An Introduction to
Lattice Quantum Chromodynamics

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The **2026 IITB-CFNS-CTEQ School on Perturbative QCD for EIC**

Lecture 1

- **Quantum Chromodynamics**
  Theory of quarks and gluons
- **Introduction to lattice QCD**
  - Gauge Action
  - Fermion Action
- **How to calculate an observable?**
  Numerical procedure
- **Some results**
- **Challenges**

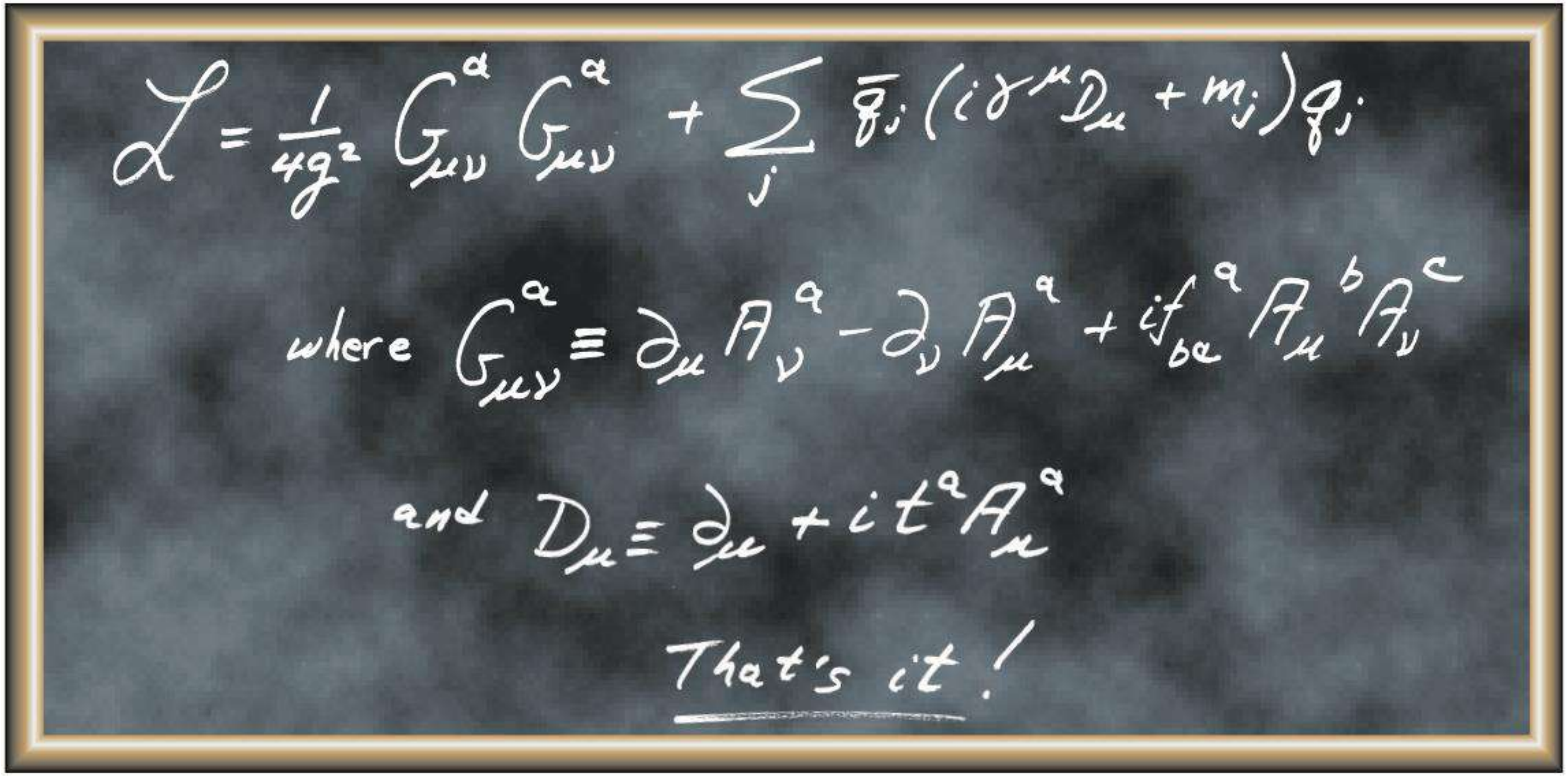
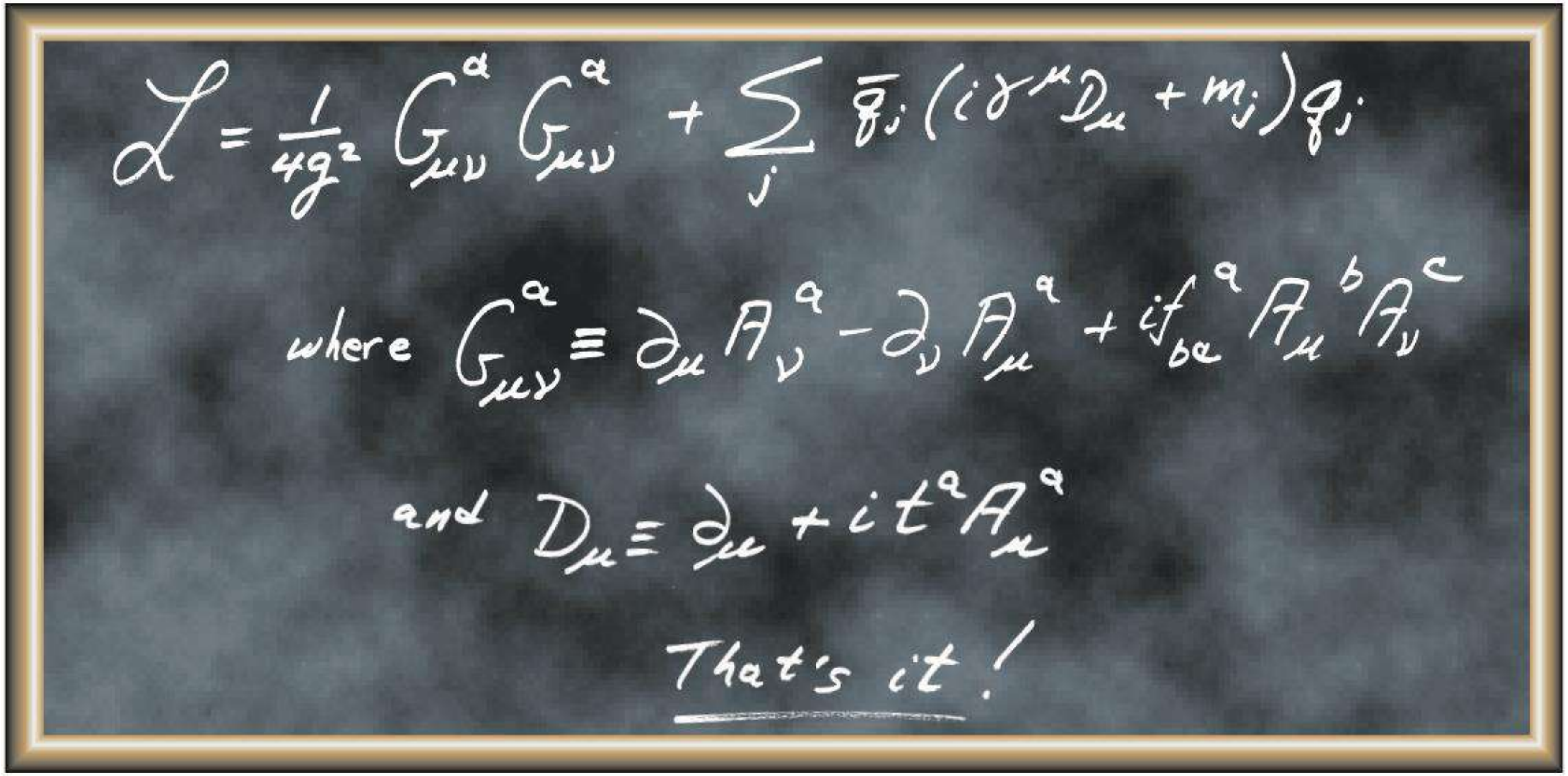


FIGURE 1. THE QCD LAGRANGIAN $\mathcal{L}$ displayed here is, in principle, a complete description of the strong interaction. But, in practice, it leads to equations that are notoriously hard to solve. Here m_j and q_j are the mass and quantum field of the quark of jth flavor, and A is the gluon field, with spacetime indices μ and ν and color indices a, b, c. The numerical coefficients f and t guarantee SU(3) color symmetry. Aside from the quark masses, the one coupling constant g is the only free parameter of the theory.

..Wilczek



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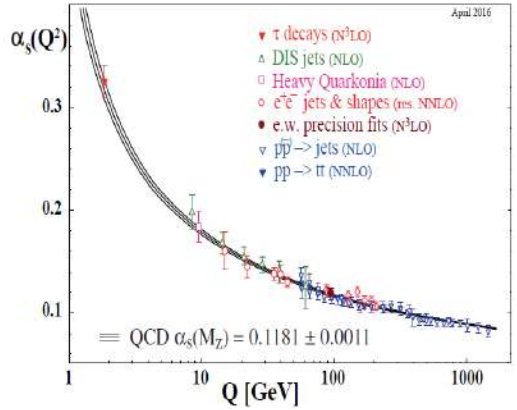
QCD

Quantum Chromodynamics is the gauge field theory of the strong interactions of nature

Strong interactions:

- Confining
- Asymptotically free
- Chirally broken



SU(3) component of the SU(3) X SU(2) X U(1) standard model of particle physics

To study the physics at hadronic scales, perturbative methods fail and we need a non-perturbative regulator

A good regulator should

- ➢ Interpret $\int D\phi$
- ➢ Regularize momentum

Lattice can do both

QCD

$$\mathcal{L} = \sum_q \bar{\psi}_{q,a}(i\gamma^\mu \partial_\mu \delta_{ab} - g_s \gamma^\mu t^C_{ab} \mathcal{A}^C_\mu - m_q \delta_{ab}) \psi_{q,b} - \frac{1}{4} F^A_{\mu\nu} F^{A\,\mu\nu}$$

$$F^A_{\mu\nu} = \partial_\mu \mathcal{A}^A_\nu - \partial_\nu \mathcal{A}^A_\mu - g_s\, f_{ABC} \mathcal{A}^B_\mu \mathcal{A}^C_\nu \qquad [t^A, t^B] = i f_{ABC} t^C$$

- Symmetries :
  - SU(3) local gauge symmetry
  - Lorentz, C, P, T and global flavor
  - $\psi \rightarrow e^{i\alpha}\psi$
  - In the m = 0 chiral limit $\psi \rightarrow e^{i\gamma_5 \alpha}\psi$

- Need to preserve as much symmetries as possible on lattice

- Fundamental free parameters in the theory :

  The coupling between quarks and the gluons—dimensionless

  The quark masses – dimensionfull

Lattice should not introduce more free parameters to be an *ab-initio* method

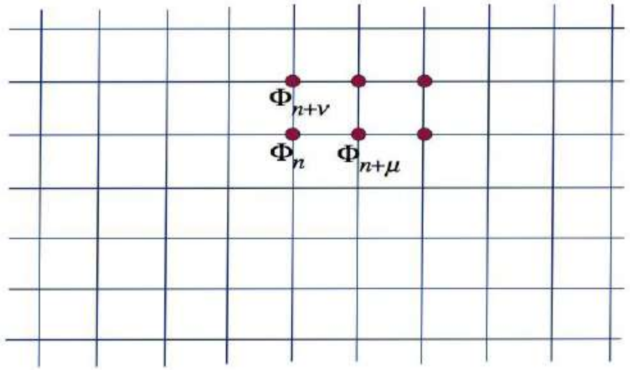
Scalar Theory

Continuum Euclidean action :

$$Z = \int \mathcal{D}\phi e^{-S[\phi]}$$

$$S[\phi] = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2} m^2 \phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right)$$

Lattice action :



- Momentum is regularized
- A discrete set of integration points, that is finite dimensional integrals for finite lattice

Discretization of fields and derivatives :

$$\phi(x) \longrightarrow \phi_n, \quad x = na$$

$$\int dx_i \longrightarrow a \sum_{n_i}$$

$$\int \mathcal{D}\phi \longrightarrow \prod_n d\phi_n$$

$$\partial_\mu \phi(x) \longrightarrow \Delta_\mu \phi_n = \frac{1}{a}(\phi_{n+\hat{\mu}} - \phi_n)$$

$$\Delta^*_\mu \phi_n = \frac{1}{a}(\phi_n - \phi_{n-\hat{\mu}})$$

$$S[\phi] = a^4 \sum_n \left(\frac{1}{2}(\Delta_\mu \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 + \frac{\lambda}{4!}\phi_n^4 \right)$$

$$= a^4 \sum_n \left(-\frac{1}{2}\phi_n(\Delta^*_\mu \Delta_\mu)\phi_n + \frac{1}{2} m^2 \phi_n^2 + \frac{\lambda}{4!}\phi_n^4 \right)$$

Fourier transformation of discrete fields

$$\phi(x) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} e^{ipx} \tilde{\phi}(p)$$

$$\tilde{\phi}(p) = a^4 \sum_x e^{-ipx} \phi(x)$$

Fields are periodic in momentum space and are restricted to the first Brillouin zone

$$\phi(p) = \phi(p + \frac{2\pi}{a} n_\mu), \; n_\mu \in \mathbb{Z}$$

$$p_\mu = \frac{2\pi}{a}\frac{n_\mu}{L} \qquad |p_\mu| \leq \pi/a = \Lambda_{\text{cutoff}}$$

$$\int \frac{d^4p}{(2\pi)^4} \to \frac{1}{a^4 L^3 T} \sum_{n_\mu}$$

L : spatial size, $T \rightarrow$ temporal size

Thus, Lattice provides a natural regulator for field theories.
Continuum limit can be obtained by taking the limit L and $T \rightarrow \infty$, and $a \rightarrow 0$

Does this method work for QCD?

Euclidean Formulation

$$\int Dx\, e^{iS(x)}$$

* Paths are weighted with an oscillating function and so is not suitable for numerical calculation!

* Change real time to imaginary time (Minkowski space to Euclidean space)

$$t \longrightarrow -i\tau,$$

$$< x_f, \tau_f \,|\, x_i, \tau_i > \;=\; \int_{x_i}^{x_f} \mathcal{D}x\, e^{-S_E[x(\tau)]}$$

Birth of Lattice QCD

Phys. Rev. D10, 2445 (1974)

PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Confinement of quarks*

Kenneth G. Wilson

Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850

(Received 12 June 1974)

A mechanism for total confinement of quarks, similar to that of Schwinger, is defined which requires the existence of Abelian or non-Abelian gauge fields. It is shown how to quantize a gauge field theory on a discrete lattice in Euclidean space-time, preserving exact gauge invariance and treating the gauge fields as angular variables (which makes a gauge-fixing term unnecessary). The lattice gauge theory has a computable strong-coupling limit; in this limit the binding mechanism applies and there are no free quarks. There is unfortunately no Lorentz (or Euclidean) invariance in the strong-coupling limit. The strong-coupling expansion involves sums over all quark paths and sums over all surfaces (on the lattice) joining quark paths. This structure is reminiscent of relativistic string models of hadrons.

I. INTRODUCTION

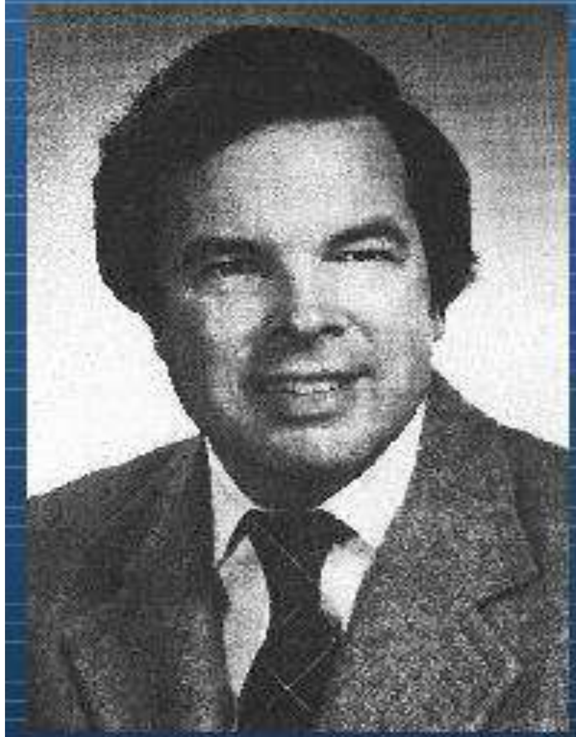
The success of the quark-constituent picture both for resonances and for deep-inelastic electron and neutrino processes makes it difficult to believe quarks do not exist. The problem is that quarks have not been seen. This suggests that quarks, for some reason, cannot appear as separate particles in a final state. A number of speculations have been offered as to how this might happen.[1]

Independently of the quark problem, Schwinger observed many years ago[2] that the vector mesons of a gauge theory can have a nonzero mass if vacuum polarization totally screens the charges in a gauge theory. Schwinger illustrated this result particles over short times and short distances. The polarization effects which prevent the appearance of electrons in the final state take place on a longer time scale (longer than $1/m_\gamma$, where m_γ is the photon mass).

A new mechanism which keeps quarks bound will be proposed in this paper. The mechanism applies to gauge theories only. The mechanism will be illustrated using the strong-coupling limit of a gauge theory in four-dimensional space-time. However, the model discussed here has a built-in ultraviolet cutoff, and in the strong-coupling limit all particle masses (including the gauge field masses) are much larger than the cutoff; in consequence the theory is far from covariant.

The confinement mechanism proposed here is

Among all time favorites



Ken Wilson

Nobel Laureate **1982**

Continuum to Lattice (Computer)

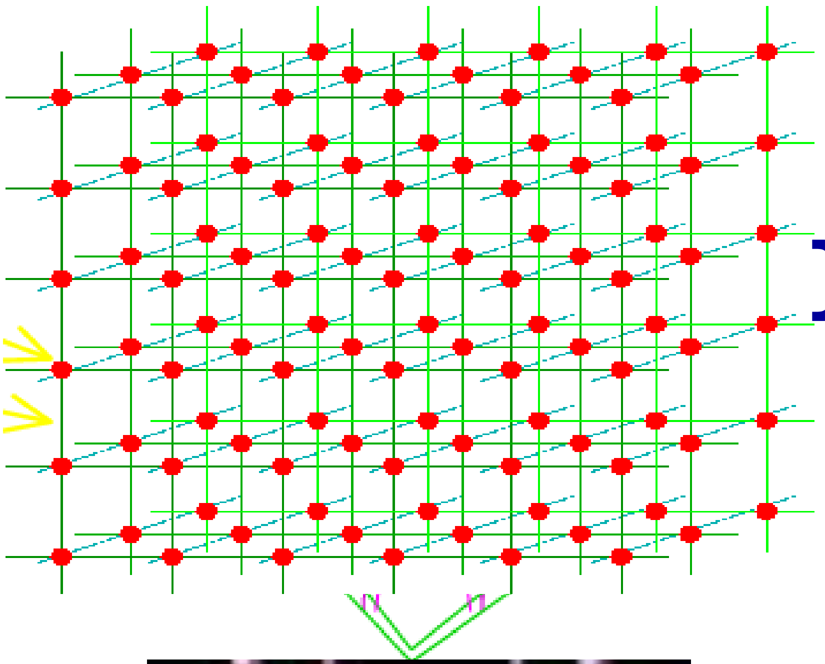
$$L_{QCD} = \frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(\gamma^{\mu}D_{\mu} + m_{q})\psi$$

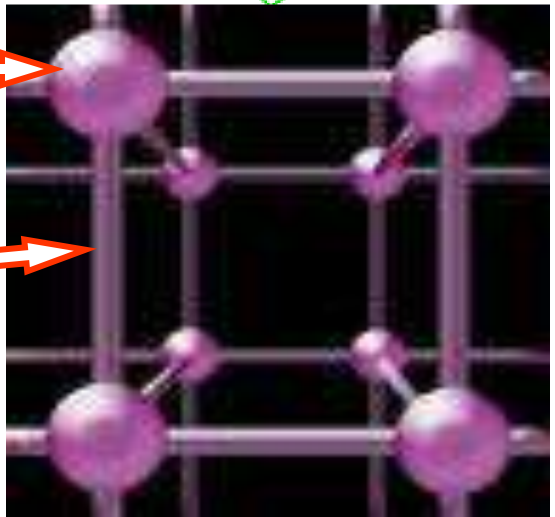
Continuous space-time

Euclidean

path integral formulation

Discrete Euclidean space-time





Quark Jungle Gym

Life on a Lattice

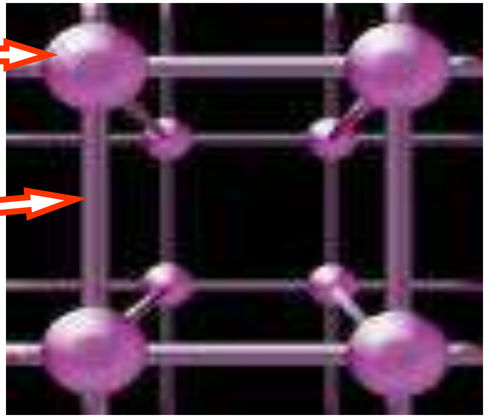
- Quarks fields are on the lattice sites and carry colors, flavors and spin as like the continuum
- Gauge fields are links between sites and are SU(3) matrices. Continuum and lattice gauge fields are related by

$$A_\mu(x) \to U(x,\mu) = e^{-iagA^b_\mu(x)t^b}$$

- Under gauge transformation

Quark (on Lattice sites)

Gluon (on Links)



Quark Jungle Gym

$$\psi_n \to V_n\psi_n, \quad \bar\psi_n \to \bar\psi_n V_n^\dagger \quad V_n \in SU(3)$$

$$\bar\psi_n \psi_{n+\hat\mu} \longrightarrow \bar\psi_n U_{n,\mu}\psi_{n+\hat\mu}$$

$$U_{n,\mu} \to V_n U_{n,\mu} V^{-1}_{n+\mu}$$

- Make derivatives gauge invariants :

$$\nabla^{\rm fwd}_\mu \psi(x) = \frac{1}{a}\left[U_\mu(x)\psi(x+a\hat\mu) - \psi(x)\right]$$

- Gauge invariant quantities are closed loops :

$$\prod_{\mathcal{P}} U_\mu(x)$$

$$\bar\psi(x)\gamma_\mu U_\mu(x)\psi(x+a\hat\mu)$$

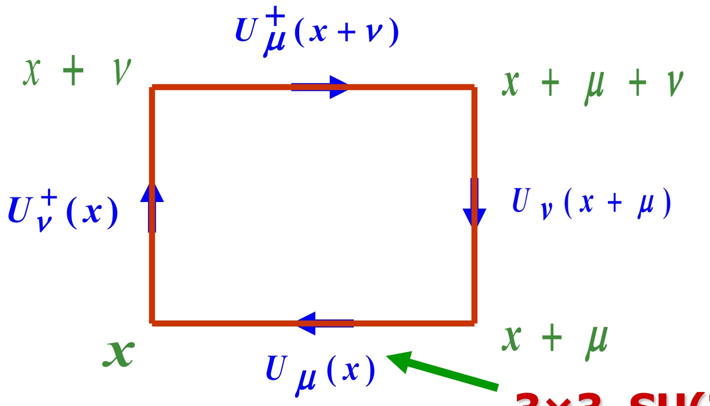
Lattice Formulation of QCD

❖ Preserve the symmetry on the lattice **(QCD has gauge symmetry)**

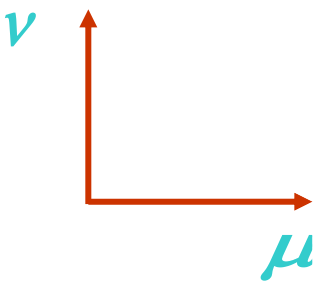
➢ Gauge Symmetry --- Wilson, 1974 ($A_\mu \rightarrow U_\mu$)

U : Link Variable

$$U_\mu(x) = e^{-iagA_\mu^a \lambda^a / 2}$$



3×3 SU(3) matrices



The simplest possible gauge invariant object (plaquette):

$$\mathrm{Tr}\,\Box_{\mu\nu}(x) \equiv \mathrm{Tr}\,U_\nu(x)U_\mu(x+\hat{\nu})U_\nu^\dagger(x+\mu)U_\mu^\dagger(x)$$

Gauge Action

The simplest possible gauge invariant object (plaquette):

$$\mathrm{Tr}\,\Box_{\mu\nu}(x) \equiv \mathrm{Tr}\, U_\nu(x)U_\mu(x+\hat\nu)U_\nu^\dagger(x+\mu)U_\mu^\dagger(x)$$

Next to leading order expansion :

$$\begin{aligned}\Box_{\mu\nu} &\simeq 1 + ia\Big(A_\nu(x+\hat\nu/2) + A_\mu(x+\hat\nu+\hat\mu/2) - A_\nu(x+\hat\nu/2+\mu) - A_\mu(x+\hat\mu/2)\Big)\\ &\simeq 1 + ia^2(\partial_\nu A_\mu - \partial_\mu A_\nu)\,.\end{aligned}$$

Next order :

$$\begin{aligned}\Box_{\mu\nu} &\simeq (\text{above}) - a^2\left(A_\mu^2 + A_\nu^2 + A_\nu A_\mu - A_\nu A_\nu - A_\nu A_\mu - A_\mu A_\nu - A_\mu A_\mu + A_\nu A_\mu)\right)\\ &= (\text{above}) - a^2\big[A_\nu\,,\,A_\mu\big]\,,\end{aligned}$$

$$\Box_{\mu\nu} \simeq -ia^2 F^a_{\mu\nu} T^a$$

Reflection symmetry : $\Box_{\mu\nu} \simeq 1 - ia^2 F^a_{\mu\nu}T^a - \frac{a^4}{2} F^a_{\mu\nu}F^b_{\mu\nu}T^aT^b + \ldots$

$$\sum_{x,\mu>\nu}[N_c - \mathrm{Tr}\,\Box_{\mu\nu}(x)] \simeq \int d^4x \frac{1}{8} F^a_{\mu\nu}F^a_{\mu\nu}$$

Wilson Action: $S = \frac{\beta}{N_c}\sum_{x,\mu>\nu}(N_c - \mathrm{Tr}\,\Box_{\mu\nu}(x))$

Improved action (1 X 2 boxes) :

$$S_{\mathrm{Symanzik}}: \sum_{\mu>\nu} \mathrm{Tr}\,\Box_{\mu\nu} \rightarrow \frac{5}{3}\sum_{\mu>\nu}\mathrm{Tr}\,\Box_{\mu\nu} - \frac{1}{12}\sum_{\mu>\nu}(\mathrm{Tr}\,\sqsubset\!\sqsupset_{\mu\nu} + \mathrm{Tr}\,\sqsubset\!\sqsupset_{\nu\mu})$$

Continuum Limit

Relation between lattice spacing and gauge coupling, $\beta = 6/g^2$ is

$$a = \frac{1}{\Lambda} e^{-1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2}(1 + \mathcal{O}(g^2))$$

Lattice regularization generates a mass scale automatically!

$$\Lambda = \frac{1}{a} e^{-1/2\beta_0 g(a)^2} g(a)^{-\beta_1/\beta_0^2}$$

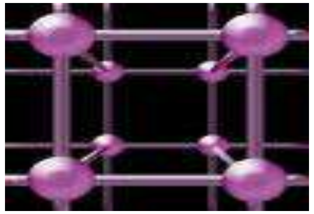
$\boldsymbol{a \to 0}$ **means** $\boldsymbol{g^2 \to 0,\ \beta \to \infty}$

$$\boldsymbol{m_{lat}(g(a)) = am_{phy} \to 0}$$

So, the correlation length $\boldsymbol{\xi \sim 1/m_{latt} \to \infty}$

- A continuum limit of the lattice theory can be obtained by tuning the bare parameters to the critical surface of a lattice system.
- Use a few physical quantities, like masses/string tension to set the physical scale and quark masses.
- All other QCD observables will be prediction

Fermion Actions



Quark Jungle Gym

$$< \hat{O} > = \mathrm{Lim}_{\beta\to\infty} \frac{1}{Z} \mathrm{Tr}[e^{-\beta H} \hat{O}(U, \bar{\psi}, \psi)]$$

$$= \mathrm{Lim}_{\beta\to\infty} \frac{\int DU\, D\bar{\psi}\, D\psi\, O[U, \bar{\psi}, \psi]\, e^{-S_g[U] - S_F[U,\bar{\psi},\psi]}}{\int DU\, D\bar{\psi}\, D\psi\, e^{-S_g[U] - S_F[U,\bar{\psi},\psi]}}$$

Integrating out the Grassmann variables is possible since $S = \sum_{x,y} \bar{\psi}(x) M(x,y) \psi(y)$

M is a complex matrix functional of the gauge fields

$$< \hat{O} > = \frac{\int DU \{\det D\}^{n_f}\, O[U, D^{-1}]\, e^{-S_g[U]}}{\int DU\, \{\det D\}^{n_f}\, e^{-S_g[U]}} = \prod_n \int \underbrace{dU_n \frac{1}{Z} \{\det D(U)\}^{n_f}\, e^{-S_g[U]}} O[U, D^{-1}]$$

- **Traslational invariance** $M(x,y) = D(x-y)$
- **Chiral symmetry** $\{\tilde{D}(p), \gamma^5\} = 0$ $\qquad D(p) = \sum_\mu \gamma_\mu d_\mu(p)$
- **Locality : D(p) is a regular function of p throughout the Brillouin zone**

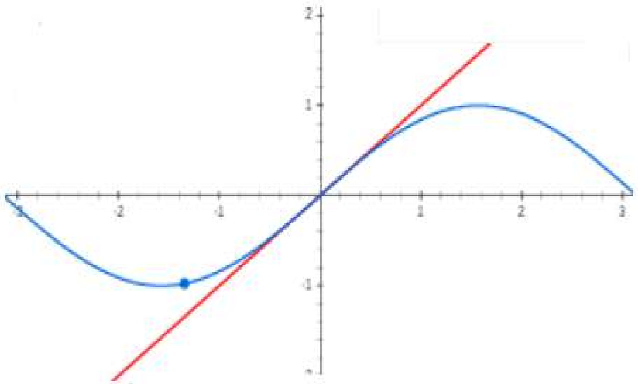
Fermion Actions

Fermion action with naïve discretization :

$$S_f = a^4 \sum_{x,\mu} \left(\bar{\psi}(x)\gamma_\mu \left(\nabla^*_\mu + \nabla_\mu\right)\psi(x) + m\bar{\psi}(x)\psi(x) \right)$$

$$= a^4 \int_p \bar{\psi}(-p)\left[\frac{i}{a}\sin(p_\mu a)\gamma_\mu + m\right]\psi(p)$$

Continuum to discrete momentum : $p_\mu \longrightarrow \sin(p_\mu a)/a$



- Very different at the Brillouin zone
- 16 (2^d) poles of the propagator!
- Tried to describe one fermion, got 16!

Fermion doubling problem

- Translational invariance $M(x,y) = D(x-y)$
- Chiral symmetry $\{\tilde{D}(p), \gamma^5\} = 0$ $D(p) = \sum_\mu \gamma_\mu d_\mu(p)$
- Locality : D(p) is a regular function of p throughout the Brillouin zone

Neilsen-Ninomiya No-Go Theorem

It is not possible to construct an action satisfying following properties :

1. $D(x)$ should be local, i.e. $||D(x)|| \leq C \exp(-\gamma x)$

2. $D(p) = i \sum_\mu \gamma_\mu p_\mu + O((ap)^2)$ (cubic symmmetry)

3. no doubler exist, i.e. $D(p)$ is invertible for $p \neq 0$

4. $\gamma_5 D + D \gamma_5 = 0$ (chiral symmetry)

That is, there is no doubler free, chirally symmetric, translation invariant, real-valued lattice action.

Wilson Fermions

Lift the fermion mass at the Brillouin zone by adding a term

$$S_W = a^4 \sum \bar{\psi}_n (D_W + m)\psi_n$$

$$D_W = \frac{1}{2}\left(\gamma_\mu(\Delta^*_\mu + \Delta_\mu) - ar\Delta^*_\mu \Delta_\mu\right)$$

Wilson term

- Like a scalar kinetic term
- Lifts the doubler
- O(a) irrelevant operator, additive mass renormalization
- Breaks chiral symmetry even in the chiral limit

Rescaling : $\psi \to \sqrt{2\kappa}\psi,\ \kappa = \frac{1}{2ma+8r}$ ← hopping parameter

$$S_W = a^4 \sum_n \left(\bar{\psi}_n\psi_n - \kappa(\bar{\psi}_n(r-\gamma_\mu)U_{n,\mu}\psi_{n+\mu} + \bar{\psi}_n(r+\gamma_\mu)U_{n,\mu}\psi_{n-\mu})\right)$$

Staggered Fermions

- Distribute four spin components of the Dirac spinors on four different sites of the lattice.

$$\psi_n \to \Omega_n \psi'_n \quad , \quad \Omega_n = \gamma_0^{n_0}\gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}$$

$$S = \frac{1}{2a}\sum_n \bar{\psi}'_n \alpha_\mu(n)[U_{n,\mu}\psi_{n+\hat{\mu}} - U^\dagger_{n-\hat{\mu},\mu}\psi_{n-\hat{\mu}}] + m\sum_n \psi'_n\psi'_n$$

Spin diagonal

phase $\alpha_\mu(n) = (-1)^{n_0+\cdots+n_{\mu-1}}$ ± 1

Removing 3 of the four components of $\psi \to \chi$

$$S = \frac{1}{2a}\sum_n \bar{\chi}_n \alpha_\mu(n)[U_{n,\mu}\chi_{n+\hat{\mu}} - U^\dagger_{n-\hat{\mu},\mu}\chi_{n-\hat{\mu}}] + m\sum_n \chi_n\chi_n$$

- Four tastes → 16 pions, only one is Goldstone boson. Taste changing interactions can be suppressed.
- Remnant U(1) X U(1) symmetry protects quark mass:

$$\chi_n \to e^{\pm i\theta}\chi_n, \quad n = \text{even or odd}$$

- However, four flavours to one flavour needs rooting. Breaking locality?

Fermion Actions

- **Wilson :** Add a term to make doublers massive (proportional to inverse lattice spacing) –Wilson action. But it breaks chiral symmetry (quark mass is not protected from additive mass renormalization)

- **Staggered :** Put four spin component of the Dirac spinor on four different sites of the lattice. No breaking of chiral symmetry. Four tastes → 16 pions, only one is Goldtone boson. Taste changing interactions (Naik term) can be suppressed. However, four flavours to one flavour needs rooting. Breaking locality?

- **Clover fermions :** Wilson fermions with symmanzik type improvement. No O(a) term

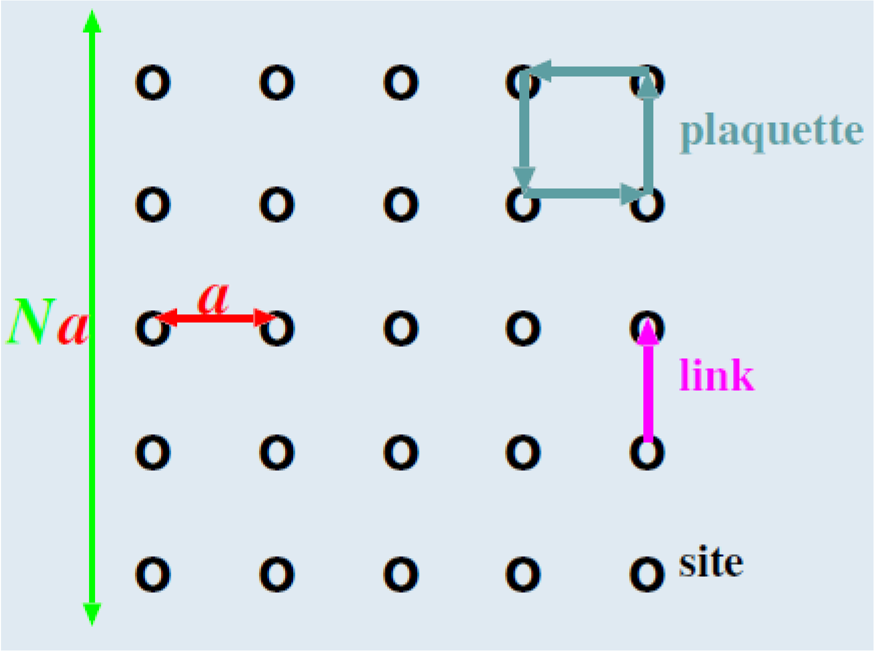
- **Twisted mass :** a variety of Wilson action with flavour dependent chiral rotation of fermion fields. No O(a) term but breaks isospin symmetry.

- **HISQ :** Highly improved staggered quarks

- **Domain wall and Overlap fermions :** Exact extend chiral symmetry on the lattice. Not ultralocal. Extremely costly compared to Staggered and Wilson actions

Fermion Actions

Action	Advantages	Disadvantages
Wilson	Computationally fast	Breaks chiral symmetry, O(a) error, additive mass renormalization
Improved Wilson (Clover)	Computationally fast	Breaks chiral symmetry, operator improvement is necessary
Twisted mass	Computationally fast	Breaks chiral symmetry, Isospin breaking
Staggered	Computationally fast	Fourth root problem, taste breaking, complication in operator construction
Highly improved staggered (HISQ)	Computationally reasonably fast, lesser taste breaking	Fourth root problem, complication in operator construction
Domain wall	Improved Chiral symmetry	Computationally expensive
Overlap	Exact chiral symmetry	Computationally very expensive

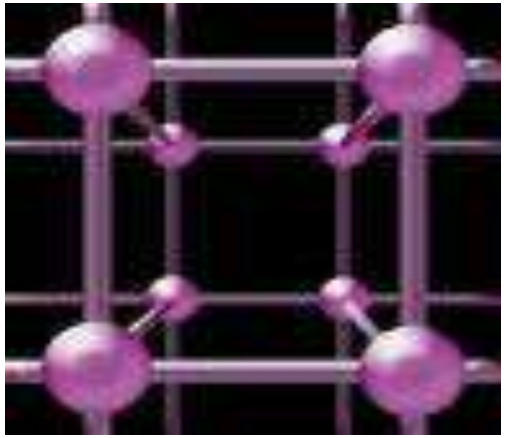


Typical values :
a = 0.15--0.04 fm
Na = 2--6 fm

Continuum limit :
$a \rightarrow 0$, Na fixed
Infinite volume : $Na \rightarrow \infty$

Inputs :
Lattice spacing (coupling) : from known masses : say M_Ω
or static quark potential
heavy meson spectrum etc

Quark mass : u(d) ← m_π
s ← m_k or m(ss)
c ← ¼(3J/Ψ + η_c)
b ← Y_b



How to calculate an observable??

From statistical mechanical
correlation functions

Too Many Integration!!

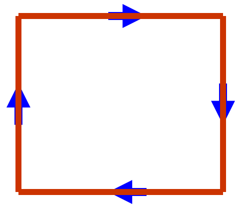
$$< 0 \,|\, \hat{Q}[\hat{\phi}(x)] \,|\, 0 > \quad \longrightarrow \quad < Q(\phi_n) > \equiv \frac{\int d\phi_n \, Q(\phi_n) \, e^{-S^{lat}(\phi_n)}}{\int d\phi_n \, e^{-S^{lat}(\phi_n)}}$$

We need lattice lattice spacing as small as possible i.e., big volume.

Suppose we have a 10^4 lattice.

This has

- $10^4 = 10,000$ sites,
- $10^4 \times 4 = 40,000$ links (due to gluons)
- $10^4 \times 4 \times 8 = \mathbf{320,000}$ dimensional integral !!
- If we take only 2 points/dimension (very crude) sum will be with $2^{320,000} = 3.8 \times 10^{96,329}$ terms!!



Age of the universe is only $\sim 10^{27}$*nanoseconds*!

Is it possible to do this integration??

Statistical Evaluation -- Monte-Carlo Method

- The appearance of such large number immediately suggests a statistical treatment.

- These integrations can be avoided if we assume that the microscopic laws are defined in terms of probabilities and it is possible to generate a sequence of statistically independent configurations with that probability distribution over the possible paths or over the possible locations of the system in configuration space.

- These large number of integrations then reduce to an ensemble average

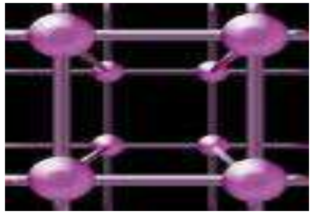
$$\langle O \rangle = \frac{1}{Z} \int [dU]\,[d\psi][d\bar{\psi}]\; O[U] e^{-S[U,\psi,\bar{\psi}]}$$

- Measurement : Average over a representative ensemble of gluon configurations $\{U_i\}$ with probability $P(U_i) \propto \int [d\psi][d\bar{\psi}] e^{-S[U,\psi,\bar{\psi}]}$

$$< O > = \frac{1}{N} \sum_{i=1}^{N} O\{U_i\} + \Delta O \qquad \Delta O \propto \frac{1}{\sqrt{N}} \xrightarrow{N \to \infty} 0$$

$\{U_i\}$'s are the configurations generated by **Monte-Carlo Method**.

Observables



Quark Jungle Gym

$$< \hat{O} > = \mathrm{Lim}_{\beta\to\infty} \frac{1}{Z} \mathrm{Tr}[e^{-\beta H} \hat{O}(U,\bar{\psi},\psi)]$$

$$= \mathrm{Lim}_{\beta\to\infty} \frac{\int DU\, D\bar{\psi}\, D\psi\, O[U,\bar{\psi},\psi]\, e^{-S_g[U]-S_F[U,\bar{\psi},\psi]}}{\int DU\, D\bar{\psi}\, D\psi\, e^{-S_g[U]-S_F[U,\bar{\psi},\psi]}}$$

Integrating out the Grassmann variables is possible since $S_F = \bar{\psi} D \psi$

$$< \hat{O} > = \frac{\int DU \{\det D\}^{n_f} O[U,D^{-1}]\, e^{-S_g[U]}}{\int DU\, \{\det D\}^{n_f}\, e^{-S_g[U]}} = \prod_n \int dU_n \frac{1}{Z} \{\det D(U)\}^{n_f}\, e^{-S_g[U]}\, O[U,D^{-1}]$$

$$< \hat{O} > = \frac{1}{N} \sum_{U \in \frac{1}{Z} e^{-S_g[U] + \ln\{\det D\}^{n_f}}} O[U, D^{-1}]$$

$\sum D^{-1}(U) D^{-1}(U) \ldots D^{-1}(U)$

Parameters : gauge coupling and quark masses

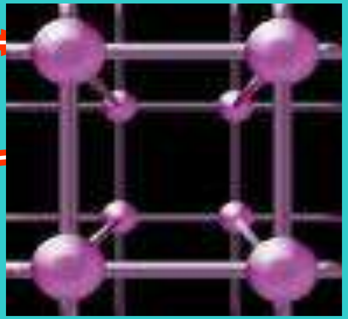
Rational Hybrid Monte Carlo Algorithm

- A state of the art Monte Carlo algorithm for lattice QCD which can generate gauge field configurations relatively faster way even at physical quark masses M. Clark: PoSLAT2006:004,2006

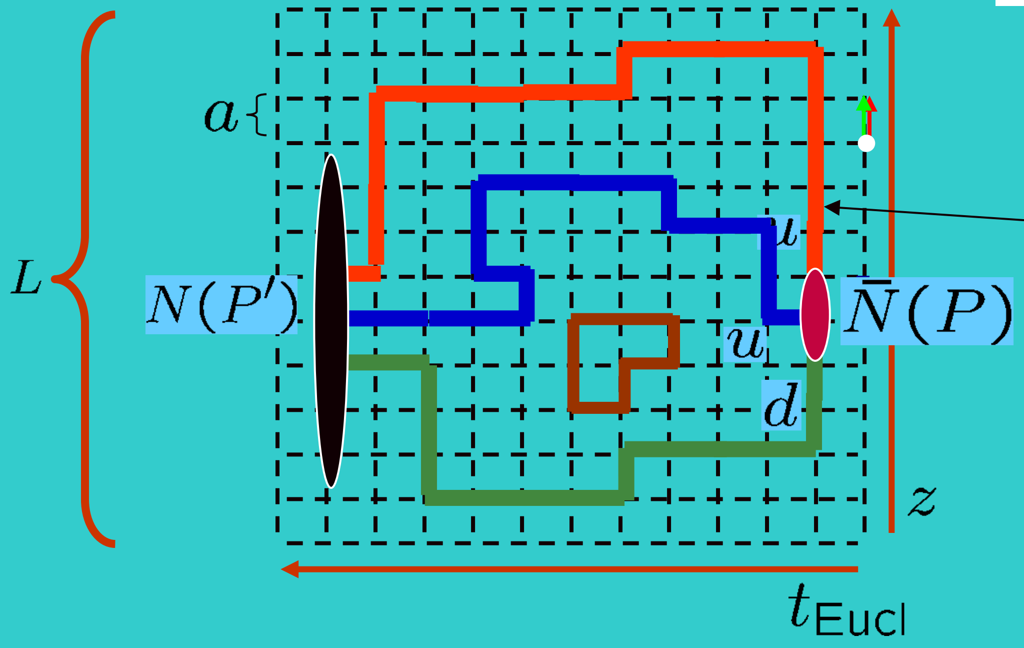
- Fermion determinant of very large matrice gets evaluated by a rational approximation

- Many supercomputers throughout the world are being used for generating these gauge configurations

- Huge amount of efforts go on to improve algorithms (by lattice gauge physicists, applied mathematician, computer scientists)



$$p_\alpha(x) = \epsilon_{abc} u^a_\alpha(x) u^b_\beta(x) (C\gamma_5)_{\beta\eta} d^c_\eta(x),$$

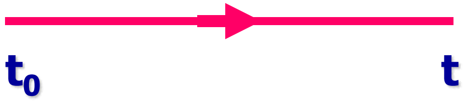


quark propagators :
Inverse of very large matrix of space-time, spin and color

Quark Propagator

$$L_{QCD} = \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(\gamma^{\mu} D_{\mu} + m_q)\psi$$

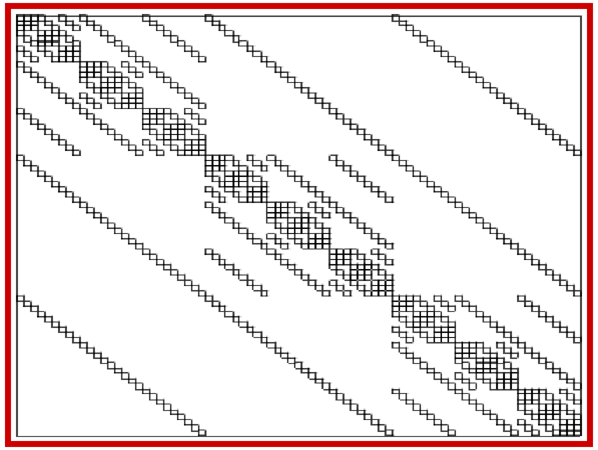
Fermion Action $$S_F = \sum_{x,y} \bar{\psi}(x) M(x,y) \psi(y)$$



Lattice size : $16^3 \times 24$

Dimension of M : $16^3 \times 24 \times 2 \times 3 \times 4$
$\approx 10^6 \times 10^6$!!

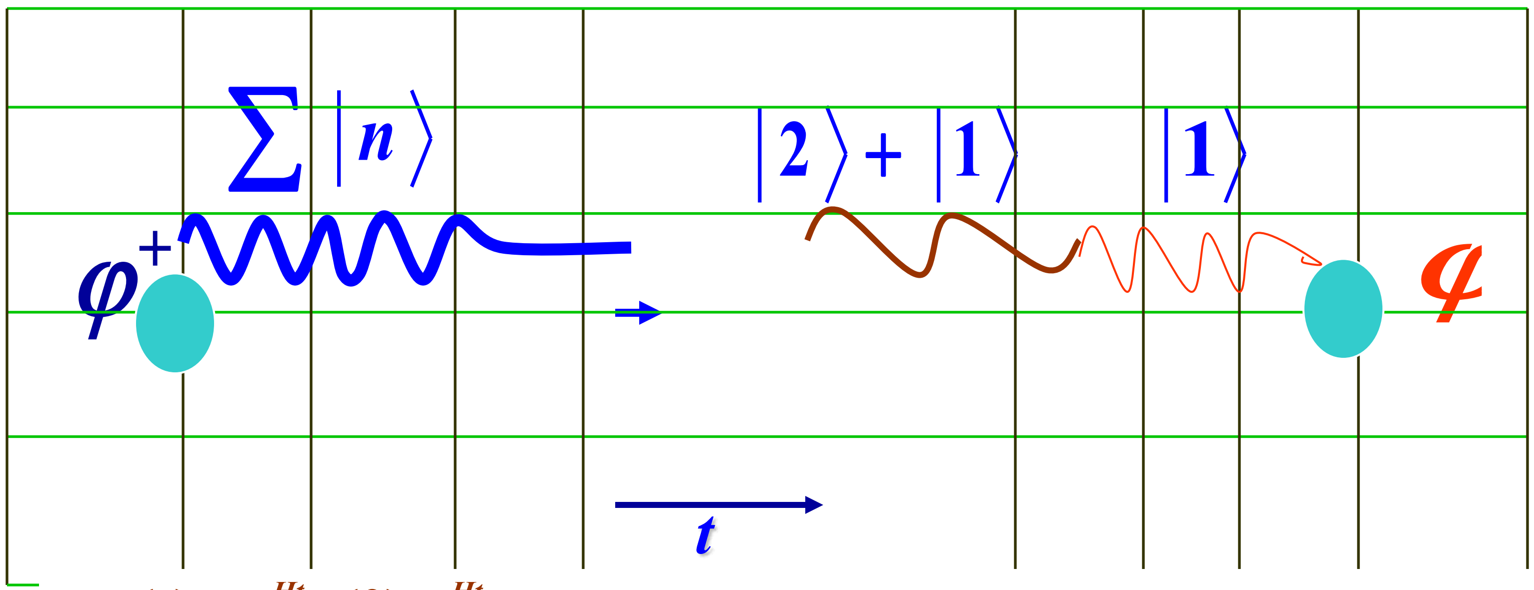
Need $M^{-1}(x,y)$, Tr[$M^{-1}(x,y)$], Det[M(x,y)]



$M^{ab}_{\alpha\beta}(x,y)$

Translational invariance : need only $M^{-1}(x,0)$

Different type of algorithms for different fermionic action
Conjugate gradient is the best for Hermitian positive definite matrix.
Here also algorithms are constantly being improved



$$\varphi(t) = e^{Ht}\varphi(0)e^{-Ht}$$

$$G(t,\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}.(\vec{x}-\vec{x}_0)} \sum_{n,\vec{q}} \langle 0|\varphi(x)|n,\vec{q}\rangle\langle n,\vec{q}|\varphi(x_0)|0\rangle$$

$$= \sum_{\vec{x}} e^{-i\vec{p}.(\vec{x}-\vec{x}_0)} \sum_{n,\vec{q}} \langle 0|e^{H(t-t_0)-i\vec{p}'.(\vec{x}-\vec{x}_0)}\varphi(x_0)e^{-H(t-t_0)+i\vec{p}'.(\vec{x}-\vec{x}_0)}|n,\vec{q}\rangle\langle n,\vec{q}|\varphi(x_0)|0\rangle$$

$$= \sum_{\vec{x}} e^{-i\vec{p}.(\vec{x}-\vec{x}_0)} \sum_{n,\vec{q}} e^{i\vec{q}.(\vec{x}-\vec{x}_0)-E_q^n(t-t_0)} \langle 0|\varphi(x_0)|n,\vec{q}\rangle\langle n,\vec{q}|\varphi(x_0)|0\rangle$$

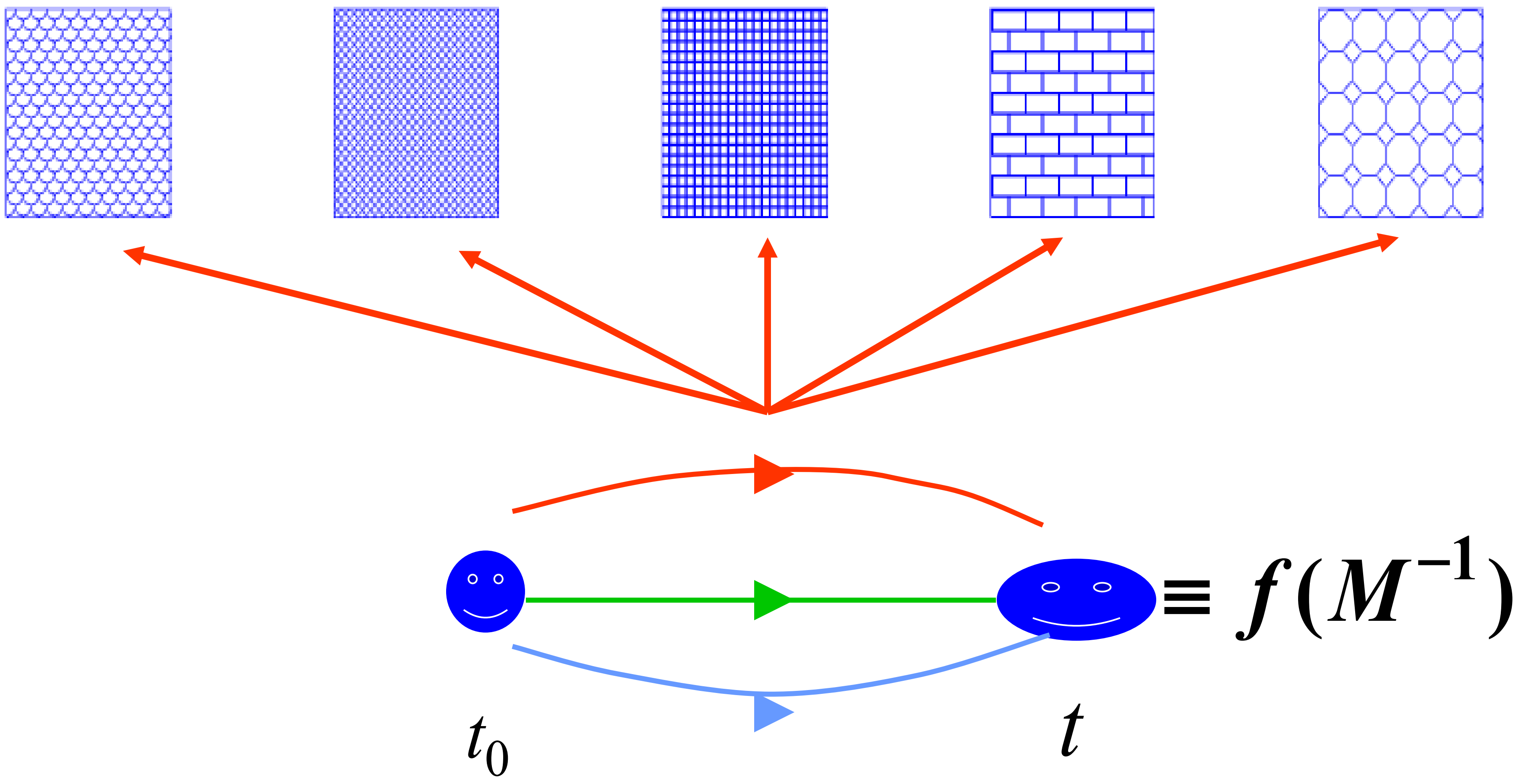
$$\approx \sum_{n,\vec{q}} \delta(\vec{p}-\vec{q}) e^{i(\vec{p}-\vec{q}).\vec{x}_0-E_q^n(t-t_0)} \langle 0|\varphi(x_0)|n,\vec{q}\rangle\langle n,\vec{q}|\varphi(x_0)|0\rangle$$

Determines how effectively this operator interpolates states 'n' from the vacuum

$$= \sum_{n} e^{-E_p^n(t-t_0)} \left|\langle 0|\varphi(x_0)|n,\vec{p}\rangle\right|^2$$

$$= \sum_{n} W_n e^{-E_p^n(t-t_0)} \xrightarrow[t\to\infty]{} W_1 e^{-E_1^n(t-t_0)}$$

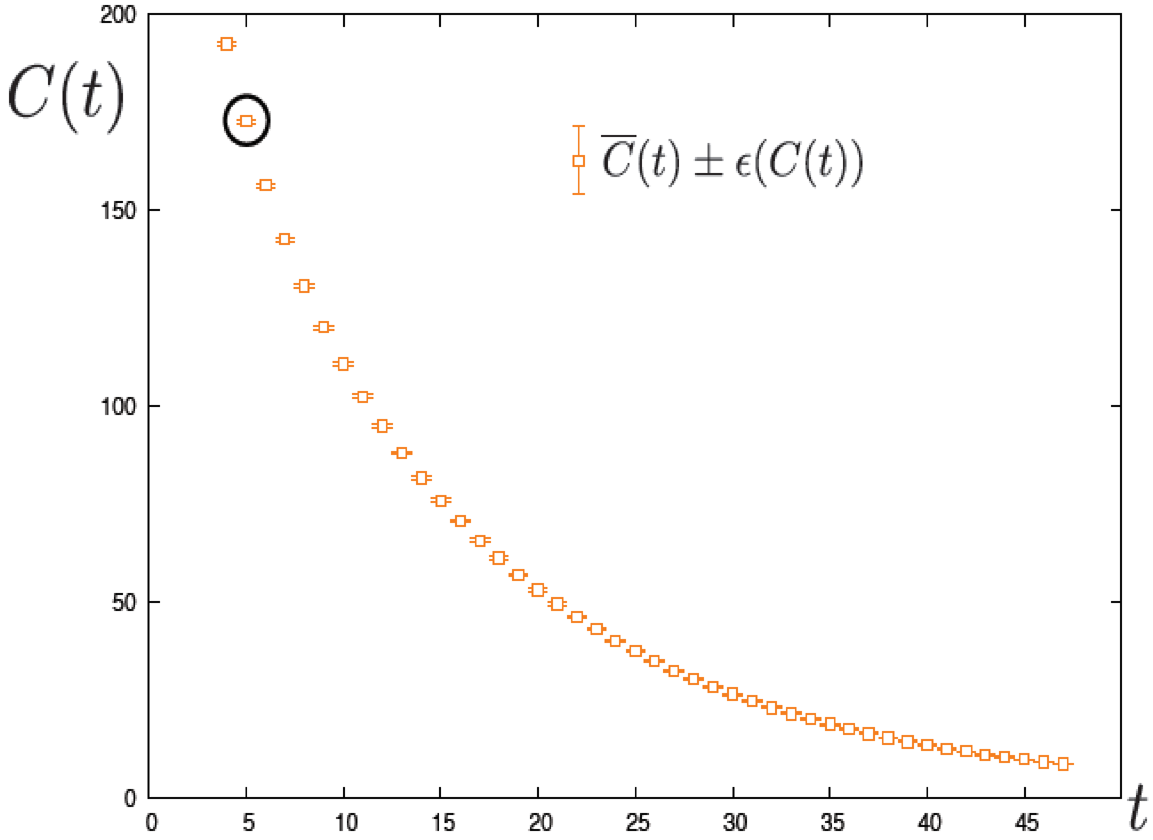
Measure observable on many independent configurations (ensemble)

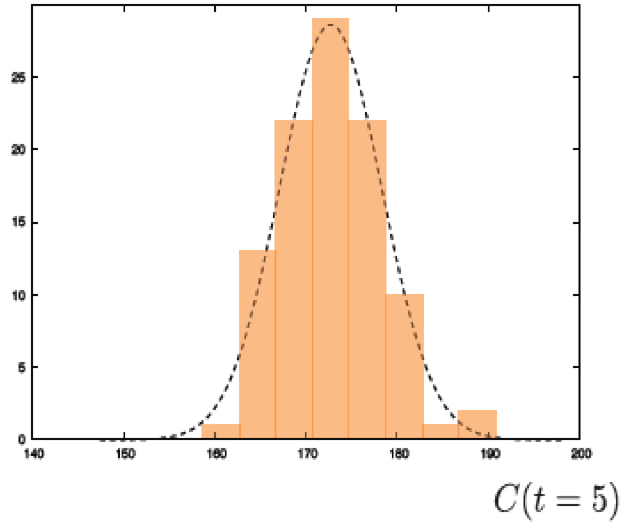


$$C(t) = \langle 0 | \sum_{\vec{x}} \bar{\psi}_{\vec{x},t}\gamma_5\psi_{\vec{x},t} \cdot \bar{\psi}_{\vec{0},0}\gamma_5\psi_{\vec{0},0} | 0 \rangle$$

so we actually obtain an ensemble $C^{(\mathrm{i})}(t)$

one entry for each gauge-field configuration $\{U_{x\mu}\}^{\mathrm{i}=1\ldots\mathrm{N}}$





… how do we relate this information to the mass of the pion ?

Analysis (Extraction of Mass)

$$G(\tau)=\sum_{i=1}^{N} W_i\, e^{-m_i\tau} \underset{\tau\to\infty}{\approx} W_1\, e^{-m_1\tau}$$

Effective mass :

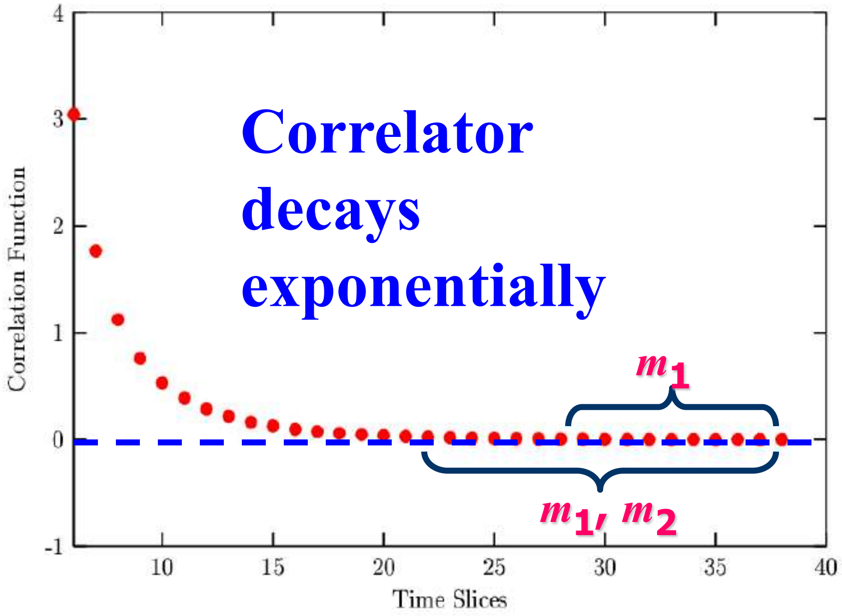
$$\frac{G(\tau)}{G(\tau+1)}=e^{-m_1\tau+m_1(\tau+1)}$$

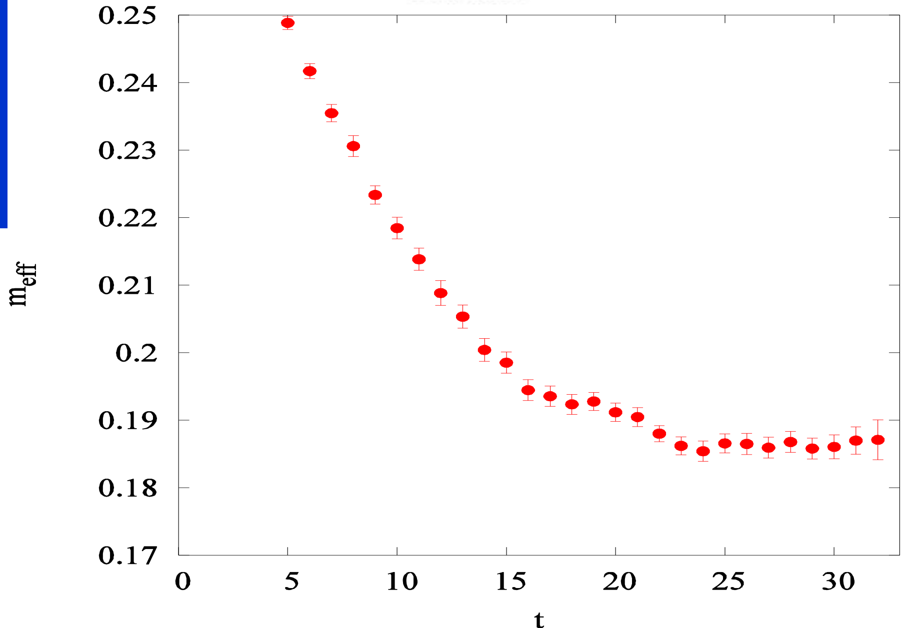
$$m(\tau)=\ln\left[\frac{G(\tau)}{G(\tau+1)}\right]$$

$$\chi^2=\sum_{i=1}^{N}\left[\frac{f(t_i)-\langle G(t_i)\rangle}{\varepsilon(t_i)}\right]$$

$$e^{-mt}=e^{-(ma)(t/a)}$$

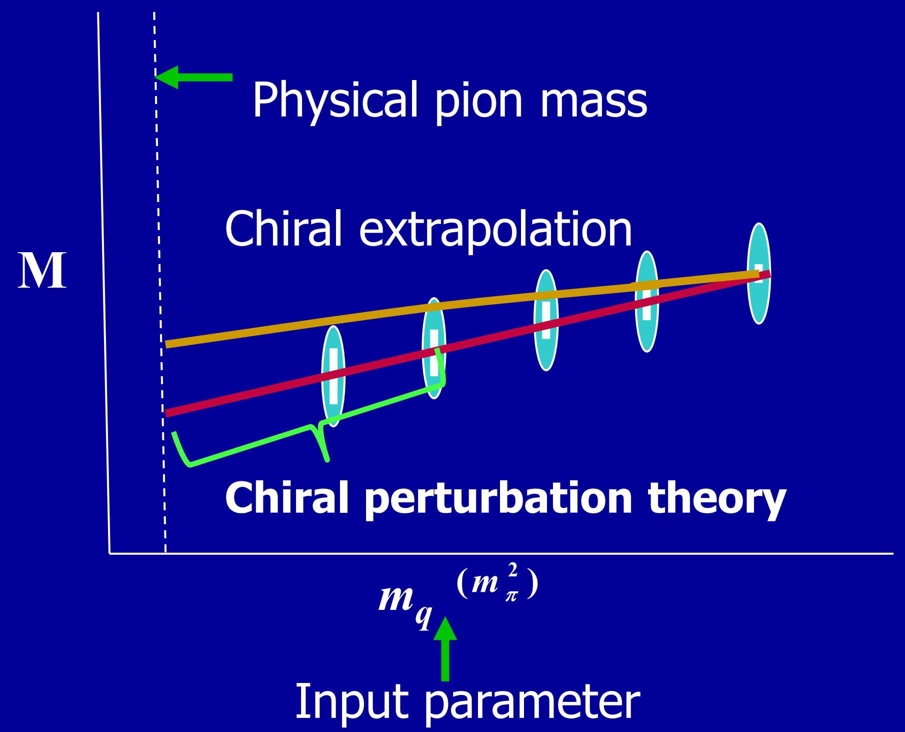
dimensionless mass   integer timeslices

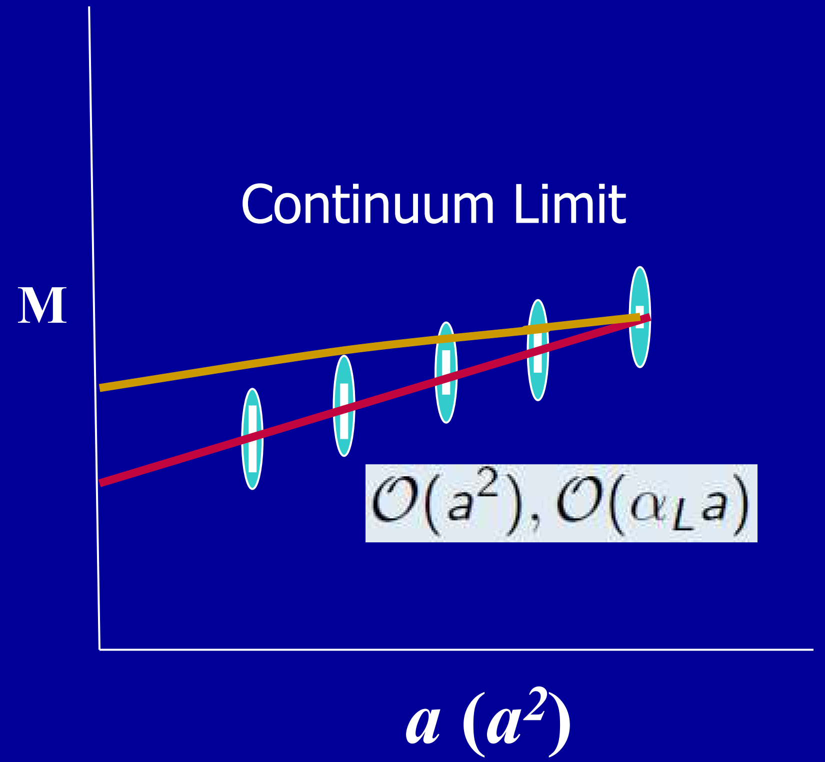


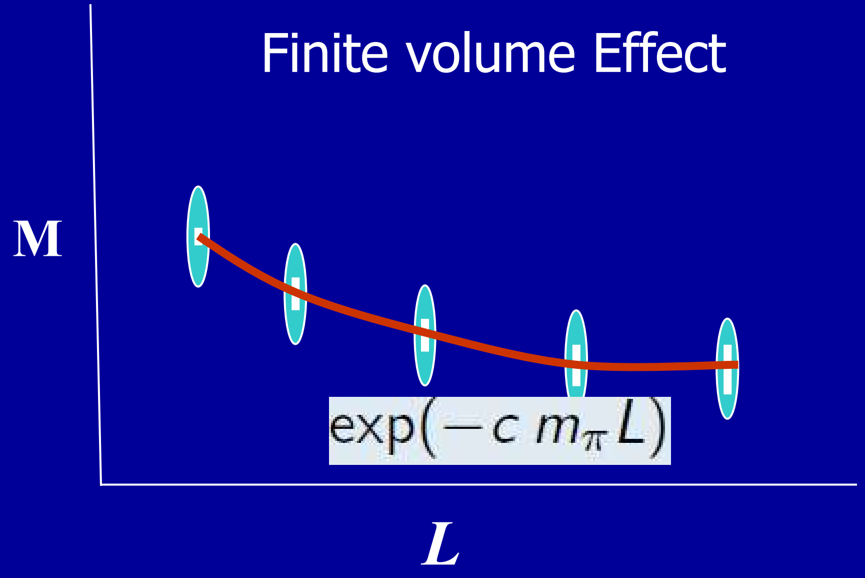


Determine *a* by measuring some physical quantity and compare that to expt, like parameter tuning in any renormalized field theory

Control of Sytemetics







Lattice QCD Workflow

Using Monte Carlo methods generate a set of gauge ensembles representing QCD dynamics, on Euclidean space-time lattices with varying quark masses, varying lattice sizes and varying lattice spacings

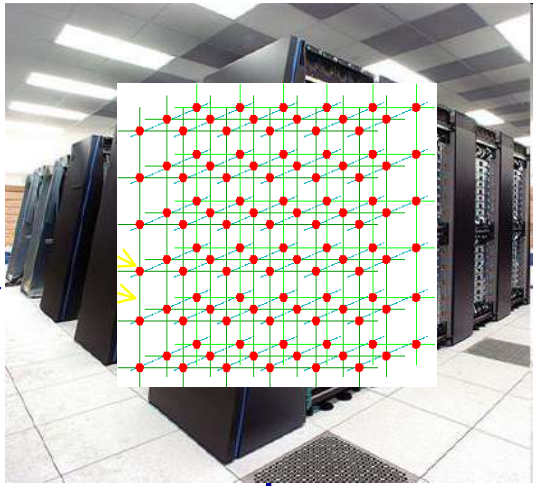
Compute Euclidean correlation functions corresponding to desired observables on those lattice ensembles

Analyse correlation functions, and finally take chiral, continuum and finite volume limits (also renormalization of lattice to physical results)

QCD → LQCD →

Euclidian time
Improved
discretization

Inputs : lattice spacing,
quark masses



Algorithmic developments
AI-ML applications, Algorithms for QC

Continuum,
Chiral,
finite volume
inter-extra-
polations

Spectra, matrix elements,
scattering lengths,
thermodynamic properties
vacuum structure etc.

Experiments :
(postdictions,
predictions)

Models (validity)

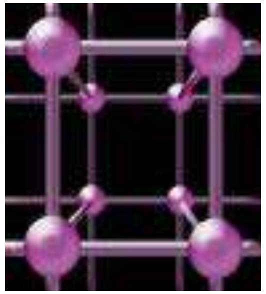
Weak decays and matrix elements

Hadron Structure

Chiral symmetry

Hadron spectroscopy and interactions

Vacuum structure and confinement



Hadron tomography

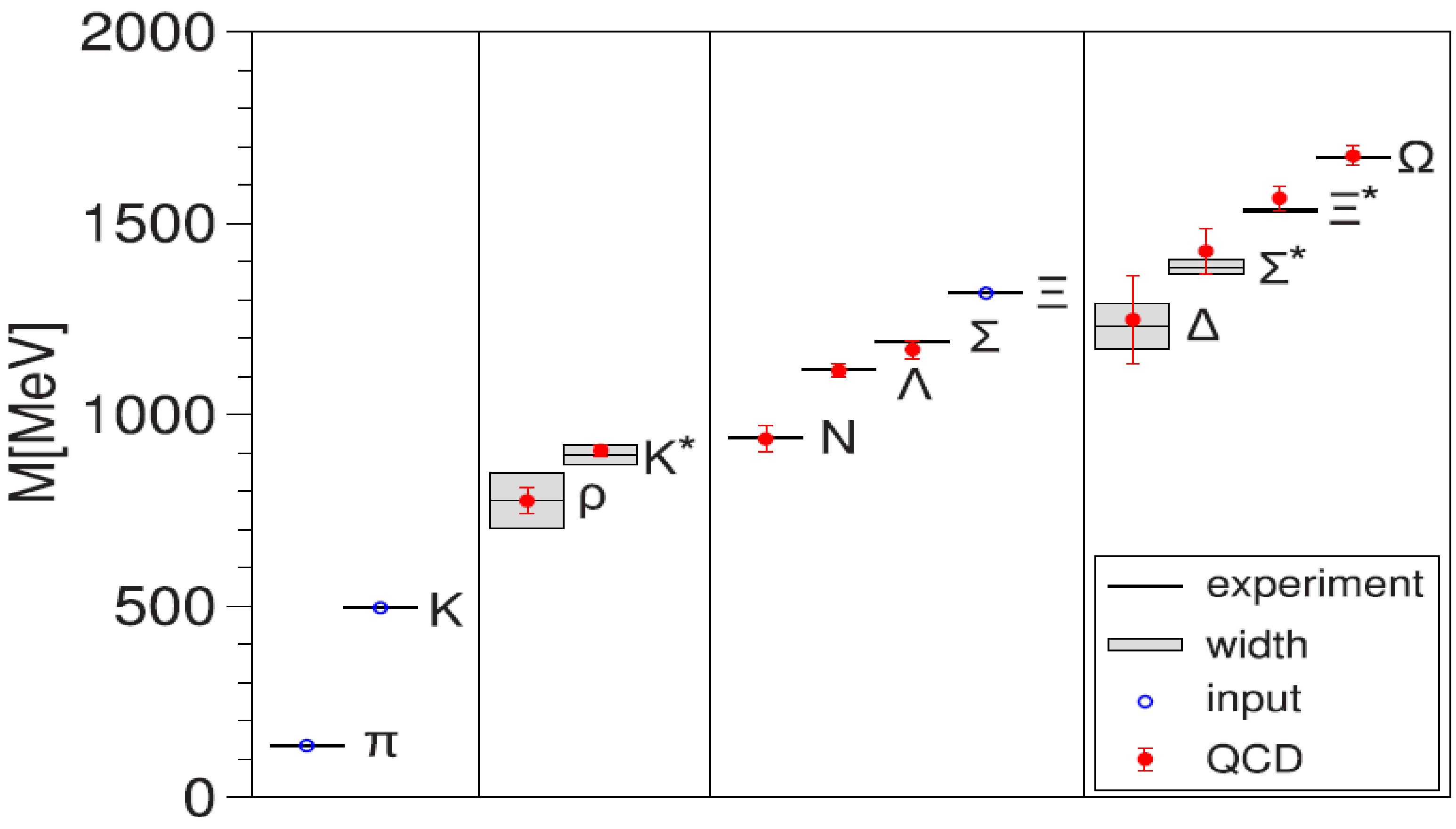
Nonzero temperature and density

Nuclear Physics

Standard model parameters and renormalization

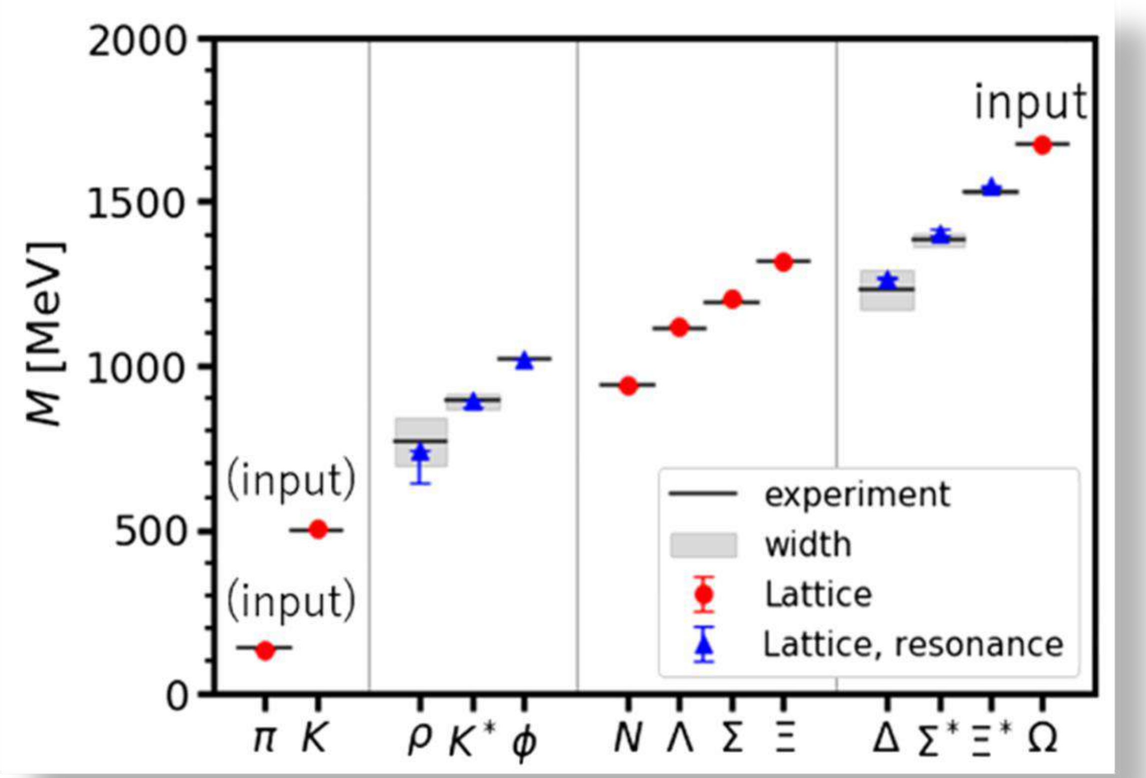
Physics Beyond the Standard Model, Supersymmetry

Hadronic mass generation

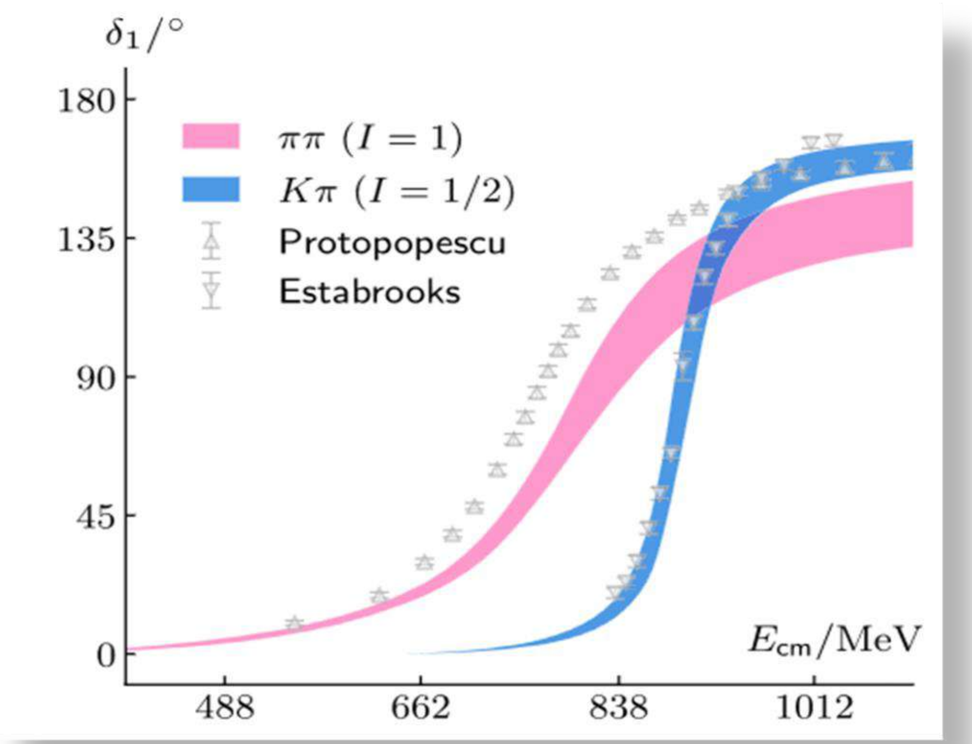


S. Durr et.al, Science 322, 1224 (2008)

Some up-to-date results (incomplete list)



HALQCD: Phys. Rev. D 110, 094502 (2024)



Boyle et al: Phys. Rev. Lett. 134, 111901 (2025)

Today's common practice

Construct the correlator matrix with your favorite interpolators

$$\begin{bmatrix} C(t)_{00} & C(t)_{01} & \cdots \\ \vdots & \ddots & \\ C(t)_{N0} & & C(t)_{NN} \end{bmatrix} v_a = \lambda_a(t,t_0) \begin{bmatrix} C(t_0)_{00} & C(t_0)_{01} & \cdots \\ \vdots & \ddots & \\ C(t_0)_{N0} & & C(t_0)_{NN} \end{bmatrix} v_a$$

Distillation:
[Peardon etal. PRD 09]
[Morningstar et al PRD 11]
CODE: Chroma, [github:paboyle/grid] [github:aportelli/hadrons]

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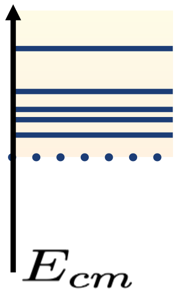
Distillation: [Peardon etal. PRD 09] [Morningstar et al PRD 11]
CODE: Chroma, [github:paboyle/grid] [github:aportelli/hadrons]

Fit the principal correlators and extract the finite volume spectra

$$\lambda_\alpha(t,t_0) \sim e^{-E_\alpha(t-t_0)}$$



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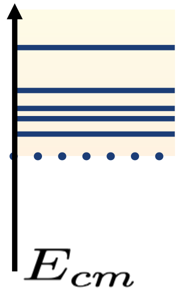


Solve for the quantization condition, Analytically continue the amplitudes, look for poles in the complex energy plane

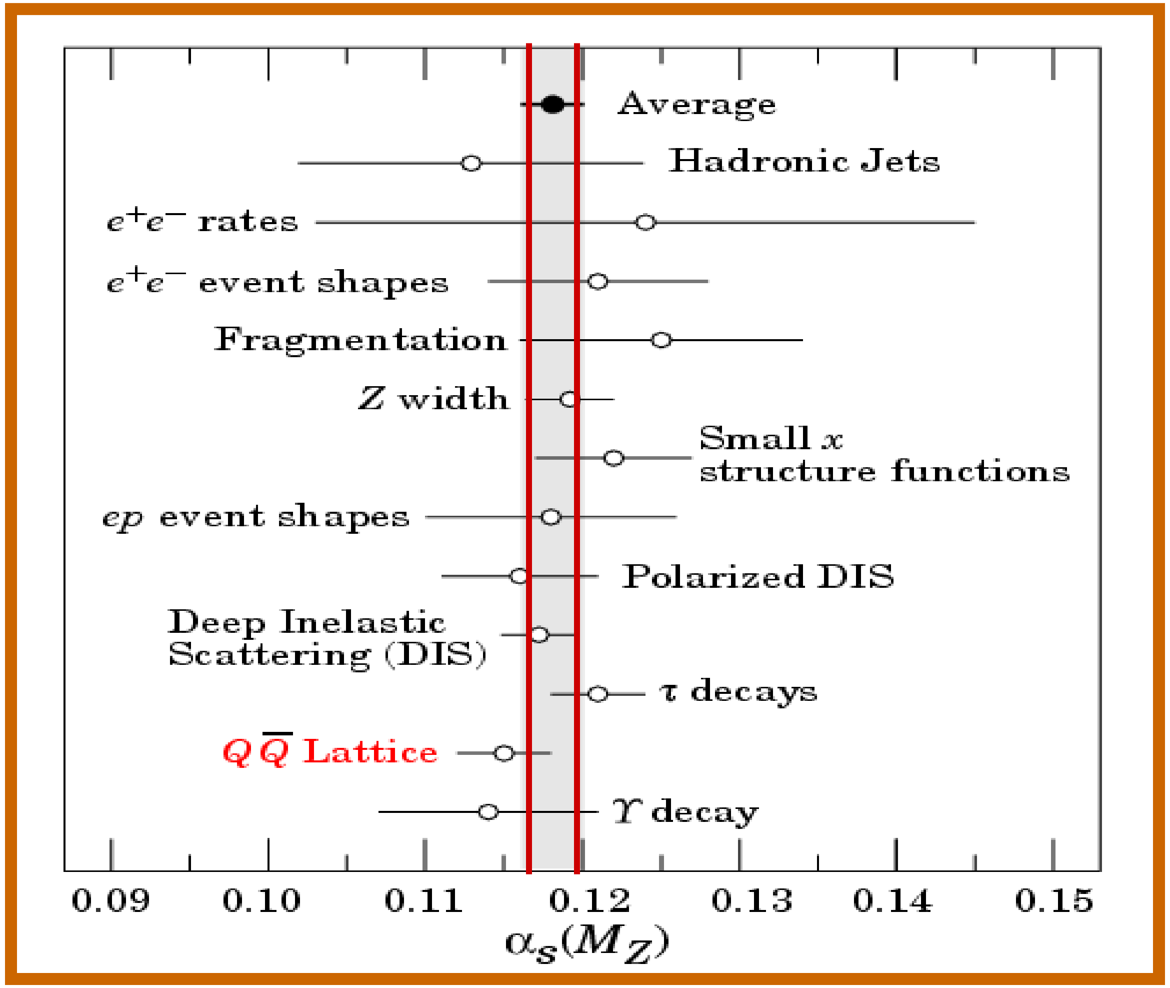
$$\det\left[F^{-1} + \mathcal{M}\right] = 0$$

Phase sifts, pole positions

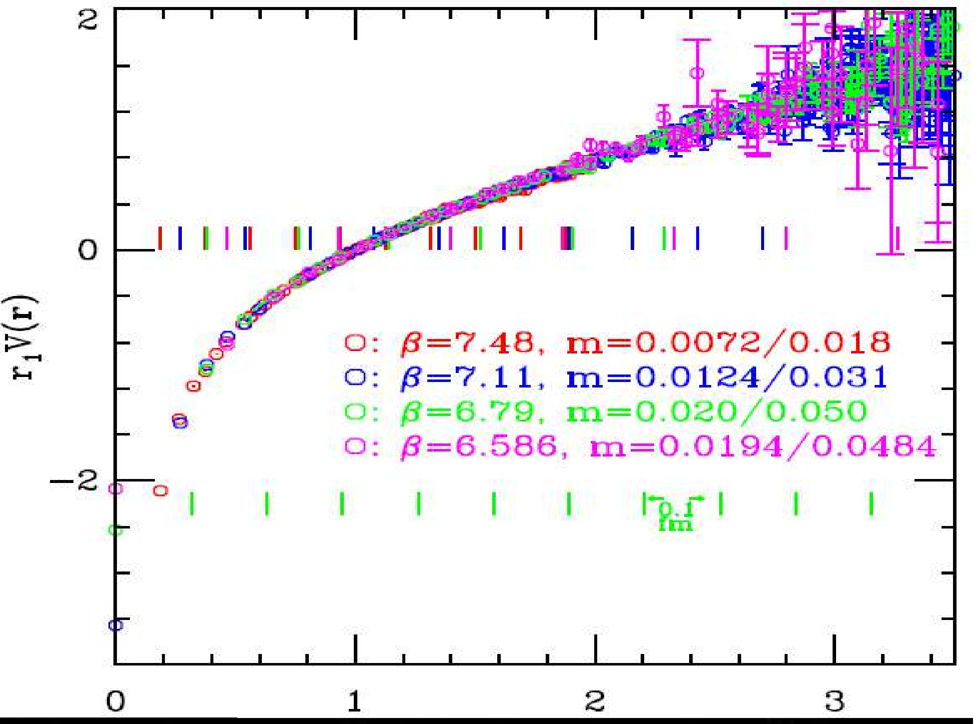
Example of codes: Morningstar et al, ``TwoHadronsInBox'', NPB924, 477 (2017); Chroma etc

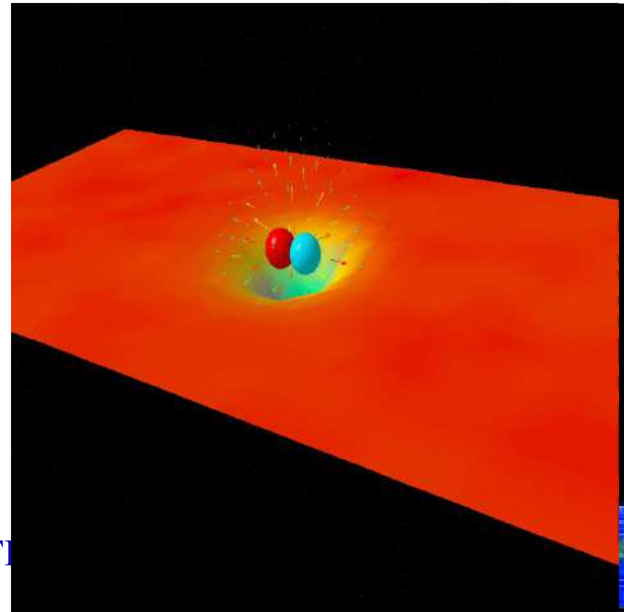
The Strong Coupling Constant

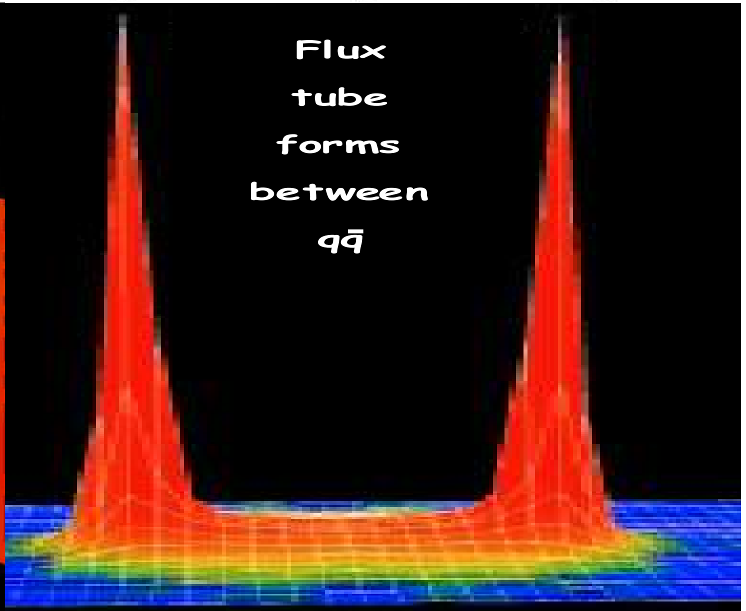
Lattice result : 0.115 ± 0.03, Combined average : 0.118 ± 0.002



Static Quark Potential



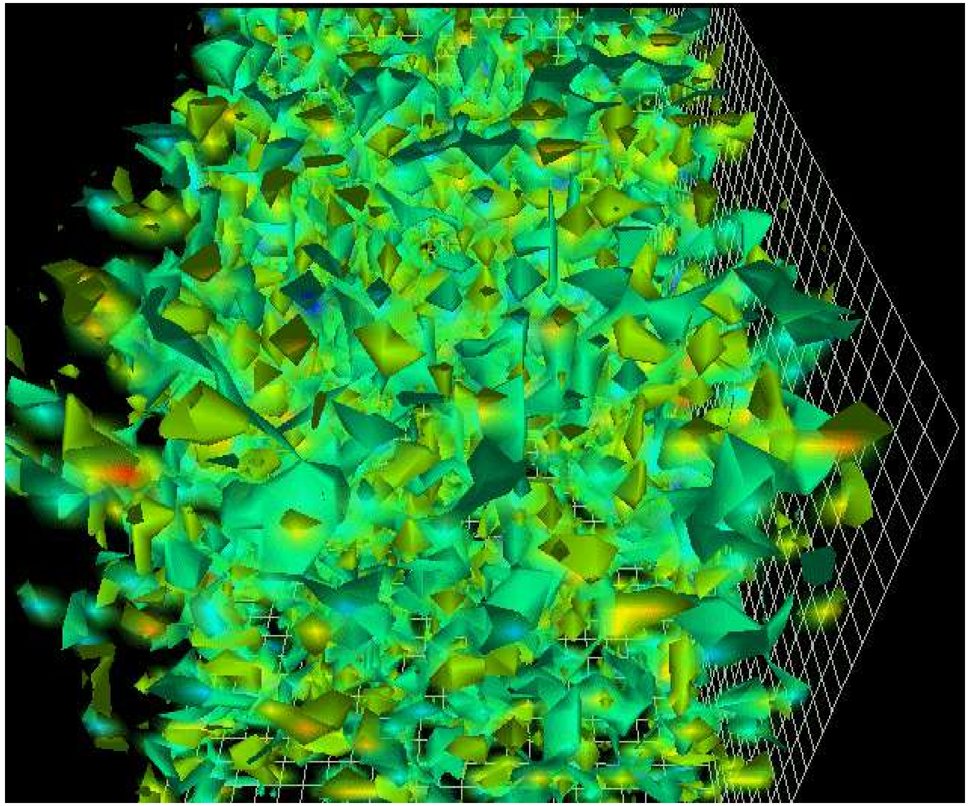




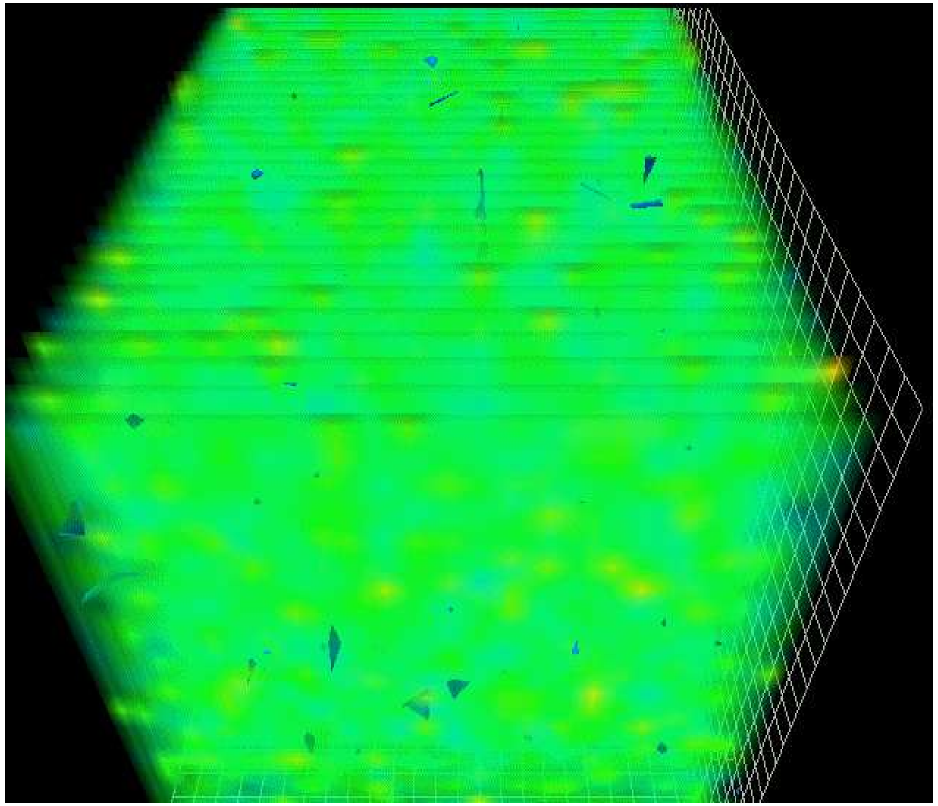
Action density

Bali et. al

Topological Structure of QCD vacuum



Topological charge density



Action density

@D. Lienweber

Topological charge can be calculated by measuring zero modes using overlap action

Decay constants from Lattice QCD

In SM :

$$\Gamma(H \to \ell\nu) = \frac{M_H}{8\pi} f_H^2 \left|G_F V_{Qq}^* m_\ell\right|^2 \left(1 - \frac{m_\ell^2}{M_H^2}\right)^2 ,$$

Pseudoscalar to vacuum matrix element of the axial current → **pseudoscalar decay constant**

$$\langle 0|\mathcal{A}^\mu|H(p)\rangle = i p^\mu f_H ,$$

$$\langle 0|\mathcal{A}^\mu|H(p)\rangle \,(M_H)^{-1/2} = i(p^\mu/M_H)\phi_H$$

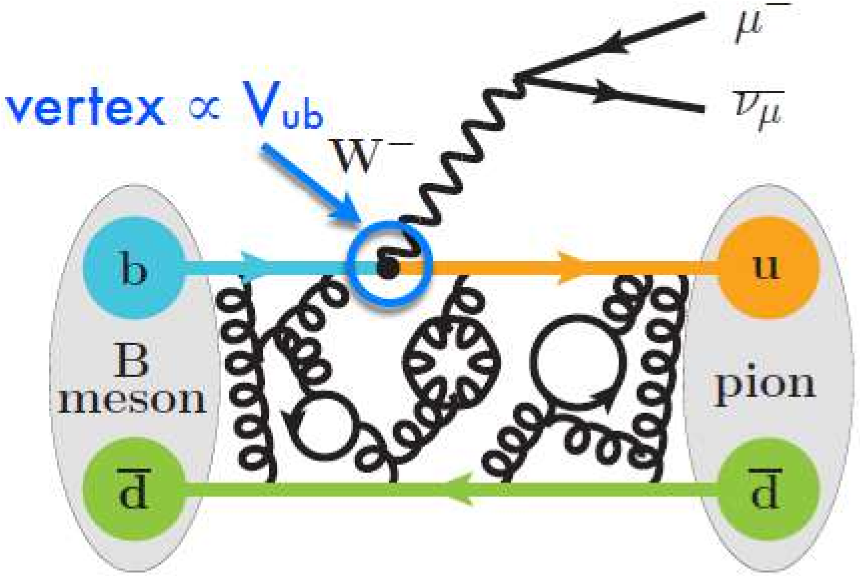
$$f_H = \phi_H/\sqrt{M_H}$$

H	$\mathcal{A}^\mu$	V
D	$\bar{d}\gamma^\mu\gamma^5 c$	V_{cd}^*
D_s	$\bar{s}\gamma^\mu\gamma^5 c$	V_{cs}^*
B	$\bar{b}\gamma^\mu\gamma^5 u$	V_{ub}
B_s	$\bar{b}\gamma^\mu\gamma^5 s$	—

Renormalization constant (to match with continuum physics) :

$$Z_{A^\mu} A^\mu \doteq \mathcal{A}^\mu + \mathrm{O}\left(\alpha_s a \Lambda f_i(m_Q a)\right) + \mathrm{O}\left(a^2 \Lambda^2 f_j(m_Q a)\right)$$

Weak matrix elements



$$\frac{d\Gamma(B \to \pi \ell \nu)}{dq^2}, \frac{d\Gamma(B \to D^{(*)} \ell \nu)}{dw}, \ldots$$

Compute nonperturbative QCD parameters (decay constants, form factors, B-parameters,...) numerically with LATTICE QCD

(Experiment) = (known) x (CKM factors) × (Hadronic Matrix Element)

@Van de Water

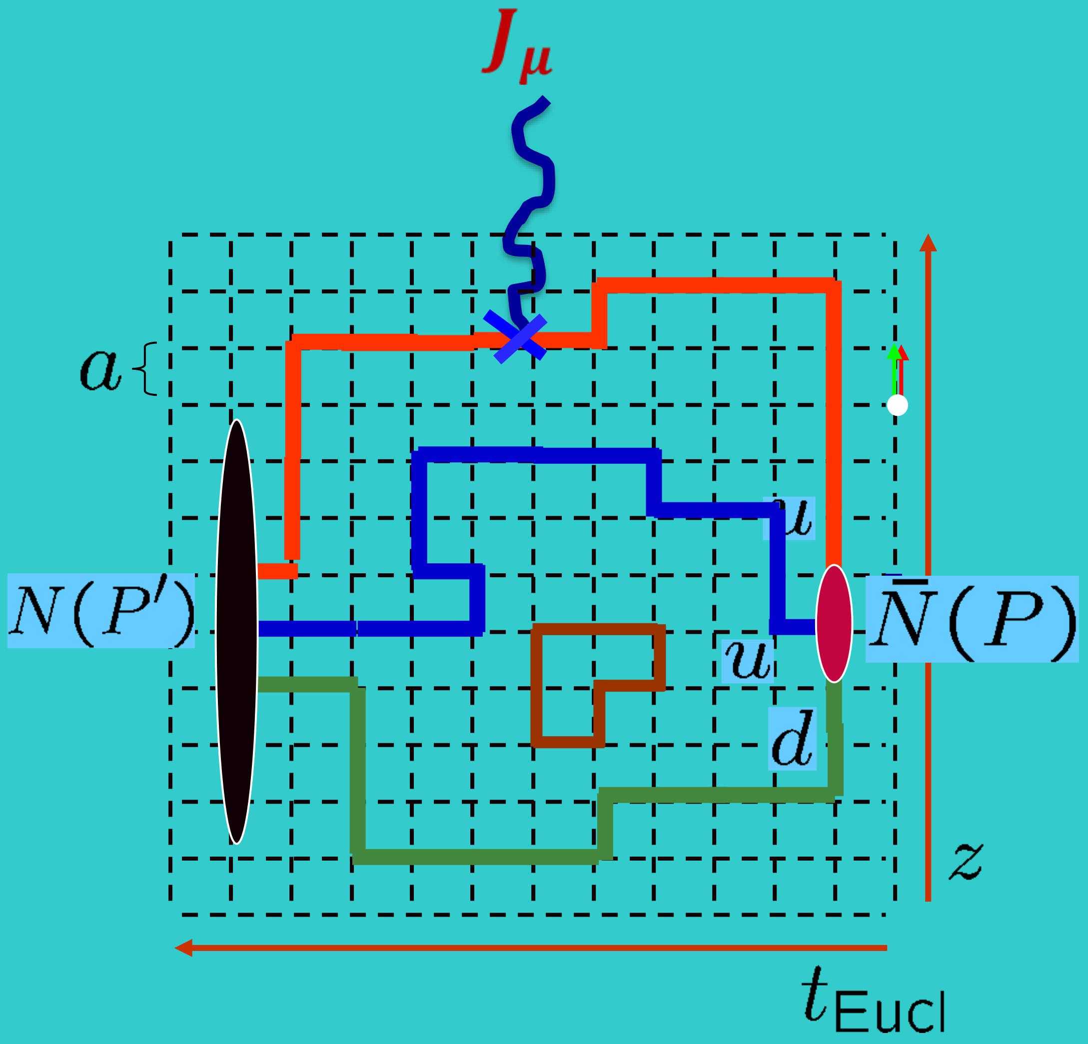
CKM matrix elements and lattice calculations

$$\begin{pmatrix} \mathbf{V_{ud}} & \mathbf{V_{us}} & \mathbf{V_{ub}} \\ \pi \to \ell\nu & K \to \ell\nu & B \to \ell\nu \\ & K \to \pi\ell\nu & B \to \pi\ell\nu \\ \mathbf{V_{cd}} & \mathbf{V_{cs}} & \mathbf{V_{cb}} \\ D \to \ell\nu & D_s \to \ell\nu & B \to D\ell\nu \\ D \to \pi\ell\nu & D \to K\ell\nu & B \to D^*\ell\nu \\ \mathbf{V_{td}} & \mathbf{V_{ts}} & \mathbf{V_{tb}} \\ \langle B_d|\bar{B}_d\rangle & \langle B_s|\bar{B}_s\rangle & \\ B \to \pi\ell\ell & B \to K\ell\ell & \end{pmatrix}$$

"Gold plated" processes on the lattice → **CKM matrix elements**

- **One hadron in the initial state and zero or one hadron in the final state**
- **Stable hadrons (that is narrow or far from threshold → easier to study on lattice)**
- **Chiral extrapolation is controllable**

Three Point Functions



Various matrix elements

- Electromagnetic, scalar, axial and tensor form factors are being investigated rigorously by various lattice groups.

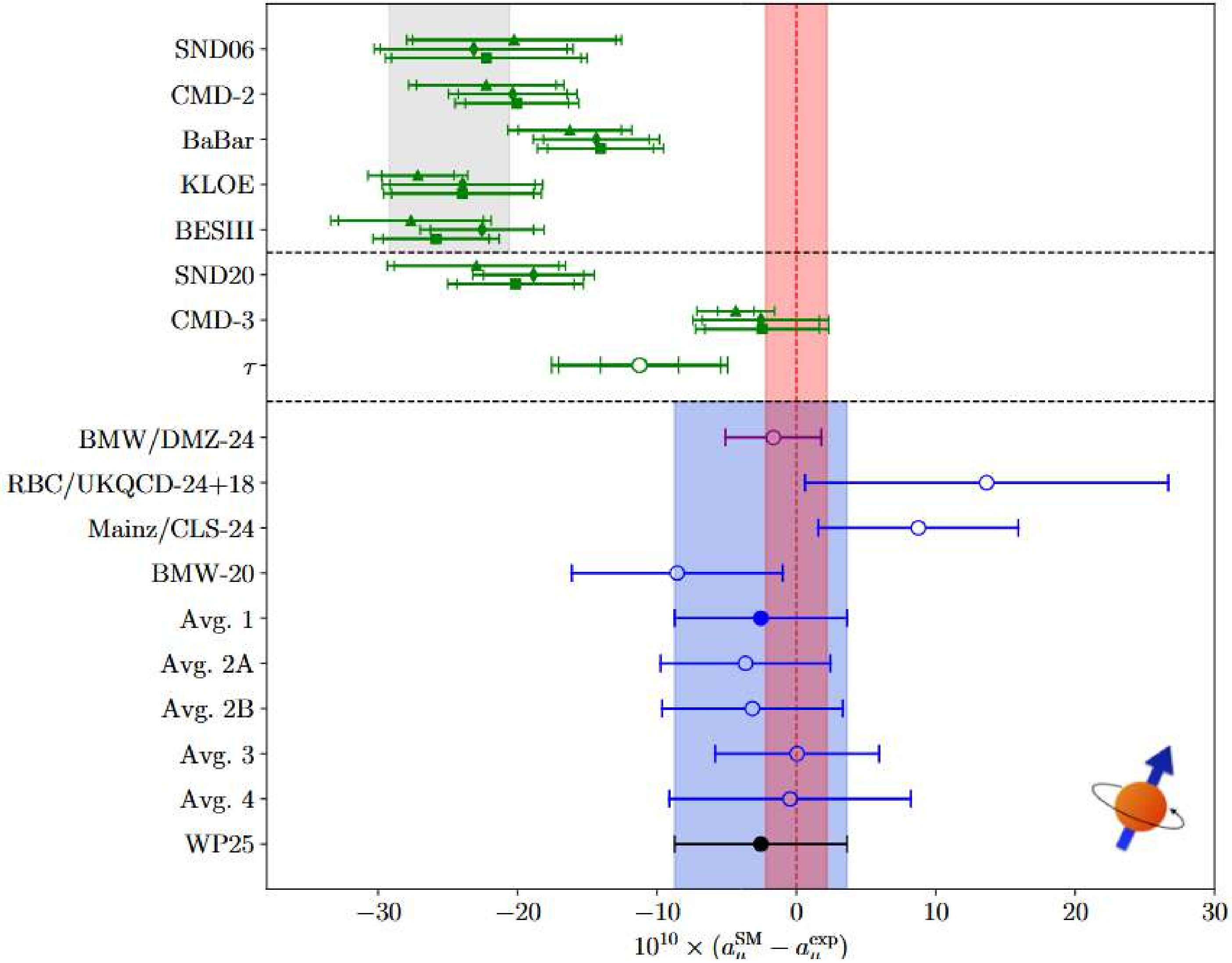
- Quark-gluon spin and orbital angular momentum are also being studied on lattice. Matrix elements of energy momentum tensor ((Liu), PRL 118, 102001(2017))

- GPD, TMD and also PDFs are being investigated using lattice QCD

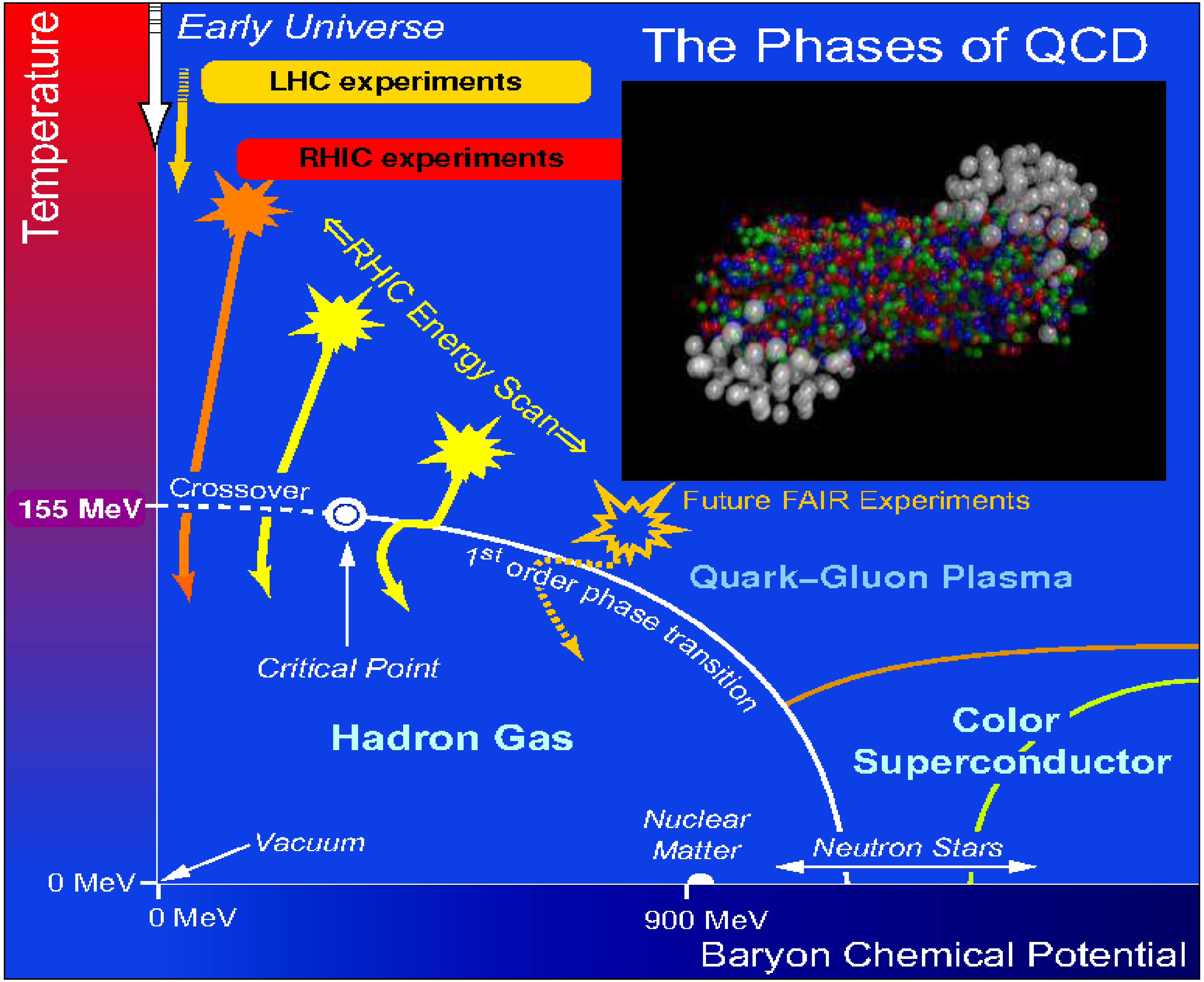
Some other lattice calculations :

- Muon g-2,
- K_0 oscillation,
- ϵ/ϵ'

Muon *g*-2



SND06
CMD-2
BaBar
KLOE
BESIII
SND20
CMD-3
τ
BMW/DMZ-24
RBC/UKQCD-24+18
Mainz/CLS-24
BMW-20
Avg. 1
Avg. 2A
Avg. 2B
Avg. 3
Avg. 4
WP25
−30
−20
−10
0
10
20
30
$10^{10} \times (a_\mu^{SM} - a_\mu^{exp})$



Temperature
Early Universe
The Phases of QCD
LHC experiments
RHIC experiments
⇐RHIC Energy Scan⇒
Future FAIR Experiments
155 MeV
Crossover
1st order phase transition
Quark–Gluon Plasma
Critical Point
Hadron Gas
Color Superconductor
Nuclear Matter
Vacuum
Neutron Stars
0 MeV
0 MeV
900 MeV
Baryon Chemical Potential

Lattice QCD and Cosmology-Astrophysics

Matter at extreme conditions

- Extreme temperature : Study of early Universe : QGP, equation of state, Phase diagram, Phase transition, Critical Point
- Extreme density : Neutron star
- Extreme magnetic fields : Magnetars, early universe?
- Search for dark matter

QCD properties

- Quark confinement :
  - No quarks and gluons in the physical spectrum
  - The physical degrees of freedom are hadrons
  - Order parameter : Polyakov Loop

- Chiral symmetry breaking :
  - Hadrons are massive
  - Order parameter : quark condensate

The relevant order parameter : Polyakov loop

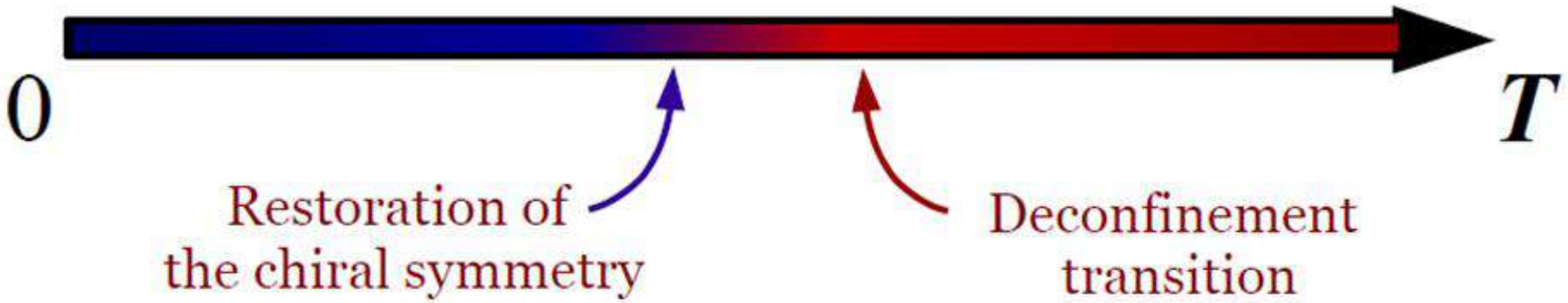
$$L(x) = \frac{1}{3} Tr\, \mathcal{P}\, exp \left[i \int_0^{1/T} d\tau\, A_4(x, \tau) \right]$$

Related to the free energy of a single quark

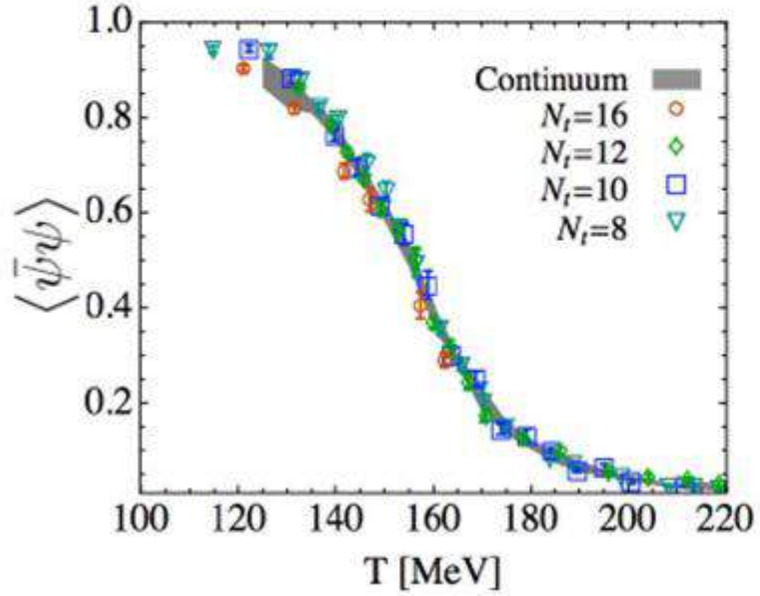
$$\langle L \rangle = exp\left(-F_q/T\right)$$

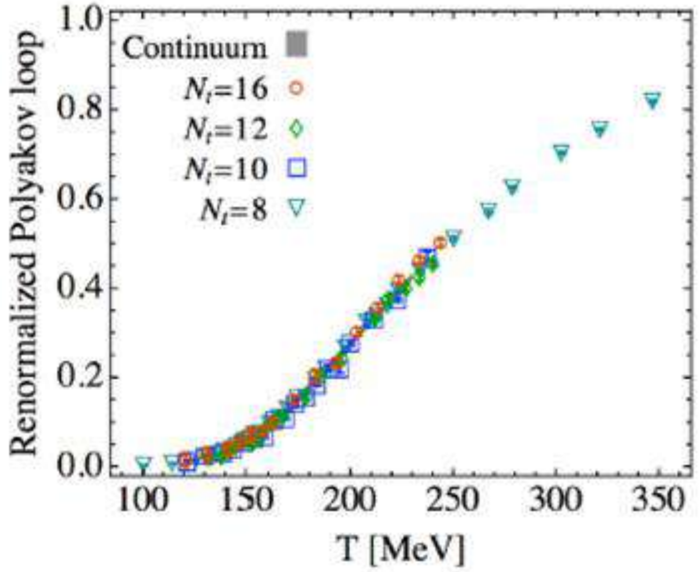
Confinement-deconfinement :

$$\langle L \rangle = 0, \qquad \text{low}\, T,$$
$$\langle L \rangle \neq 0, \qquad \text{high}\, T$$



$T_{\text{chiral}} \approx 155$ MeV $\qquad T_{\text{deconf}} \approx 170$ MeV





@BMW, HOTQCD

$T_c(\bar{\psi}\psi) = 155(3)(3)$ MeV$= 1.95(5) \cdot 10^{12}$ K.

(Cross-over: other quantities may have different pseudocritical temperatures.)

Finite density – sign problem

Partition function with chemical potential :

$$Z = Tr\, e^{-\beta(H-\mu N)} = \int DU D\bar{\psi} D\psi e^{-(S_G+\bar{\psi}\Delta\psi)}$$
$$= \int DU \det\Delta\, e^{-S_G}$$

$$\Delta = D_\nu\gamma_\nu + m + \mu\gamma_0$$

Now,

$$U_\mu(x) = e^{iA_\mu(x)}$$
$$U_t(x) \to e^{\mu}U_t(x)$$
$$U_t^\dagger(x) \to e^{-\mu}U_t^\dagger(x)$$

At $\mu = 0$

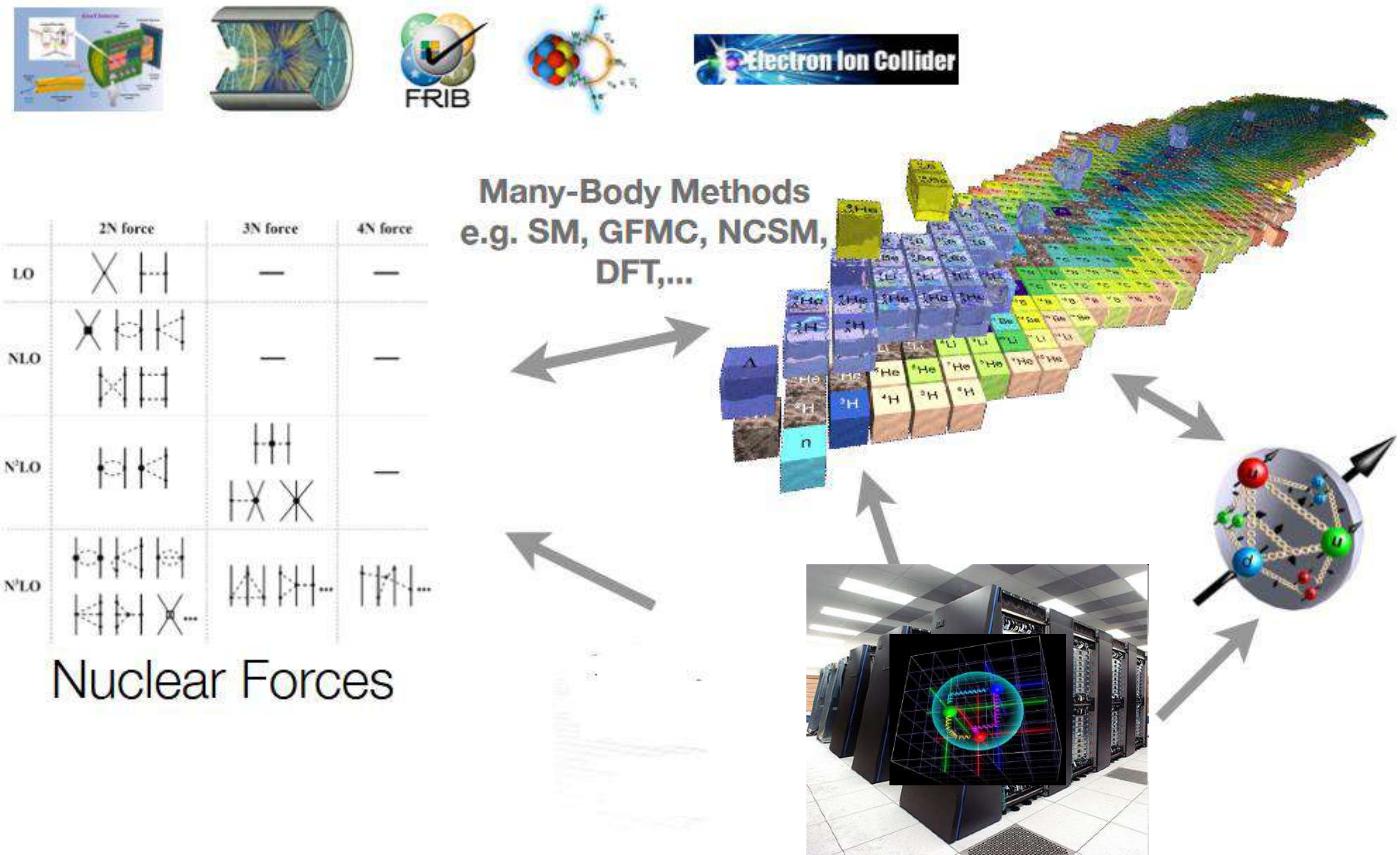
$$(\det\Delta)^* = \det\Delta^\dagger = \det\gamma_5\Delta\gamma_5 = \det\Delta$$ $\Longrightarrow$ $\det\Delta$ **: real**

However, at $\mu \neq 0$

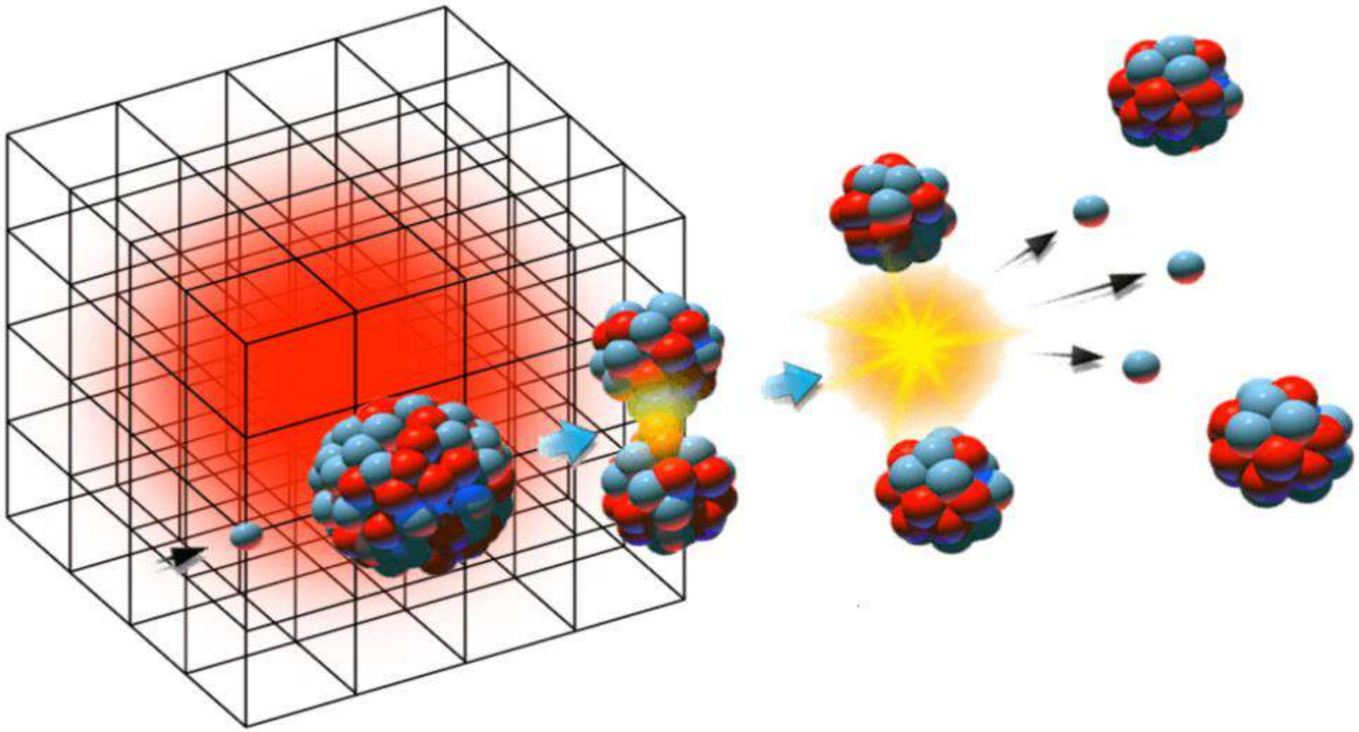
$$\Delta^\dagger = -D_\nu\gamma_\nu + m + \mu\gamma_0 \neq \gamma_5\Delta\gamma_5$$ $\Longrightarrow$ $\det\Delta$ **: complex**

Cannot do Monte-Carlo : Sign problem

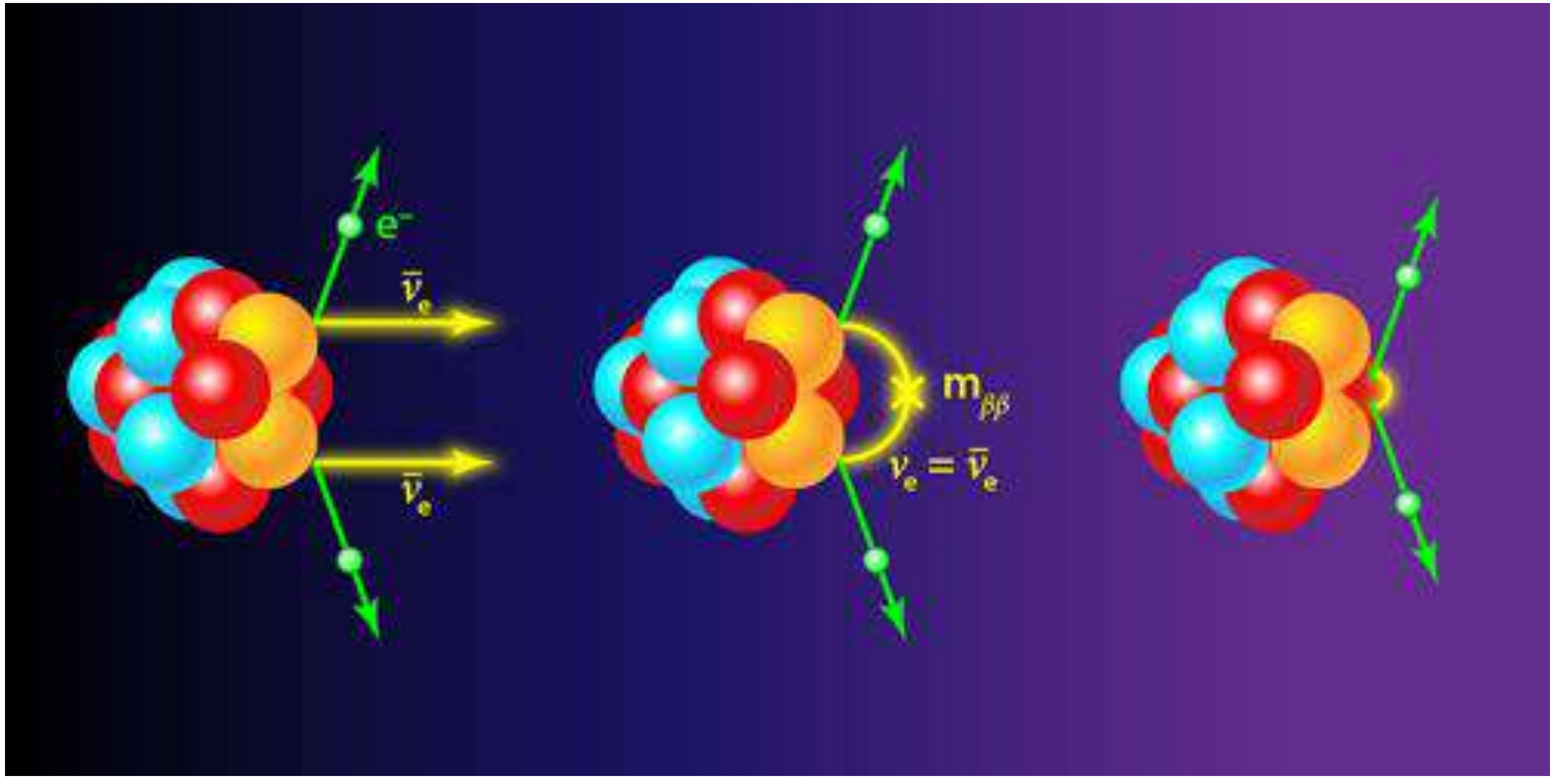
Emergence of Nuclei from QCD



Can we study light nuclei and their properties from first-principles?



Weak nuclear processes: BSM Physics and Various nuclear matrix elements



Challenges in Lattice QCD

- How to go to smaller lattice spacings (freezing topology) with reasonable computing time? AI-ML
- How to compute nuclear tomography with precision?
- How to solve sign problem? (physics of neutron star, QCD phase diagram, condensed matter systems with fermions). Is it an unsolvable problem with classical computers? Quantum computing?
- How to calculate precision spectra of multi-hadron states in nuclear physics, Deuteron to Carbon, with reasonable computing time? Quantum computing?
- How to deal nuclear physics reactions with precision?
- How to deal with chiral gauge theories?

Computing resources for LQCD

Books

- **M. Creutz:** ***Quarks, Gluons and Lattices;*** **(Cambridge Univ. Press: Cambridge, 1983)**
- **C. Rebbi:** ***Lattice Gauge Theories and Monte Carlo Simulations;*** **(World Scientific:Singapore, 1983)**
- **M. Creutz:** ***Quantum Fields on the Computer;*** **(World Scientific: Singapore, 1992)**
- **H. J. Rothe:** ***Lattice Gauge Theories - An Introduction;*** **(World Scientific: Singapore, 1992)**
- **I. Montvay and G. Münster:** ***Quantum Fields on a Lattice;*** **(Cambridge University Press: Cambridge, 1994)**
- **J. Smit :** ***Introduction to Quantum Fields on a Lattice;*** **(Cambridge University Press: Cambridge, 2002)**
- **T. DeGrand and C. DeTar :** ***Lattice Methods for Quantum Chromodynamics;*** **(World Scientific: Singapore, 2007)**
- **C. Gattringer and C. B. Lang : Quantum Chromodynamics on the Lattice (An introductory presentation); (Springer: Berlin, Heidelberg, New York, 2009)**