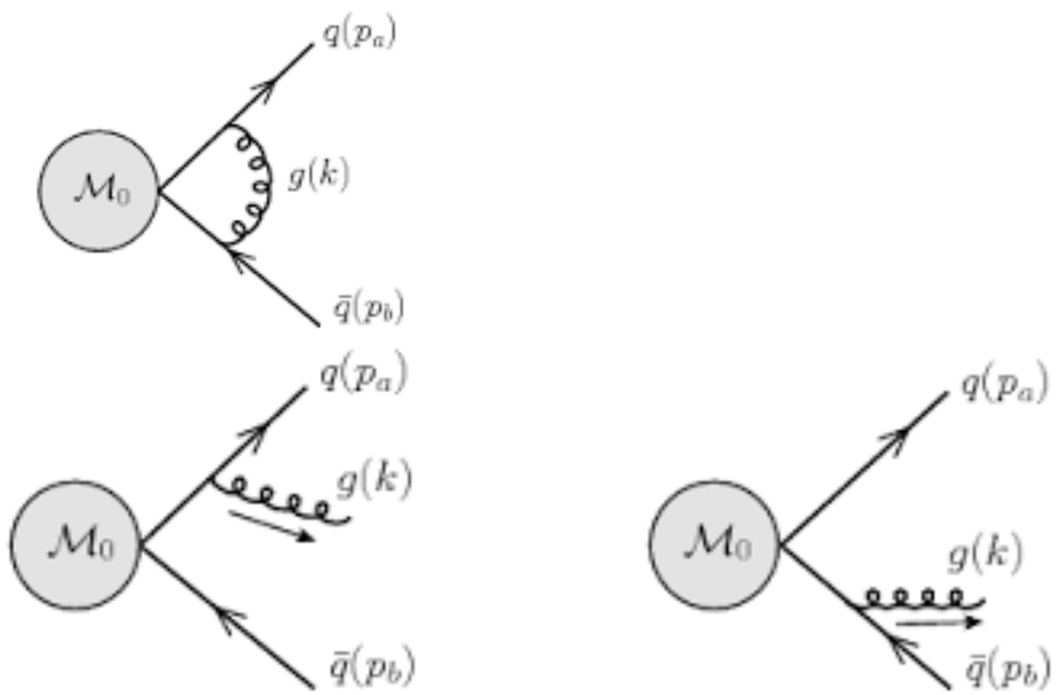
"Weinberg Fear"

Indistinguishable states

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

If the detector is not sensitive to photons below certain energy Es (soft ones)

Below this energy the Detector can not distinguish these two processes when the gluons are soft/collinear



Indistinguishable when soft or collinear

Sum their contributions and it is finite but dependent on Es !

Infrared Safety

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

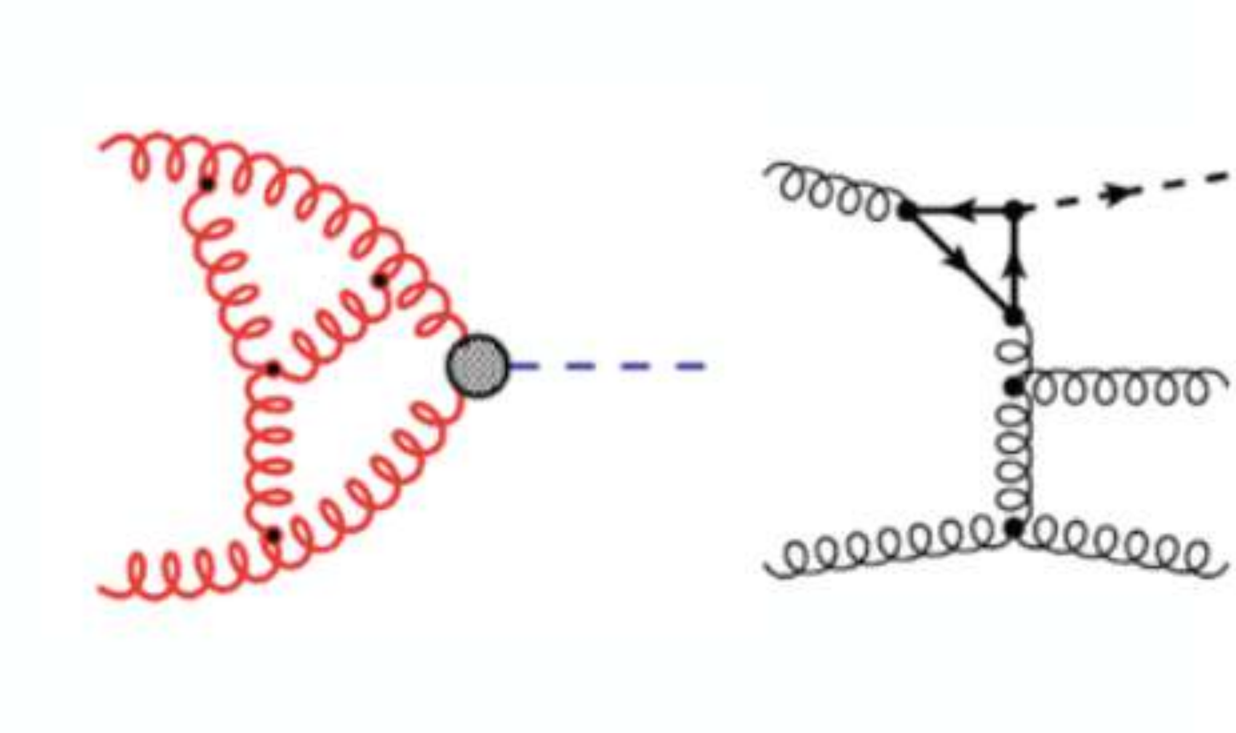
Physical processes that happen at Long distances are responsible for these divergences.

Measurable quantities are not sensitive to soft and Collinear divergences

REASON

Long distance physics is associated to configurations that are experimentally indistinguishable

Multi-loops and Multi-legs

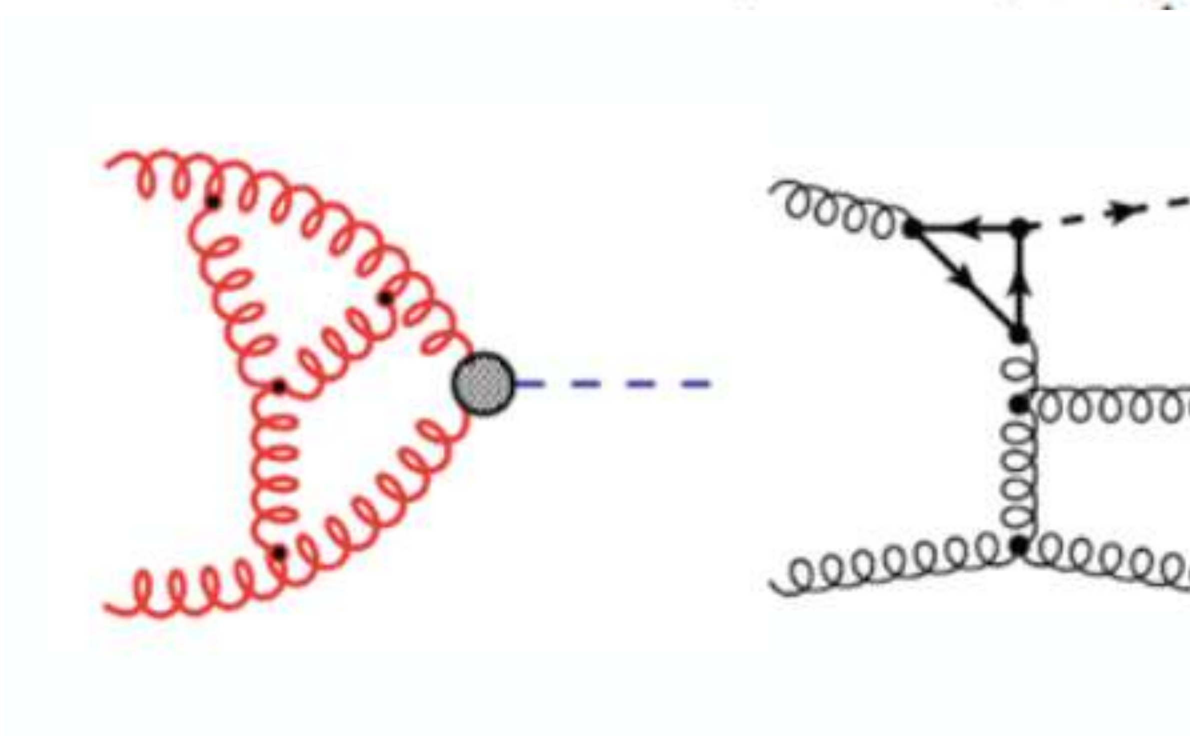


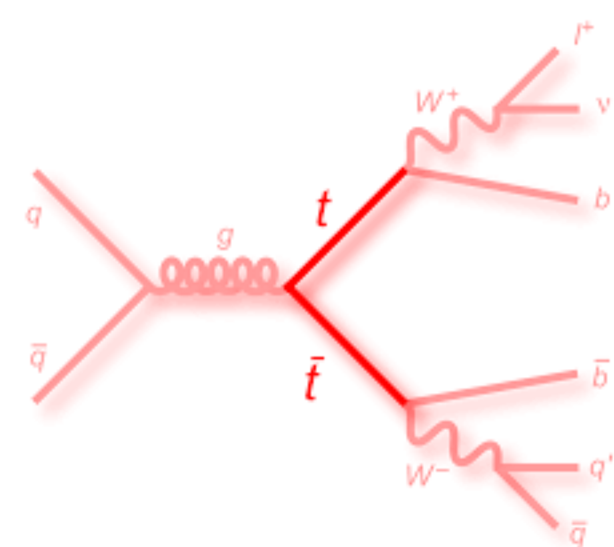
Catani's proposal

[Yennie, Frautschi, Subram; Weinberg]

UV Renormalised on-shell QCD amplitudes

$$|\mathcal{M}_n(\epsilon, \{p\})\rangle$$





Universal Infrared Structure

Catani's proposal

[S. Catani]

Universal IR Subtraction Operator

Up to Two loop !

$$\left[1-\frac{\alpha_s}{2\pi}\boldsymbol{I}^{(1)}(\epsilon)-\left(\frac{\alpha_s}{2\pi}\right)^2\boldsymbol{I}^{(2)}(\epsilon)\right]|\mathcal{M}_n(\epsilon,\{p\})\rangle$$

IR Finite

$$\boldsymbol{I}^{(1)}(\epsilon)=\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_i\left(\frac{1}{\epsilon^2}+\frac{g_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right)\sum_{j\neq i}\frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon}$$

$$\boldsymbol{I}^{(2)}(\epsilon)=\frac{e^{-\epsilon\gamma_E}\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(K+\frac{\beta_0}{2\epsilon}\right)\boldsymbol{I}^{(1)}(2\epsilon)-\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon)\left(\boldsymbol{I}^{(1)}(\epsilon)+\frac{\beta_0}{\epsilon}\right)+\boldsymbol{H}^{(2)}_{\mathrm{R.S.}}(\epsilon)$$

Colour matrices satisfy

$$\sum_i \boldsymbol{T}_i\,|\mathcal{M}_n(\epsilon,\{p\})\rangle=0.$$

Catani's proposal

[Catani]

Upto Two loop !

$$\left[1 - \frac{\alpha_s}{2\pi} \boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon)\right] |\mathcal{M}_n(\epsilon, \{p\})\rangle$$

Universal IR Subtraction Operators
depend only on
Process independent

Soft and Collinear

Anomalous Dimensions

NNLO n-Jettiness:

$$\mathcal{T}_N = \sum_k \min_i \left\{ \frac{2p_i \cdot q_k}{Q_i} \right\}.$$

$\mathcal{T}_N < \mathcal{T}_N^{cut}$

$$\sigma(\mathcal{T}_N < \mathcal{T}_N^{cut}) = \int H \otimes B \otimes B \otimes S \otimes \left[\prod_n^N J_n \right] + \cdots .$$

NNLO - analytical

- H- pure virtual
- B - Initial state beam fns.
- S - Soft distribution fns.
- J - Final state jet fns.

$\mathcal{T}_N \geq \mathcal{T}_N^{cut}$

$$\sigma\left(\mathcal{T}_N \geq \mathcal{T}_N^{cut}\right)$$

NLO with two jets

finite- numerically

NNLO n-Jettiness:

$\mathcal{T}_N < \mathcal{T}_N^{cut}$ and $\mathcal{T}_N \geq \mathcal{T}_N^{cut}$

$$\mathcal{T}_N^{\mathrm{cut}} \to \mathcal{T}_\delta = \delta_{\mathrm{IR}} Q,$$

$$\begin{aligned}
\sigma(X) &= \int_0^{\mathcal{T}_\delta} \mathrm{d}\mathcal{T}_N \, \frac{\mathrm{d}\sigma(X)}{\mathrm{d}\mathcal{T}_N} + \int_{\mathcal{T}_\delta} \mathrm{d}\mathcal{T}_N \, \frac{\mathrm{d}\sigma(X)}{\mathrm{d}\mathcal{T}_N} \\
&= \sigma^{\mathrm{sing}}(X, \mathcal{T}_\delta) + \int_{\mathcal{T}_\delta} \mathrm{d}\mathcal{T}_N \, \frac{\mathrm{d}\sigma(X)}{\mathrm{d}\mathcal{T}_N} + \mathcal{O}(\delta_{\mathrm{IR}}) \, .
\end{aligned}$$

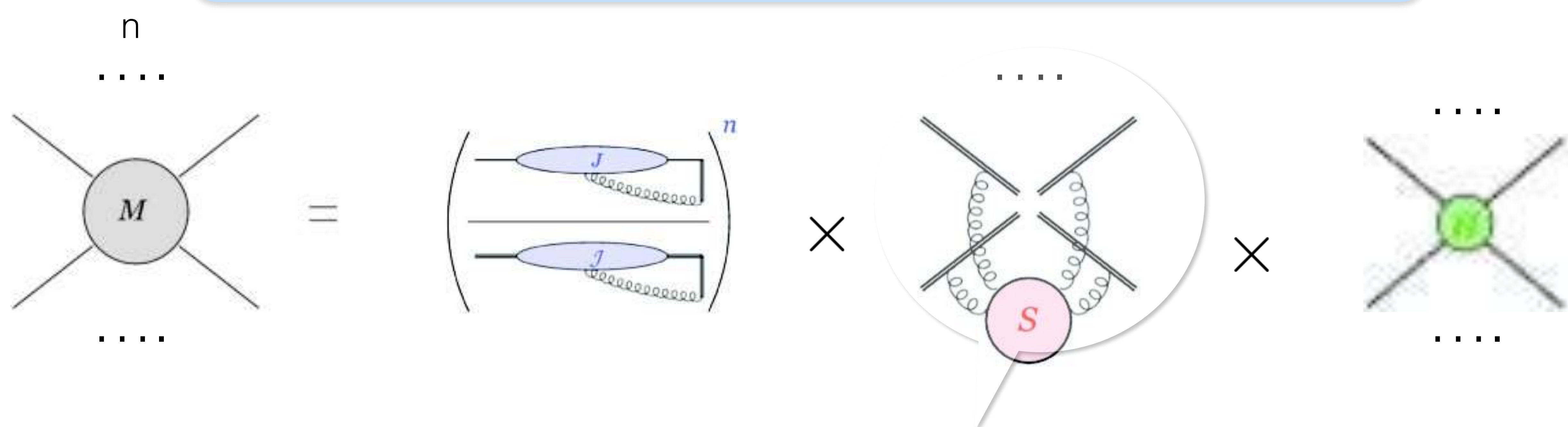
$\sigma^{\mathrm{nons}}(X, \mathcal{T}_\delta)$ is of $\mathcal{O}(\mathcal{T}_\delta / Q) = \mathcal{O}(\delta_{\mathrm{IR}})$.

Sterman's proof using factorisation

On-shell QCD amplitude in color basis:

[G. Sterman, M Tejeda-Yeomans]

$$\mathcal{M}^{[f]}_{\{r_i\}}\left(\beta_j,\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)=\sum_{L=1}^{N^{[f]}}\mathcal{M}^{[f]}_L\left(\beta_j,\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)(c_L)_{\{r_i\}}$$



$$|\mathcal{M}_n(\epsilon,\{p\})\rangle = \prod_{i=1}^{n+2} J^{[i]}\left(\frac{Q'^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) S^{[f]}_{LI}\left(\beta_j,\frac{Q'^2}{\mu^2},\frac{Q'^2}{Q^2},\alpha_s(\mu^2),\epsilon\right) H^{[f]}_I\left(\beta_j,\frac{Q^2}{\mu^2},\frac{Q'^2}{Q^2},\alpha_s(\mu^2)\right)$$

Collinear Soft Hard

qT subtraction at N3LO:

$$d\sigma^F_{(N)NLO} = \mathcal{H}^F_{(N)NLO} \otimes d\sigma^F_{LO} + \left[d\sigma^{F+\text{jets}}_{(N)LO} - d\sigma^{CT}_{(N)LO} \right]$$

$$d\sigma^{CT} = d\sigma^F_{LO} \otimes \Sigma^F(q_T/Q)\, d^2\mathbf{q}_T .$$

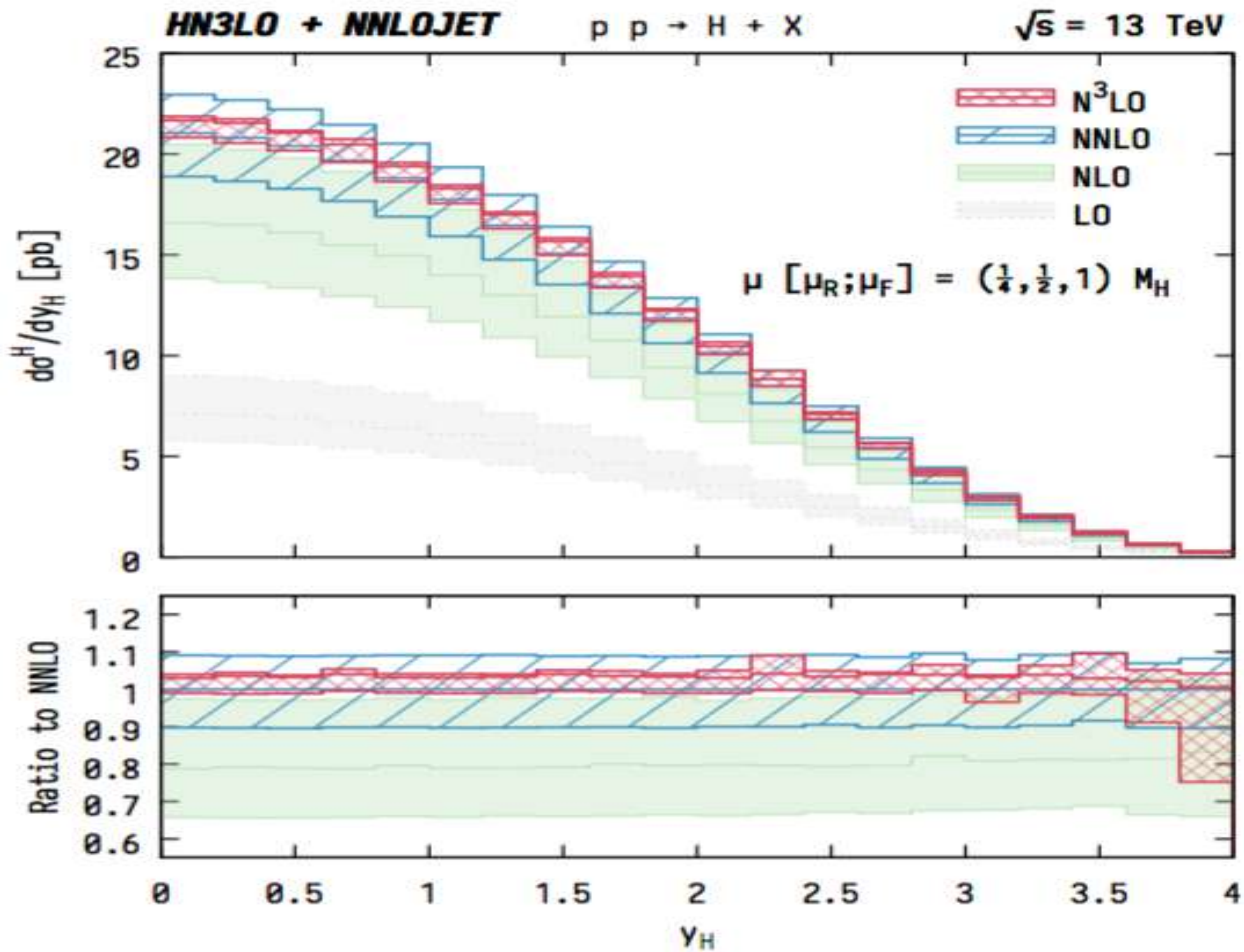
Note that

$$d\sigma^F_{(N)NLO}\big|_{q_T \neq 0} = d\sigma^{F+\text{jets}}_{(N)LO}$$

qT resummation gives

$$\Sigma^F(q_T/Q) \xrightarrow[q_T \to 0]{} \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n \sum_{k=1}^{2n} \Sigma^{F(n;k)} \frac{Q^2}{q_T^2} \ln^{k-1} \frac{Q^2}{q_T^2} \quad .$$

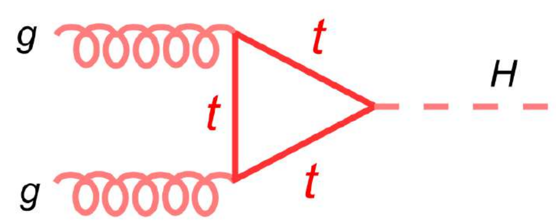
qT subtraction at N3LO:

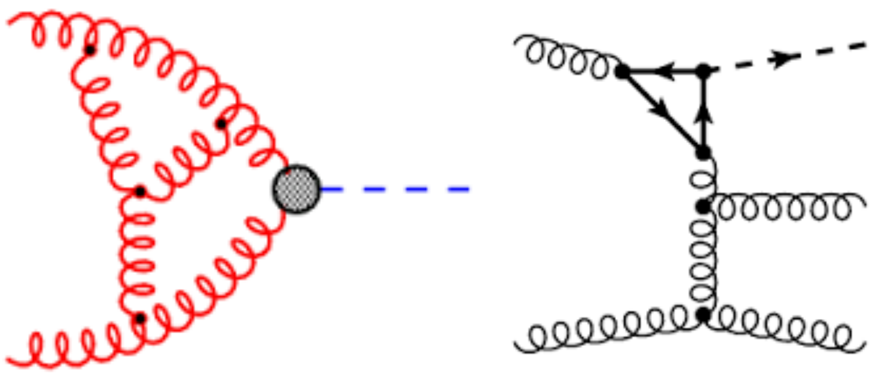


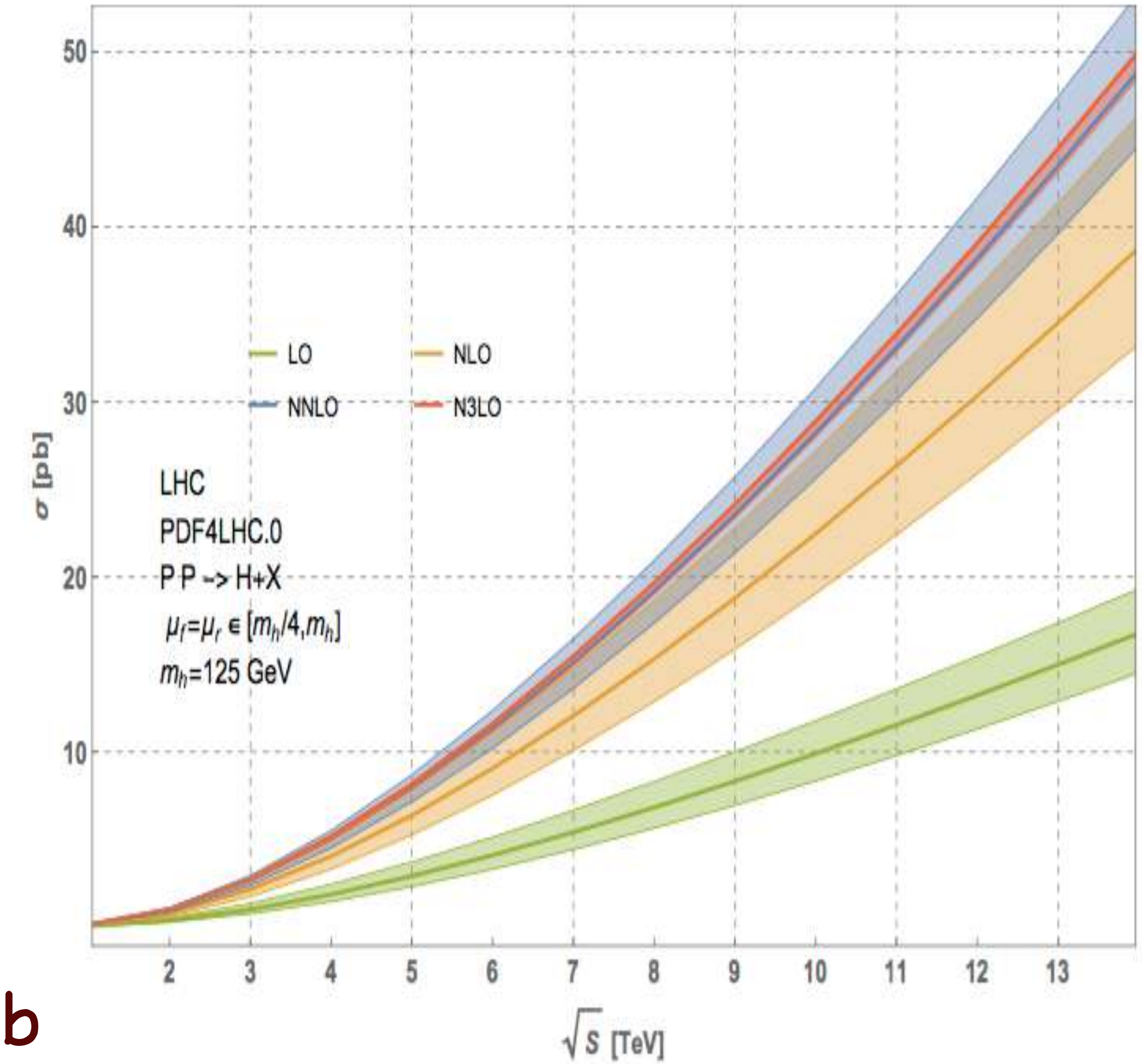
True Result for Higgs

$$\hat{\sigma}(z) = \hat{\sigma}^{LO}(z) + \alpha_S \hat{\sigma}^{NLO}(z) + \alpha_S^2 \hat{\sigma}^{NNLO}(z) + \alpha_S^3 \hat{\sigma}^{N3LO}(z) + \mathcal{O}(\alpha_S^4)$$

✓ ✓ ✓ ✓





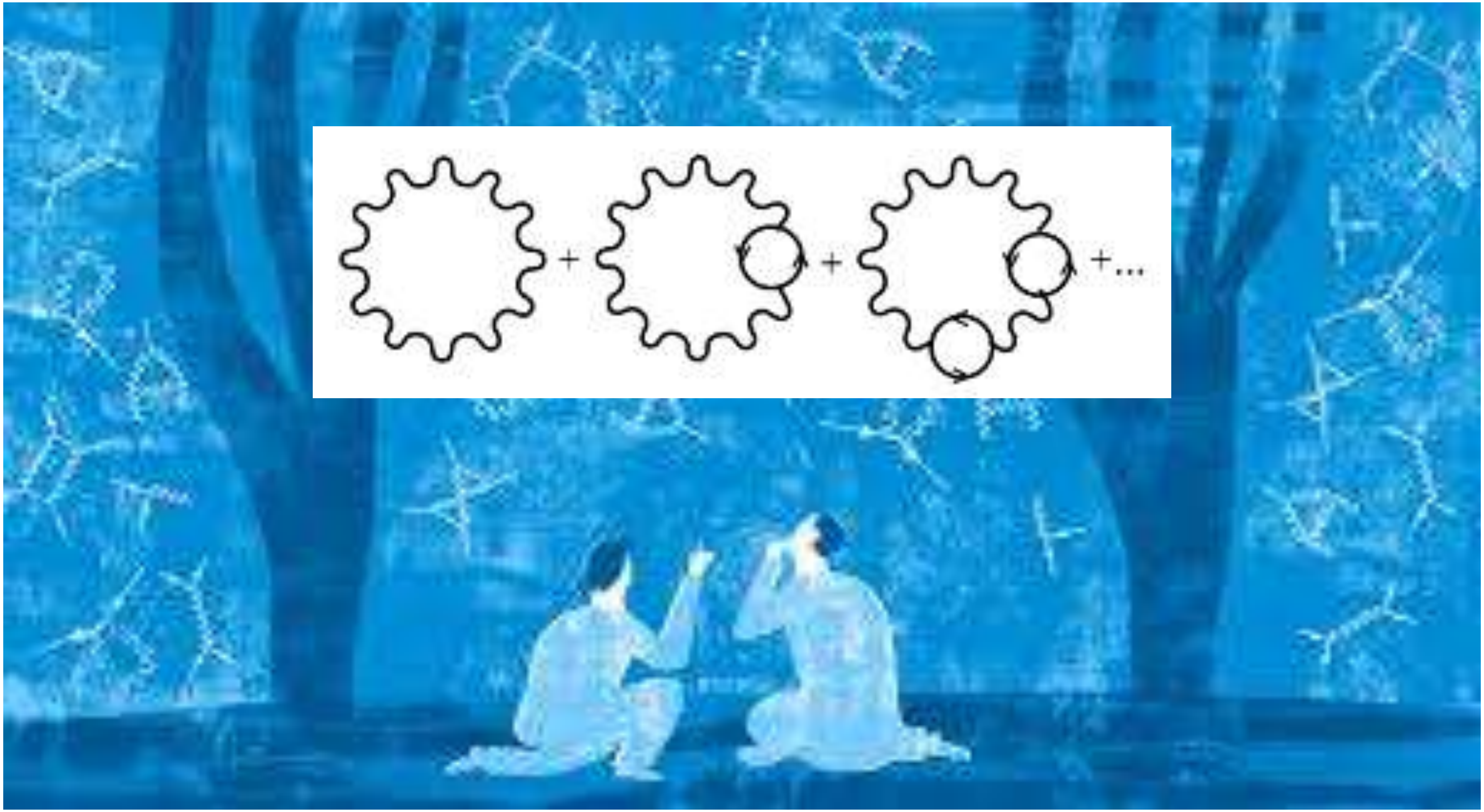


LO	$15.05 \pm 14.8\%$
NLO	$38.2 \pm 16.6\%$
NNLO	$45.1 \pm 8.8\%$
N3LO	$45.2 \pm 1.9\%$

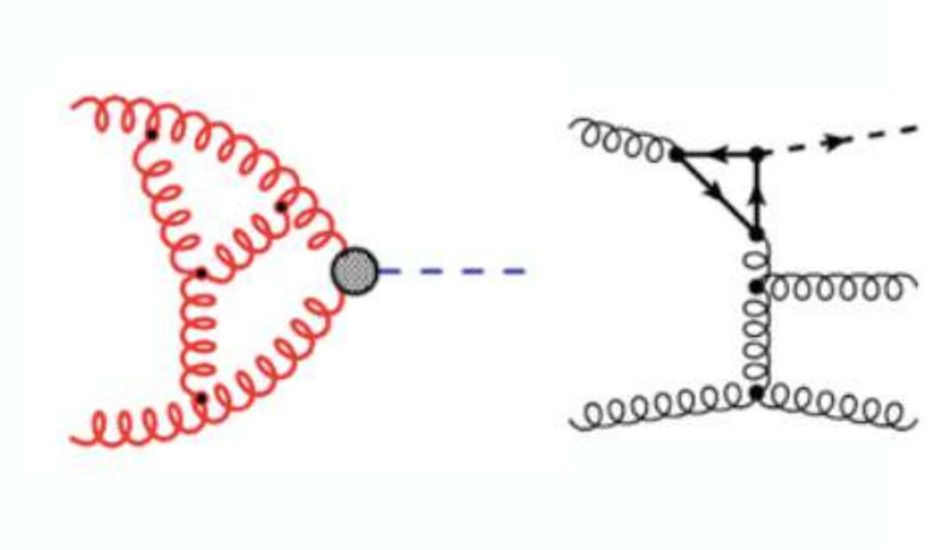
pb

Anastasiou, Duhr, Dulat and Mistleberger

IR structure, Resummation



Multi-loops and Multi-legs

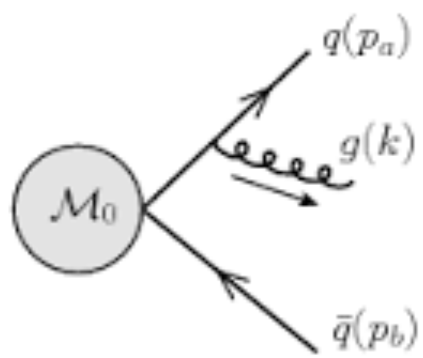


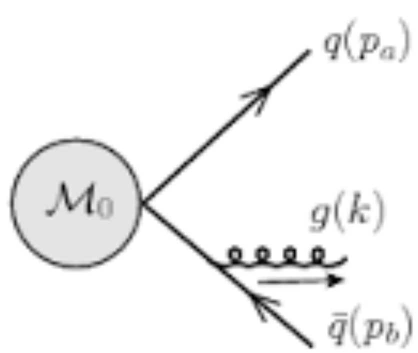
Infrared divergences

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

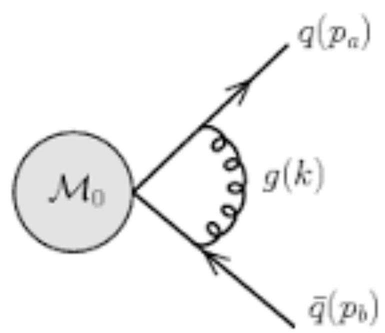
In the Limit $k \to p \quad (p_a \text{ or } \mathrm{p_b})$ $\quad m_a, m_b << Q$

Real emission





Virtual



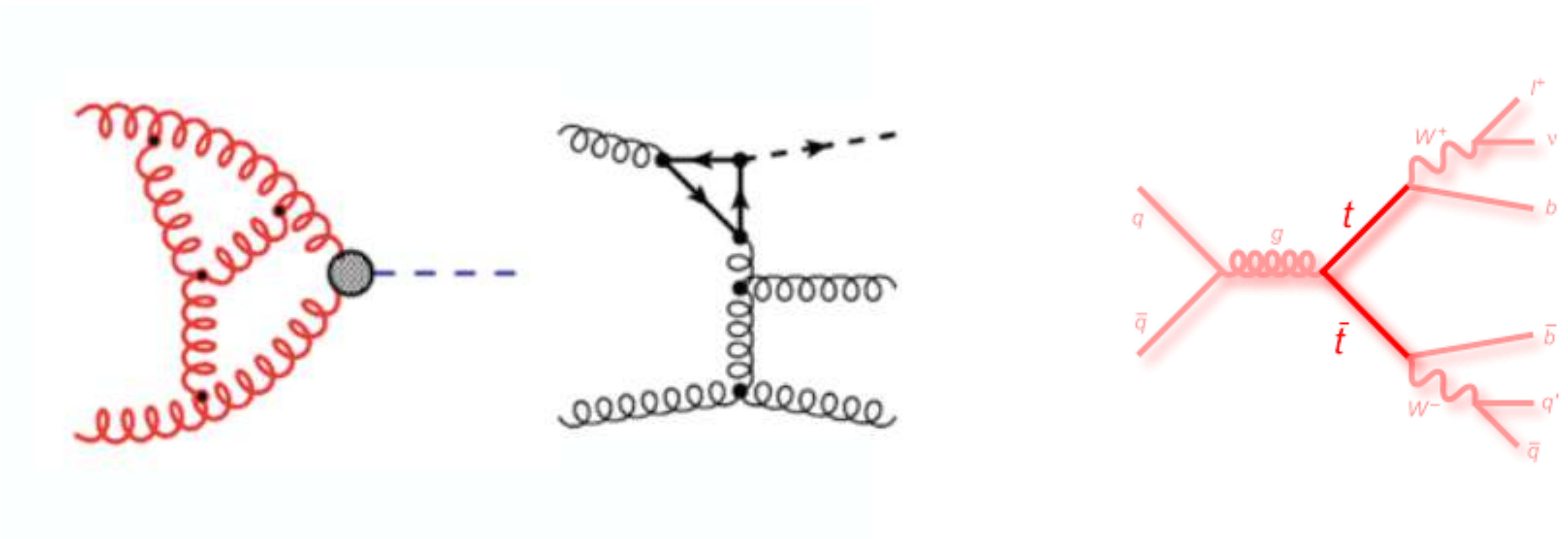
$$\frac{1}{(p+k)^2} = \frac{1}{2p^0 k^0 \, (1 - \cos\theta)}$$

$k^0 \to 0$	Soft divergence
$\cos\theta \to 0$	Collinear divergence

Catani's Proposal

[Yennie, Frautschi, Suura; Weinberg]

UV Renormalised on-shell QCD amplitudes $|\mathcal{M}_n(\epsilon, \{p\})\rangle$



Universal Infrared Structure

Catani's Proposal

[S. Catani]

Universal IR Subtraction Operator is **Universal up to Two loop !**

$$\left[1 - \frac{\alpha_s}{2\pi}\,\boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon)\right] |\mathcal{M}_n(\epsilon,\{p\})\rangle \;= |\mathcal{M}_n(\epsilon,\{p\}) >_{finite}$$

Singular

IR Finite

At one and Two loops

$$\boldsymbol{I}^{(1)}(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} + \frac{g_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right) \sum_{j\neq i} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2} \left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon}$$

$$\boldsymbol{I}^{(2)}(\epsilon) = \frac{e^{-\epsilon\gamma_E}\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(K + \frac{\beta_0}{2\epsilon}\right) \boldsymbol{I}^{(1)}(2\epsilon) - \frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon)\left(\boldsymbol{I}^{(1)}(\epsilon) + \frac{\beta_0}{\epsilon}\right) + \boldsymbol{H}^{(2)}_{\rm R.S.}(\epsilon)$$

Colour matrices satisfy $\sum_i \boldsymbol{T}_i\,|\mathcal{M}_n(\epsilon,\{p\})\rangle = 0.$

Catani's Proposal

[Catani]

$$\left[1 - \frac{\alpha_s}{2\pi}\, \boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon)\right] |\mathcal{M}_n(\epsilon, \{p\})\rangle$$

Upto Two loop !

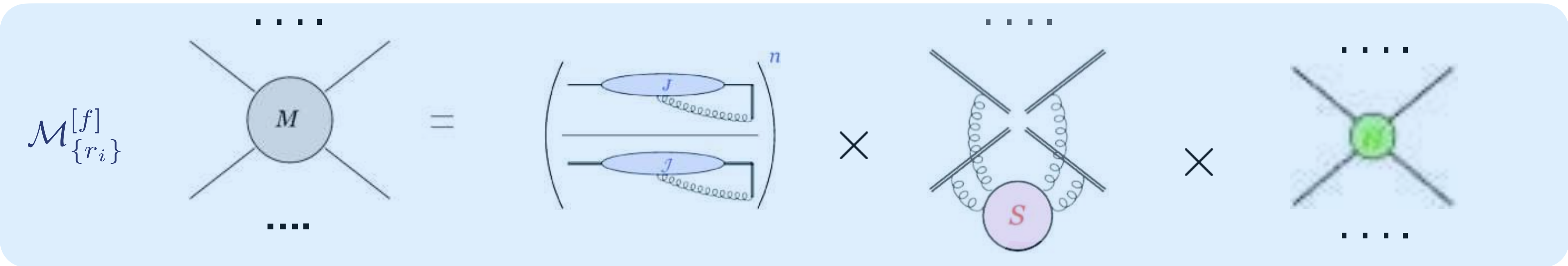
Universal IR Subtraction operators - Process independent

Depend only on ``Soft and Collinear" Anomalous Dimensions

Sterman and Tejeda-Yeomans

[G. Sterman, M Tejeda-Yeomans]

$$\mathcal{M}^{[\mathrm{f}]}_{\{r_i\}}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \sum_{L=1}^{N^{[\mathrm{f}]}} \mathcal{M}^{[\mathrm{f}]}_{L}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) (c_L)_{\{r_i\}}$$



Factorisation

$$\mathcal{M}^{[\mathrm{f}]}_{L}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \prod_{i=1}^{n+2} J^{[i]}\left(\frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) S^{[\mathrm{f}]}_{LI}\left(\beta_j, \frac{Q'^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2), \epsilon\right) H^{[\mathrm{f}]}_{I}\left(\beta_j, \frac{Q^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2)\right)$$

Collinear **Soft** **Hard**

Renormalisation Group Equation for IR

[Sterman]

Factorisation of IR singularities:

IR singular

$$\mathcal{M}\left(\frac{p_i\cdot p_j}{\mu^2},\alpha_s(\mu^2),\epsilon\right)=\mathcal{Z}\left(\frac{p_i\cdot p_j}{\mu_f^2},\alpha_s(\mu_f^2),\epsilon\right)\times\mathcal{H}\left(\frac{p_i\cdot p_j}{\mu^2},\frac{\mu^2}{\mu_f^2},\alpha_s(\mu^2)\right)$$

Introduces Arbitrary Factorisation Scale μ_F

Amplitudes are independent of this scale

Renormalisation Group Invariance (RGE)

$$\mu_f\frac{d}{d\mu_f}\mathcal{Z}\left(\frac{p_i.p_j}{\mu_f^2},\alpha_s(\mu_f^2),\epsilon\right)=-\mathcal{Z}\left(\frac{p_i.p_j}{\mu_f^2},\alpha_s(\mu_f^2),\epsilon\right)\Gamma\left(\frac{p_i.p_j}{\mu_f^2},\alpha_s(\mu_f^2)\right)$$

Three loop conjecture

[Becher, Neubert, Gardi, Magnea]

Matrix valued solution

$$\mathcal{Z}\left(\frac{p_i.p_j}{\mu_f^2}, \alpha_s(\mu_f^2), \epsilon\right) = \mathcal{P}exp\left[-\int_0^{\mu_f^2} \frac{d\lambda}{\lambda}\Gamma\left(\frac{p_i.p_j}{\lambda}, \alpha_s(\lambda)\right)\right]$$

Conjecture for IR anomalous dimension in QCD

$$\boldsymbol{\Gamma} = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

Di-pole

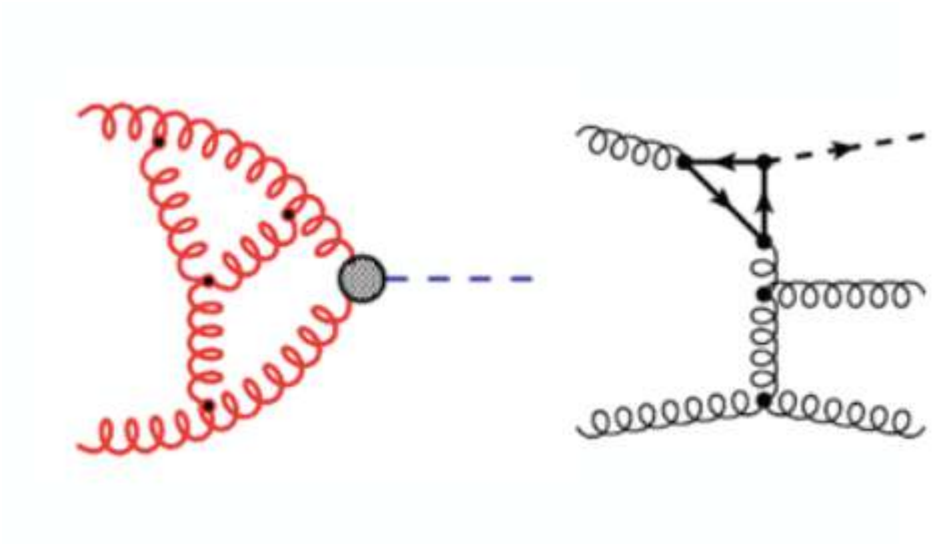
Soft

Soft
+Collinear

Scale dependent !

Only Di-pole part Depends on Kinematics

Multi-parton amplitude



$$\Gamma = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

$$\gamma_{cusp} \to A_I(a_s)$$

$$\gamma^i \to 2B_I(a_s) + f_I(a_s)$$

Universal Anomalous dimension

$\gamma_q, \quad \gamma_g$	UV
$A_q = \frac{C_F}{C_A} A_g$	Cusp
$B_q, \quad B_g$	Collinear
$f_q = \frac{C_F}{C_A} f_g$	Soft

Three loop conjecture

All order Conjecture of QCD:

$\gamma^i(\alpha_s)$ are independent of s_{ij}

$$\Gamma = \sum_{(i,j)} \frac{T_i \cdot T_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

Three loop non-planar in $\mathcal{N} = 4$ SYM

known

$$\frac{1}{4}\sum_{L=1}^{\infty} \alpha^L \left[\frac{\gamma_c^{(L)}}{L^2\epsilon^2}\mathbf{D_0} - \frac{\gamma_c^{(L)}}{L\epsilon}\mathbf{D} + \frac{4}{L\epsilon}\gamma_J^{(L)}\mathbb{I} + \frac{1}{L\epsilon}\mathbf{\Delta}^{(L)} \right]$$

$$\mathbf{\Delta}^{(1)} = \mathbf{\Delta}^{(2)} = \mathbf{0}$$

$$\mathbf{\Delta}_4^{(3)} = \frac{1}{4} f_{abe} f_{cde} \Big[\mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \, \mathcal{S}(x) + \mathbf{T}_4^a \mathbf{T}_1^b \mathbf{T}_2^c \mathbf{T}_3^d \, \mathcal{S}(1/x) \Big],$$

$$\mathbf{\Delta}_3^{(3)} = -C\, f_{abe} f_{cde} \sum_{\substack{i=1\ldots4 \\ 1\le j<k\le 4 \\ j,k\neq i}} \{\mathbf{T}_i^a, \mathbf{T}_i^d\}\, \mathbf{T}_j^b \mathbf{T}_k^c \,.$$

Breakdown of Conjucture

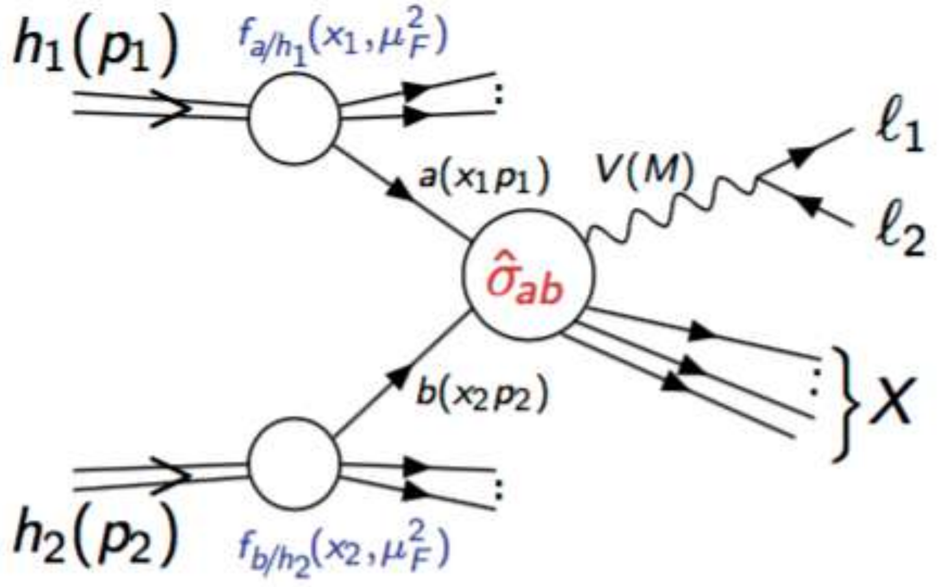
$\mathcal{S}(x)$ are dependent on s_{ij} at three loop level

All order Resumed Prediction:

$$\mathcal{H}(\mu^2) = \mathcal{H}(\mu_0^2)\mathcal{P}\exp\left[\int_{\mu_0^2}^{\mu_f^2} \frac{d\lambda}{\lambda}\Gamma(\lambda)\right]$$

Small qT Resummation for DY

$$\frac{d\sigma}{dq_T^2}(q_T,M,s)=\sum_{a,b}\int_0^1 dx_1 \int_0^1 dx_2\, f_{a/h_1}(x_1,\mu_F^2)\, f_{b/h_2}(x_2,\mu_F^2)\, \frac{d\hat{\sigma}_{ab}}{dq_T^2}(q_T,M,\hat{s};\alpha_S,\mu_R^2,\mu_F^2).$$



For $q_T \to 0$, $\alpha_S^n \log^m(M^2/q_T^2) \gg 1$

$$\frac{d\hat{\sigma}}{dq_T^2} = \frac{d\hat{\sigma}^{(res)}}{dq_T^2} + \frac{d\hat{\sigma}^{(fin)}}{dq_T^2}$$

$$\frac{d\hat{\sigma}^{(res)}}{dq_T^2} = \frac{M^2}{\hat{s}}\int_0^\infty db\,\frac{b}{2}J_0(bq_T)\,\mathcal{W}(b,M)$$

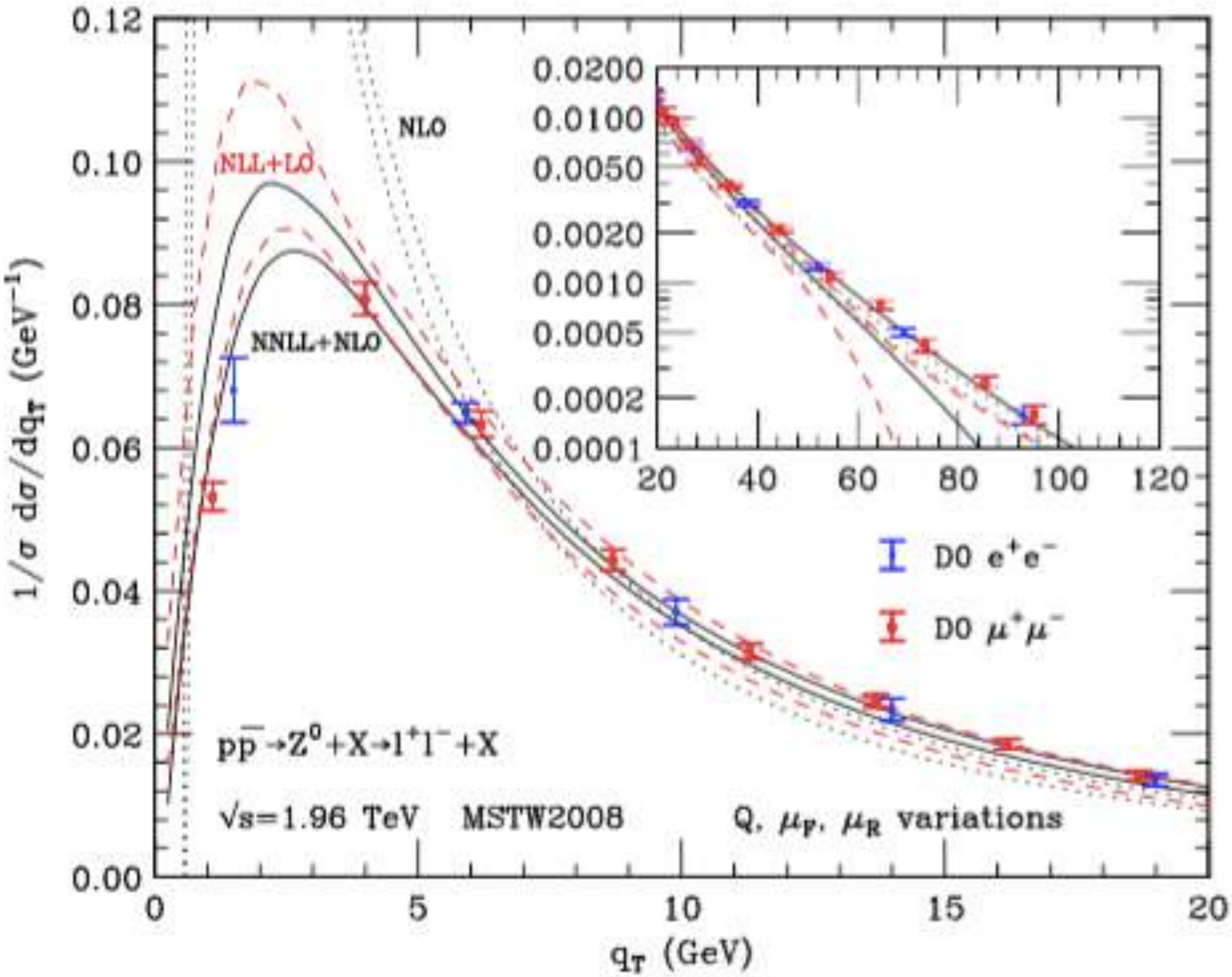
$$\mathcal{W}_N(b,M) = \mathcal{H}_N(\alpha_S) \times \exp\{\mathcal{G}_N(\alpha_S,L)\}$$

$$L \equiv \log(M^2b^2)$$

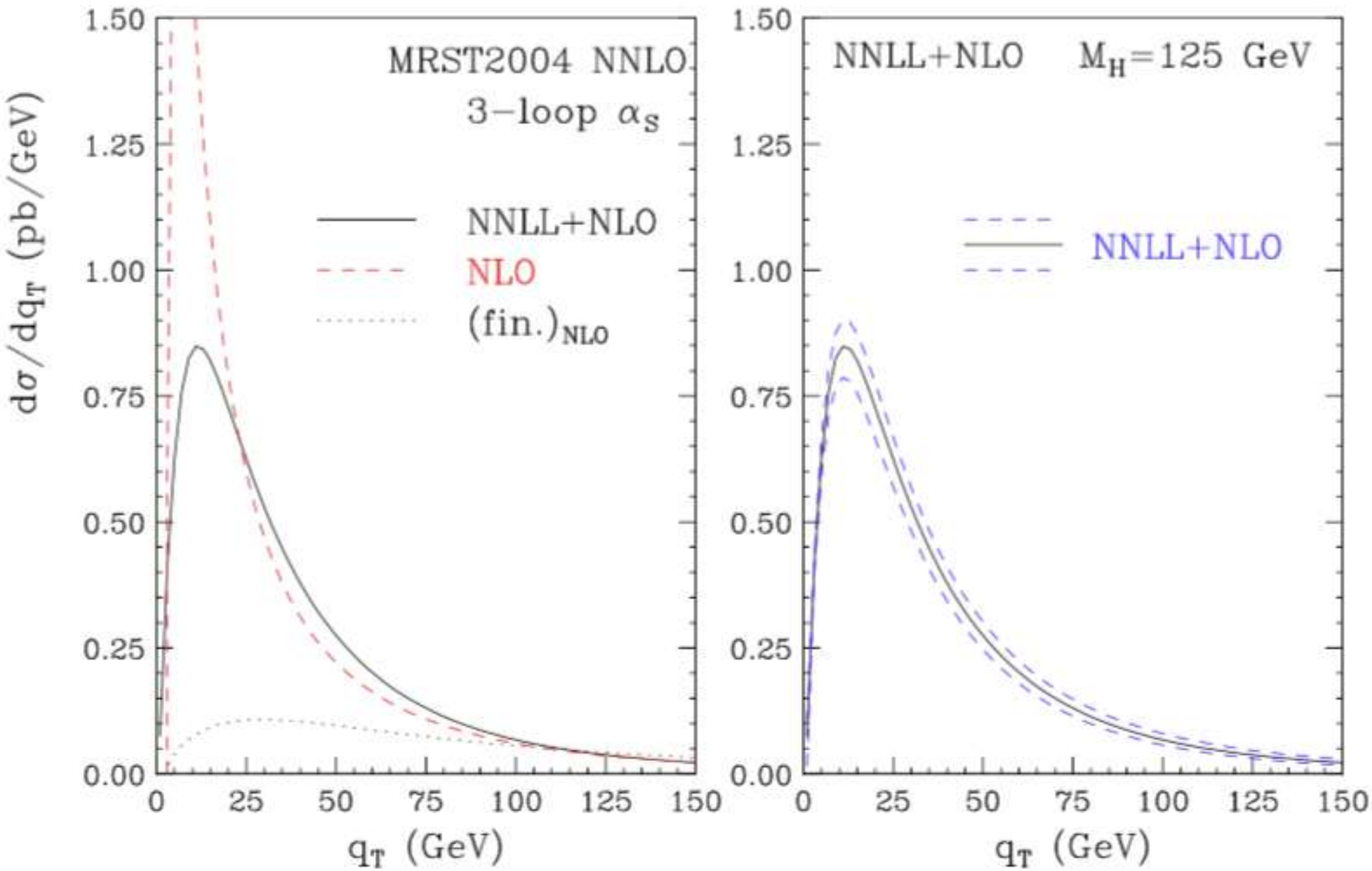
$$\mathcal{G}_N(\alpha_S,L) = L\,g^{(1)}(\alpha_S L)+g_N^{(2)}(\alpha_S L)+\frac{\alpha_S}{\pi}g_N^{(3)}(\alpha_S L)+\cdots$$

Small qT Resummation for DY

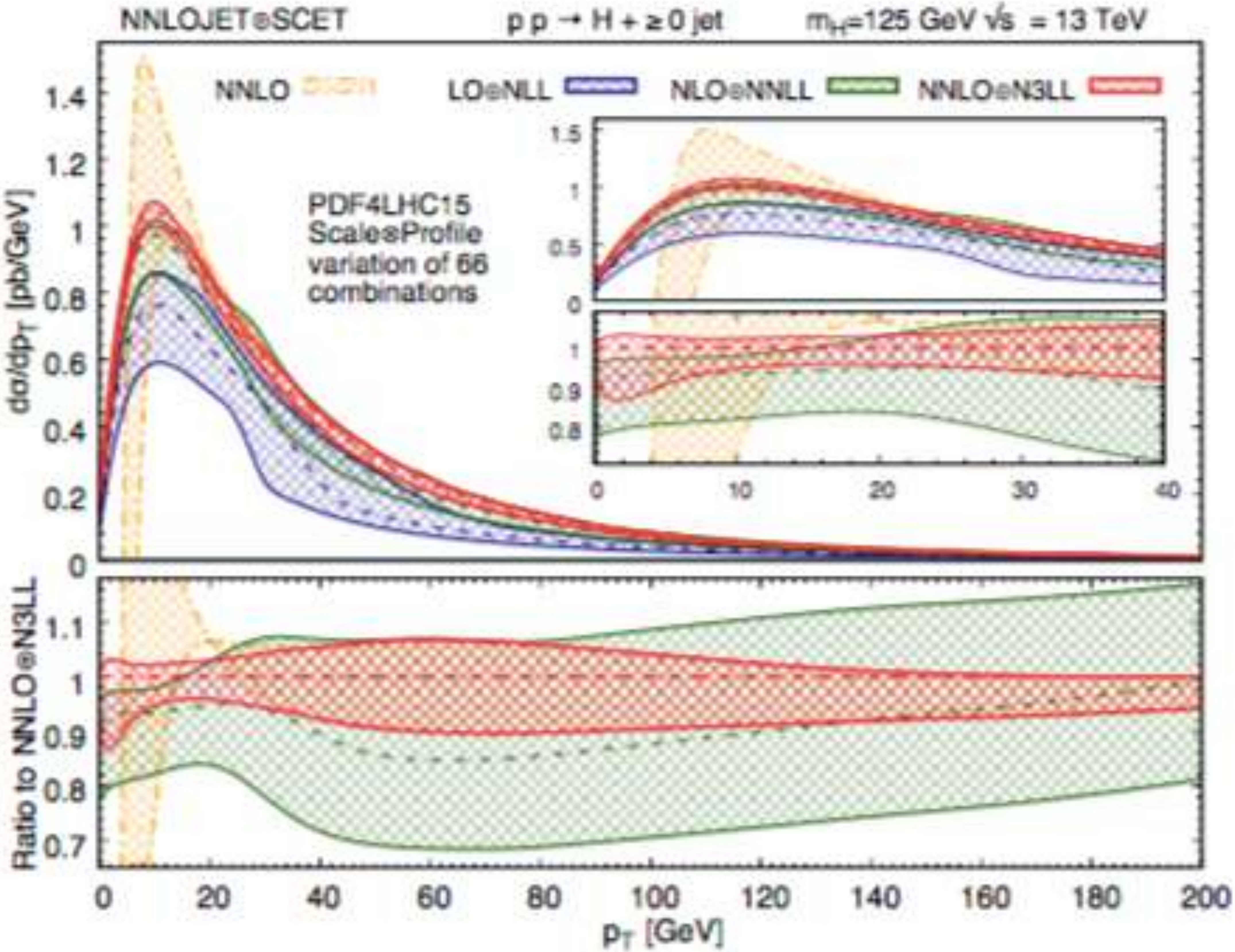
D0 data for the Z q_T spectrum compared with perturbative results.



Small qT Resummation for Higgs



Small qT Resummation for Higgs

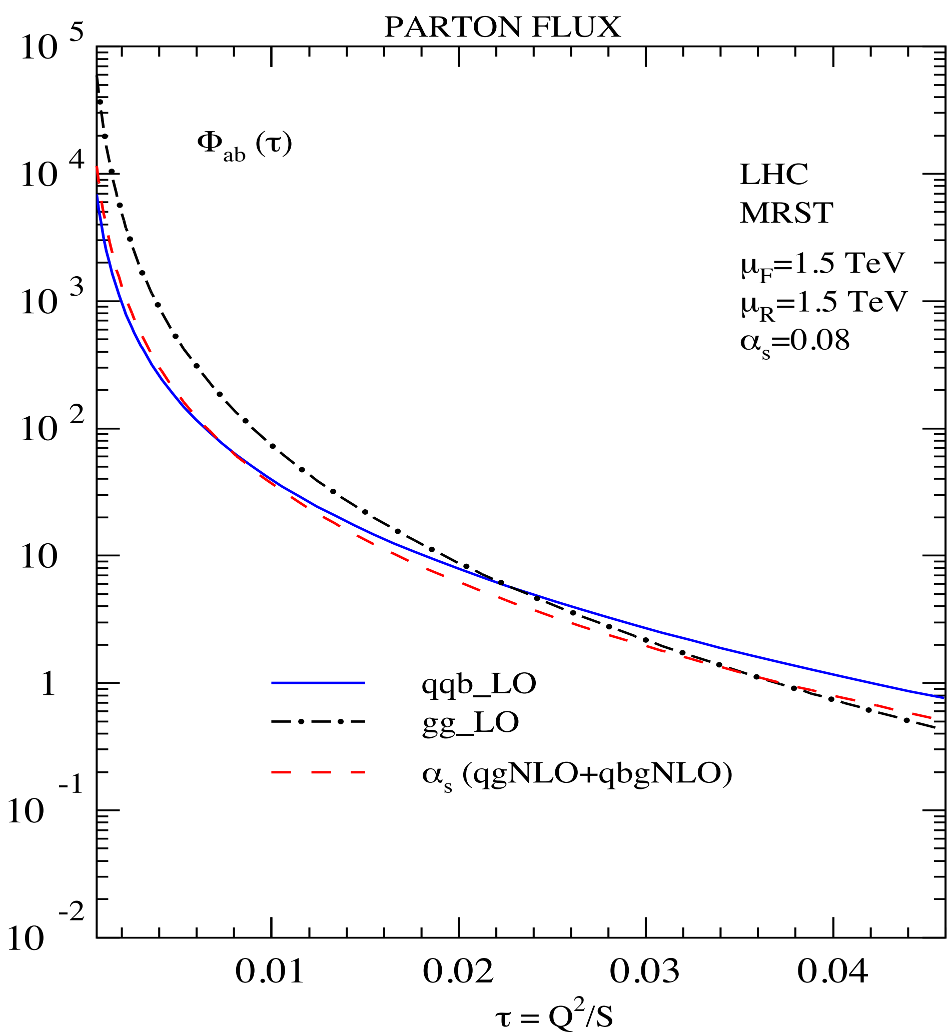


Soft gluons dominance

Why Threshold corrections?

Catani et al, Harlander, Kilgore

$$2S\, d\sigma^{P_1 P_2}(\tau, m_h) = \sum_{ab} \int_\tau^1 \frac{dx}{x} \Phi_{ab}(x)\; 2\hat{s}\, d\hat{\sigma}^{ab}\left(\frac{\tau}{x}, m_h\right) \qquad \tau = \frac{m_h^2}{S}$$



Gluon flux is largest at LHC

- Parton flux $\Phi_{ab}(x)$ becomes large when $x \to x_{min} = \tau$
- Dominant contribution to Higgs production comes from the region when $x \to \tau$
- It is sufficient if we know the partonic cross section when $x \to \tau$
- $x \to \tau$ is called $soft\ limit$.
- Expand the partonic cross section around $x = \tau$.
- Dominantly come from virtual and soft gluon emission processes (SV)

Rapidity Distribution

Rapidity Distribution of any colorless particle:

$$\frac{d\sigma^I}{dy} = \hat{\sigma}^I_B \sum_{ab=q,\bar{q},g} \int_{x_1^0}^1 \frac{dz_1}{z_1} \int_{x_2^0}^1 \frac{dz_2}{z_2} \hat{\mathcal{H}}^I_{ab}\left(\frac{x_1^0}{z_1}, \frac{x_2^0}{z_2}, \mu^2\right) \hat{\Delta}^I_{d,ab}\left(z_1, z_2, q^2, \mu^2\right)$$

DY production of lepton pairs $\sigma^I = \frac{d\sigma^q\left(\tau, q^2, y\right)}{dq^2}$.

Higgs through gluon (bottom anti-bottom), $\sigma^I = \sigma^{g(b)}\left(\tau, q^2, y\right)$.

Rapidity: $y = \frac{1}{2}\ln\left(\frac{p_2\cdot q}{p_1\cdot q}\right) = \ln\left(\frac{x_1^0}{x_2^0}\right)$, $\tau = x_1^0 x_2^0$

Partonic Scaling variables:

$$z_1 = \frac{x_1^0}{x_1}, \qquad z_1 = \frac{x_2^0}{x_2}$$

Soft and Virtual terms

$$\Delta_d^I = \delta(1-z_1)\delta(1-z_2) + a_s \Bigg\{ c_1^{(1)} \delta(1-z_1)\delta(1-z_2)$$

$$+c_2^{(1)} \left(\frac{\ln(1-z_1)}{1-z_1}\right)_+ + R^{(1)}(z_1,z_2) + z_1 \leftrightarrow z_2 \Bigg\} + \mathcal{O}(a_s^2)$$

$$\Delta_d^I(z_1,z_2) = \Delta_d^{I,SV}(z_1.z_2) + \Delta_d^{I,hard}(z_1,z_2)$$

Virtual , Soft $\delta(1-z_i)$ $\left(\frac{\ln(1-z_i)}{(1-z_i)}\right)_+$

$$\tilde{\Delta}_d^{I,\mathrm{SV}}(\omega) = \int_0^1 dz_1 z_1^{N_1-1} \int_0^1 dz_2 z_2^{N_2-1} \Delta_d^{I,\mathrm{SV}}(z_1,z_2)$$

Resummation to NNLL

Logarithms that are resummed in g_d^I

	LL	NLL	NNLL
$\mathcal{O}(a_s)$	$\ln^2(\bar{N}_1\bar{N}_2)$	$\ln(\bar{N}_1\bar{N}_2)$	
$\mathcal{O}(a_s^2)$	$\ln^3(\bar{N}_1\bar{N}_2)$	$\ln^2(\bar{N}_1\bar{N}_2)$	$\ln(\bar{N}_1\bar{N}_2)$
$\mathcal{O}(a_s^3)$	$\ln^4(\bar{N}_1\bar{N}_2)$	$\ln^3(\bar{N}_1\bar{N}_2)$	$\ln^2(\bar{N}_1\bar{N}_2)$
Resummed terms:	$a_s^m \ln^{m+1}(\bar{N}_1\bar{N}_2)$	$a_s^m \ln^m(\bar{N}_1\bar{N}_2)$	$a_s^{m+1} \ln^m(\bar{N}_1\bar{N}_2)$
Function that resums :	$g_{d,1}^I \ln(\bar{N}_1\bar{N}_2)$	$g_{d,2}^I$	$a_s g_{d,3}^I$

Soft Gluon Resummation

Double Mellin Transformation:

$$\tilde{\Delta}_d^{I,\mathrm{SV}}(\omega) = \int_0^1 dz_1 z_1^{N_1-1} \int_0^1 dz_2 z_2^{N_2-1} \Delta_d^{I,\mathrm{SV}}(z_1, z_2)$$

Resumed Rapidity distribution:

Ni dependent

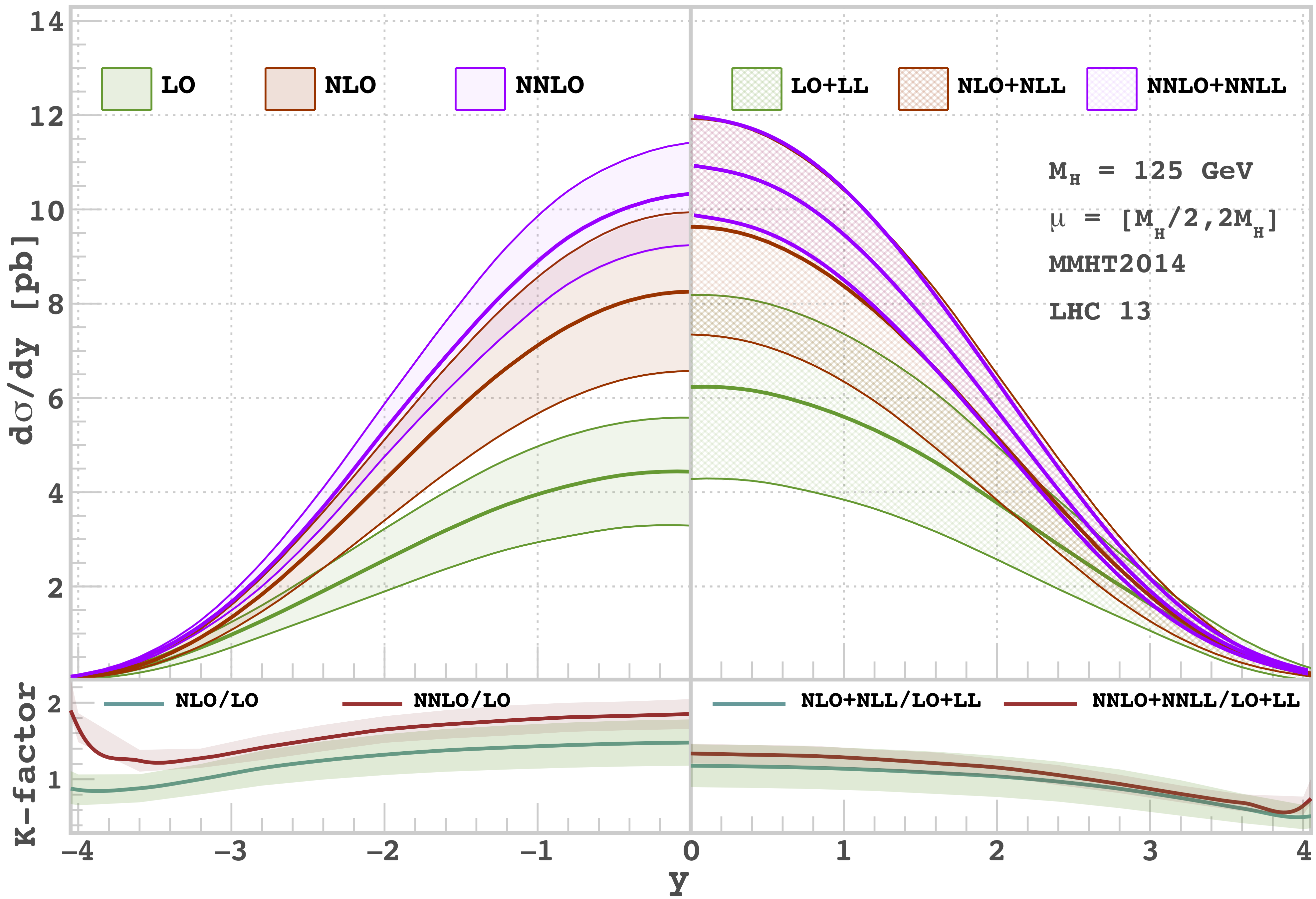
$$\tilde{\Delta}_d^{\mathrm{SV},I}(\omega) = \tilde{g}_{d,0}^{I}(a_s) \exp\left(g_d^I(a_s, \omega)\right)$$

Ni independent

$$\omega = a_s \beta_0 \ln\left(\overline{N}_1 \overline{N}_2\right)$$

$$\overline{N}_i = e^{\gamma_E} N_i$$

Rapidity of Higgs at NNLO + NNLL



Fixed order CS

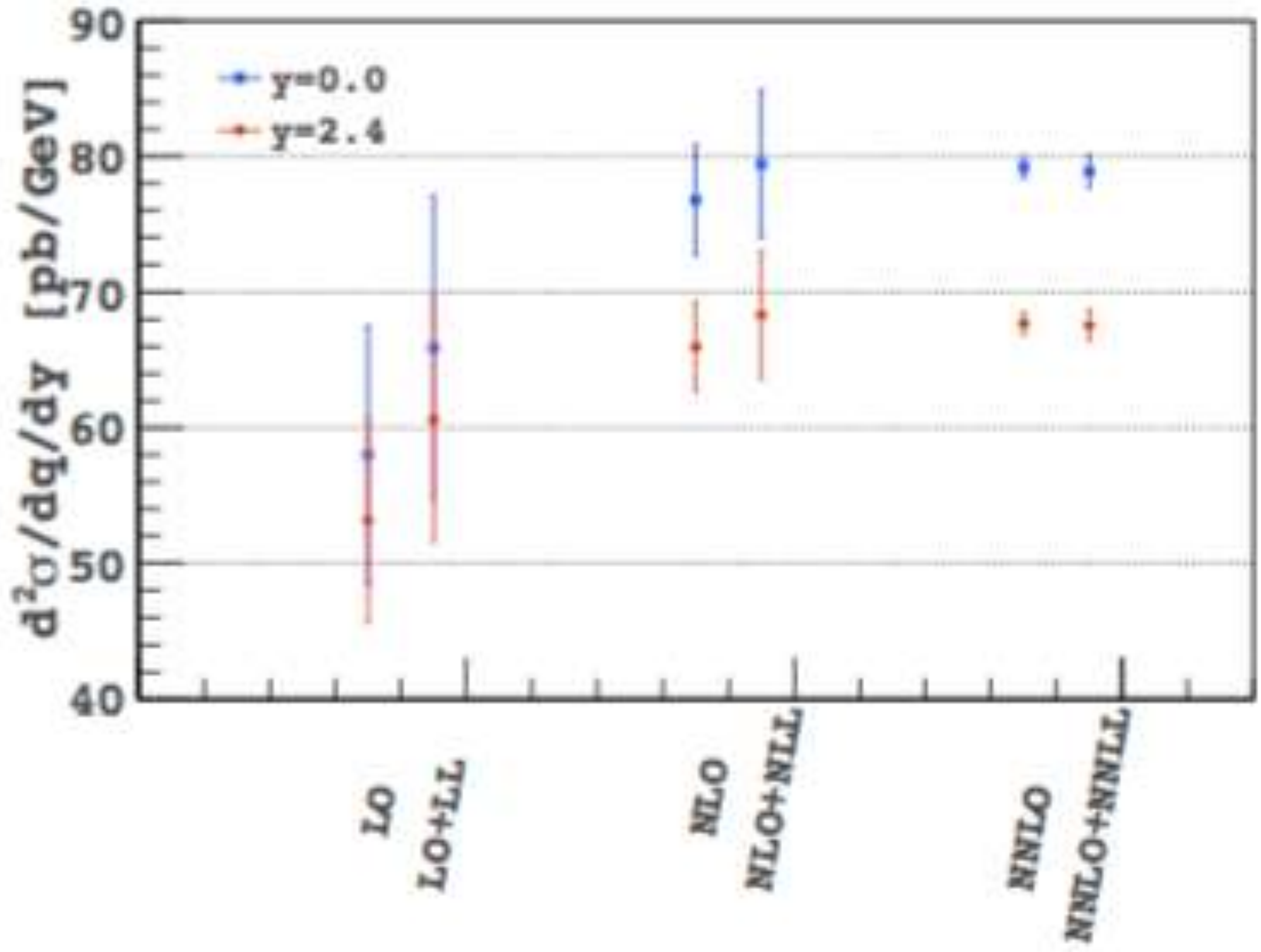
$$M_H/2 \leq \mu_{R,F} \leq 2M_H$$

Resummed CS

$$\Longrightarrow$$

At NNLO+NNLL result stabilises convergence of perturbation series !

Rapidity of DY at NNLO + NNLL



Parton Model in QCD

Inclusive cross section:

$$\sigma^A(\tau, m_A^2) = \sigma^{A,(0)}(\mu_R^2) \sum_{a,b=q,\bar{q},g} \int_\tau^1 dy\, \Phi_{ab}(y, \mu_F^2) \Delta_{ab}^A\left(\frac{\tau}{y}, m_A^2, \mu_R^2, \mu_F^2\right)$$

Partonic Flux:

$$\Phi_{ab}(y, \mu_F^2) = \int_y^1 \frac{dx}{x} f_a(x, \mu_F^2) f_b\left(\frac{y}{x}, \mu_F^2\right),$$

Partonic cross section:

$$\Delta_{ab}^A(z, q^2, \mu_R^2, \mu_F^2) = \Delta_{ab}^{A,\mathrm{SV}}(z, q^2, \mu_R^2, \mu_F^2) + \Delta_{ab}^{A,\mathrm{hard}}(z, q^2, \mu_R^2, \mu_F^2)$$

Soft + Virtual

Hard

Soft + Virtual (SV)

[VR]

Factorisation of universal IR configuration leads to Exponentiation

$$\Delta_g^{A,\mathrm{SV}}(z,q^2,\mu_R^2,\mu_F^2) = \mathcal{C}\exp\left(\Psi_g^A\left(z,q^2,\mu_R^2,\mu_F^2,\epsilon\right)\right)\Big|_{\epsilon=0}$$

Soft+Virtual:

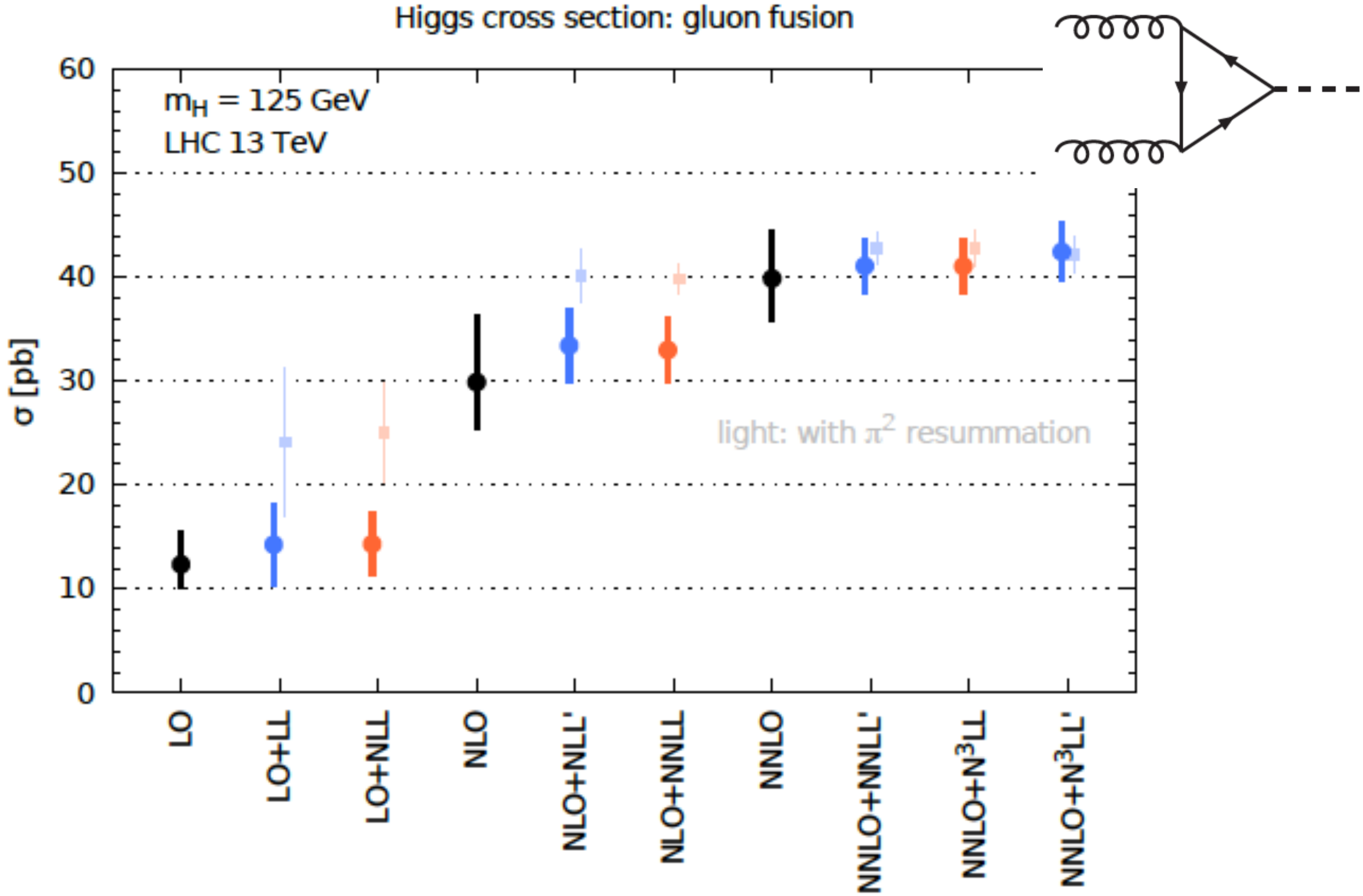
$$\Delta_{g,i}^{A,\mathrm{SV}} = \Delta_{g,i}^{A,\mathrm{SV}}|_\delta \delta(1-z) + \sum_{j=0}^{2i-1} \Delta_{g,i}^{A,\mathrm{SV}}|_{\mathcal{D}_j} \mathcal{D}_j \, .$$

Mellin Convolution in z-space:

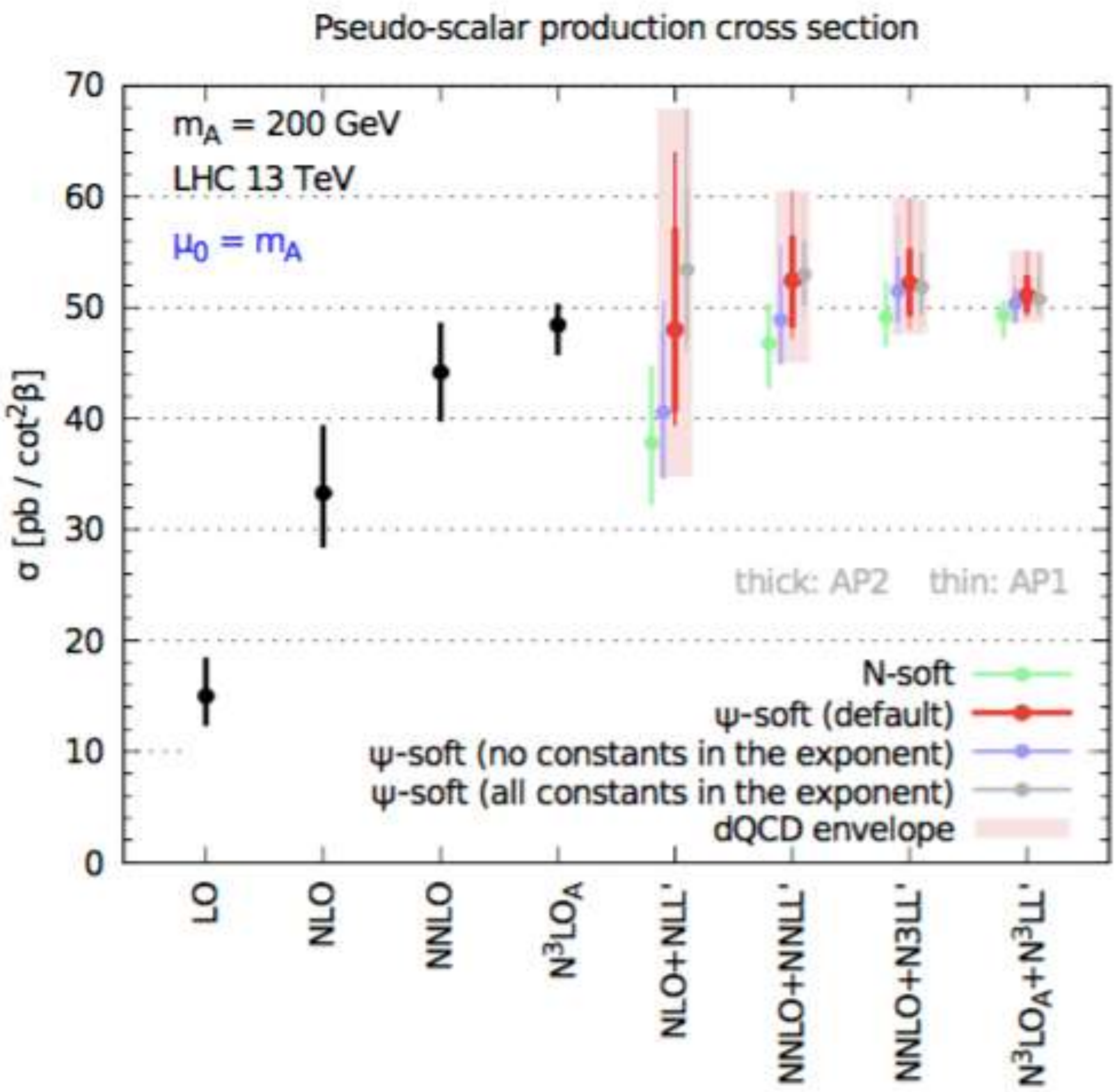
$$\mathcal{D}_i \equiv \left[\frac{\ln^i(1-z)}{1-z}\right]_+$$

$$\mathcal{C}e^{f(z)} = \delta(1-z) + \frac{1}{1!}f(z) + \frac{1}{2!}f(z)\otimes f(z) + \cdots \, .$$

Resummation at N3LL for Higgs



Pseudoscalar Higgs at N3LO(A) + N3LL



Amplitudes in N=4 Super YM theory

N = 4 Super Yang-Mills (SYM) theory

- A renormalizable gauge theory in four dimensional Minkowski space.
- Apart from having all the symmetries of QCD there are two extra features, namely supersymmetry and conformal symmetry.
- The on-shell amplitudes have a simpler analytical structure in comparison to QCD amplitudes.

AdS/CFT correspondence

- one to one correspondence between maximally SYM theory in four dimensions and gravity in five-dimensional anti-de Sitter space
- quantities in the strong coupling limit of N = 4 SYM can be related to those in the weakly coupled gravity
- quantities computed in a perturbative expansion should add up to some simple expression

BDS conjecture:

Bern-Dixon-Smirnov (BDS) conjecture:

Based on the results of 4-point three loop and 5-point two loop amplitudes N=4 SYM,

BDS conjectured that "n-point m-loop amplitudes" is related to their one loop counterpart.

Alday and Maldacena provided a connection between on-shell scattering amplitudes in the strong coupling and Wilson loops defined in dual coordinate space with light like segments.

Drummond, Korchemsky, and Sokatchev demonstrated the equality of results of Wilson loop with four light-like segments and the one-loop four point maximally helicity violating (MHV) amplitude and the equality was established between n-sided polygon and one-loop n-point MHV amplitude.

Later Alday and Maldacena showed that their Wilson loop calculation does not agree with BDS ansatz when the number of legs is large

K+G Sudakov Equation

K+G equation:

$$\frac{d}{d\ln Q^2}\ln \mathcal{F}_f^\rho = \frac{1}{2}\left[K_f^\rho + G_f^\rho\right]$$

Dimension regularization

$d = 4 + \varepsilon$

Solution

$$\ln \mathcal{F}_f^\rho(a, Q^2, \mu^2, \epsilon) = \sum_{j=1}^{\infty} a^j \left(\frac{Q^2}{\mu^2}\right)^{j\frac{\epsilon}{2}} \mathcal{L}_{f,j}^\rho(\epsilon)$$

where

$$\mathcal{L}_{f,j}^\rho(\epsilon) = \frac{1}{\epsilon^2}\left\{-\frac{2}{j^2}A_j\right\} + \frac{1}{\epsilon}\left\{\frac{1}{j}G_{f,j}^\rho(\epsilon)\right\}$$

Cusp Anomalous dimension

Universal - operator independent

Collinear Anomalous dimension

Process dependent

Leading Transcendentality principle

Example-1 Half-BPS operators

UV finite to all orders, protected by SUSY

IR divergent due to massless particles

Transcendentality in $d = 4 + \varepsilon$

$$T[\varepsilon^{\alpha}] = -\alpha$$
$$T[\log^{\alpha}(x)] = \alpha$$
$$T[\zeta(\alpha)] = \alpha$$

In d-dimensions:

Uniform transcendental structure at every order in perturbative expansion

In on-supersymmetric SU(N) when Cf=Ca=nf, the leading transcendental terms agree with N=4 SYM

Example-2: The Konishi operator

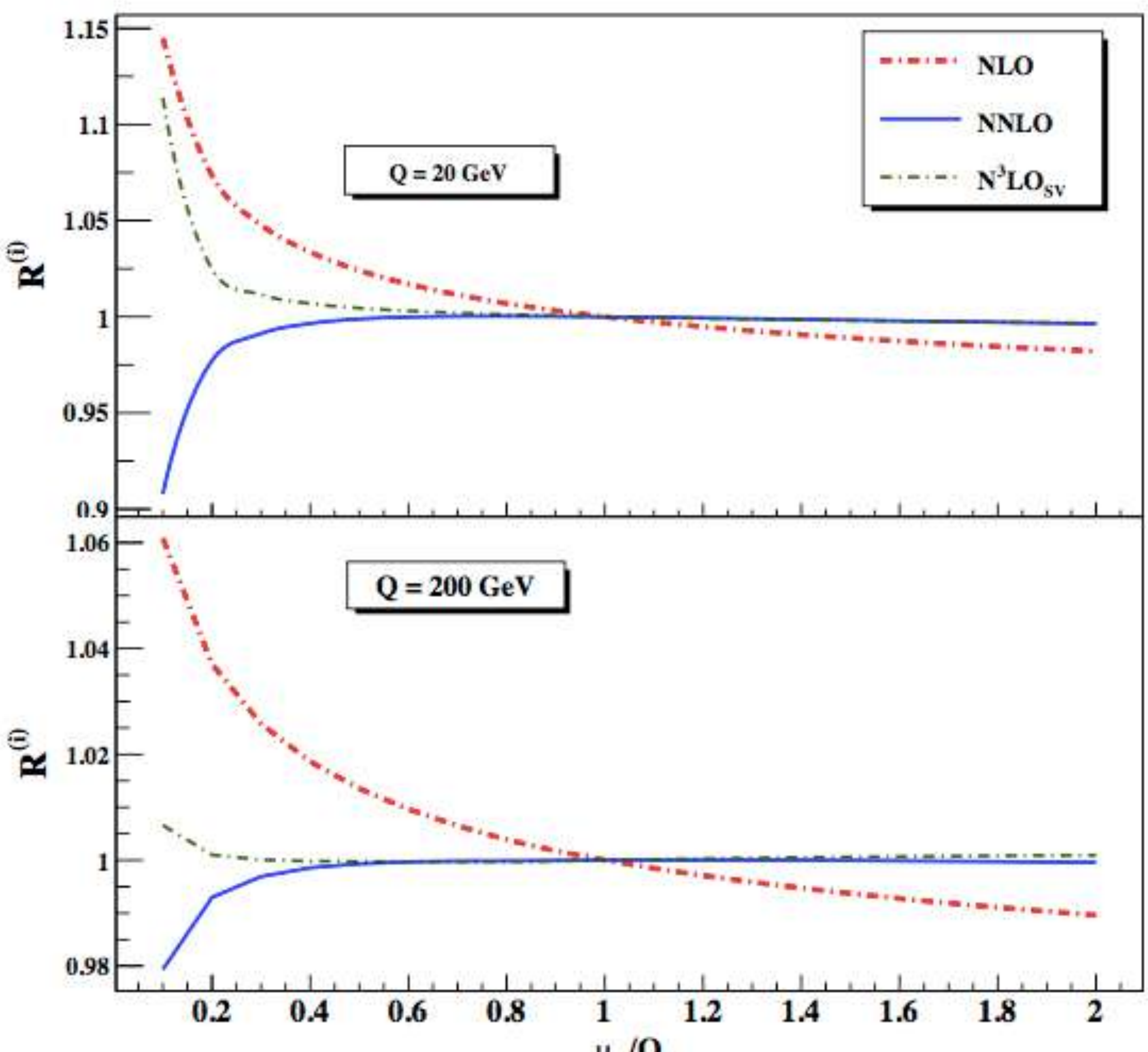
UV as well as IR divergent

At 3-point 3-loop and 4-point 2-loops,the leading transcendental terms agree with those of counterparts with Half-BPS FF.

Conclusions:

- Perturbative predictions from QFT is important for physics studies at the collider experiments
- Large number of Feynman diagrams
  * multi-leg , multi-loop amplitudes,
  * Feynman integrals, Phase space integrals
- Integration By Parts (IBP) identities to get Master Integrals
- Solving MIs using the method of Differential Equations
- Infrared Structure of on-shell Amplitudes in QCD and SYM
  * Catani's proposal
  * BDS conjecture
  * Leading Transcendentality principle

Drell-Yan at N3LO (SV)



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