An Introduction to Lattice QCD:
Applications to EIC related physics

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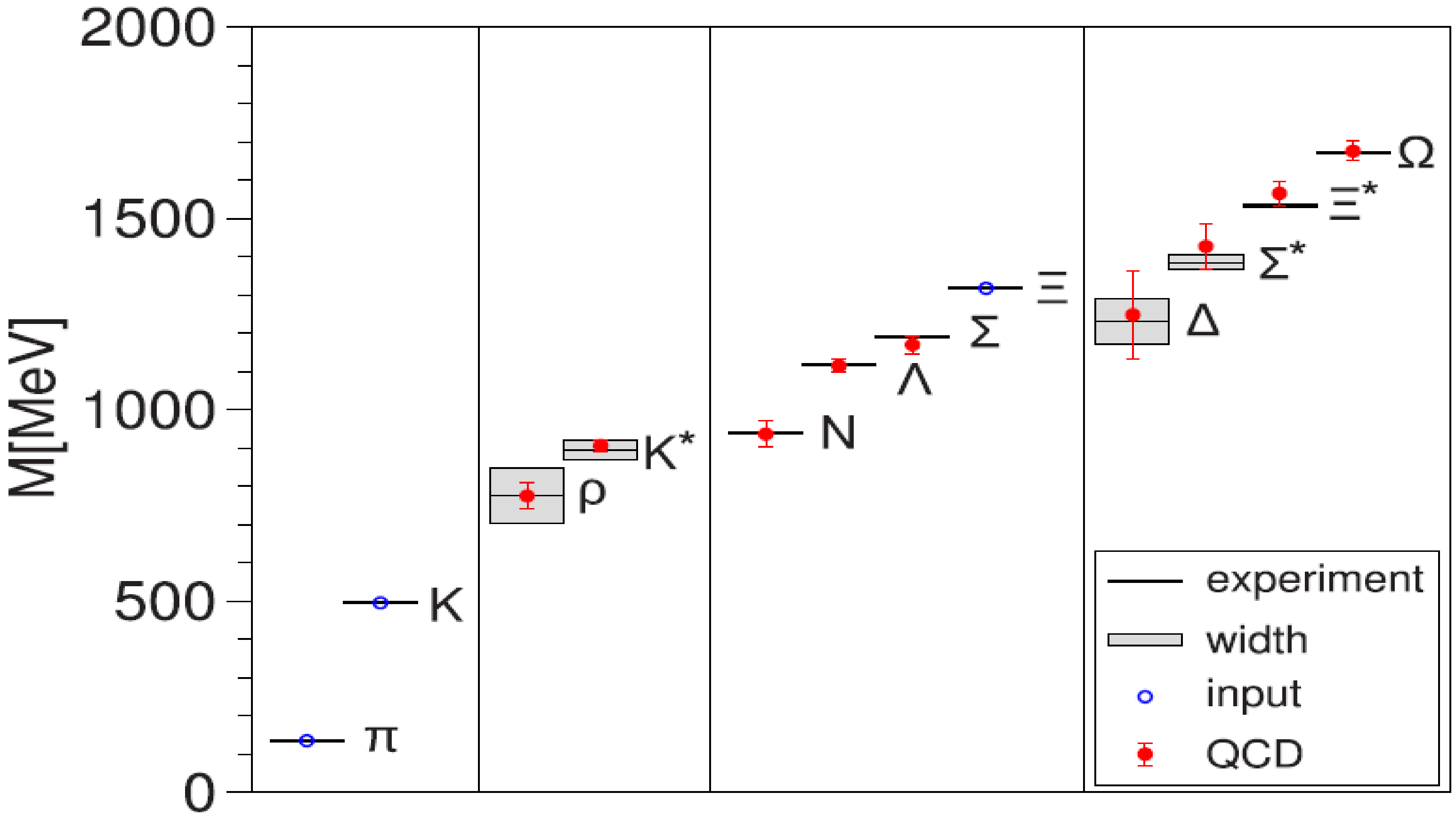
tata institute of fundamental research

The **2026 IITB-CFNS-CTEQ School on Perturbative QCD for EIC**

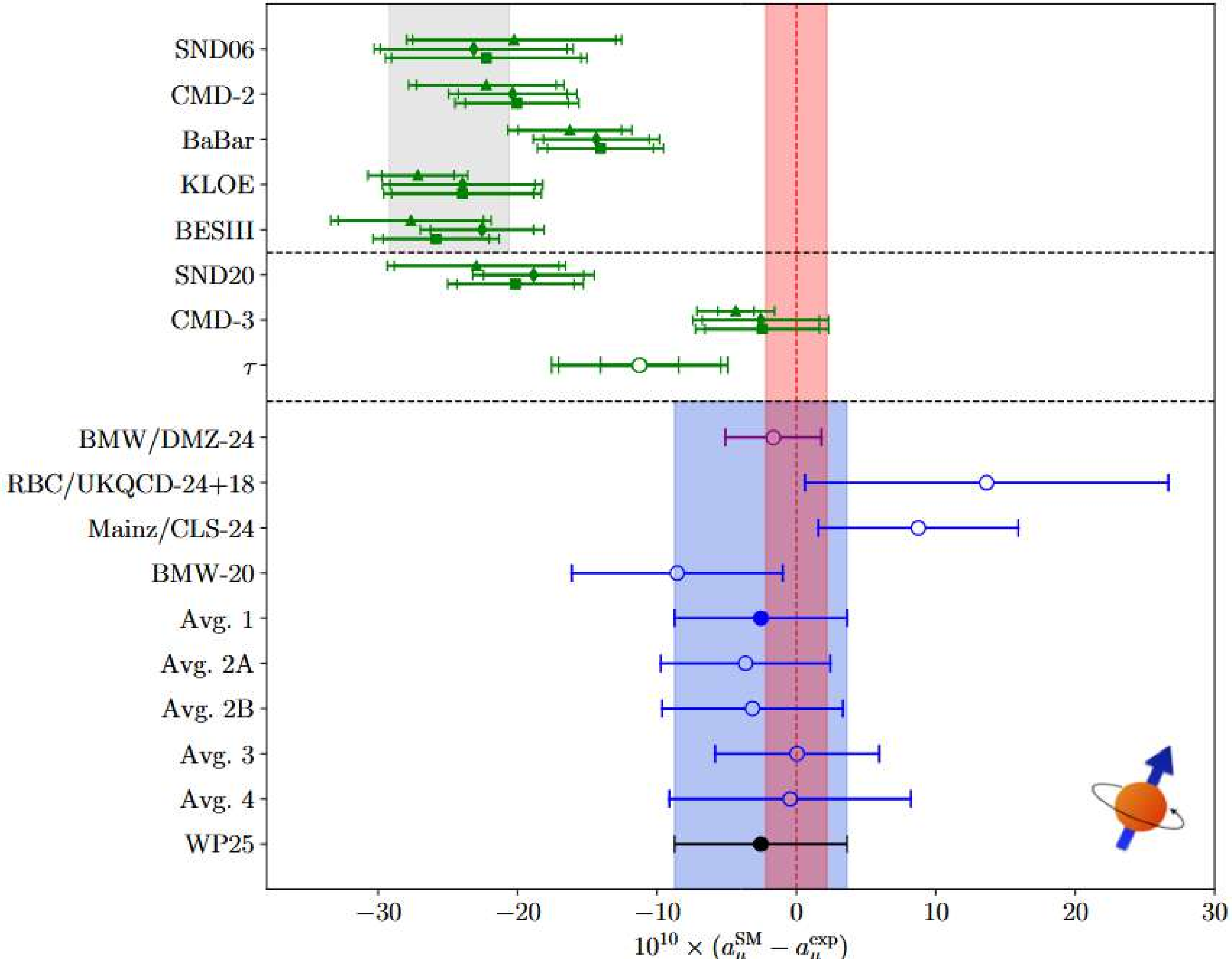
Lecture 2

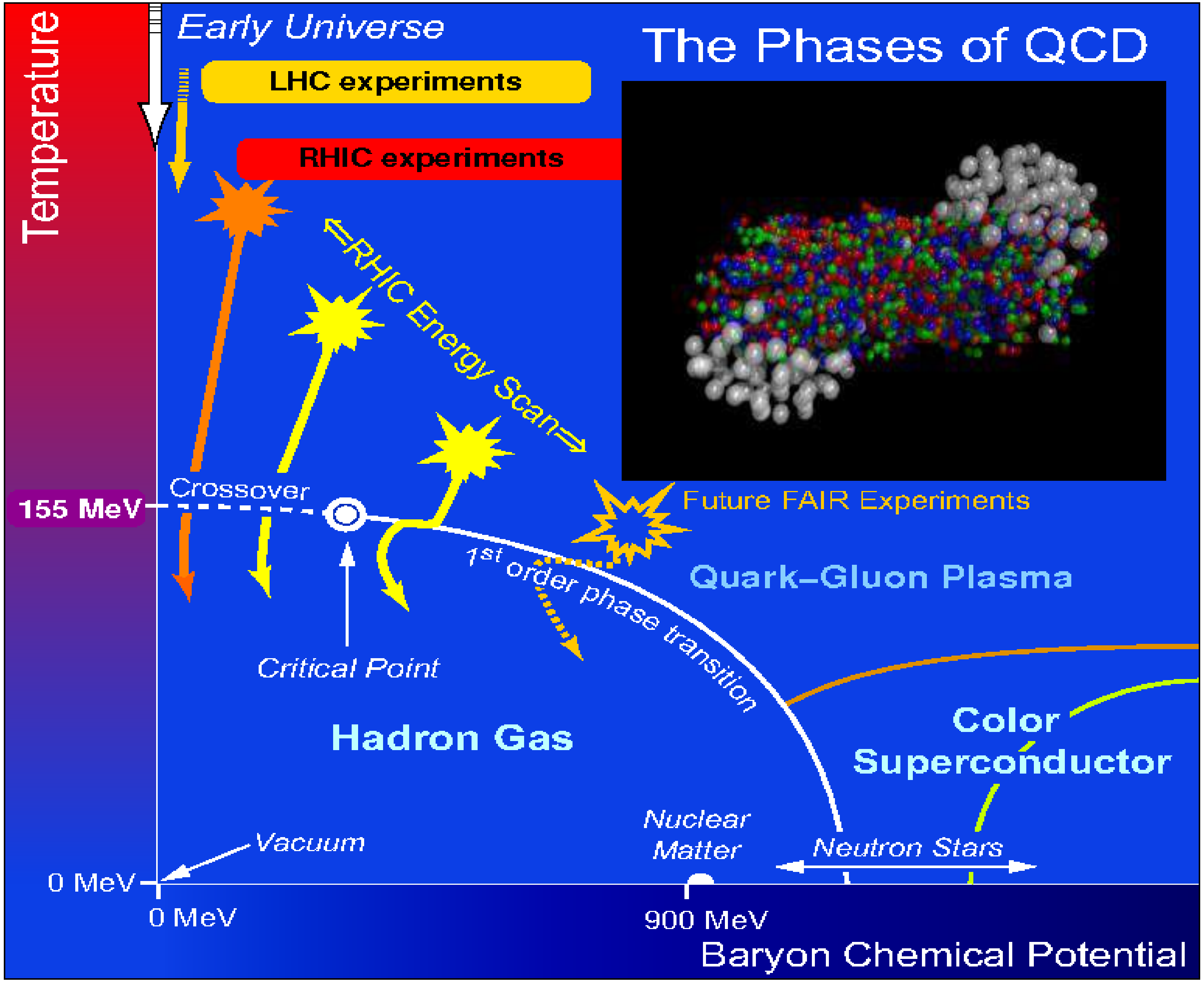
Hadronic mass generation



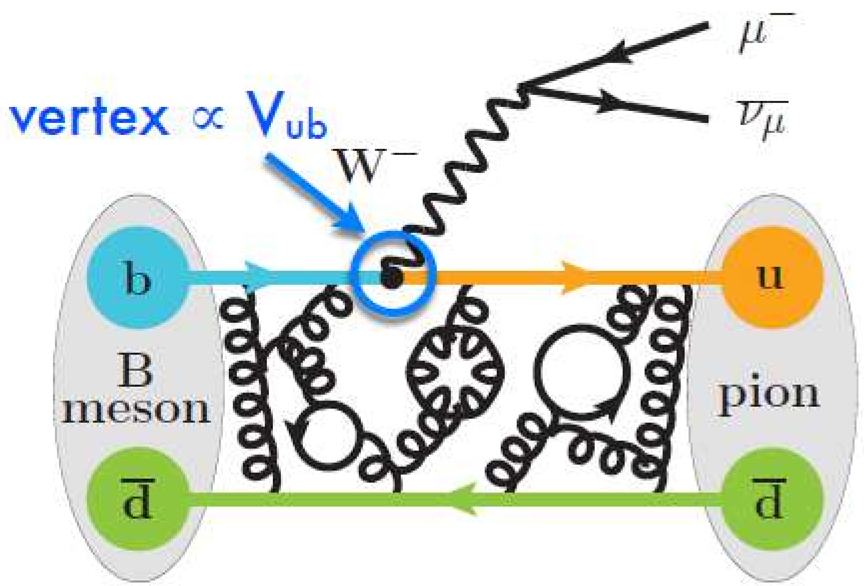
S. Durr et.al, Science 322, 1224 (2008)

Muon *g*-2


SND06
CMD-2
BaBar
KLOE
BESIII
SND20
CMD-3
τ
BMW/DMZ-24
RBC/UKQCD-24+18
Mainz/CLS-24
BMW-20
Avg. 1
Avg. 2A
Avg. 2B
Avg. 3
Avg. 4
WP25
−30
−20
−10
0
10
20
30
$10^{10} \times (a_\mu^{\rm SM} - a_\mu^{\rm exp})$


The Phases of QCD
Temperature
Early Universe
LHC experiments
RHIC experiments
⇐RHIC Energy Scan⇒
Future FAIR Experiments
155 MeV
Crossover
Critical Point
1st order phase transition
Quark–Gluon Plasma
Hadron Gas
Color Superconductor
Nuclear Matter
Neutron Stars
Vacuum
0 MeV
0 MeV
900 MeV
Baryon Chemical Potential

Weak matrix elements



$$\frac{d\Gamma(B \to \pi \ell \nu)}{dq^2}, \frac{d\Gamma(B \to D^{(*)} \ell \nu)}{dw}, \dots$$

Compute nonperturbative QCD parameters (decay constants, form factors, B-parameters,...) numerically with **LATTICE QCD**

(Experiment) = (known) x (CKM factors) × (Hadronic Matrix Element)

@Van de Water

Decay constants from Lattice QCD

In SM :

$$\Gamma(H \to \ell\nu) = \frac{M_H}{8\pi} f_H^2 \left|G_F V_{Qq}^* m_\ell\right|^2 \left(1 - \frac{m_\ell^2}{M_H^2}\right)^2 ,$$

Pseudoscalar to vacuum matrix element of the axial current ⟹ **pseudoscalar decay constant**

$$\langle 0|\mathcal{A}^\mu|H(p)\rangle = ip^\mu f_H ,$$

$$\langle 0|\mathcal{A}^\mu|H(p)\rangle \, (M_H)^{-1/2} = i(p^\mu/M_H)\phi_H$$

$$f_H = \phi_H/\sqrt{M_H}$$

H	$\mathcal{A}^\mu$	V
D	$\bar{d}\gamma^\mu\gamma^5 c$	V_{cd}^*
D_s	$\bar{s}\gamma^\mu\gamma^5 c$	V_{cs}^*
B	$\bar{b}\gamma^\mu\gamma^5 u$	V_{ub}
B_s	$\bar{b}\gamma^\mu\gamma^5 s$	—

Renormalization constant (to match with continuum physics) :

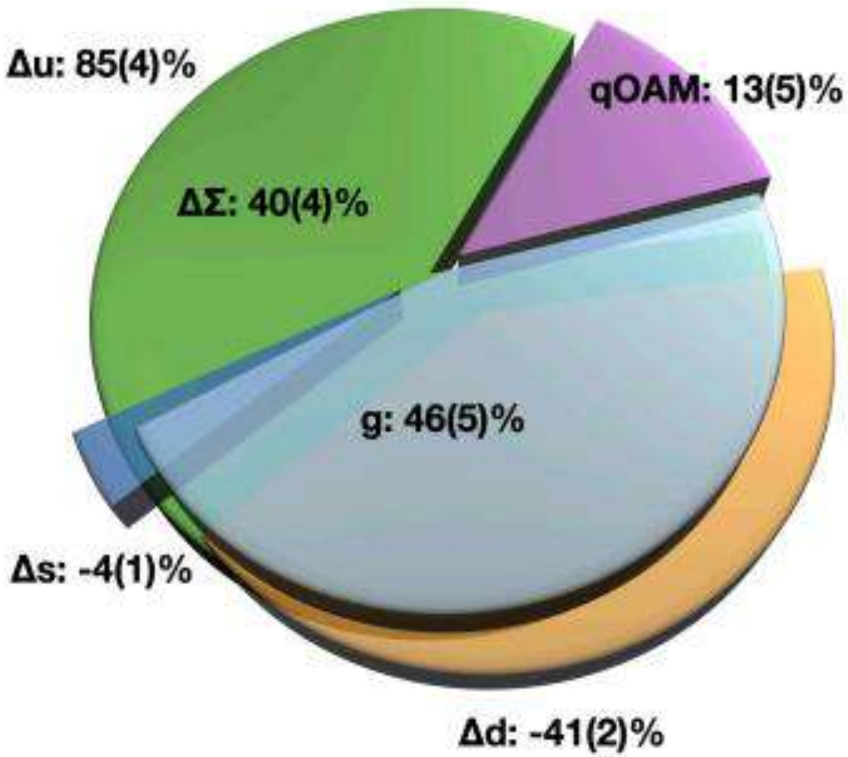
$$Z_{A^\mu} A^\mu \doteq \mathcal{A}^\mu + \mathrm{O}\left(\alpha_s a \Lambda f_i(m_Q a)\right) + \mathrm{O}\left(a^2 \Lambda^2 f_j(m_Q a)\right)$$

CKM matrix elements and lattice calculations

$$\begin{pmatrix} \mathbf{V}_{\mathbf{ud}} & \mathbf{V}_{\mathbf{us}} & \mathbf{V}_{\mathbf{ub}} \\ \pi \to \ell\nu & K \to \ell\nu & B \to \ell\nu \\ & K \to \pi\ell\nu & B \to \pi\ell\nu \\ \mathbf{V}_{\mathbf{cd}} & \mathbf{V}_{\mathbf{cs}} & \mathbf{V}_{\mathbf{cb}} \\ D \to \ell\nu & D_s \to \ell\nu & B \to D\ell\nu \\ D \to \pi\ell\nu & D \to K\ell\nu & B \to D^*\ell\nu \\ \mathbf{V}_{\mathbf{td}} & \mathbf{V}_{\mathbf{ts}} & \mathbf{V}_{\mathbf{tb}} \\ \langle B_d|\bar{B}_d\rangle & \langle B_s|\bar{B}_s\rangle & \\ B \to \pi\ell\ell & B \to K\ell\ell & \end{pmatrix}$$

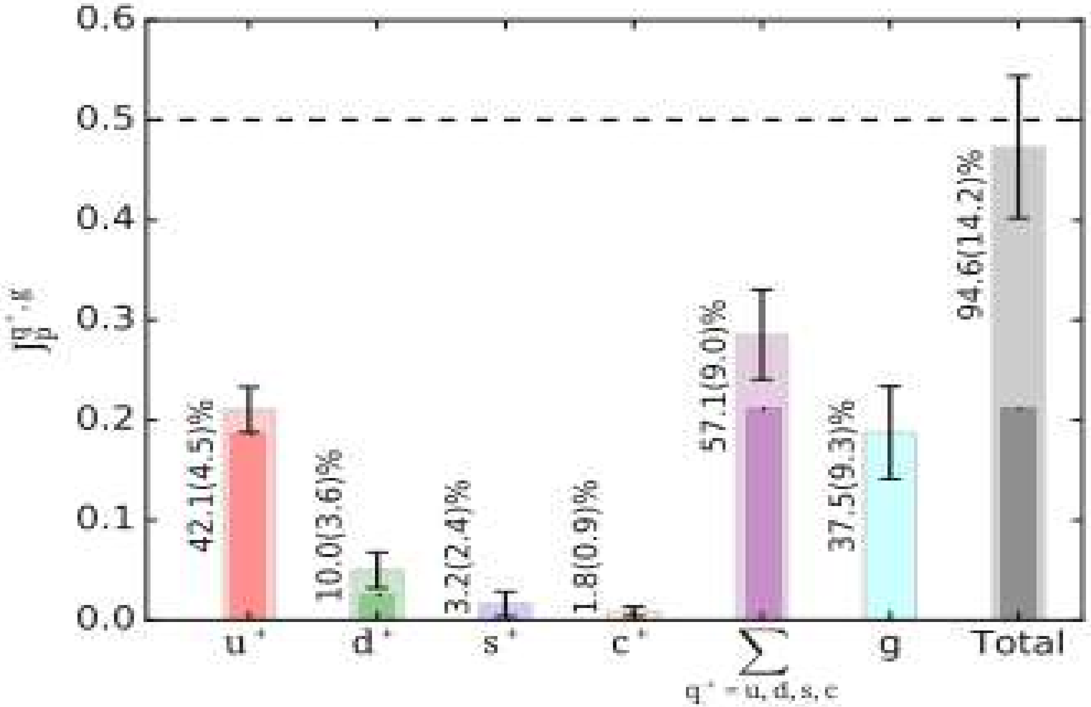
"Gold plated" processes on the lattice → **CKM matrix elements**

- **One hadron in the initial state and zero or one hadron in the final state**
- **Stable hadrons (that is narrow or far from threshold → easier to study on lattice)**
- **Chiral extrapolation is controllable**



χQCD (2022)

Phys. Rev. D 106, 014512 (2022)



ETMC (2020)

Phys. Rev. D 101, 094513 (2020)

Lattice QCD Workflow

Using Monte Carlo methods
generate a set of gauge ensembles
representing QCD dynamics,
on Euclidean space-time lattices
with varying quark masses,
varying lattice sizes and
varying lattice spacings

Compute Euclidean correlation functions
corresponding to desired observables
on those lattice ensembles

Analyse correlation functions, and
finally take chiral, continuum and
finite volume limits (also
renormalization of lattice to
physical results)

Lattice QCD Workflow

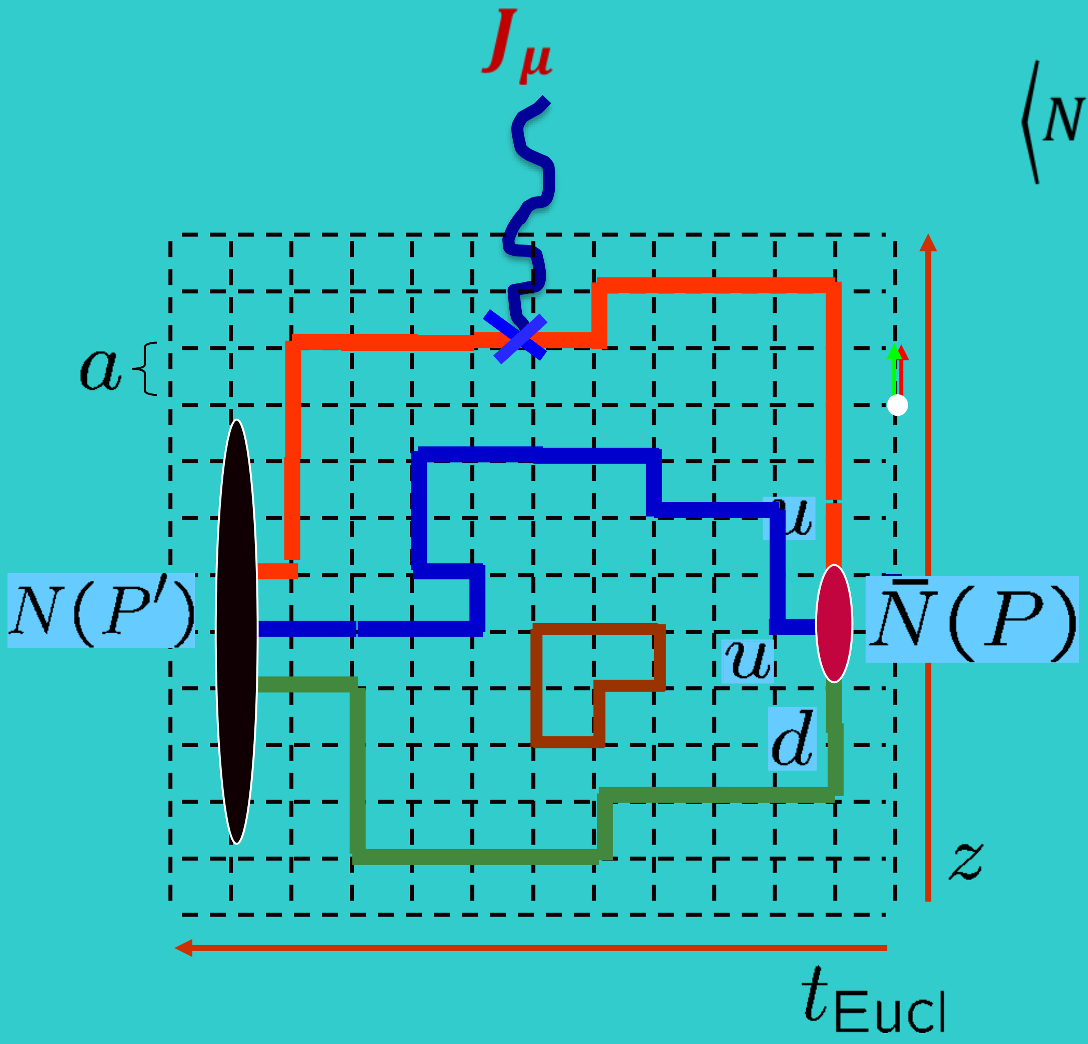
Using Monte Carlo methods generate a set of gauge ensembles representing QCD dynamics, on Euclidean space-time lattices with varying quark masses, varying lattice sizes and varying lattice spacings

Angular momentum of proton:
Need appropriate operators and then to compute their correlation functions

Compute Euclidean correlation functions corresponding to desired observables on those lattice ensembles

Analyse correlation functions, and finally take chiral, continuum and finite volume limits (also renormalization of lattice to physical results)

Three Point Functions



$$\left\langle N(p',s') \middle| \mathcal{T}_{q,g}^{\{\mu\nu\}} \middle| N(p,s) \right\rangle$$

Operators for Angular momentum and spin sum rules

- Energy momentum tensor (Belinfante):

$$T^{\mu\nu} = \overline{T}^{\mu\nu} + \widehat{T}^{\mu\nu}$$

$$\overline{T}^{\mu\nu} = \overline{T}_q^{\mu\nu} + \overline{T}_g^{\mu\nu}$$

$$\overline{T}_q^{\mu\nu} = \overline{q} i\gamma \overleftrightarrow{D}^{\{\mu\nu\}} q \qquad \overline{T}_g^{\mu\nu} = F^{\{\mu\rho} F_{\rho}^{\nu\}}$$

$$\overleftrightarrow{D} = \frac{1}{2}\left[\overrightarrow{D} + \overleftarrow{D}\right] \qquad \{\,\} \Rightarrow \text{symetrization}$$

- Angular momentum density:

$$M^{\alpha\mu\nu} = \overline{T}^{\alpha\nu} x^{\mu} - \overline{T}^{\alpha\mu} x^{\nu}$$

- Angular momentum:

$$J_i = \frac{1}{2} \epsilon^{ijk} \int d^3x \, M^{0jk}(x)$$

$$\vec{J}^g = \int d^3x \left[\vec{x} \times \left(\vec{E} \times \vec{B}\right)\right]$$

- Angular momentum density:

$$M^{\alpha\mu\nu} = \overline{T}^{\alpha\nu} x^{\mu} - \overline{T}^{\alpha\mu} x^{\nu}$$

- Angular momentum of quarks:

$$J_i = \frac{1}{2} \epsilon^{ijk} \int d^3x \, M^{0jk}(x)$$

$$\vec{J}^q(\mu) = \int d^3x \left[\overline{q} \frac{\vec{\gamma}\gamma_5}{2} q + \overline{q}(\vec{x} \times i\vec{D}) q \right]$$

$$= \frac{1}{2} \Sigma_q(\mu) + \vec{L}_q(\mu)$$

Proton spin decomposition

Frame Independent (Ji)

Phys. Rev. Lett., 78:610–613, 1997

$$J_P = J_q + J_g$$

$$= \sum_{q=u,d,s,c} \left(\frac{1}{2}\Sigma_q + L_q \right) + J_g$$

$\frac{1}{2}\Sigma_q$	L_q	J_g
Quark spin	Quark orbital angular momentum	Total gluon angular momentum

Each term is gauge invariant.

Expt: JLab, COMPASS, HERMES, J-PARC, EIC

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Total gluon angular momentum

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Expt: JLab, COMPASS, HERMES, J-PARC, EIC

Infinite momentum frame

(Jaffe-Manohar) Nucl. Phys., B337:509–546, 1990

$$J_P = \sum_{q=u,d,s,c} \frac{1}{2}\Sigma_q + \Delta G + \mathcal{L}_q + \mathcal{L}_g$$

Quark spin

Gluon helicity $\sim \epsilon^{ij} F^{+i} A^j$

Quark orbital angular momentum $\bar{q}(x \times i\partial)\psi$

Gluon orbital angular momentum $F(x \times \partial)A$

Not gauge invariant. Use light-cone gauge. Also from GPDs, GTMD

Expt: PHENIX, STAR, COMPASS, HERMES, EIC

These decompositions are not unique. There are other ways, and each can have their legitimate meanings

$$\left\langle N(p',s')\middle|\mathcal{T}_{q,g}^{\{\mu\nu\}}\middle|N(p,s)\right\rangle = \frac{1}{2}\bar{u}_N(p',s')\Big[\boldsymbol{T_1^{q,g}}(q^2)(\gamma^\mu\bar{p}^\nu + \gamma^\nu\bar{p}^\mu)$$
$$+ \frac{1}{2m_N}\boldsymbol{T_2^{q,g}}(q^2)\{iq_\alpha(\bar{p}^\mu\sigma^{\nu\alpha} + \bar{p}^\nu\sigma^{\mu\alpha})\}$$
$$+ \boldsymbol{D_{q,g}}(q^2)\frac{q^\mu q^\nu - g_{\mu\nu}q^2}{m_N} + \boldsymbol{\bar{C}_{q,g}}(q^2)m_N g_{\mu\nu}\Big]\, u_N(p,s)$$

$q = p' - p$: momentum transfer $\quad \bar{p} = (p + p')/2$

$\boldsymbol{T_1, T_2, D, \bar{C}}$: Gravitational form factors

$$T_1^q(0) = \int_0^1 dx\, x(q(x) + \bar{q}(x)) \qquad T_1^g(0) = \int_0^1 dx\, xg(x)$$

Momentum fraction

$$\left\langle N(p',s') \middle| \mathcal{T}_{q,g}^{\{\mu\nu\}} \middle| N(p,s) \right\rangle = \frac{1}{2} \bar{u}_N(p',s') \Big[\boldsymbol{T_1^{q,g}}(q^2)(\gamma^\mu \bar{p}^\nu + \gamma^\nu \bar{p}^\mu) + \frac{1}{2m_N} \boldsymbol{T_2^{q,g}}(q^2)\{ i q_\alpha (\bar{p}^\mu \sigma^{\nu\alpha} + \bar{p}^\nu \sigma^{\mu\alpha})\} + \boldsymbol{D_{q,g}}(q^2) \frac{q^\mu q^\nu - g_{\mu\nu} q^2}{m_N} + \boldsymbol{\bar{C}_{q,g}}(q^2) m_N g_{\mu\nu} \Big] u_N(p,s)$$

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Anomalous gravitomagntic moment

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Anomalous gravitomagntic moment

$$+ \boldsymbol{D_{q,g}}(q^2) \frac{q^\mu q^\nu - g_{\mu\nu} q^2}{m_N} + \boldsymbol{\bar{C}_{q,g}}(q^2) m_N g_{\mu\nu} \Big] u_N(p,s)$$

Pressure

$q = p' - p$: momentum transfer $\qquad \bar{p} = (p + p')/2$

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$$+ \boldsymbol{D_{q,g}}(q^2)\frac{q^\mu q^\nu - g_{\mu\nu}q^2}{m_N} + \boldsymbol{\bar{C}_{q,g}}(q^2)m_N g_{\mu\nu}]\, u_N(p,s)$$

Anomalous gravitomagntic moment

Pressure

Trace anomaly

$q = p' - p$: momentum transfer $\qquad \bar{p} = (p + p')/2$

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At $q^2 \to 0$ $\qquad \boldsymbol{J^{q,g} = \frac{1}{2}[T_1(0) + T_2(0)]^{q,g}} \qquad \boldsymbol{\langle x\rangle^{q,g} = T_1(0)^{q,g}}$

Momentum fraction (second moment of the PDF)

Sum rule: $\boldsymbol{\langle x\rangle^q + \langle x\rangle^g = T_1(0)^q + T_1(0)^g = 1}$

$$\boldsymbol{J^q + J^g = \frac{1}{2} = \frac{1}{2}[T_1(0) + T_2(0)]^q + \frac{1}{2}[T_1(0) + T_2(0)]^g}$$

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$$\boldsymbol{T_2(0)^q + T_2(0)^g = 0}$$

- Angular momentum :

$$\vec{J}^q(\mu) = \int d^3x \left[\bar{q}\frac{\vec{\gamma}\gamma_5}{2}q + \bar{q}(\vec{x}\times i\vec{D})q\right]$$

$$= \frac{1}{2}\,\Sigma_q(\mu) + \vec{L}_q(\mu)$$

$$\vec{J}^g = \int d^3x\,[\vec{x}\times(\vec{E}\times\vec{B})]$$

- **Operators:**

$$\langle N(p',s)|\mathcal{O}_A^{\mu}|N(p,s)\rangle\,; \qquad \mathcal{O}_A^{\mu} = \bar{q}\gamma^{\mu}\gamma_5 q$$

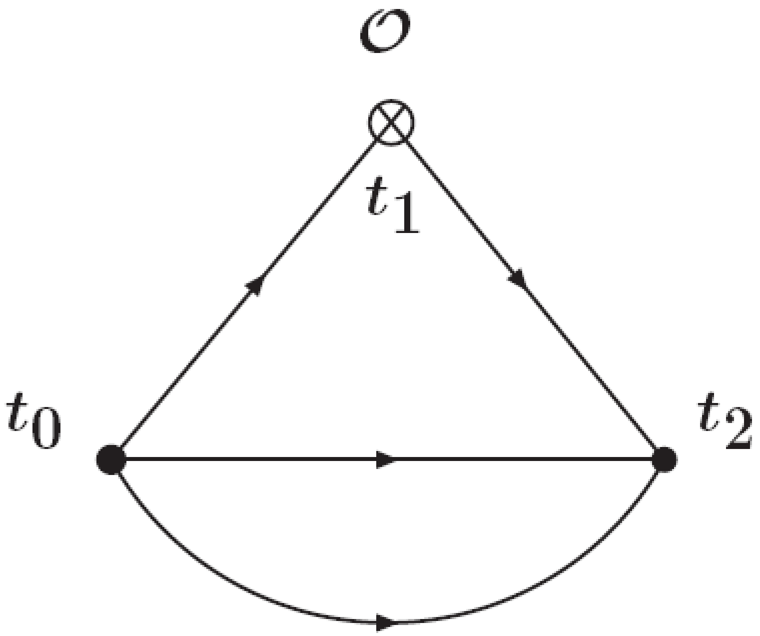
$$\langle N(p',s)|\mathcal{O}_{J_q}^{\mu\nu}|N(p,s)\rangle\,; \qquad \mathcal{O}_{J_q}^{\mu\nu} = \bar{q}\,\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}q$$

$$\langle N(p',s)|\mathcal{O}_{J_g}^{\mu\nu}|N(p,s)\rangle\,; \qquad \mathcal{O}_{J_g}^{\mu\nu} = 2\,\mathrm{Tr}\,[\,G_{\mu\sigma}G_{\nu\sigma}]$$

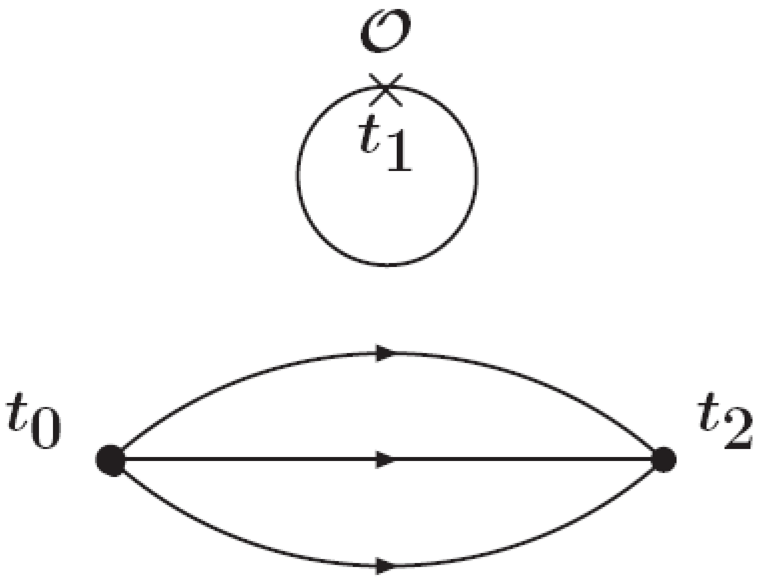
Physical observables

$\langle x \rangle_{q,g}$	Momentum fraction	$\gamma_{\mu} \overleftrightarrow{D}_{\nu}$
$\langle x^2 \rangle_{q,g}$	Second moment	$\gamma_{\mu} \overleftrightarrow{D}_{\nu} \overleftrightarrow{D}_{\delta}$
$\Delta u - \Delta d = g_A$	Axial charge	$\gamma_5 \gamma_{\mu}$
$\Delta u + \Delta d = \Delta\Sigma_{u,d}$	Spin content	$\gamma_5 \gamma_{\mu}$
$\delta u - \delta d = g_T$	Tensor charge	$\gamma_5 \sigma_{\mu\nu}$

$$G^{\alpha\beta}_{N\mathcal{O}N}(t_2, t_1, \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}.(\vec{x}_2 - \vec{x}_0)} \langle 0 \,|\, \mathrm{T}\,(\chi^{\alpha}(x_2)\, \mathcal{O}(x_1)\, \bar{\chi}^{\beta}(x_0))\,|\, 0 \rangle$$



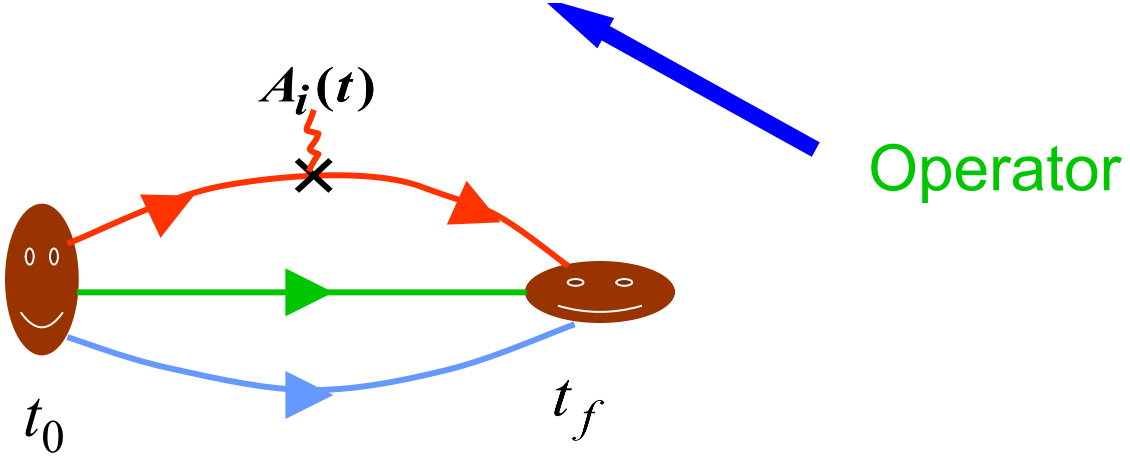
Connected insertion



Disconnected insertion

Three Point Correlation Function

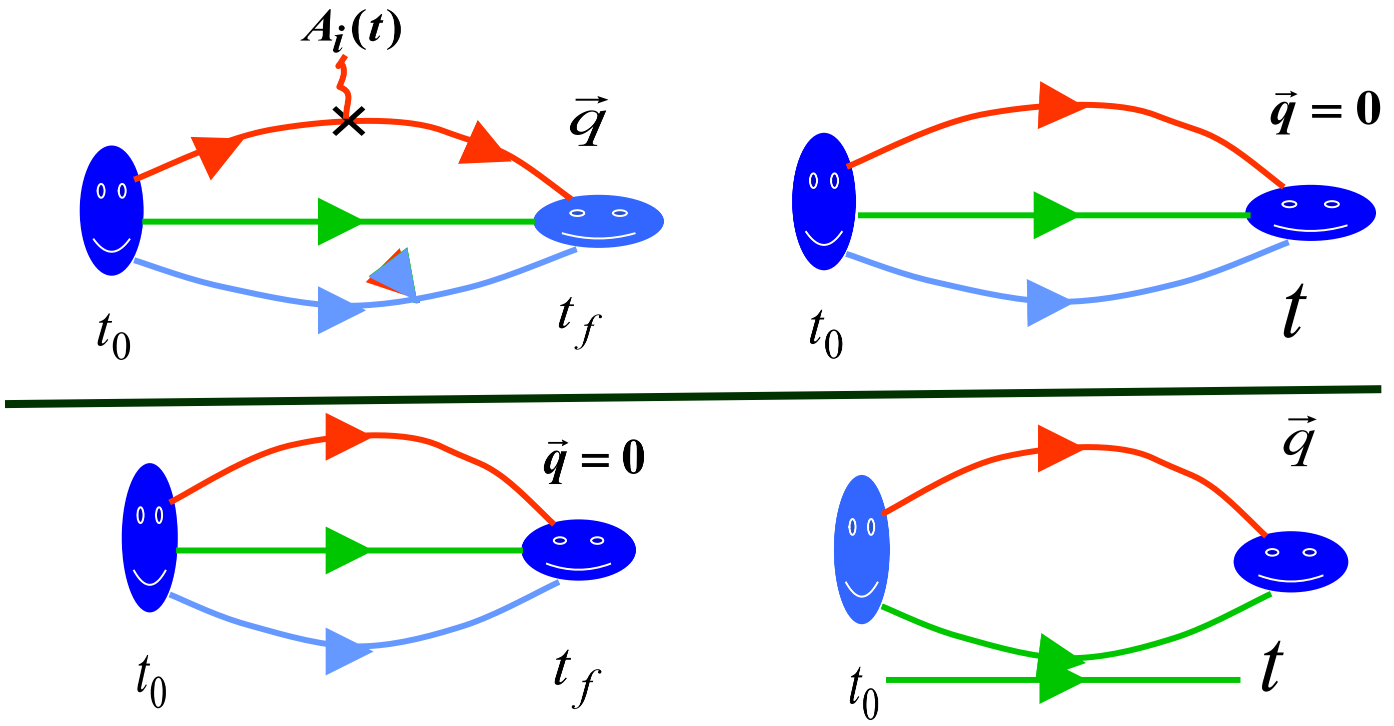
$$G^{\alpha\beta}_{PT_{\mu\nu}P}(t_2, t_1, \vec{p}, \vec{p'}) = \sum_{\vec{x_2}, \vec{x_1}} e^{-i\vec{p}\cdot\vec{x_2}} e^{-i\vec{q}\cdot\vec{x_1}} < 0|\, \mathrm{T} \left(\chi^{\alpha}(x_2)\, \mathrm{O}_{\mu\nu}(x_1)\, \bar{\chi}^{\beta}(0)\right) |0 >$$



$$\Gamma_{\alpha\beta}\; G^{\beta\alpha}_{NA_iN}(t_f, t, t_0, \bar{p}, \vec{q}) \equiv \Gamma_{\alpha\beta} \sum_{\vec{x}, \vec{x}_f} e^{i\vec{q}\cdot\vec{x}} \left\langle T(\chi^{\alpha}(x_f)\, A_i\, \bar{\chi}^{\beta}(x_0)) \right\rangle$$

$$\xrightarrow{t_f - t,\, t - t_0 >> 1,} \frac{E_q + m}{E_q} |\phi|^2\, e^{-m(t_f - t) - E_q(t - t_0)} \left[g_A(q^2) - h_A(q^2) \frac{q_i^2}{E_q + m} \right] \;;\; \Gamma = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}$$

Form Factors



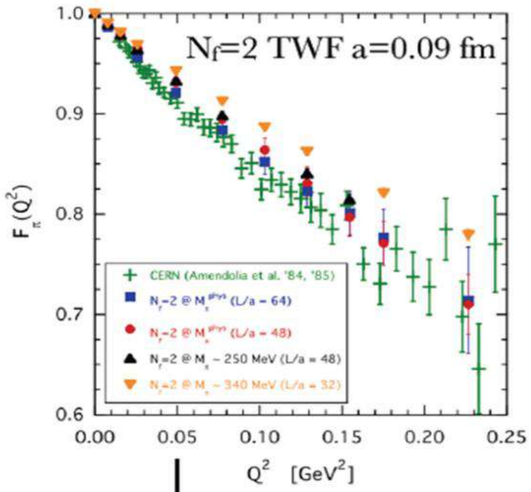
The combined ratios leads to the form factors

$$g_A(q^2) - h_A(q^2)\frac{q_i^2}{E_q + m} \xrightarrow{q_{i=0}} g_A(q^2)$$

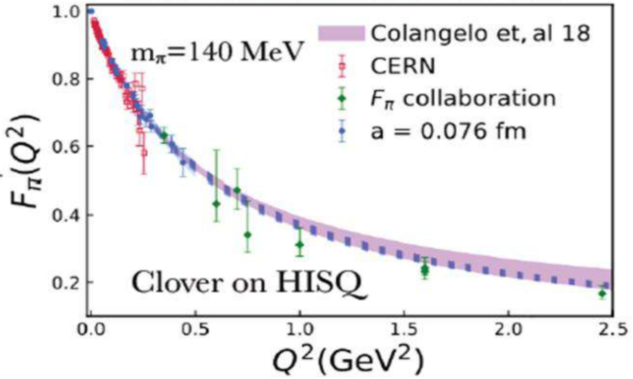
Vector form factor of the pion

- Three recent computations that include at least one ensemble with physical pion mass

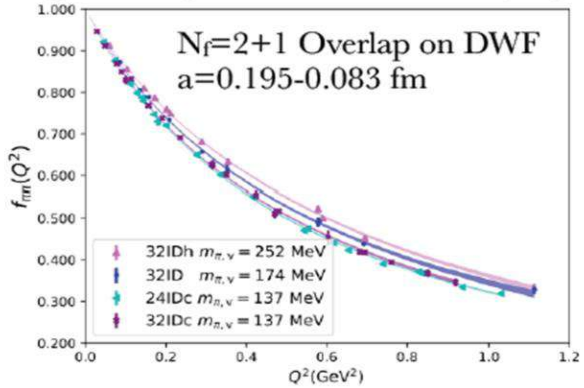
ETMC: C.A. *et al.*, PHYS. REV. D 97, 014508 (2018)



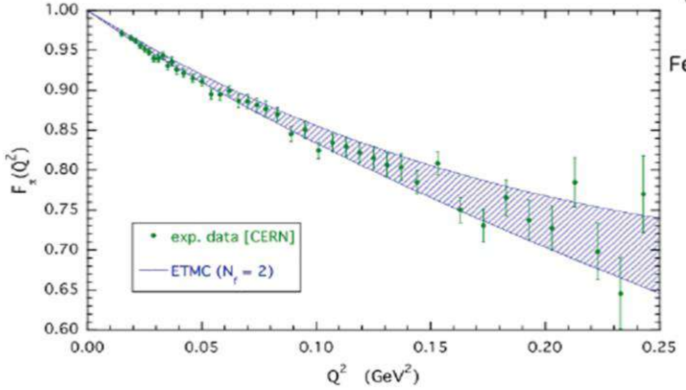
X. Gao *et al*, PHYS. REV. D 104, 114515 (2021)



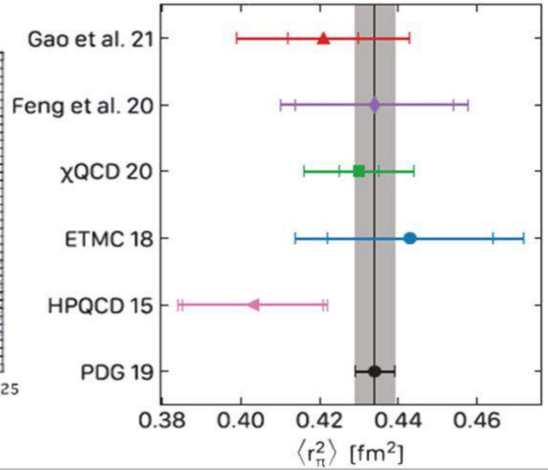
χQCD: G. Weng *et al.*, PHYS. REV. D 104, 074502 (2021)



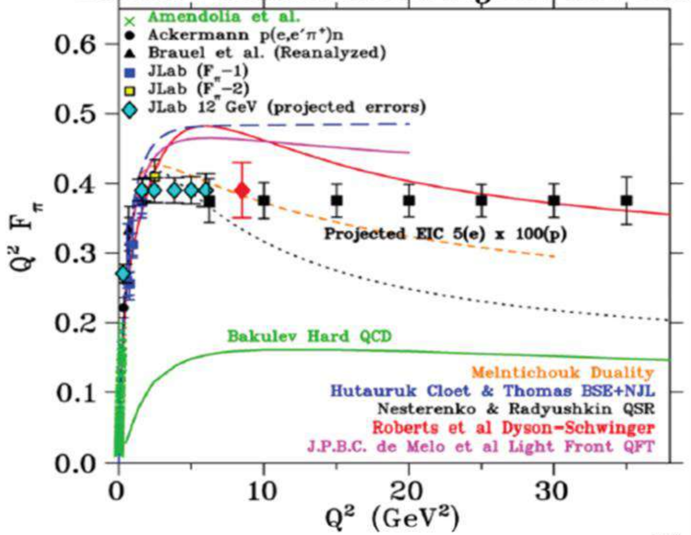
Chiral & infinite volume extrapolations



$$\langle r^2\rangle_\Gamma = -\frac{6}{F_\Gamma(0)}\frac{\partial F_\Gamma(Q^2)}{\partial Q^2}\bigg|_{Q^2=}$$



Planned measurements at JLab and EIC



Quark spin contribution $\Delta\Sigma$

- Flavor-singlet axial vector current

$$A_\mu^0 = \sum_{f=u,d,s,c} \bar{q}_f i\gamma_\mu\gamma_5 q_f$$

- On the lattice we can compute the matrix element of the flavor-singlet axial vector current

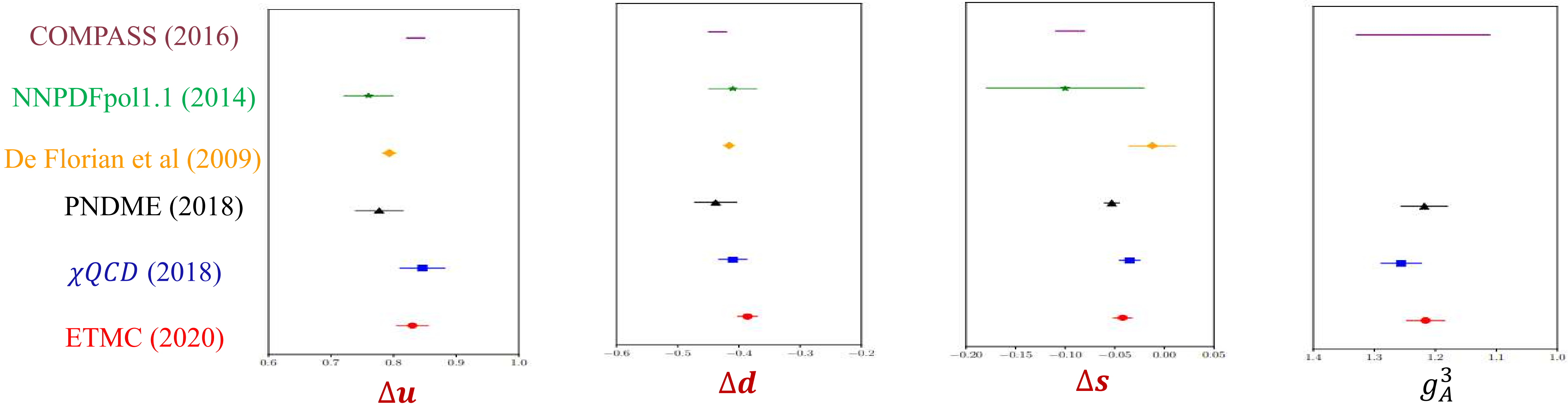
$$\langle N(p,s)|A_\mu^0|N(p,s)\rangle = s_\mu g_A^0$$

s_μ Polarization vector

$$g_A^0 = \Delta\Sigma = \Delta u + \Delta d + \Delta s + \Delta c$$ Quark spin contribution of the *u,d,s* and *c* quarks

$$= \Delta(u+d)_{\mathrm{CI}} + \Delta(u+d)_{\mathrm{DI}} + \Delta s_{\mathrm{D}I} + \Delta c_{\mathrm{D}I}$$

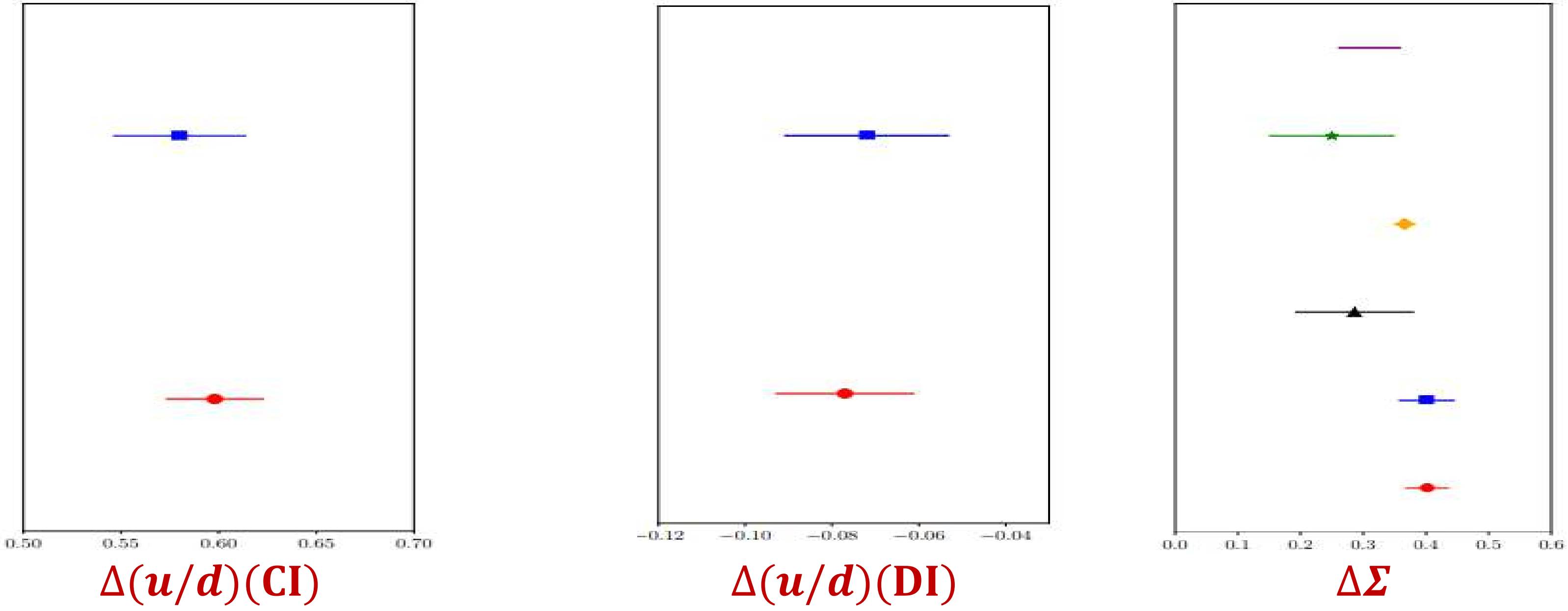
Quark spin contribution



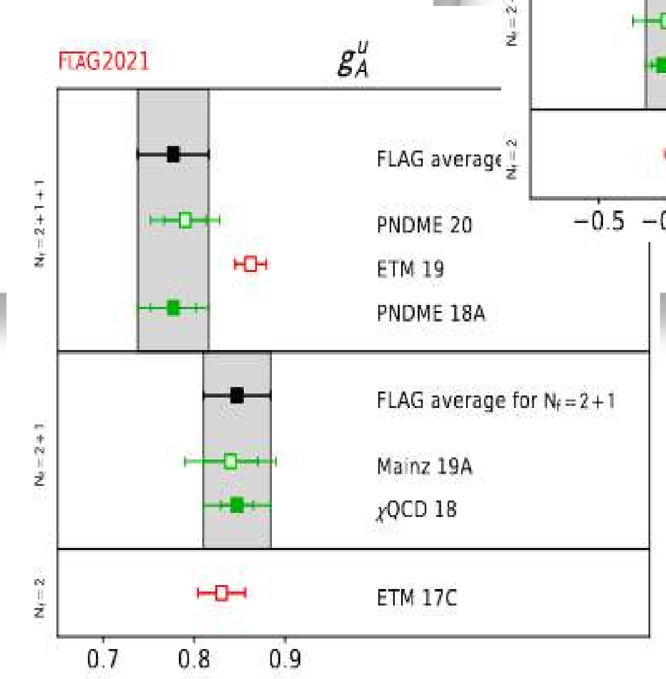
De Florian et al (2009):
Phys. Rev., D80:034030, 2009

NNPDFpol1.1 (2014):
Nucl. Phys. B, 887:276–308, 2014

COMPASS (2016)
Phys. Lett. B, 753:18–28, 2016



Summary from: K. F. Liu :arXiv 2112.08416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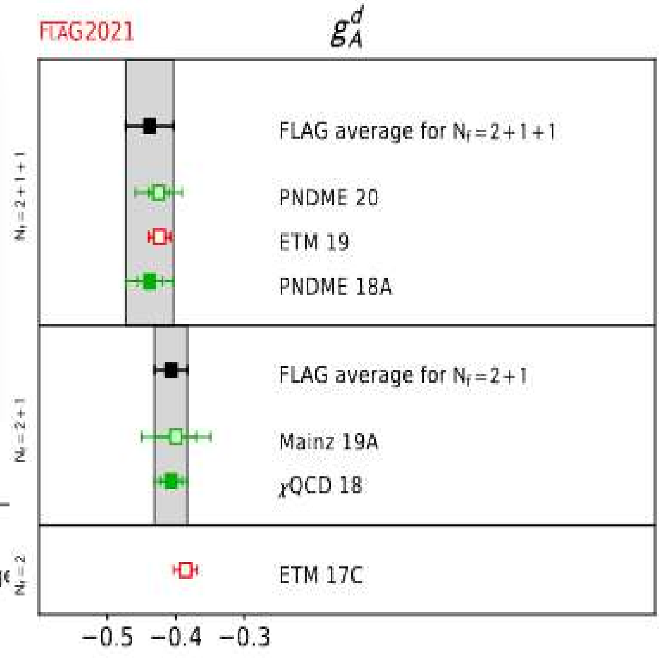
g_A^u
FLAG average
PNDME 20
ETM 19
PNDME 18A
FLAG average for $N_f=2+1$
Mainz 19A
χQCD 18
ETM 17C
$N_f=2+1+1$
$N_f=2+1$
$N_f=2$
0.7
0.8
0.9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ETM 19
PNDME 18A
FLAG average for $N_f=2+1$
Mainz 19A
χQCD 18
ETM 17C
$N_f=2+1+1$
$N_f=2+1$
$N_f=2$
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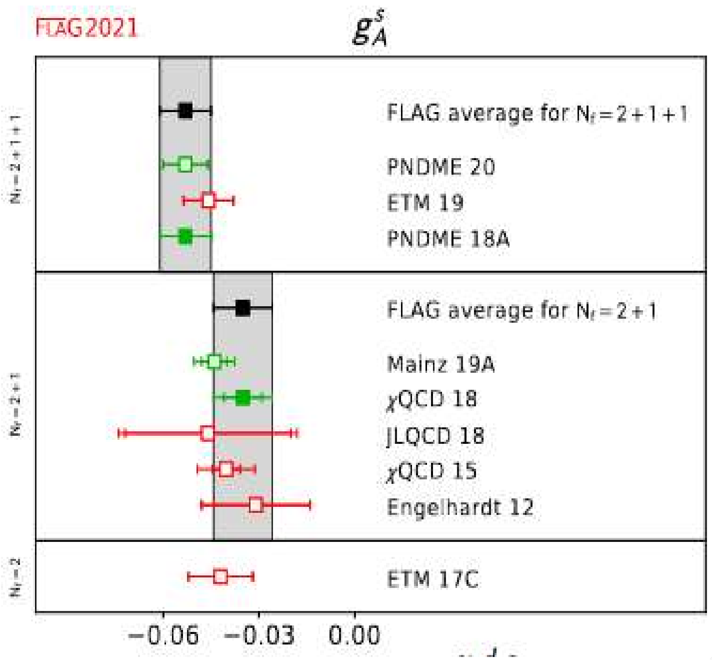
−0.4
−0.3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PNDME 18A
FLAG average for $N_f=2+1$
Mainz 19A
χQCD 18
JLQCD 18
χQCD 15
Engelhardt 12
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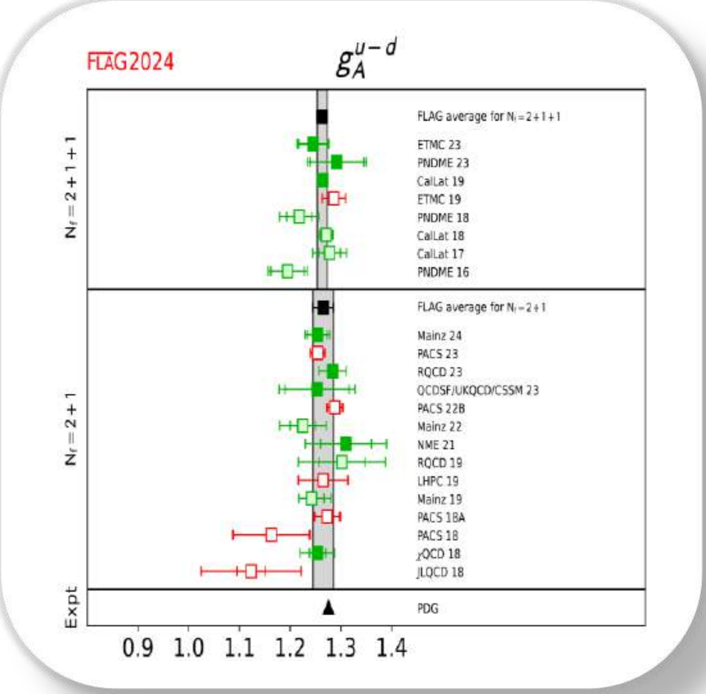
$N_f=2+1$
$N_f=2$
−0.06
−0.03
0.00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g_A^{u-d}
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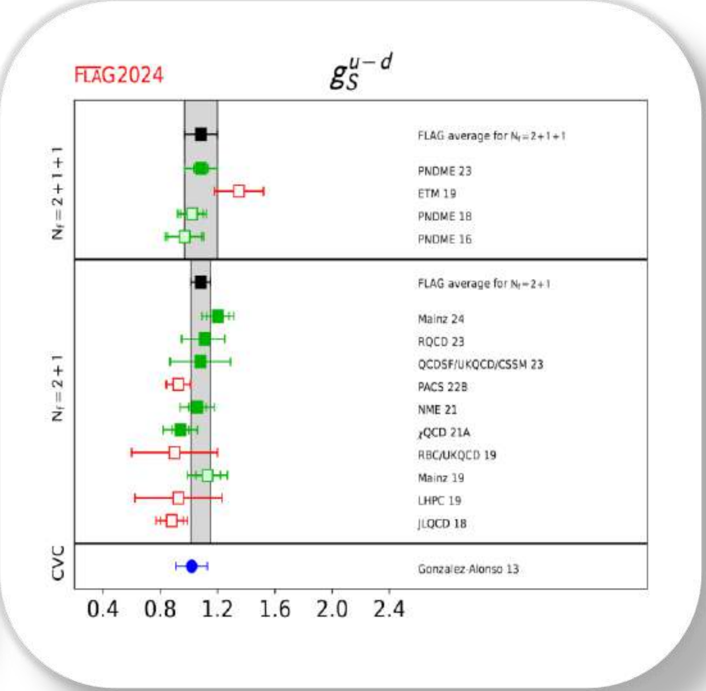
ETMC 23
PNDME 23
CalLat 19
ETMC 19
PNDME 18
CalLat 18
CalLat 17
PNDME 16
FLAG average for N_f = 2+1
Mainz 24
PACS 23
RQCD 23
QCDSF/UKQCD/CSSM 23
PACS 22B
Mainz 22
NME 21
RQCD 19
LHPC 19
Mainz 19
PACS 18A
PACS 18
χQCD 18
JLQCD 18
PDG
N_f = 2+1+1
N_f = 2+1
Expt
0.9 1.0 1.1 1.2 1.3 1.4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ETM 19
PNDME 18
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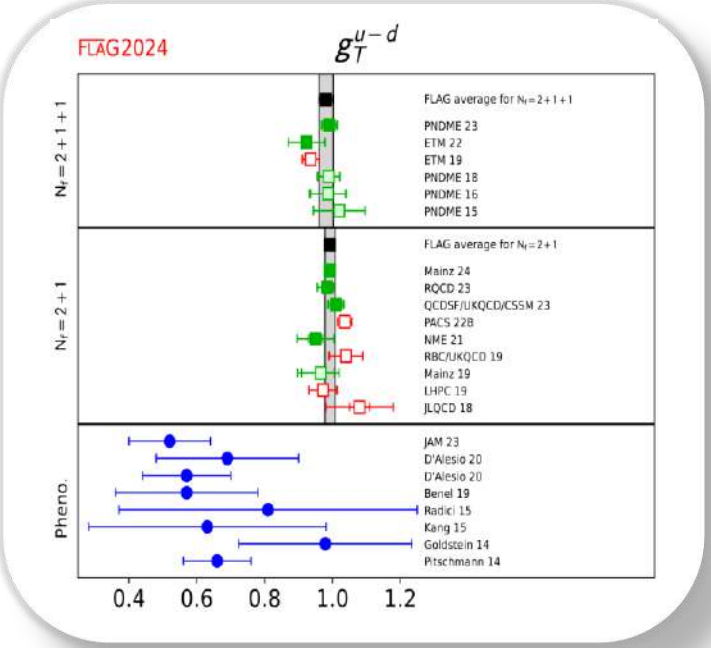
FLAG average for N_f = 2+1
Mainz 24
RQCD 23
QCDSF/UKQCD/CSSM 23
PACS 22B
NME 21
χQCD 21A
RBC/UKQCD 19
Mainz 19
LHPC 19
JLQCD 18
Gonzalez-Alonso 13
N_f = 2+1+1
N_f = 2+1
CVC
0.4 0.8 1.2 1.6 2.0 2.4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ETM 22
ETM 19
PNDME 18
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PNDME 15
FLAG average for N_f = 2+1
Mainz 24
RQCD 23
QCDSF/UKQCD/CSSM 23
PACS 22B
NME 21
RBC/UKQCD 19
Mainz 19
LHPC 19
JLQCD 18
JAM 23
D'Alesio 20
D'Alesio 20
Benel 19
Radici 15
Kang 15
Goldstein 14
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N_f = 2+1+1
N_f = 2+1
Pheno.
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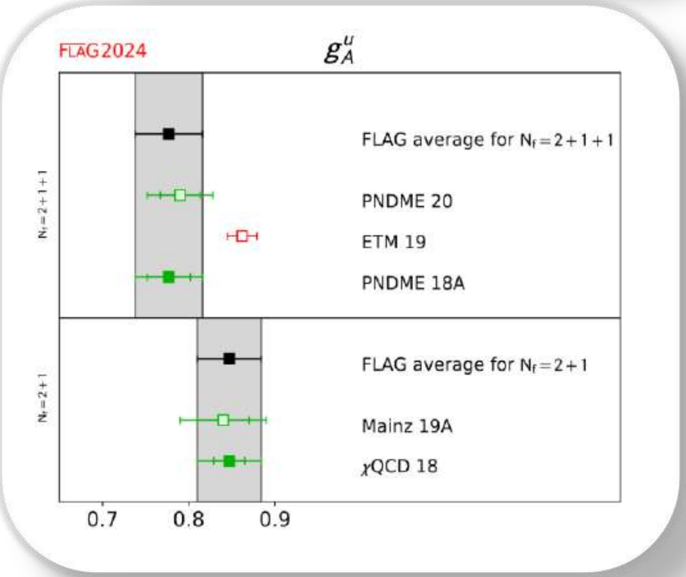
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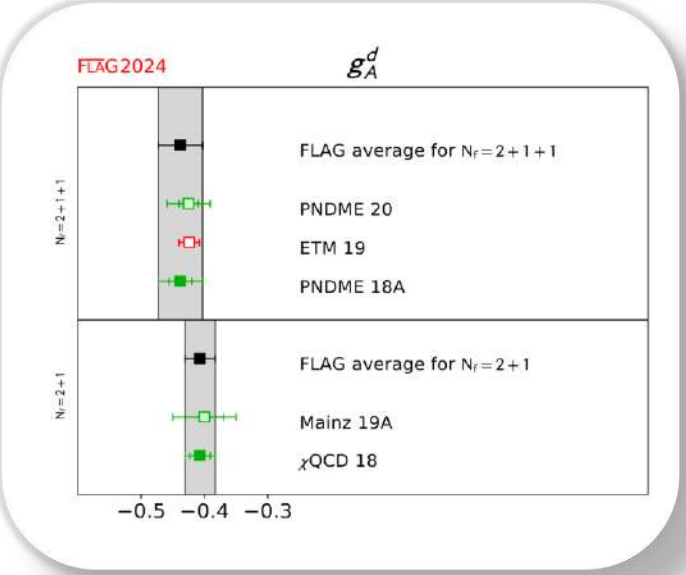
N_f = 2+1
0.7 0.8 0.9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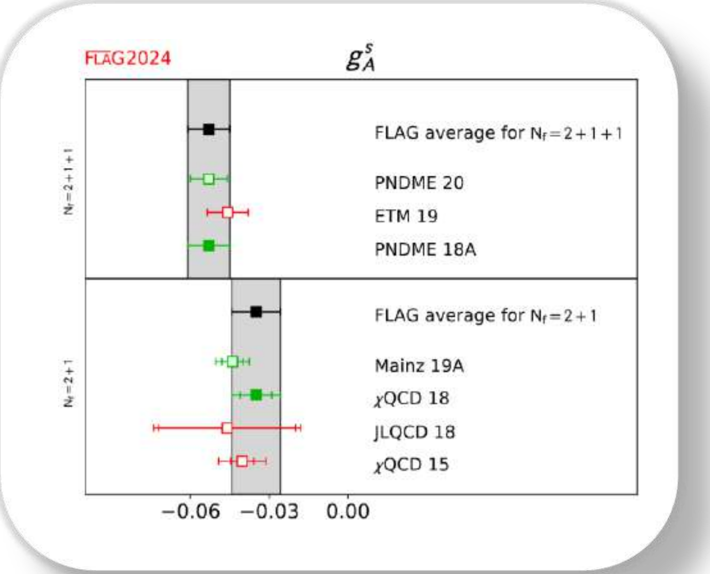
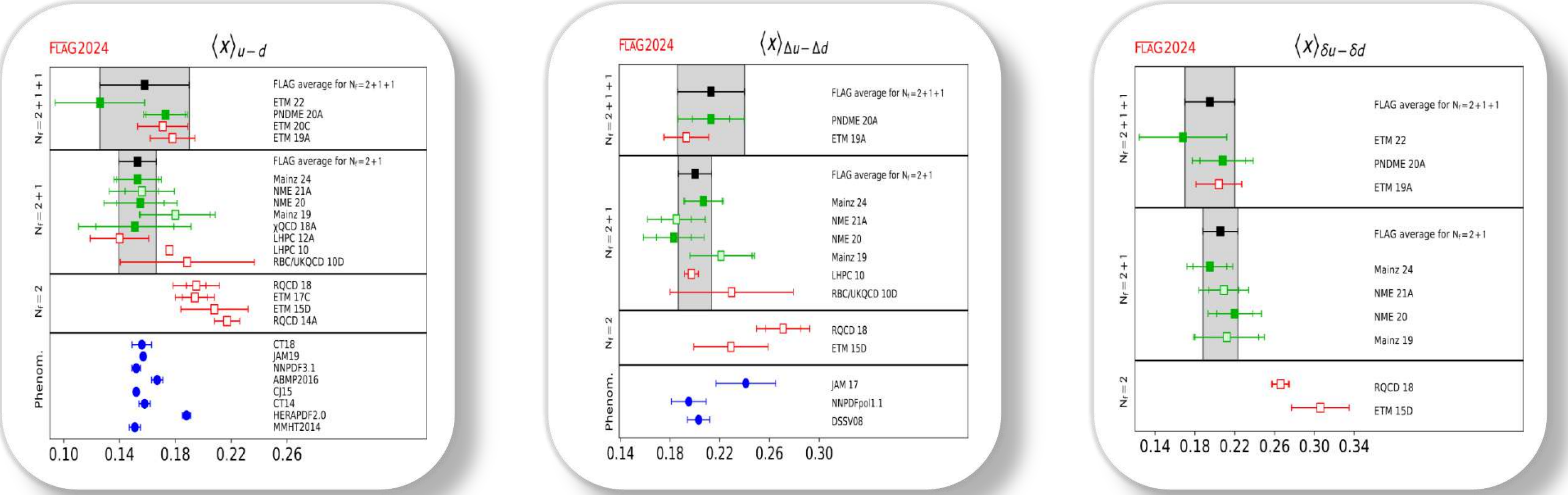
g_A^d
FLAG average for N_f = 2+1+1
PNDME 20
ETM 19
PNDME 18A
FLAG average for N_f = 2+1
Mainz 19A
χQCD 18
N_f = 2+1+1
N_f = 2+1
−0.5 −0.4 −0.3


FLAG2024
g_A^s
FLAG average for N_f = 2+1+1
PNDME 20
ETM 19
PNDME 18A
FLAG average for N_f = 2+1
Mainz 19A
χQCD 18
JLQCD 18
χQCD 15
N_f = 2+1+1
N_f = 2+1
−0.06 −0.03 0.00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⟨x⟩u−d
FLAG average for Nf=2+1+1
ETM 22
PNDME 20A
ETM 20C
ETM 19A
FLAG average for Nf=2+1
Mainz 24
NME 21A
NME 20
Mainz 19
χQCD 18A
LHPC 12A
LHPC 10
RBC/UKQCD 10D
RQCD 18
ETM 17C
ETM 15D
RQCD 14A
CT18
JAM19
NNPDF3.1
ABMP2016
CJ15
CT14
HERAPDF2.0
MMHT2014
Nf=2+1+1
Nf=2+1
Nf=2
Phenom.
0.10 0.14 0.18 0.22 0.26
FLAG2024
⟨x⟩Δu−Δd
FLAG average for Nf=2+1+1
PNDME 20A
ETM 19A
FLAG average for Nf=2+1
Mainz 24
NME 21A
NME 20
Mainz 19
LHPC 10
RBC/UKQCD 10D
RQCD 18
ETM 15D
JAM 17
NNPDFpol1.1
DSSV08
0.14 0.18 0.22 0.26 0.30
FLAG2024
⟨x⟩δu−δd
FLAG average for Nf=2+1+1
ETM 22
PNDME 20A
ETM 19A
FLAG average for Nf=2+1
Mainz 24
NME 21A
NME 20
Mainz 19
RQCD 18
ETM 15D
0.14 0.18 0.22 0.26 0.30 0.34

$$\left\langle N(p',s')\middle|\mathcal{T}_{q,g}^{\{\mu\nu\}}\middle|N(p,s)\right\rangle = \frac{1}{2}\bar{u}_N(p',s')\Big[T_1^{q,g}(q^2)(\gamma^\mu\bar{p}^\nu + \gamma^\nu\bar{p}^\mu)$$

$$+ \frac{1}{2m_N}T_2^{q,g}(q^2)\{iq_\alpha(\bar{p}^\mu\sigma^{\nu\alpha} + \bar{p}^\nu\sigma^{\mu\alpha})\}$$

$$+ D_{q,g}(q^2)\frac{q^\mu q^\nu - g_{\mu\nu}q^2}{m_N} + \bar{C}_{q,g}(q^2)m_N g_{\mu\nu}\Big]\,u_N(p,s)$$

Anomalous gravitomagntic moment

Pressure

Trace anomaly

$q = p' - p$: momentum transfer $\quad \bar{p} = (p + p')/2$

$\boldsymbol{T_1, T_2, D, \bar{C}}$: Gravitational form factors

$$\boldsymbol{J^{q,g} = \frac{1}{2}[T_1(0) + T_2(0)]^{q,g}}$$

PHYSICAL REVIEW D, VOLUME 62, 114504

Quark orbital angular momentum from lattice QCD

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[5]Andrzej Sołtan Institute for Nuclear Studies, Warsaw, Poland
(Received 10 December 1999; published 25 October 2000)

On an Euclidean space-time lattice,

$$\mathcal{T}_{4i}^{q(E)} = (-1)\frac{i}{4}\sum_f \bar{\psi}_f [\gamma_4 \vec{D}_i + \gamma_i \vec{D}_4 - \gamma_4 \overleftarrow{D}_i - \gamma_i \overleftarrow{D}_4]\psi_f$$

$$\vec{D}_\mu \psi(x) = \frac{1}{2a}[U_\mu(x)\psi(x+a_\mu) - U_\mu^\dagger(x-a_\mu)\psi(x-a_\mu)],$$

$$\bar{\psi}(x)\overleftarrow{D}_\mu = \frac{1}{2a}[\bar{\psi}(x+a_\mu)U_\mu^\dagger(x) - \bar{\psi}(x-a_\mu)U_\mu^\dagger(x-a_\mu)]$$

$$\mathcal{T}_{4i}^{g(E)} = (+i)\left[-\frac{1}{2}\sum_{k=1}^{3} 2\mathrm{Tr}^{\mathrm{color}}[G_{4k}G_{ki} + G_{ik}G_{k4}]\right]$$

$$G_{\mu\nu}^{(E)}(x) = \frac{1}{8}(P_{\mu\nu}(x) - P_{\mu\nu}^\dagger(x)) \quad P_{\mu\nu} = U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)$$
$$+ U_\nu(x)U_\mu^\dagger(x-\mu+\nu)U_\nu^\dagger(x-\mu)U_\mu(x-\mu)$$
$$+ U_\mu^\dagger(x-\mu)U_\nu^\dagger(x-\mu-\nu)U_\mu(x-\mu-\nu)U_\nu(x-\nu)$$
$$+ U_\nu^\dagger(x-\nu)U_\mu(x-\nu)U_\nu(x-\nu+\mu)U_\mu^\dagger(x)$$

Correlation functions for OAM

- Time ordered two-point correlation function of nucleon:

$$G^{NN}_{\alpha\beta}(\vec{p},t) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle 0|T[\chi_\alpha(\vec{x},t)\bar{\chi}_\beta(\vec{0},0)]|0\rangle$$

Nucleon interpolating field:

$$\chi_\alpha(x) = \epsilon_{abc} u(x)^a_\alpha [u(x)^b \tilde{C} d(x)^c]$$

$$\tilde{C} = C\gamma_5 \qquad C \equiv \gamma_2\gamma_4$$

$$C_{2\text{pt}}(\vec{p},t) \equiv \text{Tr}[\Gamma_0 G^{NN}(\vec{p},t)] \xrightarrow{t\gg 1} \frac{Z_p^2}{(La)^3}\frac{E_p+m}{E_p} e^{-E_p(t-t_0)} + A e^{-E_p^1(t-t_0)}$$

$$\Gamma_0 = P_+ = \tfrac{1+\gamma_4}{2}$$

- Matrix element of tensor current can be obtained using three-point correlation functions:

$$G^{T^{q,g}_{4i}}_{\alpha\beta}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i}) = \sum_{\vec{x}_\text{f},\vec{z}} e^{-i\vec{p}_\text{f}\cdot(\vec{x}_\text{f}-\vec{z})} e^{i\vec{p}_\text{i}\cdot\vec{z}} \langle 0|T[\chi_\alpha(\vec{x}_\text{f},t_\text{f}) T^{q,g}_{4i}(\vec{z},\tau)\bar{\chi}_\beta(\vec{0},0)]|0\rangle$$

- With a definition of $C^{4i}_{3\text{pt},\Gamma_\alpha}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i}) \equiv \text{Tr}[\Gamma_\alpha G^{T^{q,g}_{4i}}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i})]$

$$R^{4i}_{\Gamma_\alpha}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i}) \equiv \frac{C^{4i}_{3\text{pt},\Gamma_\alpha}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i})}{C_{2\text{pt}}(\vec{p}_\text{f},t_\text{f})} \sqrt{\frac{C_{2\text{pt}}(\vec{p}_\text{i},t_\text{f}-\tau)C_{2\text{pt}}(\vec{p}_\text{f},\tau)C_{2\text{pt}}(\vec{p}_\text{f},t_\text{f})}{C_{2\text{pt}}(\vec{p}_\text{f},t_\text{f}-\tau)C_{2\text{pt}}(\vec{p}_\text{i},\tau)C_{2\text{pt}}(\vec{p}_\text{i},t_\text{f})}}$$

Required ratios

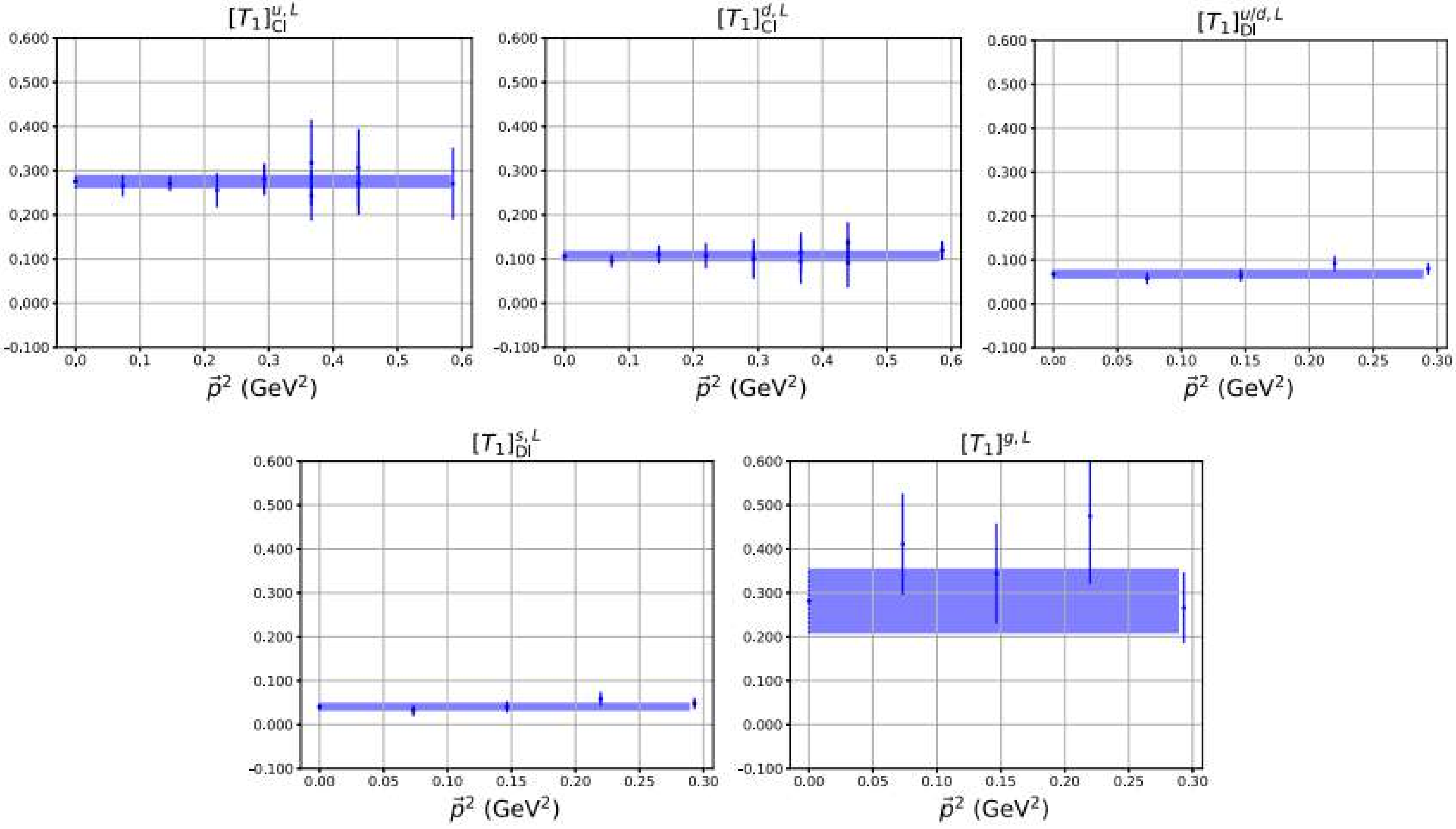
$$\xrightarrow[t_\text{f}-\tau\gg 1]{t_\text{f}\gg 1} \frac{a_1 T_1(Q^2) + a_2 T_2(Q^2) + a_3 D(Q^2)}{4\sqrt{E_{p'}(E_{p'}+m)E_p(E_p+m)}}$$

1. $$R^{4i}_{\Gamma_0}(t_f, \tau, \vec{p}, \vec{p}) = p_i T_1(0)$$

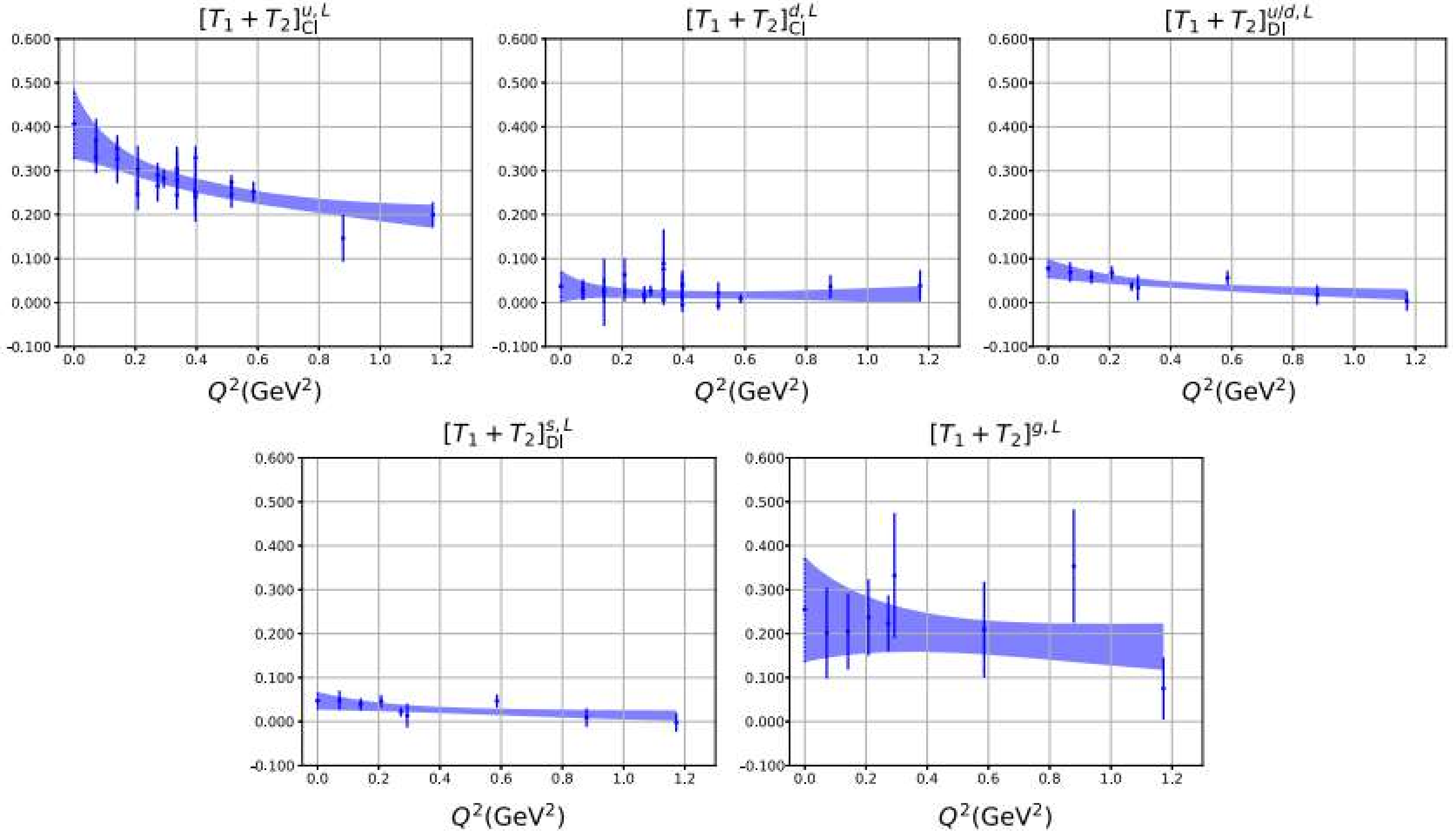
2. $$R^{4i}_{\Gamma_j}(t_f, t, \vec{p}, \vec{0}) = \frac{-i}{4}\sqrt{\frac{E_p + m}{2E_p}}\epsilon_{i,j,k} p_k [T_1 + T_2](Q^2)$$

3. $$R^{4i}_{\Gamma_j}(t_f, t, \vec{0}, \vec{p}) = \frac{-i}{4}\sqrt{\frac{E_p + m}{2E_p}}\epsilon_{i,j,k} p_k [T_1 + T_2](Q^2)$$

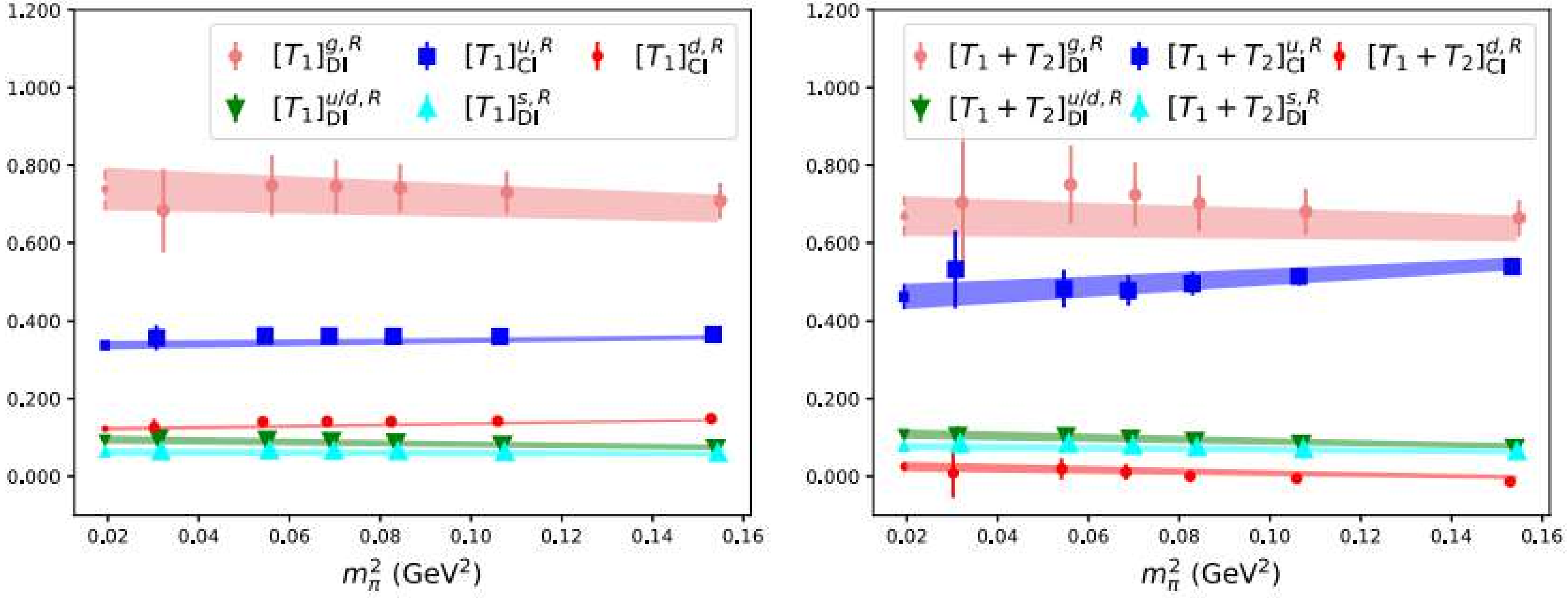
4. $$R^{4i}_{\Gamma_j}(t_f, t, \vec{p}, -\vec{p}) = \frac{-i}{2}\epsilon_{i,j,k} p_k [T_1 + T_2](Q^2)$$

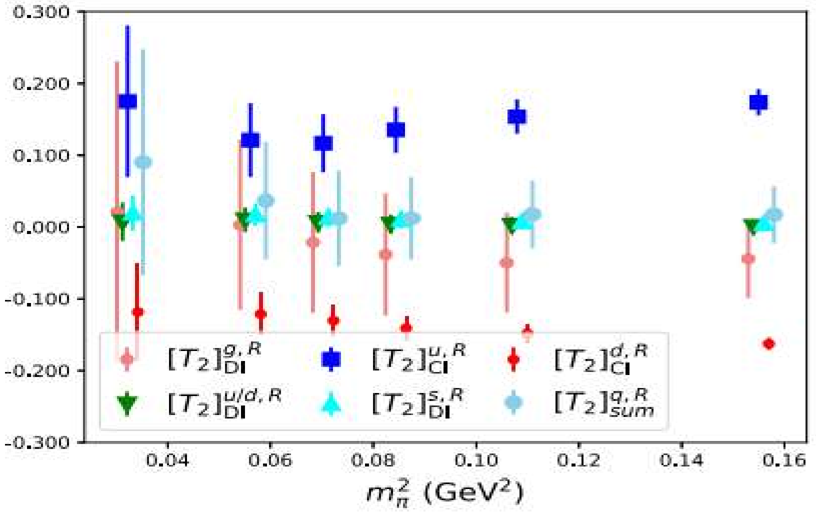


χQCD: Phys. Rev. D 106, 014512 (2022)



χQCD: Phys. Rev. D 106, 014512 (2022)





χQCD: Phys. Rev. D 106, 014512 (2022)

Renormalization

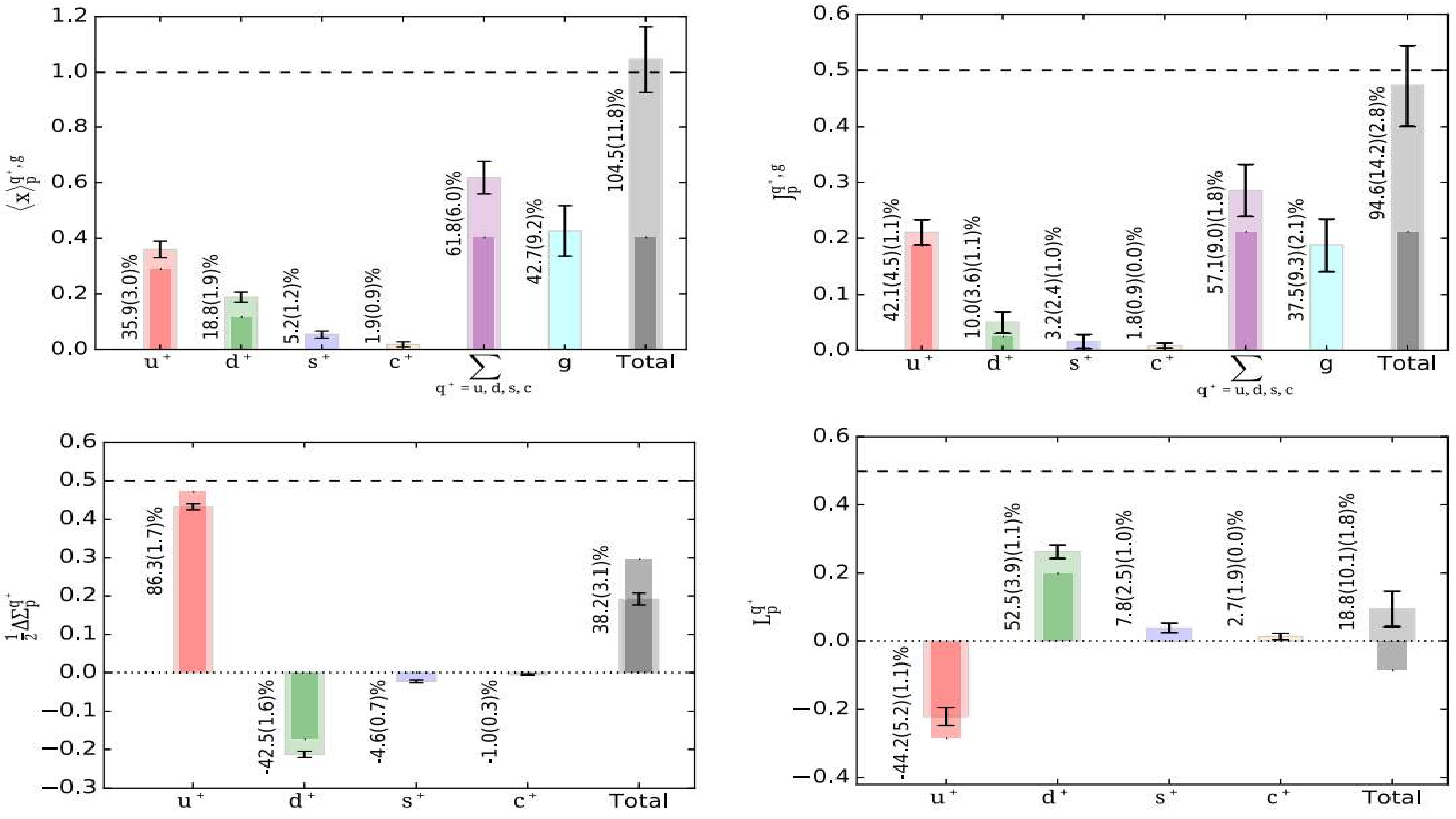
$$\mathcal{O}^{S}(M) = Z^{S}_{\mathcal{O};\mathrm{bare}}(M)\mathcal{O}_{\mathrm{bare}},$$

$$\mathcal{O}^{\mathrm{RGI}} = \Delta Z^{S}_{\mathcal{O}}(M)\mathcal{O}^{S}(M) \equiv Z^{\mathrm{RGI}}_{\mathcal{O}}\mathcal{O}_{\mathrm{bare}}.$$

For example:

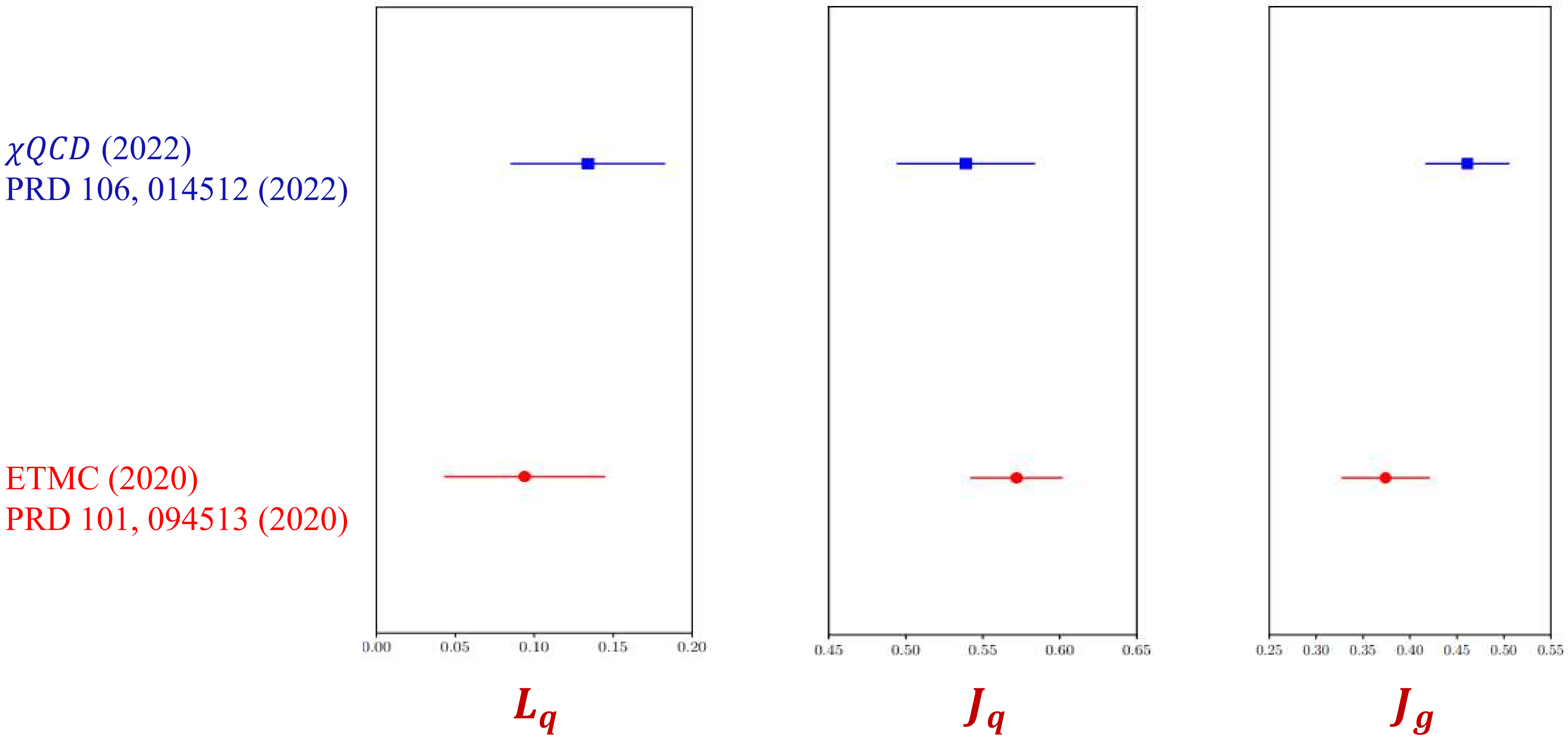
$$\langle x\rangle_R^{q^+} = Z_{qq}\langle x\rangle_B^{q^+} + Z_{qg}\langle x\rangle_B^{g}$$

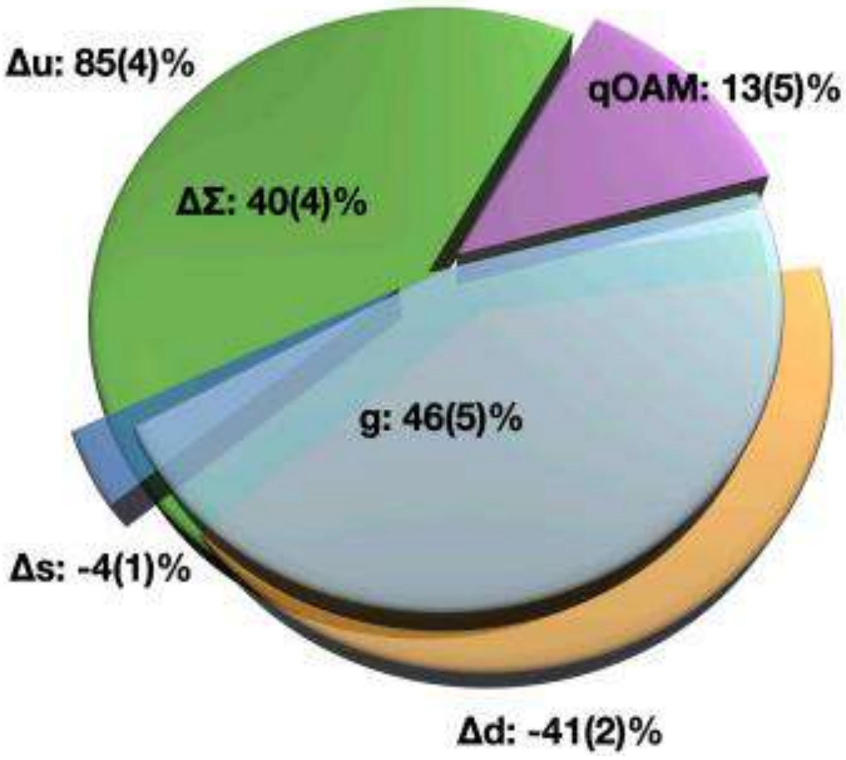
$$\langle x\rangle_R^{g} = Z_{gg}\langle x\rangle_B^{g} + Z_{gq}\langle x\rangle_B^{q^+}$$



ETMC: Phys. Rev. D 101, 094513 (2020)

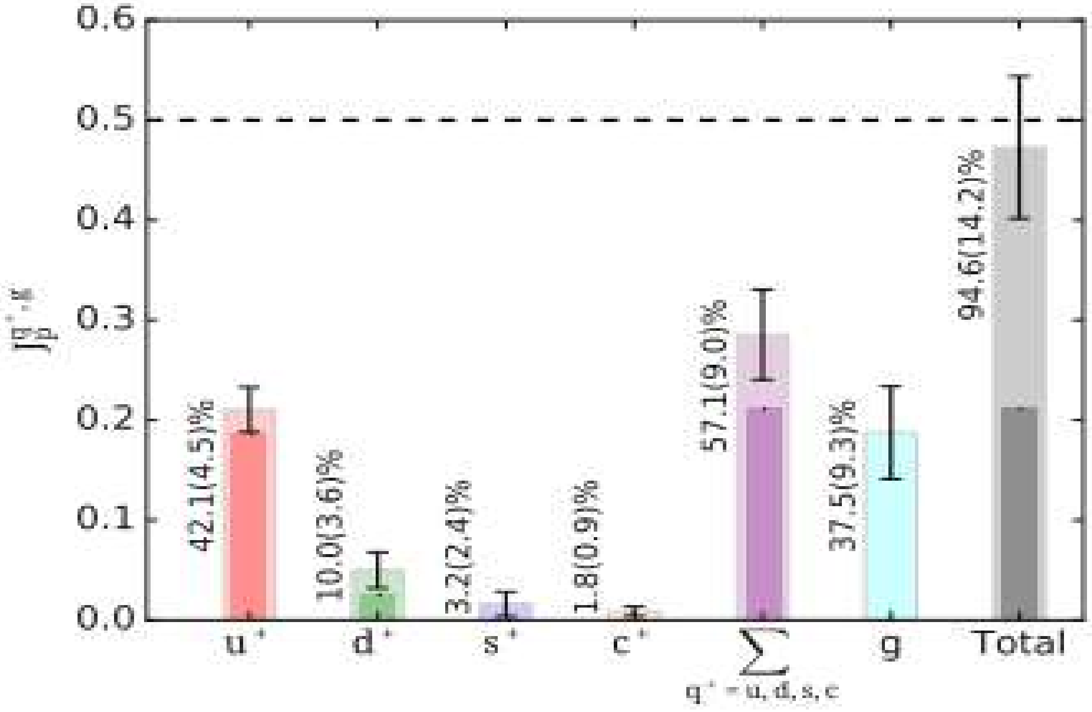
Angular momentum components from LQCD calculations





χQCD (2022)

Phys. Rev. D 106, 014512 (2022)



ETMC (2020)

Phys. Rev. D 101, 094513 (2020)

OAM from GTMD

Longitudinal quark OAM in z-dir $\quad L_3^U = \int dx \int d^2b_T \int d^2k_T (b_T \times k_T)_3 \mathcal{W}^U(x, b_T, k_T)$

$$L_3^U = -\int dx \int d^2k_T \frac{k_T^2}{m^2} F_{14}\Big|_{\Delta_T = 0}$$

C. Lorcé and B. Pasquini, Phys. Rev. D 84, 014015 (2011).

$$\frac{L_3}{n} = \frac{1}{a}\epsilon_{ij} \frac{\frac{\partial}{\partial \Delta_{T,j}}(\Phi(a\vec{e}_i) - \Phi(-a\vec{e}_i))}{\Phi(a\vec{e}_i) + \Phi(-a\vec{e}_i)}\Bigg|_{\Delta_T = 0}$$

$$\Phi(z_T) = \langle P + \Delta_T/2, S = \vec{e}_3 | \bar{\psi}(-z_T/2)\gamma^+ U[-z_T/2, z_T/2]\psi(z_T/2) | P - \Delta_T/2, S = \vec{e}_3 \rangle$$

M. Engelhardt, PRD 95, 094505 (2017)

n : number of valence quarks

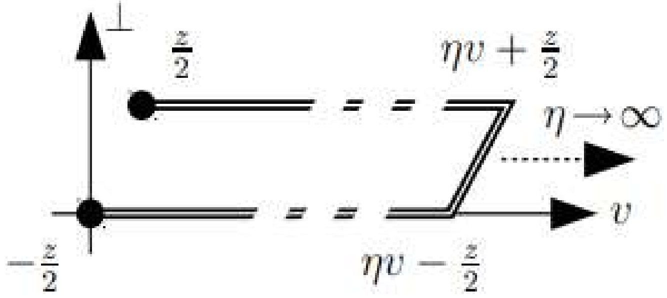
$\boldsymbol{e_3}$: Unit vector in the longitudinal direction

$\boldsymbol{e_i}$: Unit vector in the transverse direction

$\boldsymbol{\Delta_T}$: Momentum transfer

$\boldsymbol{z_T}$: Operator separation

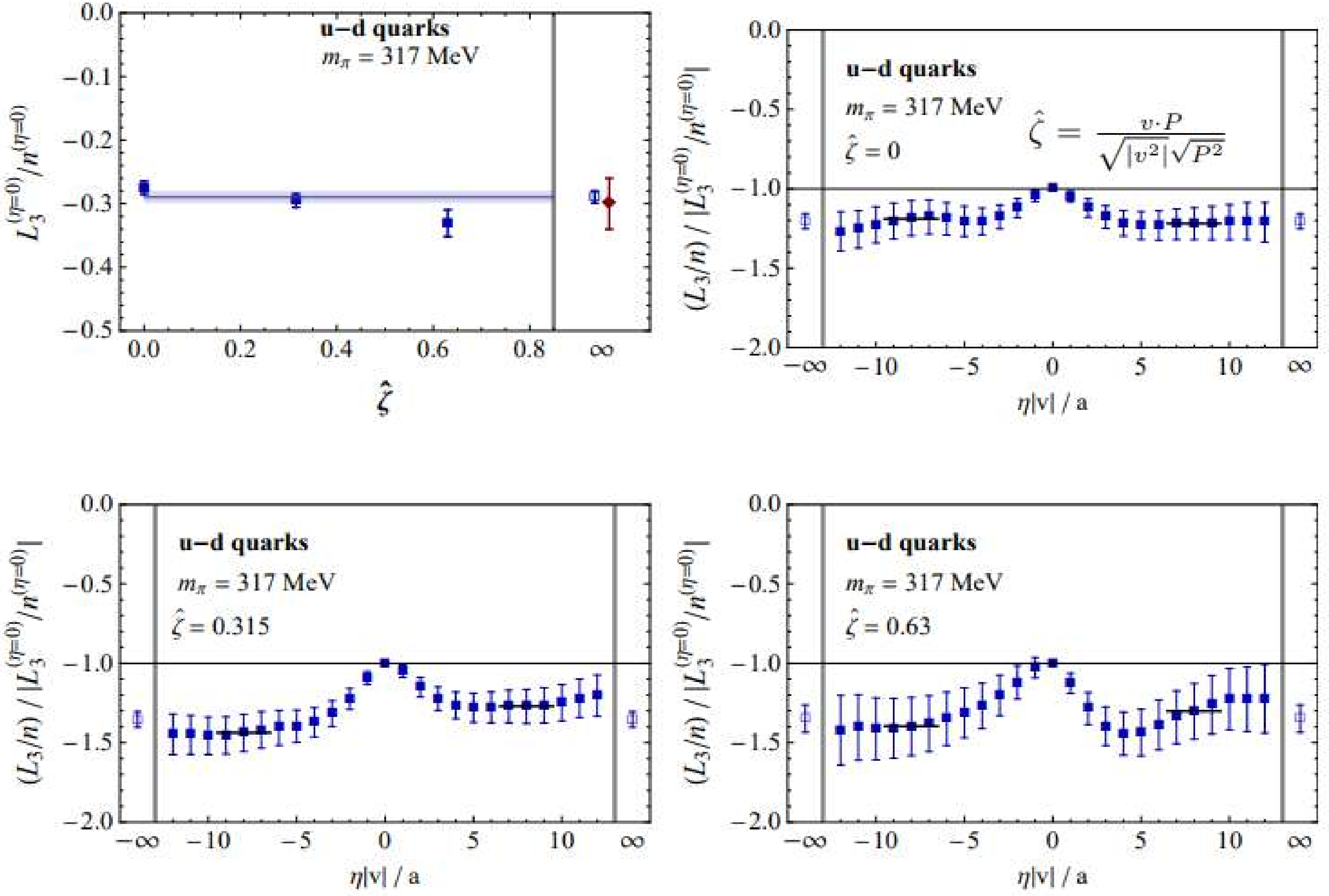
U : Wilson line between ψ and $\bar{\psi}$



- Straight $U\left[-\frac{z}{2}, \frac{z}{2}\right]$ → Ji OAM
- Staple-shaped $U\left[-\frac{z}{2}, \frac{z}{2}\right]$ → JM OAM
- Torque accumulation due to final state interaction causes the difference

M. Burkardt, Phys. Rev. D 88, 014014 (2013)

OAM from GTMD



M. Engelhardt, PRD 95, 094505 (2017)   Phys. Rev. D, 102(7) 074505, 2020

OAM from GTMD

Quark OAM can also be calculated using the second Mellin moment of the twist-3 generalized parton distribution in the forward limit

Engelhardt@SPIN 2023

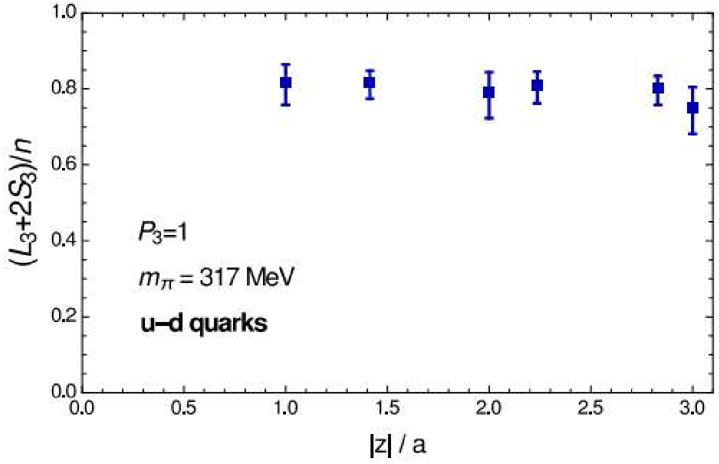
$$L_3 = (L_3 + 2S_3) - 2S_3 = -\int dx\, x\tilde{E}_{2T} - \int dx\, \tilde{H}$$

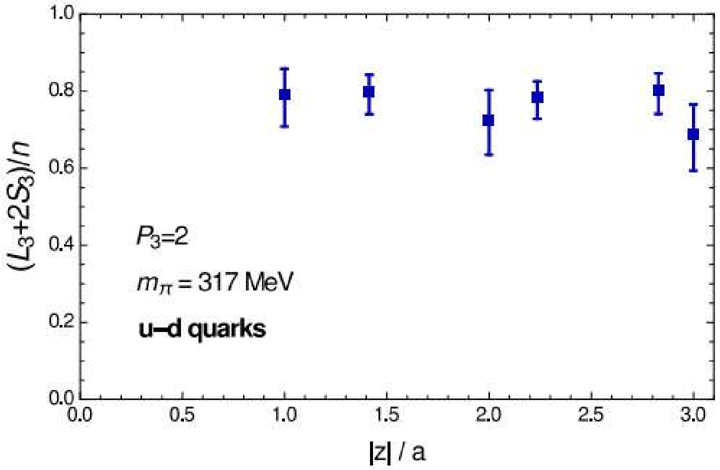
$$L_3 + 2S_3 = \epsilon_{ij}\, \frac{1}{2}\, \frac{\partial}{\partial(z\cdot P)}\, \frac{\partial}{\partial \Delta^i} \langle P + \Delta_T/2, +|\, \bar{\psi}(-z/2)\gamma^j\, \mathcal{U}\, \psi(z/2)\, |P - \Delta_T/2, +\rangle|_{z^+=z^-=0, \Delta_T=0, z_T\to 0}$$

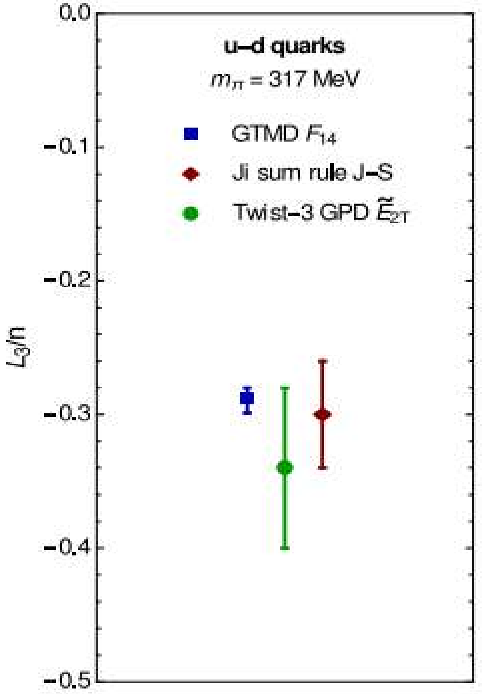
$$2\, P^j\, n = \langle P, +|\, \bar{\psi}(-z/2)\gamma^j\, \mathcal{U}\, \psi(z/2)\, |P, +\rangle\Big|_{z^+=z^-=0, z_T\to 0}$$

Original frame: $z^+ = 0, \quad z\cdot P = z^- P^+, \quad z^2 = -z_T^2$

Lattice frame: $z_0 = 0, \quad z\cdot P = -z_3 P_3, \quad z^2 = -z_3^2 - z_T^2$







Gluon helicity from Lattice QCD

- $\boldsymbol{\Delta G}$: High energy PP collision

  $\mathcal{L}_q, \mathcal{L}_g$: Generalized parton distributions (GPDs)
  Wigner distributions (GTMD)

Phys. Rev. Lett., 91:062001, 2003
Phys. Rev. D, 69:074014, 2004
JHEP, 08:056, 2009, JHEP, 05:041, 2011

- Matrix elements of appropriate equal-time local operator operators
  → boost to the infinite momentum frame
  ~ non-local gauge invariant from light-cone
  Lattice + LMET

Phys. Rev. Lett., 111:112002, 2013
Phys. Rev. D, 89(8):085030, 2014
Phys. Lett., B743:180–183, 2015
Phys. Rev., D93(5):054006, 2016

- $\boldsymbol{\Delta G \sim \vec{E} \times \vec{A}_{phys}}$

$$\boldsymbol{\mathcal{D}^i A^i_{phys} \equiv \partial^i A^i_{phys} - ig\left[A^i, A^i_{phys}\right] = 0}$$

$$\boldsymbol{A^i = A^i_{phys} + A^i_{ngi}}$$

$\boldsymbol{A^i_{phys}}$: similar to the transverse gauge-invariant part of the gauge potential $\boldsymbol{A_\perp}$ in QED

ΔG From Lattice QCD

- First moment of the gluon helicity distribution:

$$\Delta G = \int dx \frac{i}{2xP^+} \int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \times \langle PS|F_a^{+\alpha}(\xi^-)\mathcal{L}^{ab}(\xi^-,0)\tilde{F}^{+}_{\alpha,b}(0)|PS\rangle$$

A. Manohar, Phys. Lett. B 255, 579 (1991)

$\xi^\pm = (\xi^0 \pm \xi^3)/\sqrt{2}$ Light-front coordinates

$\mathcal{L}(\xi^-,0) = P\exp[-ig\int_0^{\xi^-} A^+(\eta^-,0_\perp)d\eta^-]$

Light-cone gauge link

- A gauge-invariant gluon helicity operator in a nonlocal form

$$\tilde{S}_g = \left[\vec{E}^a(0) \times \left(\vec{A}^a(0) - \frac{1}{\nabla^+}(\vec{\nabla}A^{+,b})\mathcal{L}^{ba}(\xi^-,0)\right)\right]^z$$

$\nabla^+ = \partial/\partial\xi^-$

Y. Hatta, Phys. Rev. D 84, 041701 (2011).
X. Ji, J.-H. Zhang, and Y. Zhao, Phys. Rev. Lett. 111, 112002 (2013)

$A^+ = 0$ LMET

$\vec{E} \times \vec{A}$

$$\vec{S}_g = 2\int d^3x \mathrm{Tr}(\vec{E}_c \times \vec{A}_c)$$

Coulomb gauge: $\vec{\partial} \cdot \vec{A} = 0$

S. Chen, X.-F. Lu, W.-M. Sun, F. Wang, and T. Goldman, Phys. Rev. Lett. 100, 232002 (2008).
X.-S. Chen, W.-M. Sun, X.-F. Lu, F. Wang, and T. Goldman, Phys. Rev. Lett. 103, 062001 (2009).
C. Lorce, Phys. Rev. D 87, 034031 (2013).
Y. Zhao, K.-F. Liu, and Y. B. Yang, Phys. Rev. D 93, 054006 (2016).

$\vec{S}_g$: not Lorentz invariant, frame dependent
Needs to calculate in rest and moving frames and needs to be matched after renormalization.
Also calculations have to be in Coulomb gauge

Lattice QCD calculation for ΔG

χ**QCD**: PRL 118, 102001 (2017)

- **Gauge fixed potential:**

$$A_{c,\mu} = \left(\frac{U^c_\mu(x) - U^{c\dagger}_\mu(x) + U^c_\mu(x-a\hat{\mu}) - U^{c\dagger}_\mu(x-a\hat{\mu})}{4iag}\right)_{\text{traceless}}$$

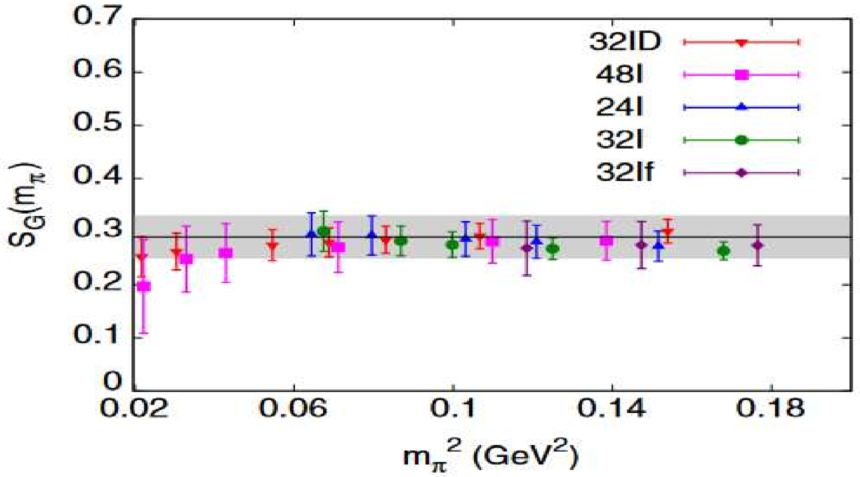
- **Coulomb gauge:**

$$\sum_{\mu=x,y,z} [U^c_\mu(x) - U^c_\mu(x-a\hat{\mu})] = 0$$

- **Chromoelectric field:**

$$F^c_{\mu\nu} = \frac{i}{8a^2 g}(\mathcal{P}_{\mu,\nu} - \mathcal{P}_{\nu,\mu} + \mathcal{P}_{\nu,-\mu} - \mathcal{P}_{-\mu,\nu} + \mathcal{P}_{-\mu,-\nu} - \mathcal{P}_{-\nu,-\mu} + \mathcal{P}_{-\nu,\mu} - \mathcal{P}_{\mu,-\nu})$$

$$\mathcal{P}_{\mu,\nu} = U^c_\mu(x) U^c_\nu(x+a\hat{\mu}) U^{c\dagger}_\mu(x+a\hat{\nu}) U^{c\dagger}_\nu(x)$$

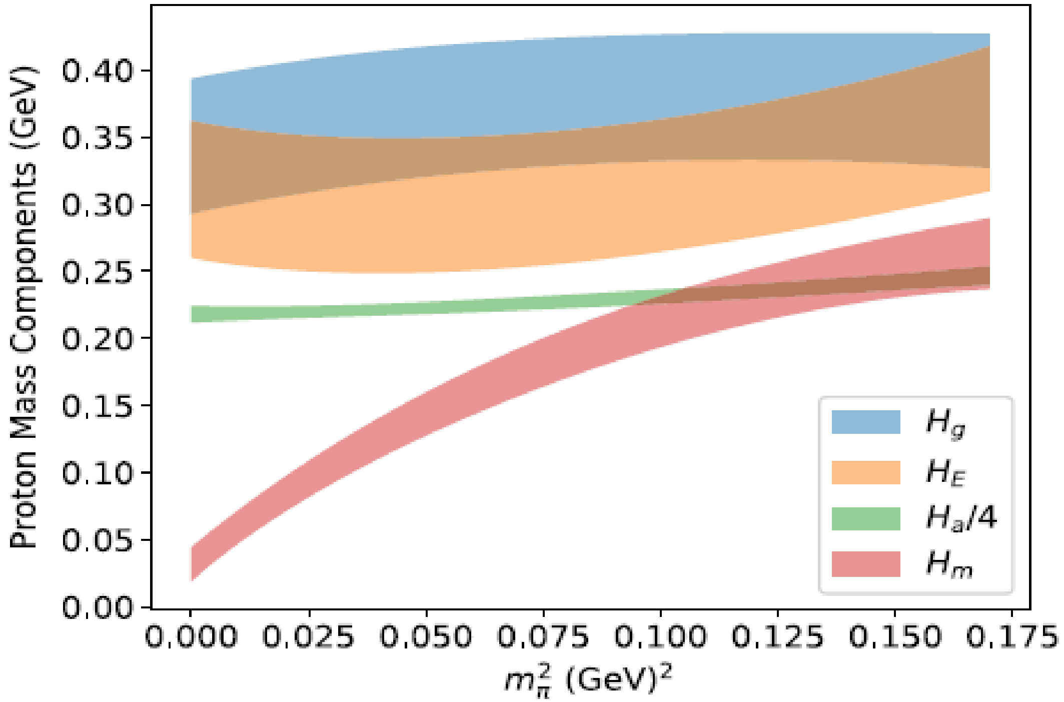


$$\Delta G\,(\mu^2 = 10\ \text{GeV}^2) \approx S_G(\infty, \mu^2 = 10\ \text{GeV}^2)$$
$$= 0.251(47)(16)$$
$$\rightarrow 50(9)(3)\%\ \text{of proton spin}$$

Caveat: convergence problem for perturbative series in LMET
systematics is not under control and more work is necessary.

Proton mass decomposition: Origin of mass

X. Ji : PRL 74, 1071 (1995).



$$M = \langle T_{00} \rangle = \langle H_m \rangle + \langle H_E(\mu) \rangle + \langle H_g(\mu) \rangle + \langle H_a \rangle / 4$$

$$H_m = \sum_{u,d,s\cdots} \int d^3x\, m\bar{\psi}\psi,$$

$$H_E = \sum_{u,d,s\ldots} \int d^3x\, \bar{\psi}(\vec{D}\cdot\vec{\gamma})\psi,$$

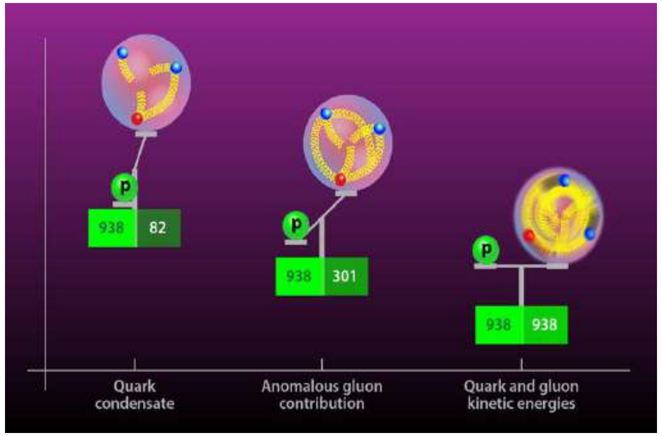
$$H_g = \int d^3x \frac{1}{2}(B^2 - E^2).$$

$$H_a = H_g^a + H_m^\gamma,$$

$$H_g^a = \int d^3x \frac{-\beta(g)}{g}(E^2 + B^2),$$

$$H_m^\gamma = \sum_{u,d,s\cdots} \int d^3x\, \gamma_m m\bar{\psi}\psi.$$

Physics Focus



$\langle H_m \rangle$: **Quark condensate 9(2)(1)%**

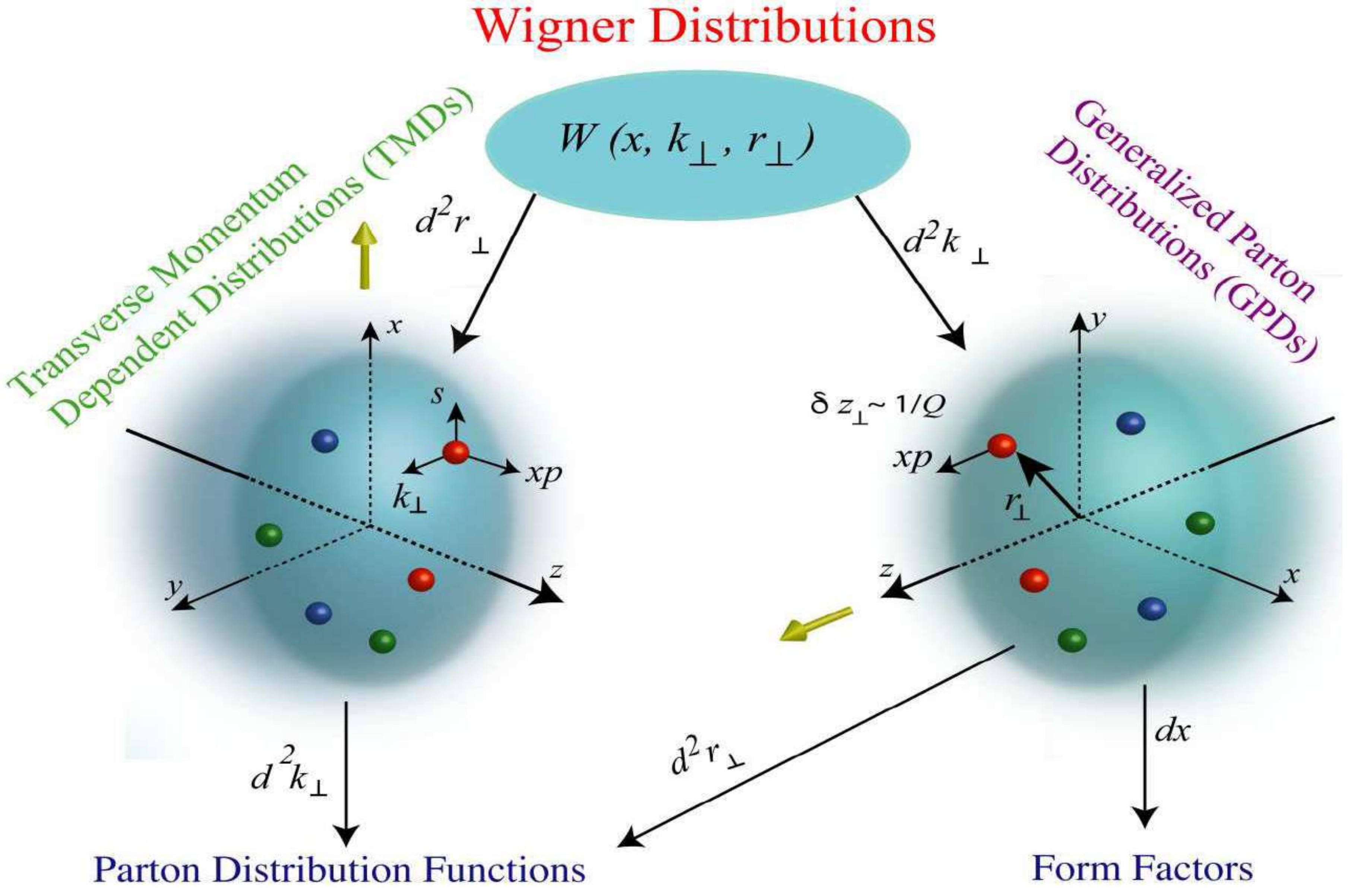
$\langle H_g \rangle$: **Glue field energy 36(5)(4)%**

$\langle H_E \rangle$: **Quark energy 32(4)(4)%**

$\langle H_{a/4} \rangle$: **Trace anomaly 23(1)(1)%**

$\overline{\text{MS}}$ = 2 GeV

χQCD: PRL 121, 212001 (2018)


Wigner Distributions
W (x, k⊥, r⊥)
Transverse Momentum Dependent Distributions (TMDs)
Generalized Parton Distributions (GPDs)
d²r⊥
d²k⊥
s
xp
k⊥
x
y
z
δ z⊥~ 1/Q
r⊥
d²k⊥
d²r⊥
dx
Parton Distribution Functions
Form Factors

Is it possible to compute these observables from Lattice QCD?

- Lattice calculation are performed in Euclidean space while these observables are in Minkowski space

- Way out: similar as experimental access to these distributions

- Solution: Use factorization

$$Q(x,\mu_R) = \int_{-1}^{1} \frac{dy}{y} C\left(\frac{x}{y},\mu_F,\mu_R\right) q(y,\mu_F)$$

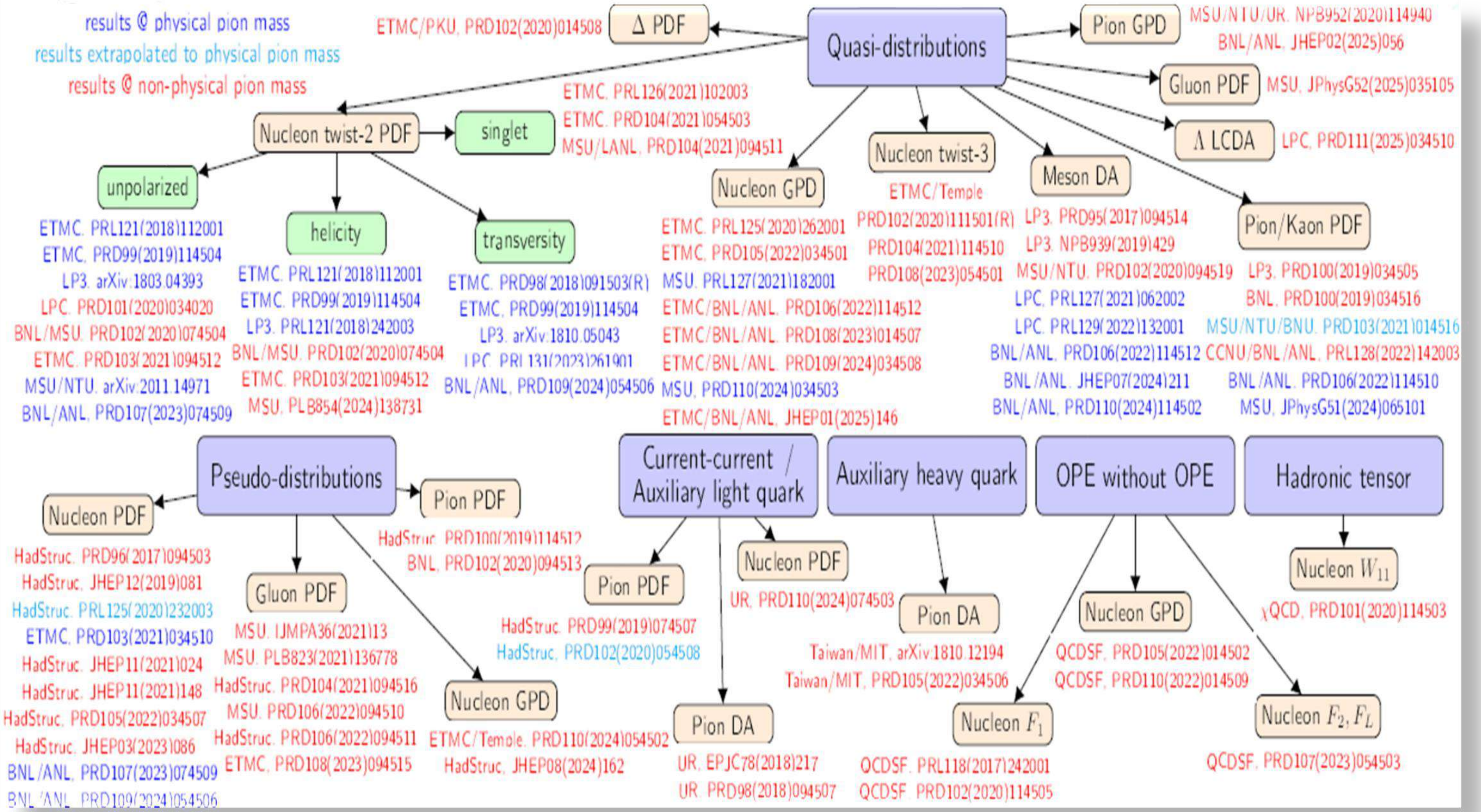
Lattice-observable = perturbative-part $*$ partonic-distribution

Two most popular approaches:

- Quasi-distributions – Xiangdong Ji, 2013
- Pseudo-distributions – Anatoly Radyushkin, 2018

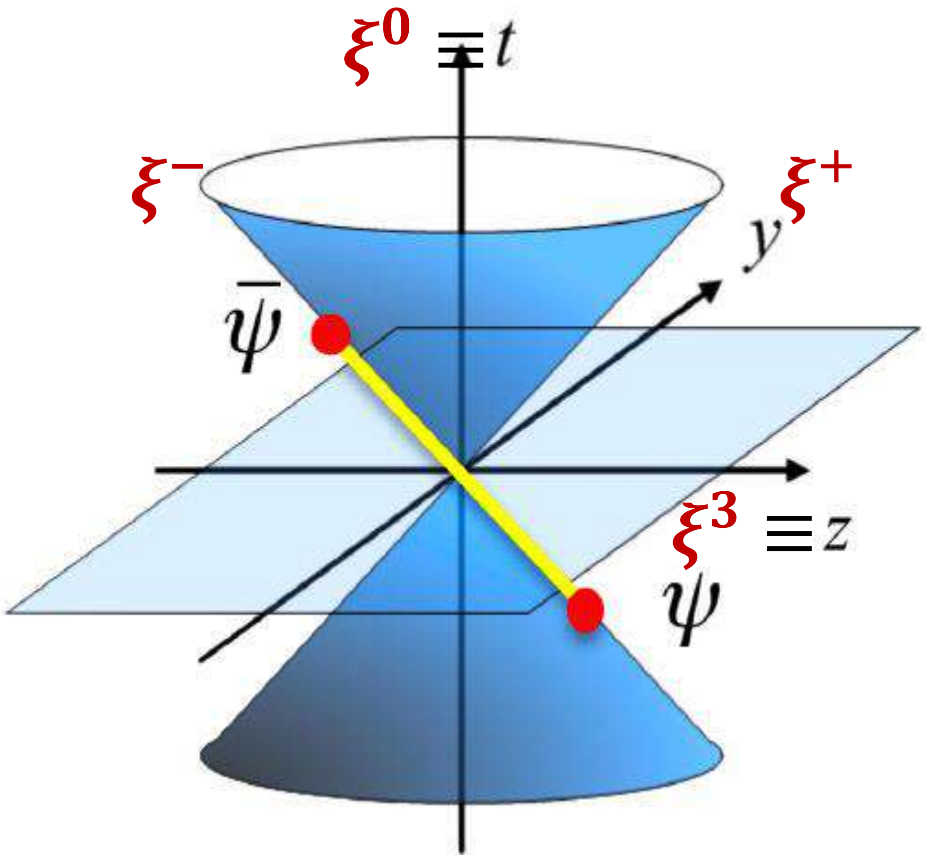
- hadronic tensor – K.-F. Liu, S.-J. Dong, 1993
- auxiliary scalar quark – U. Aglietti et al., 1998
- auxiliary heavy quark (HOPE) – W. Detmold, C.-J. D. Lin, 2005
- auxiliary light quark – V. Braun, D. Müller, 2007
- “good lattice cross sections” – Y.-Q. Ma, J.-W. Qiu, 2014,2017
- “OPE without OPE” – QCDSF, 2017

A compilation of LQCD results on PDF and GPD



C. Alexandrou:Lattice 2025, K. Cichy: EINN 2025

Quasi-PDFs

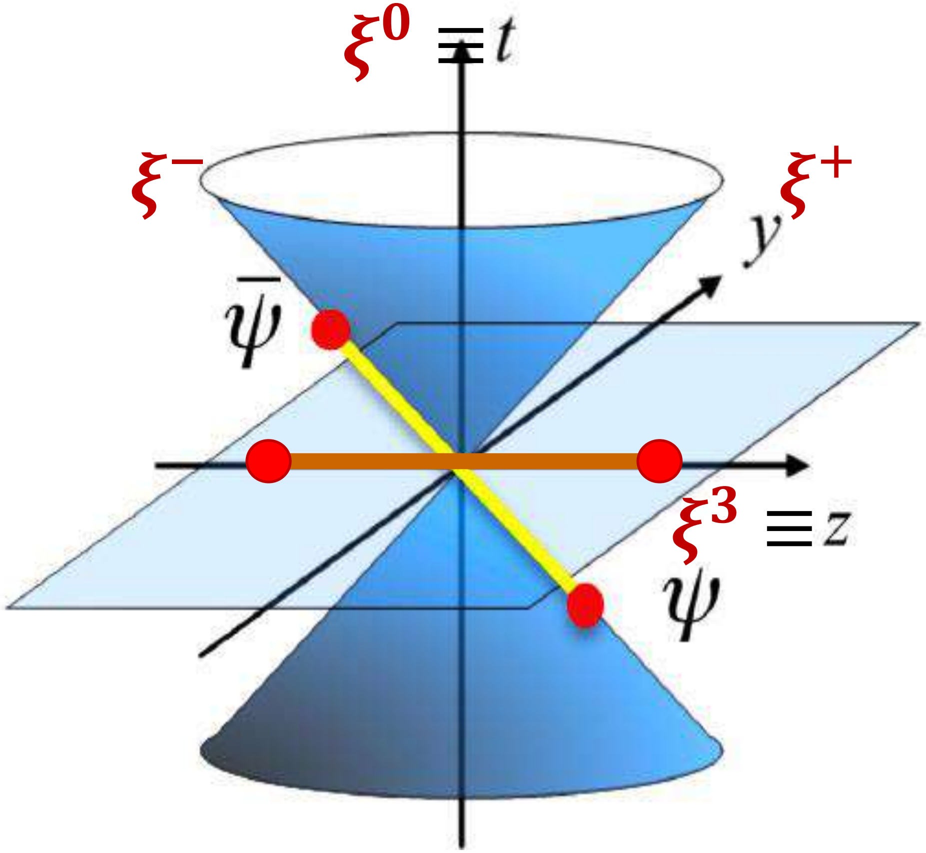


Light-cone correlation along the ξ^-direction

$$q(x) = \frac{1}{2\pi}\int d\xi^- e^{-ixp^+\xi^-}\langle N|\bar{\psi}(\xi^-)\Gamma \boldsymbol{A}(\boldsymbol{\xi}^-, \mathbf{0})\psi(0)|N\rangle$$

$|N>$ nucleon at rest in the light-cone frame

Quasi-PDFs



Light-cone correlation along the ξ^-direction

$$q(x) = \frac{1}{2\pi}\int d\xi^- e^{-ixp^+\xi^-}\langle N|\bar{\psi}(\xi^-)\Gamma \boldsymbol{A}(\boldsymbol{\xi^-},\boldsymbol{0})\psi(0)|N\rangle$$

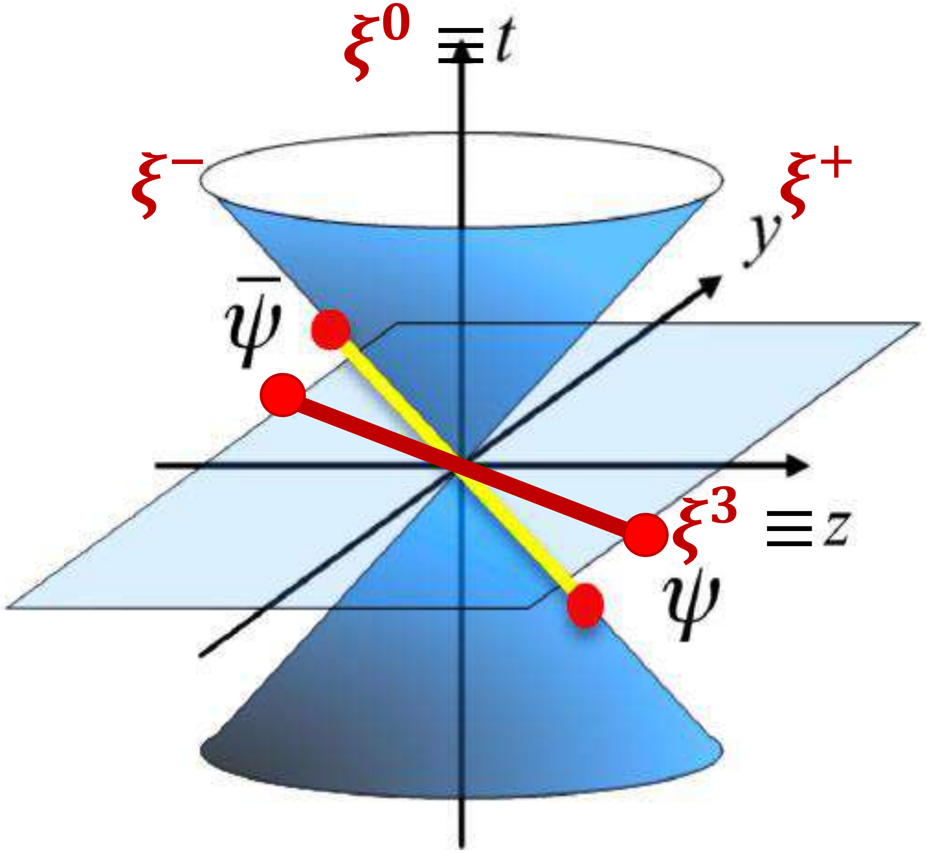
$|N>$ nucleon at rest in the light-cone frame

Spatial correlation along the $\xi^3 \equiv z$ −direction

$$\tilde{q}(x) = \frac{1}{2\pi}\int dz e^{-ixp_3 z}\langle N|\bar{\psi}(z)\Gamma \boldsymbol{A}(\boldsymbol{z},\boldsymbol{0})\psi(0)|N\rangle$$

$|N>$ nucleon at rest in the standard frame

Quasi-PDFs



Light-cone correlation along the ξ^- direction

$$q(x) = \frac{1}{2\pi}\int d\xi^- e^{-ixp^+\xi^-}\langle N|\bar{\psi}(\xi^-)\Gamma \boldsymbol{A}(\boldsymbol{\xi^-},\boldsymbol{0})\psi(0)|N\rangle$$

$|N>$ nucleon at rest in the light-cone frame

Spatial correlation along the $\xi^3 \equiv z$ −direction

$$\tilde{q}(x) = \frac{1}{2\pi}\int dz e^{-ixp_3 z}\langle N|\bar{\psi}(z)\Gamma \boldsymbol{A}(\boldsymbol{z},\boldsymbol{0})\psi(0)|N\rangle$$

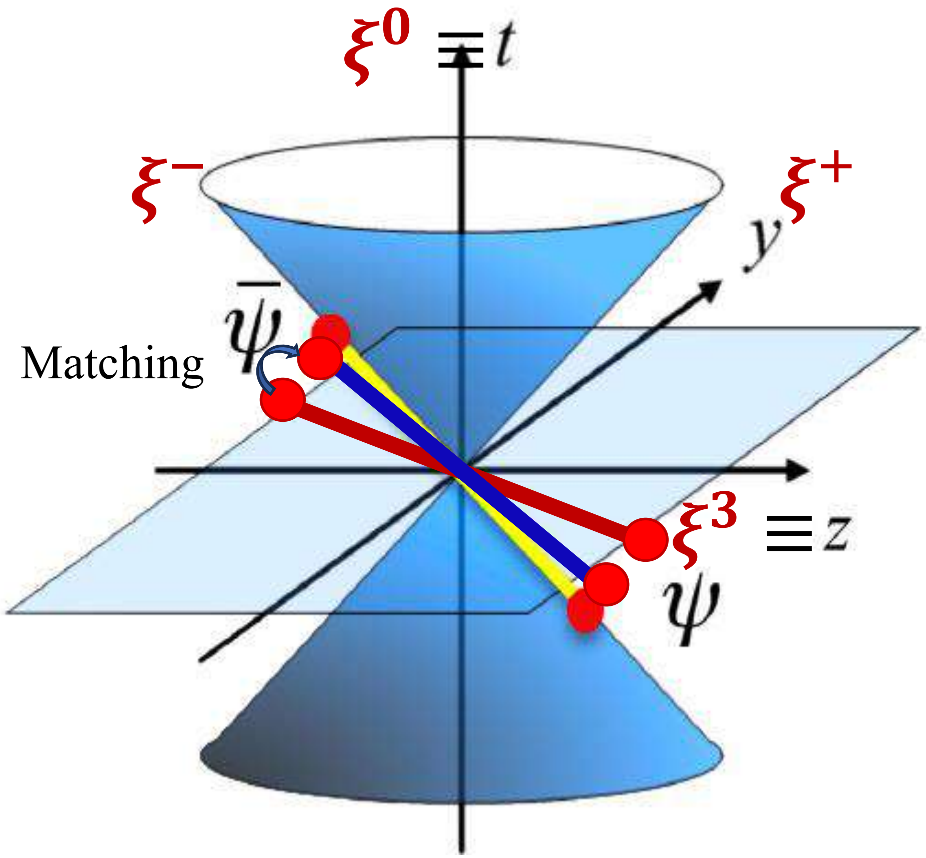
$|N>$ nucleon at rest in the standard frame

Boosted Ccorrelation along the $\xi^3 \equiv z$ −direction

$$\tilde{q}(x) = \frac{1}{2\pi}\int dz e^{-ixP_3 z}\langle P|\bar{\psi}(z)\Gamma \boldsymbol{A}(\boldsymbol{z},\boldsymbol{0})\psi(0)|P\rangle$$

$|P>$ boosted nucleon

Quasi-PDFs



Light-cone correlation along the ξ^- direction

$$q(x) = \frac{1}{2\pi}\int d\xi^- e^{-ixp^+\xi^-}\langle N|\bar{\psi}(\xi^-)\Gamma \boldsymbol{A}(\boldsymbol{\xi^-},\boldsymbol{0})\psi(0)|N\rangle$$

$|N>$ nucleon at rest in the light-cone frame

Spatial correlation along the $\xi^3 \equiv z$ −direction

$$\tilde{q}(x) = \frac{1}{2\pi}\int dz e^{-ixp_3 z}\langle N|\bar{\psi}(z)\Gamma \boldsymbol{A}(\boldsymbol{z},\boldsymbol{0})\psi(0)|N\rangle$$

$|N>$ nucleon at rest in the standard frame

Boosted correlation along the $\xi^3 \equiv z$ −direction

$$\tilde{q}(x) = \frac{1}{2\pi}\int dz e^{-ixP_3 z}\langle P|\bar{\psi}(z)\Gamma \boldsymbol{A}(\boldsymbol{z},\boldsymbol{0})\psi(0)|P\rangle$$

$|P>$ boosted nucleon

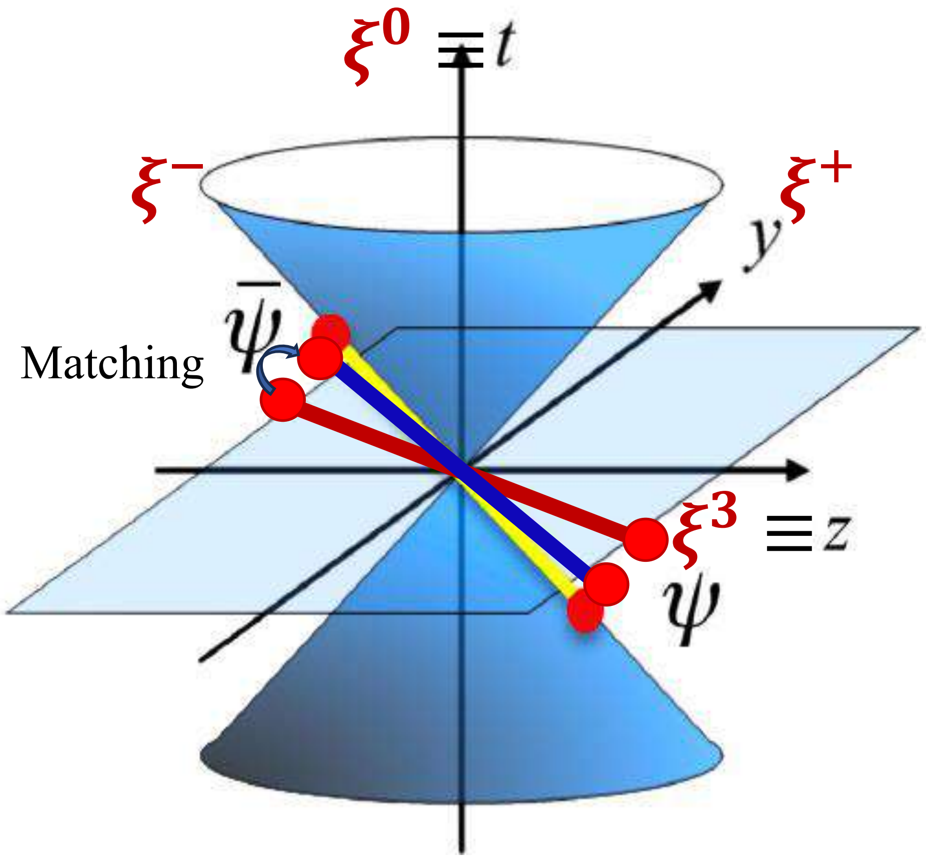
Matching [Large Momentum Effective Theory (LaMET)]

X. Ji, Parton Physics from Large-Momentum Effective Field Theory, Sci.China Phys.Mech.Astron. 57 (2014) 1407

Quasi-distribution to the light-cone distribution, up to power-suppressed effects:

$$\tilde{q}(x,\mu,P_3) = \int_{-1}^{1}\frac{dy}{|y|}\; C\left(\frac{x}{y},\frac{\mu}{P_3}\right)\; q(\mu,y)\;\; \mathcal{O}(\Lambda^2_{QCD}/P_3^2\,, M_n^2/P_3^2)$$

Quasi-PDFs



Light-cone correlation along the ξ^- direction

$$F_\Gamma(x) = \frac{1}{2\pi}\int d\xi^- e^{-ixp^+\xi^-}\langle N|\bar{\psi}(\xi^-)\Gamma \boldsymbol{A}(\boldsymbol{\xi}^-,\boldsymbol{0})\psi(0)|N\rangle$$

$|N>$ nucleon at rest in the light-cone frame

Spatial correlation along the $\xi^3 \equiv z$ –direction

$$\widetilde{F_\Gamma}(x) = \frac{1}{2\pi}\int dz e^{-ixp_3 z}\langle N|\bar{\psi}(z)\Gamma \boldsymbol{A}(\boldsymbol{z},\boldsymbol{0})\psi(0)|N\rangle$$

$|N>$ nucleon at rest in the standard frame

Boosted correlation along the $\xi^3 \equiv z$ –direction

$$\widetilde{F_\Gamma}(x) = \frac{1}{2\pi}\int dz e^{-ixP_3 z}\langle P|\bar{\psi}(z)\Gamma \boldsymbol{A}(\boldsymbol{z},\boldsymbol{0})\psi(0)|P\rangle$$

$|P>$ boosted nucleon

Matching [Large Momentum Effective Theory (LaMET)] Need to eliminate both UV and exponential divergences

X. Ji, Parton Physics from Large-Momentum Effective Field Theory, Sci.China Phys.Mech.Astron. 57 (2014) 1407

Quasi-distribution to the light-cone distribution, up to power-suppressed effects:

$$\widetilde{F_\Gamma}(x,\mu,P_3) = \int_{-1}^{1}\frac{dy}{|y|}\; C\left(\frac{x}{y},\frac{\mu}{yP_3}\right)\; F_\Gamma(\mu,y)\;\; \mathcal{O}(\Lambda_{QCD}^2/x^2P_3^2\,,M_n^2/(1-x^2)P_3^2)$$

Quasi-pdf — Perturb kernel — pdf — Higher twist effects

Dirac structures of Γ

- Vector
  - $\gamma_0, \gamma_3: H, E$ (twist-2 unpolarized)
  - $\gamma_1, \gamma_2: G_{i=1,4}$ (twist-2 vector)
- Axial-Vector
  - $\gamma_5\gamma_0, \gamma_5\gamma_3: H, E$ (helicity twist-2)
  - $\gamma_5\gamma_1, \gamma_5\gamma_2: \widetilde{G}_{i=1,4}$ (axial vector, twist-3)
- Tensor
  - $\gamma_1\gamma_3, \gamma_2\gamma_3: H_T, E_T, \widetilde{H}_T, \widetilde{E}_T$ (twist-2 transversity)
  - $\gamma_1\gamma_2: H_2', E_2'$ (tensor twist-3)

Projection operators

- Unpolarized:

$$\mathcal{P} = \frac{1+\gamma_0}{2}$$

- Polarized:

$$\mathcal{P}_k = \frac{1+\gamma_0}{2} i\gamma_5\gamma_k$$

Pseudo-PDFs

Fourier transform of Ioffe-time pseudo-distributions

A. Radyushkin, Phys. Rev. D96 (2017) 034025
A. Radyushkin, *Theory and applications of parton pseudo-distributions*, Int. J. Mod. Phys. A35 (2020) 2030002

- The same matrix elements that define quasi-distributions can also be utilized to construct pseudo-distributions in terms of $\boldsymbol{\nu = z.P}$ Ioffe time.

- Compute space-like matrix element with different initial and final nucleon boosts e.g. in the Breit frame
- Renormalize matrix element, $\widetilde{\mathcal{M}}_\Gamma(\nu, z^2) = \frac{M_\Gamma(\nu,z^2)}{M_\Gamma(\nu,z^2)}$ $\quad M_\Gamma(z, P_3) = \langle P|\bar{\psi}(z)\Gamma \boldsymbol{A}(\boldsymbol{z},\boldsymbol{0})\psi(0)|P\rangle$

- Match in coordinate space via short distance facotrization (SDF)

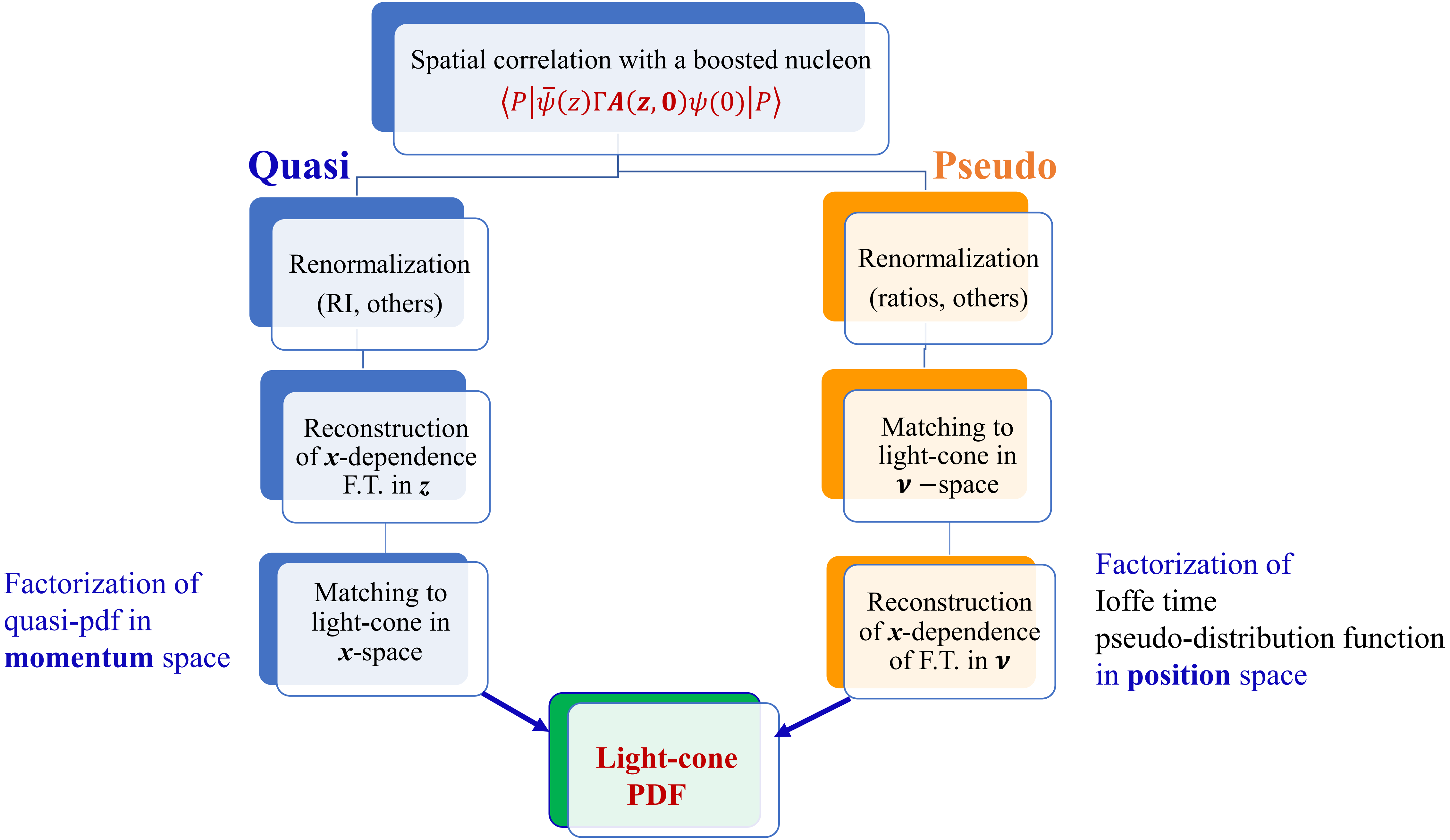
$$\widetilde{\mathcal{M}}_\Gamma(\nu, z^2) = \int_{-1}^{1} dy\ C_\nu(y)\mathcal{M}_\Gamma(y\nu,\mu) + \mathcal{O}(z^2\Lambda_{QCD}^2)$$

Ioffe time distribution

$$\mathcal{M}_\Gamma(\nu,\mu) = \int_{-1}^{1} d\nu\ e^{ix\nu}F_\Gamma(x,\mu)$$

Quasi and pseudo are complementary X. Ji, Research 8 (2025) 0695

Quasi and pseudo-PDFs



Quasi and pseudo-PDFs

quasi – **fully-reliable** in a **limited** range of $x \in [x_{\rm min}, x_{\rm max}] \approx [0.2, 0.8]$

reason: power corrections of $\mathcal{O}(\Lambda^2_{\rm QCD}/x^2 P_3^2)$, $\mathcal{O}(\Lambda^2_{\rm QCD}/(1-x)^2 P_3^2)$.

pseudo – x-dependence is **model-dependent** (assumed fitting ansatz)

reason: power corrections of $\mathcal{O}(z^2 \Lambda^2_{\rm QCD}) \Rightarrow$ limited range of ν-space data

model-independent – **fully-reliable** GPDs moments ($\nu_{\rm max} \Rightarrow \langle x^{n_{\rm max}} \rangle$)

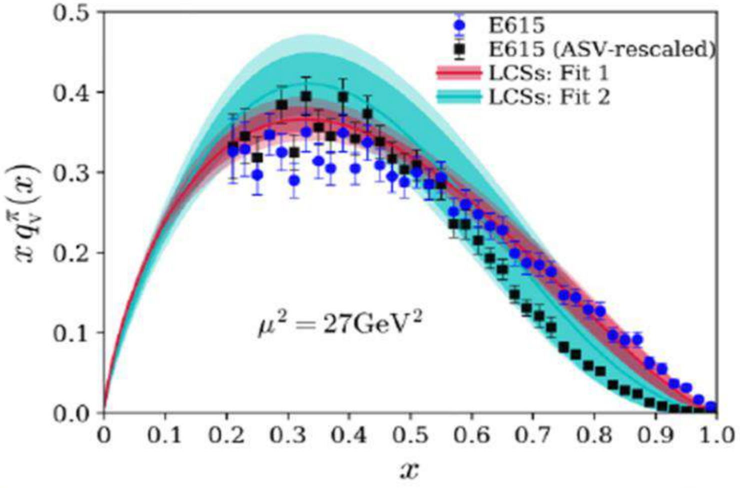
COMPLEMENTARITY – e.g. extract $x \in [0.2, 0.8]$ from quasi
+ add constraints at short distances from pseudo

Quasi and pseudo are complementary

X. Ji, Research 8 (2025) 0695

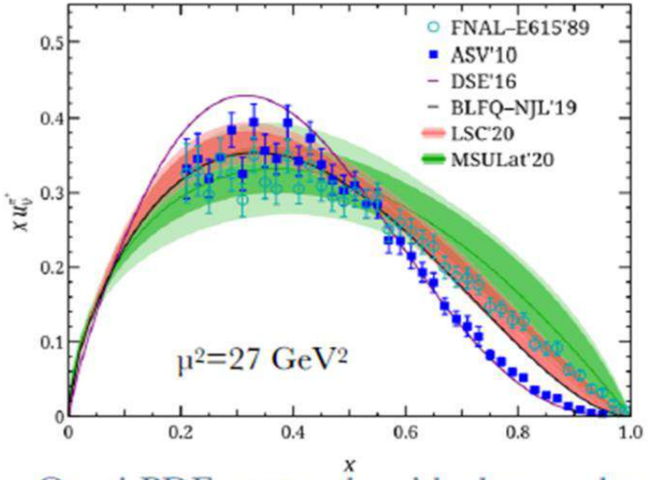
✷ Pion

R. S. Sufian *et al.*, Phys. Rev. D 102 (2020) 054508



"Good lattice cross-sections" approach with N_f=2+1 clover, m_π~410-280 MeV and a=0.127 and 0.094

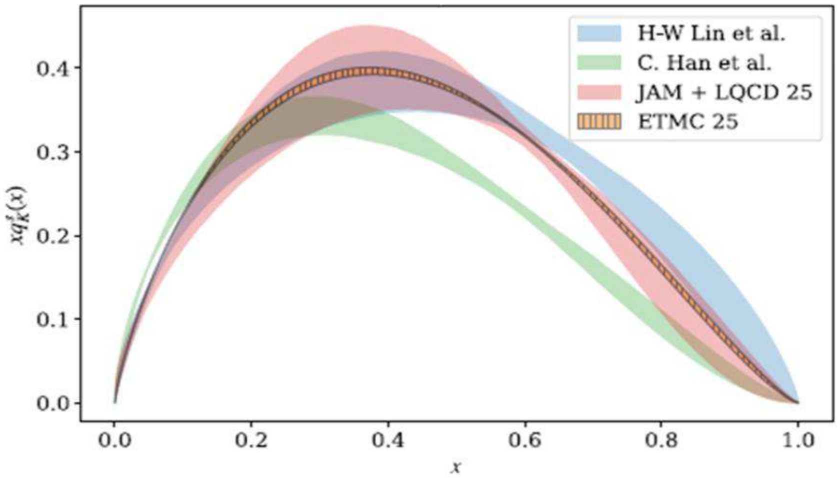
H.-W. Lin *et al.*, Phys.Rev.D 103 (2021) 014516

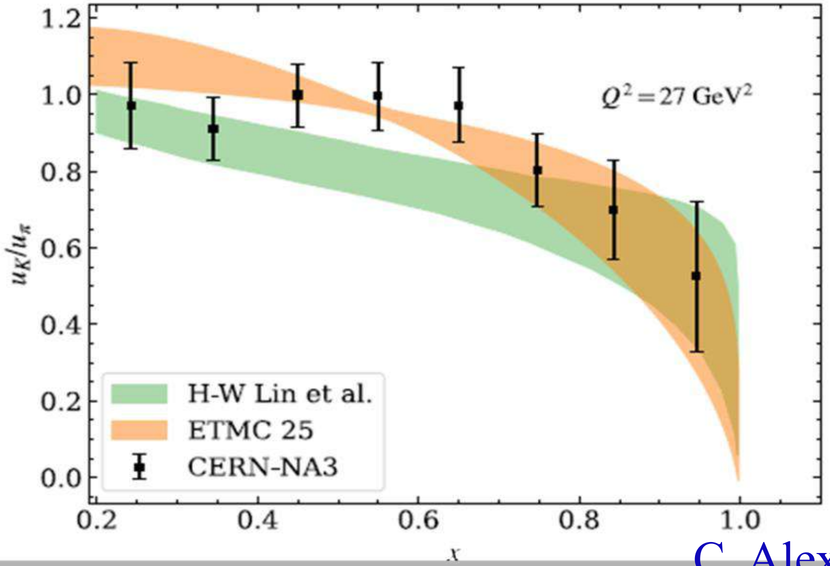


Quasi-PDF approach with clover valence on staggered sea with m_π=220, 310, 690 MeV and a=0.12, 0.06 fm with extrapolation to the continuum and physical m_π via $c_0 + c_1 m_\pi^2 + c_3 a^2$

✷ Kaon

H.-W. Lin *et al.*, Phys.Rev.D 103 (2021) 014516



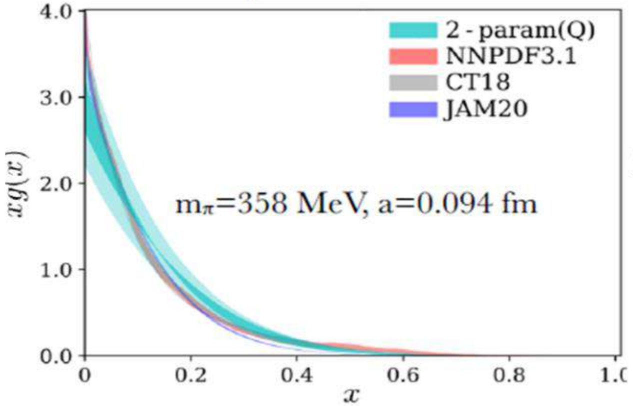


C. Alexandrou:Lattice 2025

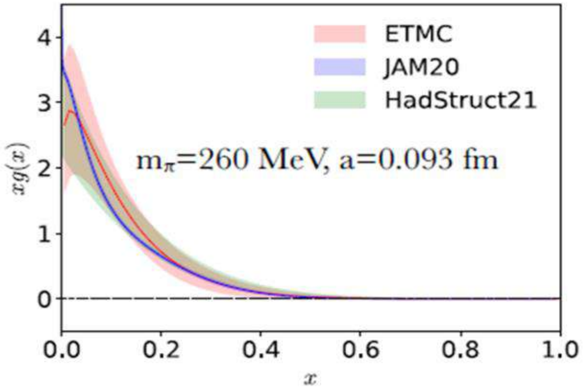
Gluon PDFs

❋ Most studies in the pseudo-pdf approach

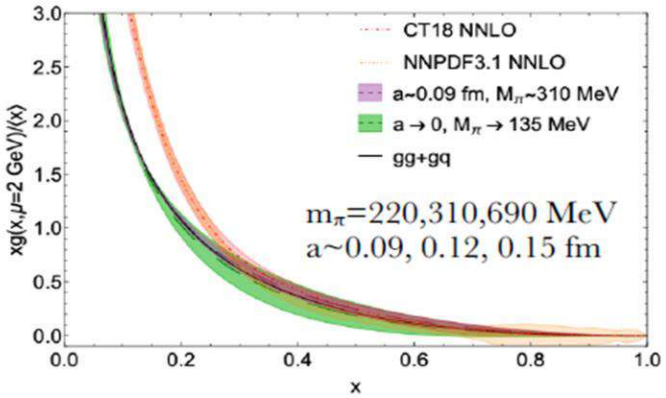
- Nucleon unpolarized PDF



HadStruc Collaboration: T. Khan *et al.*, Phys. Rev. D 104, (2021) 094516

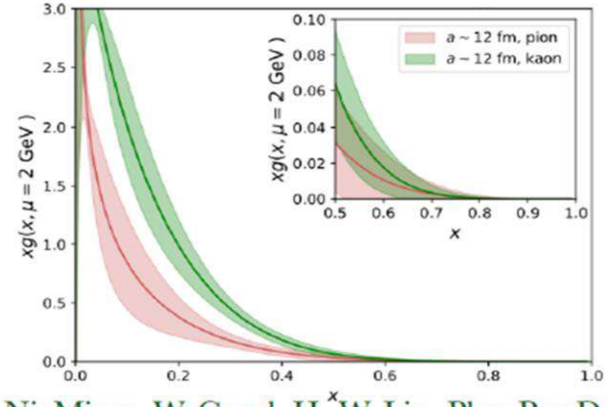


ETMC: J. Delmar *et al.*, Phys. Rev. D 108 (2023) 094515



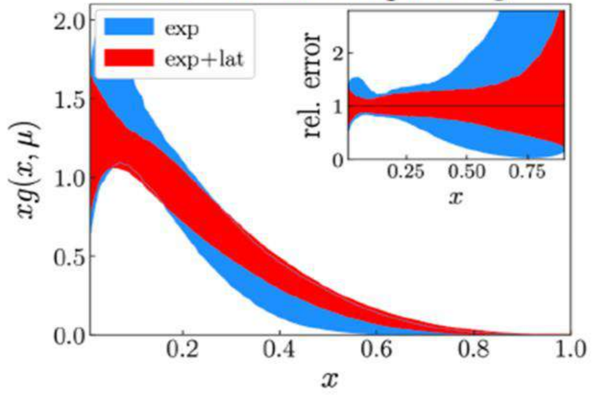
MSULAT: Z. Fan, W. Good, H.-W. Lin, Phys. Rev. D 108 (2023) 014508

- Pion and kaon gluon PDFs by MSULAT



A. NieMiera, W. Good, H.-W. Lin, Phys.Rev.D 112 (2025) 074504

MSULAT data on pion + JAM



W. Good *et al.* arXiv: 2507.22730

- Nucleon gluon helicity

HadStruc Collaboration: C. Egerer *et al.*, Phys.Rev.D 106 (2022) 094511
T. Khan, T. Liu, R. S. Sufian, Phys. Rev. D 108, (2023) 094502

Proton Helicity and Flavor-Dependent Unpolarized TMDPDF

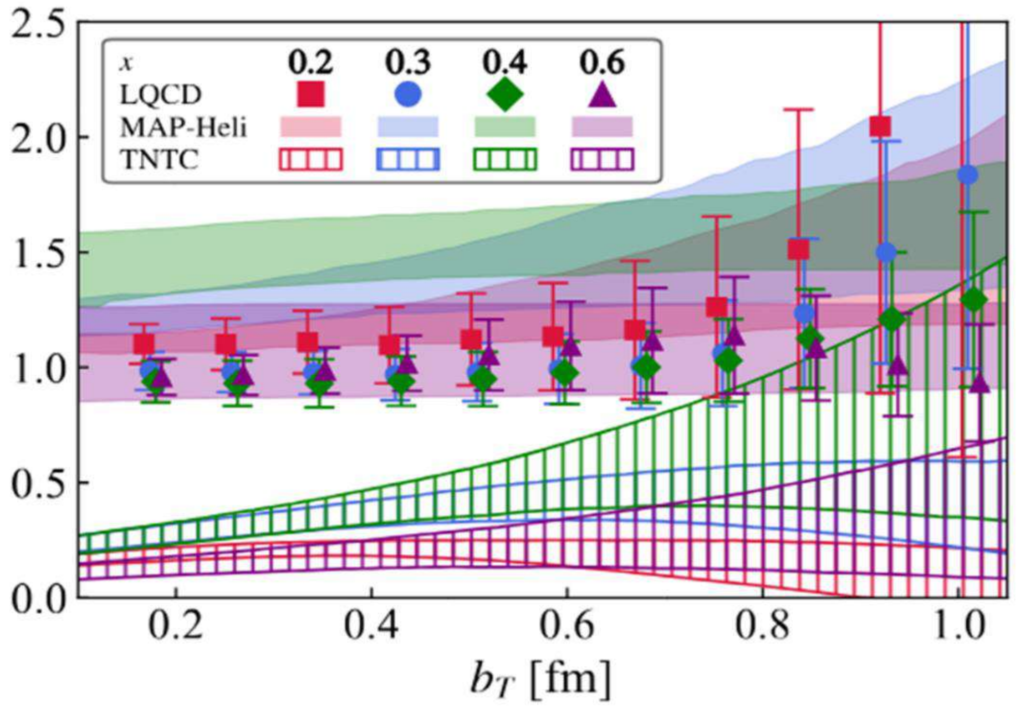


FIG. 3. Lattice QCD results (filled symbols with error bars) for the ratio $R^{u-d}_{g_{1L}/f_1}(x, b_T)$ of isovector helicity to isovector unpolarized TMDPDFs in Eq. (3), shown as a function of b_T at several values of x. The hatched bands indicate the corresponding global fits from MAP-Heli [33] and TNTC [32].

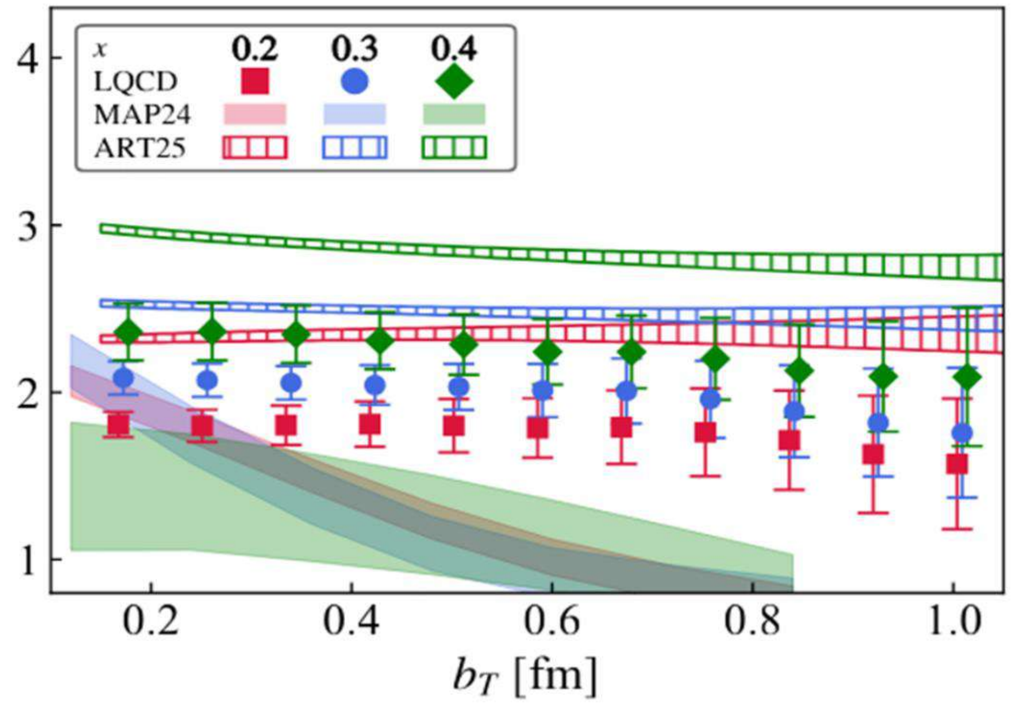


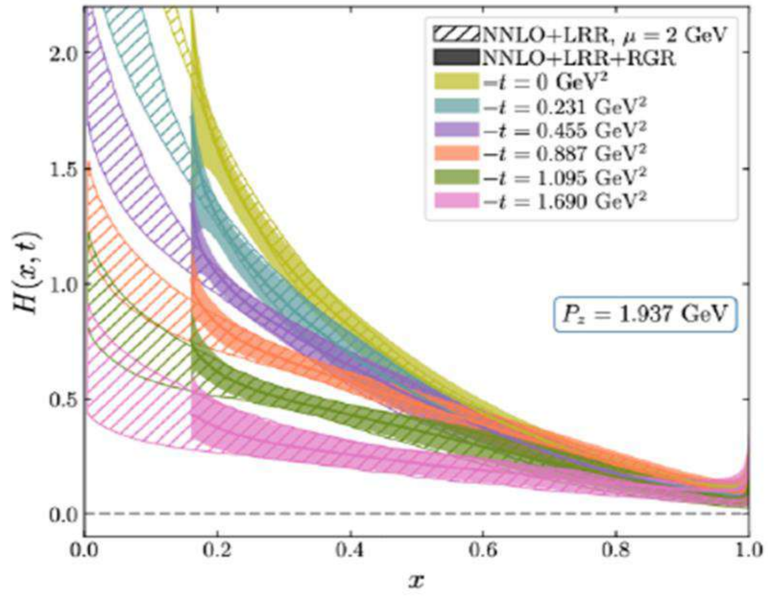
FIG. 4. Lattice QCD results (filled symbols with error bars) for the ratio $R^{u/d}_{f_1}(x, b_T)$ of up- to down-quark unpolarized TMDPDFs in Eq. (4), shown as a function of b_T at several values of x. The hatched bands denote the corresponding global fits from MAP24 [31] and ART25 [29].

D. Bollweg, X. Gao, S. Mukherjee, Y. Zhao, *Phy. Rev. Lett.* 135, 20 (2025)

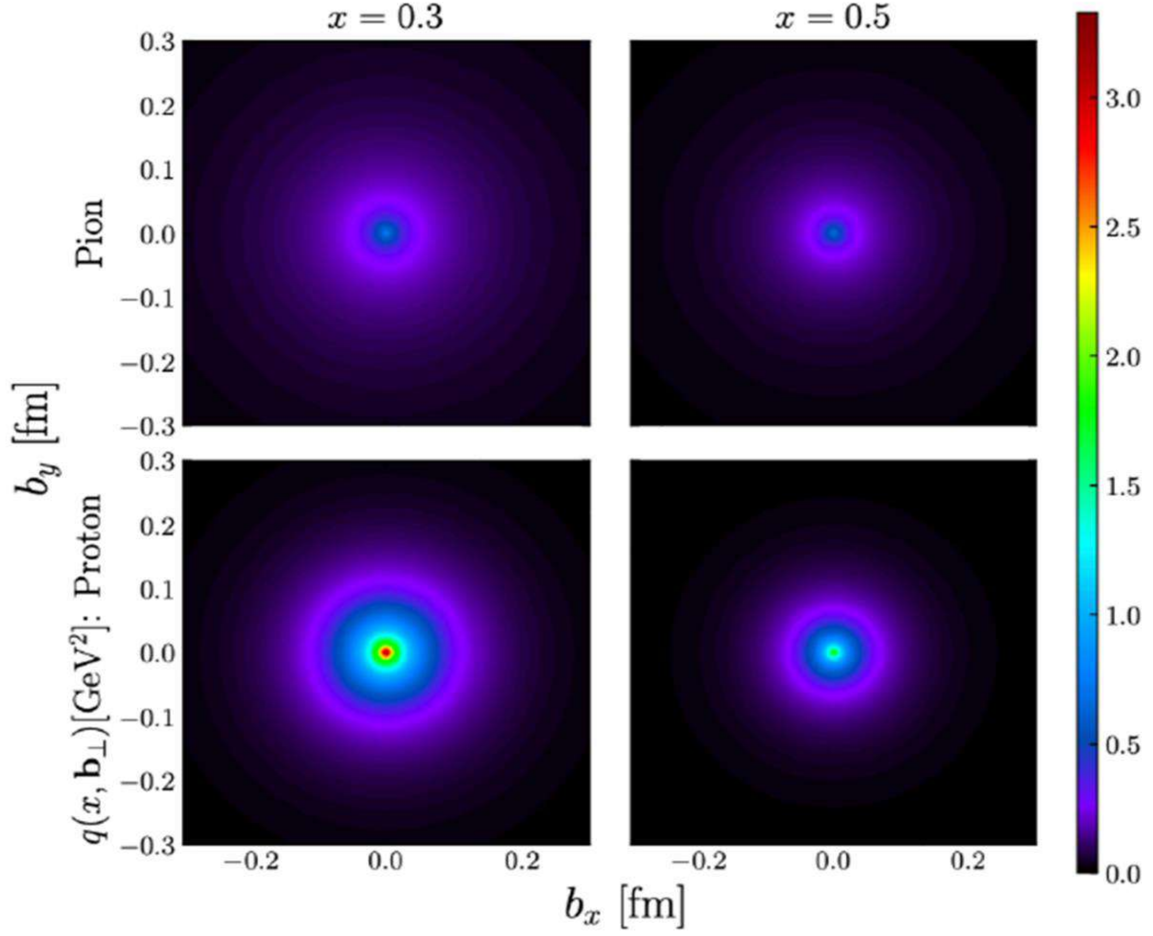
3-D imaging of pion and nucleon

Pion unpolarized GPD for ξ=0

Hybrid action, m_π=300 MeV, a=0.04 fm



$$q(x, b_\perp) = \int_{-\infty}^{\infty} \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\Delta_\perp . b_\perp} H(x, \xi = 0, \tau)$$



H.-T. Ding *et al.*, JHEP 02 (2025) 056, arXiv:2407.03516

Nucleon data are from ETMC: S. Bhattacharya et al. Phys. Rev. D 106 (2022) 11, 114512

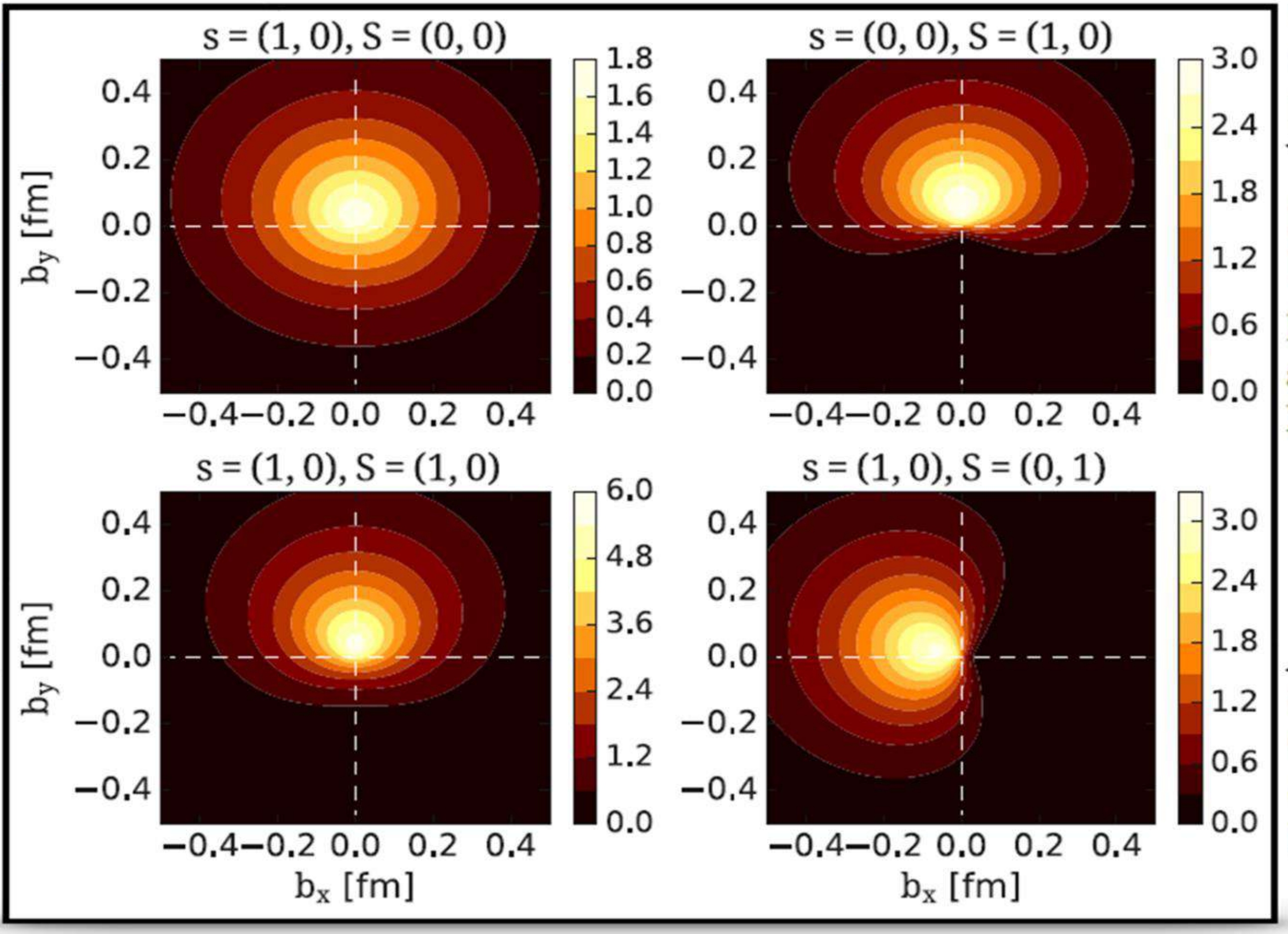
Transverse density distribution for n=1 (isovector)

- Contours of the probability density for the **first moment** as a function of b_x and b_y

$$\langle x^{n-1}\rangle_\rho(\mathbf{b}_\perp,\mathbf{s}_\perp,\mathbf{S}_\perp) \equiv \int_{-1}^{1} dx\; x^{n-1}\rho(x,\mathbf{b}_\perp,\mathbf{s}_\perp,\mathbf{S}_\perp),$$

$$\rho(x,\mathbf{b}_\perp,\mathbf{s}_\perp,\mathbf{S}_\perp) = \frac{1}{2}\left[H(x,b_\perp^2) + \frac{\mathbf{b}_\perp^j\epsilon^{ji}}{m_N}\left(\mathbf{S}_\perp^i E'(x,b_\perp^2) + \mathbf{s}_\perp^i \bar{E}'_T(x,b_\perp^2)\right) + \mathbf{s}_\perp^i\mathbf{S}_\perp^i\left(H_T(x,b_\perp^2) - \frac{\Delta_{b_\perp}\tilde{H}_T(x,b_\perp^2)}{4m_N^2}\right) + \mathbf{s}_\perp^i(2\mathbf{b}_\perp^i\mathbf{b}_\perp^j - \delta^{ij}b_\perp^2)\mathbf{S}_\perp^j\frac{\tilde{H}''_T(x,b_\perp^2)}{m_N^2}\right]$$

transversely polarized quarks in an unpolarized nucleon



unpolarized quarks in a transversely polarized nucleon

Large effect due to B_{10}

Related to Sivers asymmetry, M. Burkardt (2004)

transversely polarized quarks in a transversely polarized nucleon

transversely polarized quarks in a perpendicularly polarized nucleon

C.A. *et al.* (ETMC) Phys.Rev.D 107 (2023) 5, 054504

Looking forward– Reliability of Lattice Calculations

❖ **General Lattice-specific systematics:**

- Reliable extraction of the energy levels of hadrons
- Chiral extrapolation
- Discretization
- Finite volume effects

❖ **Specific systematics for PDFs, GPDs:**

- Large boost (controlling signal-to-noise)
- Reconstruction of the x-dependence
- Non-perturbative renormalization
- Truncation effects: conversion, evolution, matching
- Higher-twist effects

Conclusions

- **EIC is coming!**

- **Along experiment, lattice calculations and phenomenology are needed.**

- **The first phase of progress on direct LQCD computations of PDFs, GPDs, and TMDs are currently underway. Results are very encouraging.**

- **A large effort is underway, and more and more precise values for these observables are expected to come in this exascale era of computing, as we move towards the onset of EIC runs.**

- **In future these calculations will also be performed for low lying nuclei.**