

# The Use of Gradient Flow in Large Momentum Effective Theory

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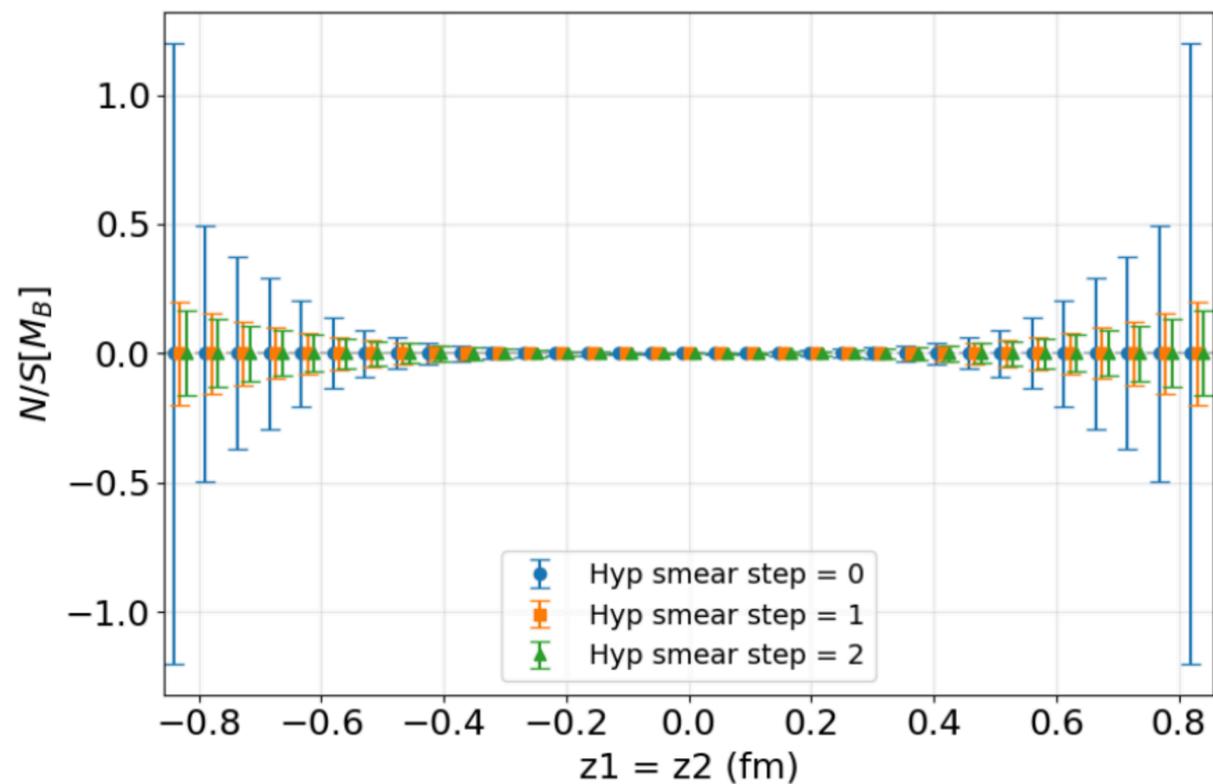
October 2025

*ArXiv:2507.18233; ...*

- **Introduction:** Key concepts and background
- **Advantages of Gradient Flow:** Why it's valuable for LaMET
- **Applications:** Baryon Quasi-DA and matching operators
- **Renormalon Effects:** Understanding UV renormalons and their resolution
- **Lattice Implementation:** A guide on how to implement on Lattice

## Poor Signal-to-Noise Ratio

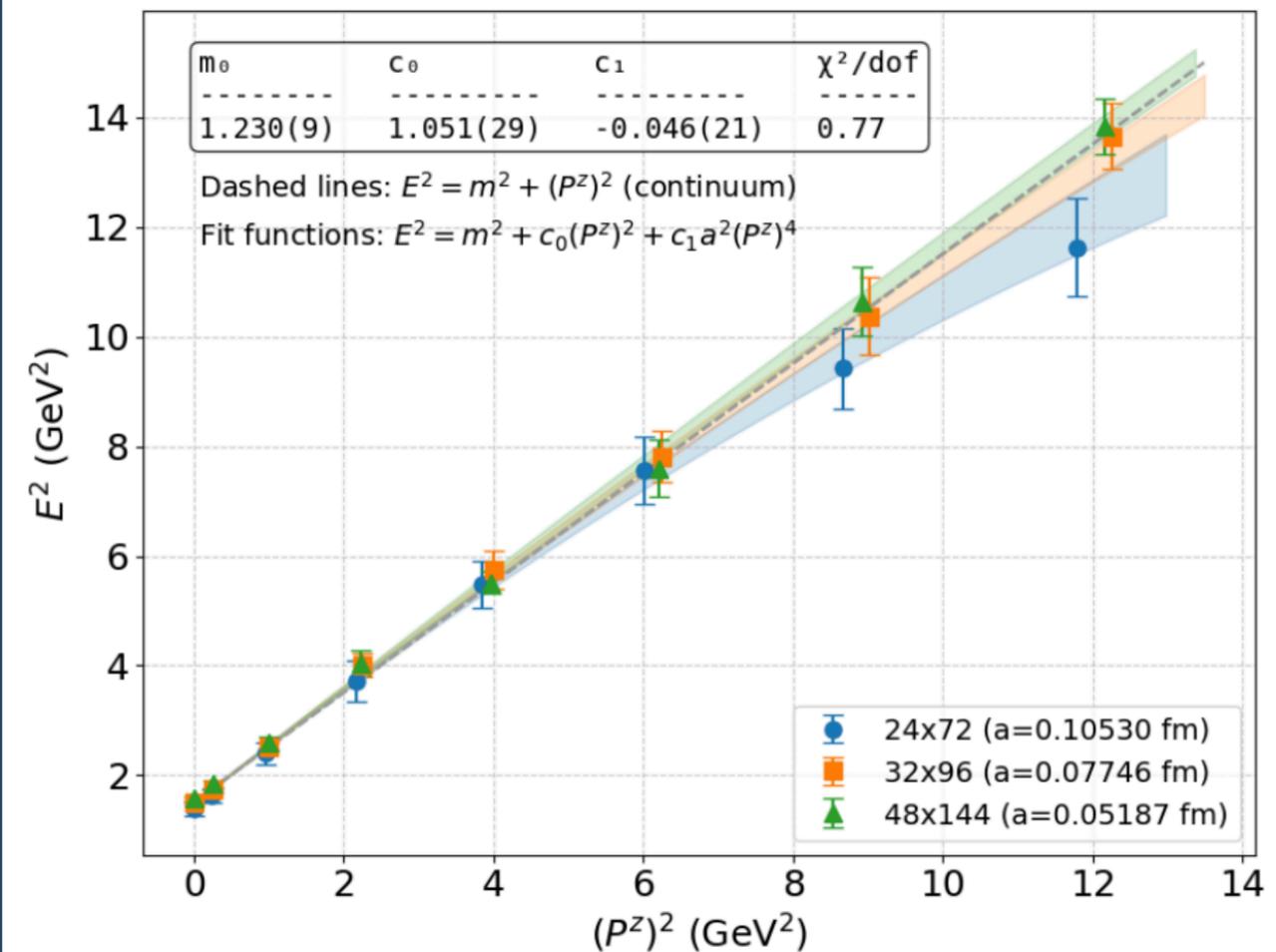
### Large Interval



(b) Noise-to-signal ratio,  $\Lambda$ ,  $P = 0$

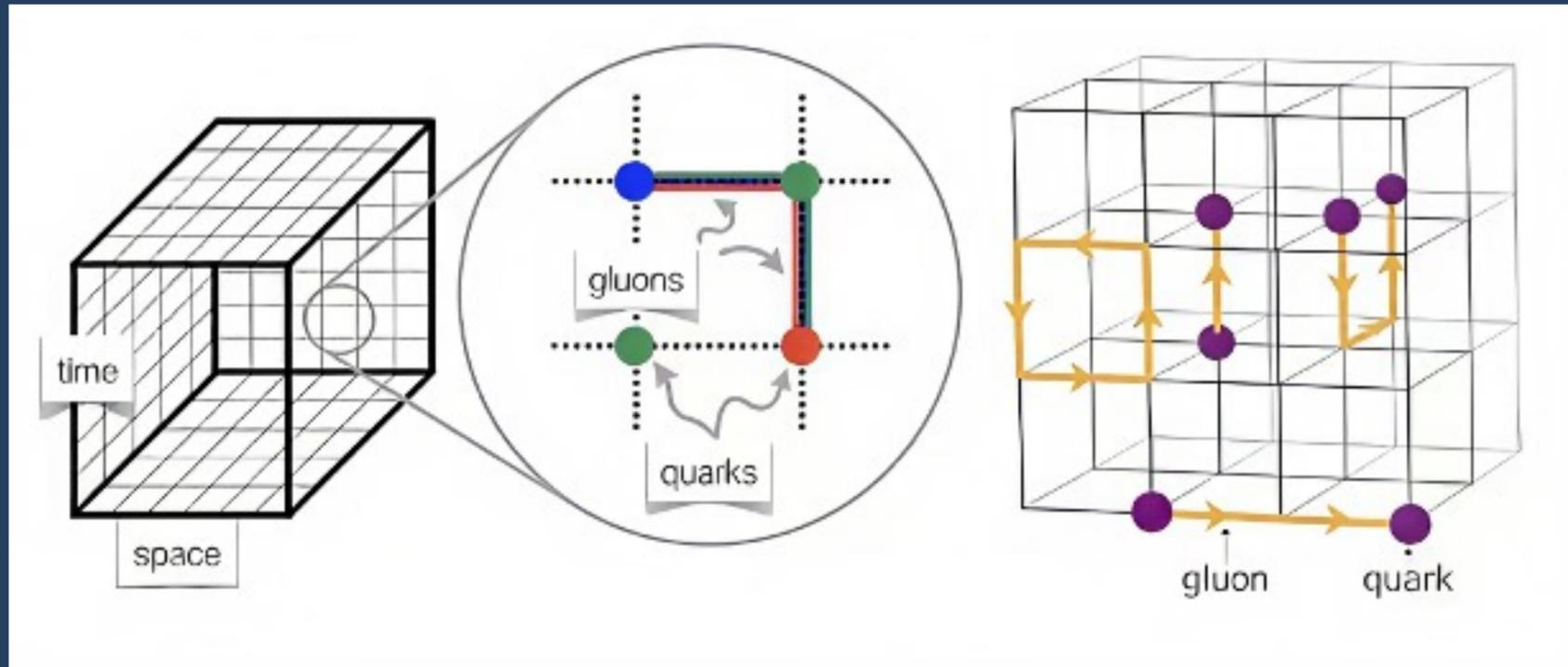
Decay in the signal

### Large Momentum



Contamination from excited states

## Difficulty in Lattice Renormalization



Lattice renormalization differs from dimensional regularization (DR) by using a discrete space-time grid, which alters the handling of ultraviolet physics.

Current solutions: RI/MOM scheme, Ratio scheme, and Self-renormalization scheme...

Their challenges:      Precise gauge fixing      IR physics      Complicated, data fitting

**A possible solution to both problems is  
Gradient flow formalism**

**The concept of Gradient Flow is related to the Stochastic quantization**

**Equivalent Definitions of QFT**

Form of Quantum Effects

**Canonical Quantization:**

$$[x, p] = i\hbar$$

**Path Integral:**

$$\langle \mathcal{O}(x) \rangle = \int d\mu e^{iS} O(x)$$

**Stochastic Quantization:**

Gaussian Noise:  $\eta(x, t)$

# What is Gradient flow

**Stochastic quantization evolves fields through a diffusion-like equation, with noise generating quantum effects.**

$$\partial_t \phi(x, t) = -\frac{\delta S}{\delta \phi} + \eta(x, t)$$

*x: spacetime 3+1*

*t: an artifact called “flow time”*

- $-\frac{\delta S}{\delta \phi}$ : The gradient of the action with respect to the scalar field, which guides the field configuration toward the classical solution  $\delta S = 0$ .
- $\eta(x, t)$ : Generate the UV fluctuations.

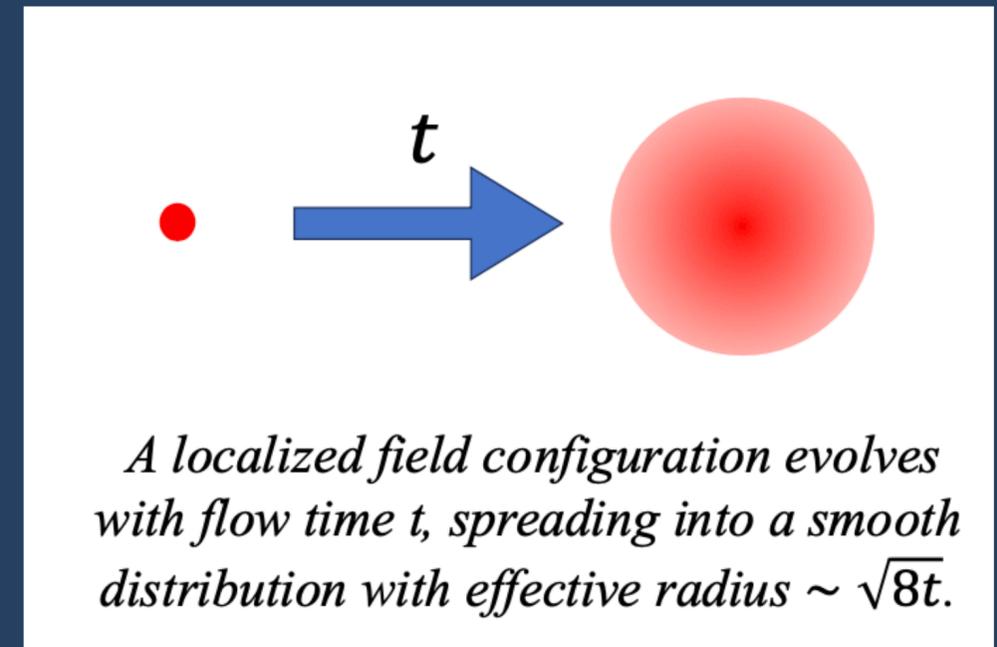
**Parisi & Wu : When  $t \rightarrow \infty$ , the probability of a certain configuration becomes  $e^{-S}$**

# What is Gradient flow

**Gradient flow:** Discard Noise term, smooth out UV physics

Gradient flow equation

$$\partial_t \phi(x, t) = - \frac{\delta S}{\delta \phi}$$



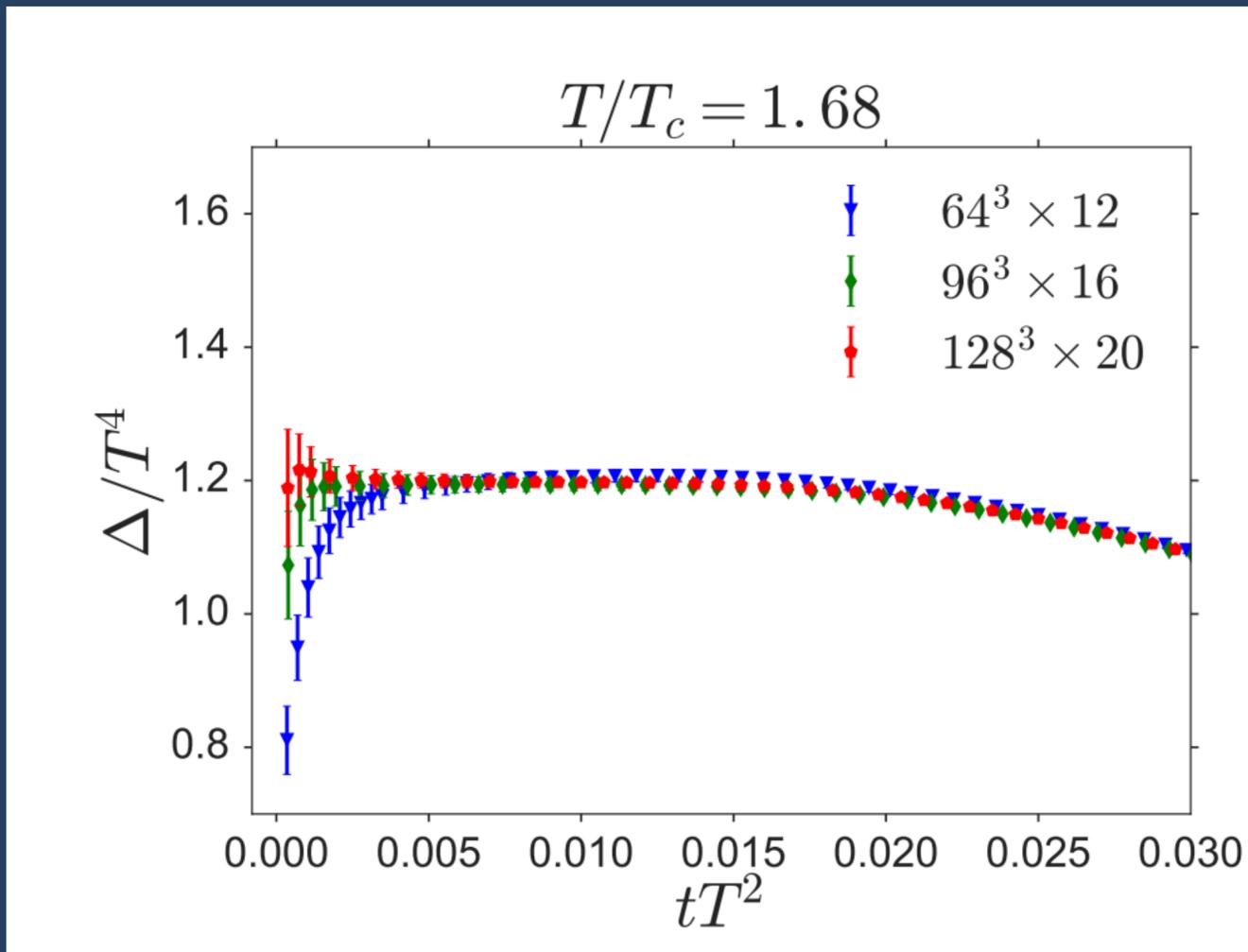
**Lattice implementation:** Starting from a set of lattice configurations which satisfies  $e^{-S}$ , we evolve each configuration according to this equation for small  $t$ .

# Why use Gradient flow

## 1. Gradient flow improves Signal-to-Noise Ratio

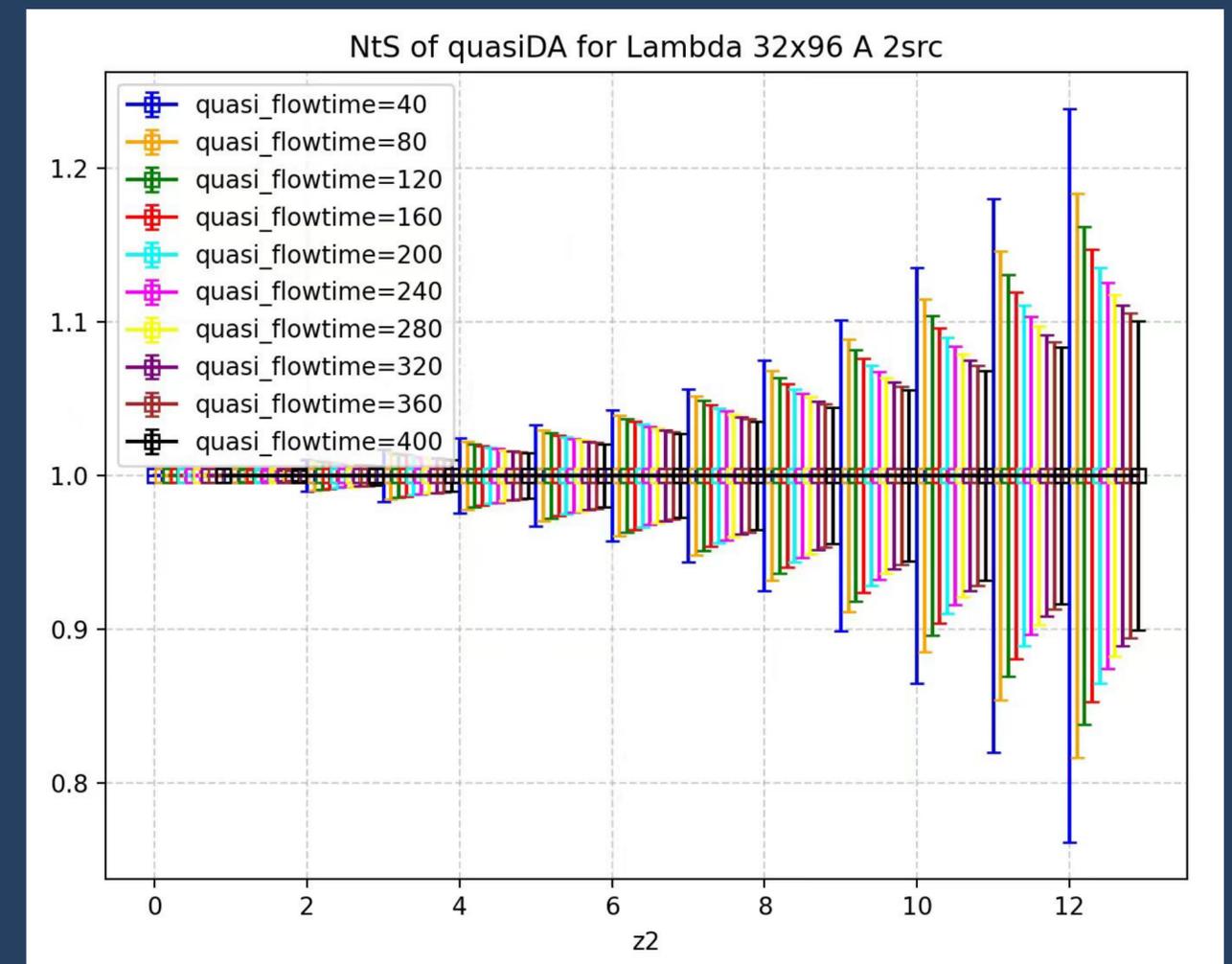
Gradient flow is a continuous version of stout smearing. *C. Morningstar, M. Peardon, Phys. Rev. D69 (2004) 054501*

Trace anomaly



*M. Kitazawa et al., Phys. Rev. D 94 no.11, 114512 (2016)*

Noise-to-Signal Ratio of Baryon quasi-DA



## 2. Gradient flow simplifies the renormalization process

Flow time is an UV regulator

Schwinger parametrization for QCD propagator:  $\frac{1}{(K^2)^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha \alpha^{n-1} e^{-K^2\alpha}$

Schwinger parametrization for gradient flow propagator:  $\frac{e^{-K^2 t}}{(K^2)^n} = \frac{1}{\Gamma(n)} \int_t^\infty d\alpha \alpha^{n-1} e^{-K^2\alpha}$

- $\alpha \rightarrow 0$ : UV part of the propagator
- $\alpha \rightarrow \infty$ : IR part of the propagator

So we try to regulate lattice observables by flow time rather than lattice spacing!

$$a \ll \sqrt{8t}$$

## 3. Converting operators into $\overline{MS}$ scheme

Flowed operator (operators evaluated with  $\phi(x, t)$ ) can be matched to operators in the  $\overline{MS}$  scheme via factorization.

Take quasi-PDF as an example:

$$\tilde{f}^R(z, t) = e^{\delta m z} C(t, z, \mu) \tilde{f}^{\overline{MS}}(z, \mu)$$

*N. Brambilla, X. Wang, JHEP 06 (2024) 210*

*C. Monahan, Phys. Rev. D 97, no.5, 054507 (2018)*

**In small flow time limit, the matching kernel is independent of space interval  $z$**

$$\tilde{f}^R(z, t) = e^{\delta m z} C(t, \mu) \tilde{f}^{\overline{MS}}(z, \mu)$$

$$C_q(t, \mu, z) = 1 + \frac{\alpha_s}{4\pi} C_F \left[ a_\Gamma - a'_\Gamma - 3 \log(2\mu^2 t e^{\gamma_E}) - \log(432) - 4e^{-z^2/8t} \right] - 3E_i(-z^2/8t),$$

$$\lim_{t \rightarrow 0} C_q(t, \mu) = 1 - \frac{\alpha_s}{4\pi} C_F \left[ 3 \log 2\mu^2 t e^{\gamma_E} + 2 + \log(432) \right]$$

## The result is reasonable

We are factorizing the flow time  $t$  out.

$z$ , which is part of IR physics, should not be in the matching kernel.

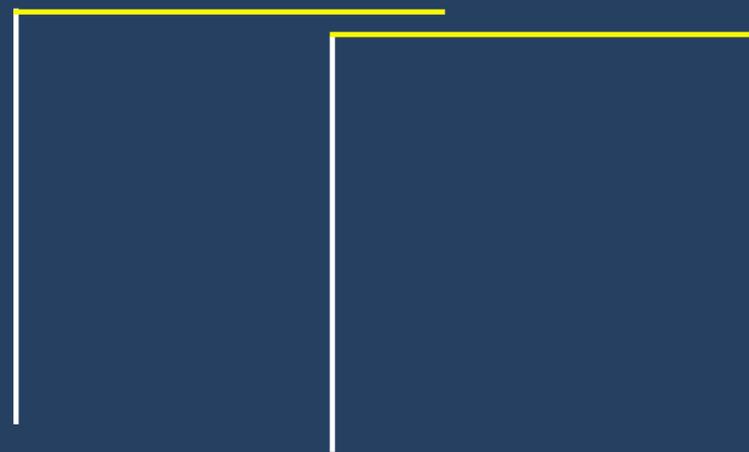
## The results can be understood using auxiliary fields

*X. Ji et al., Phys. Rev. Lett. 120, 11, 112001 (2018)*

$$\bar{\psi}(z)W(z,0)\psi(0)$$



$$\bar{\psi}(z)Q(z) \bar{Q}(0)\psi(0)$$



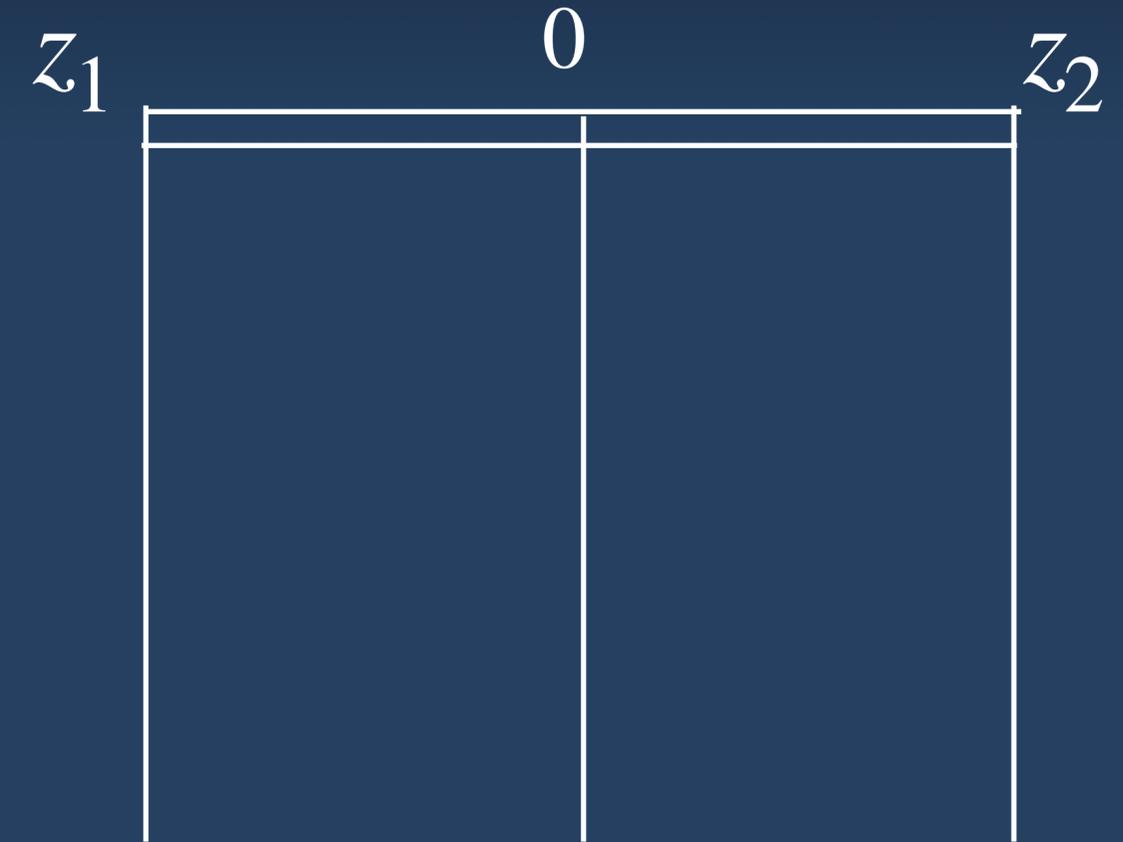
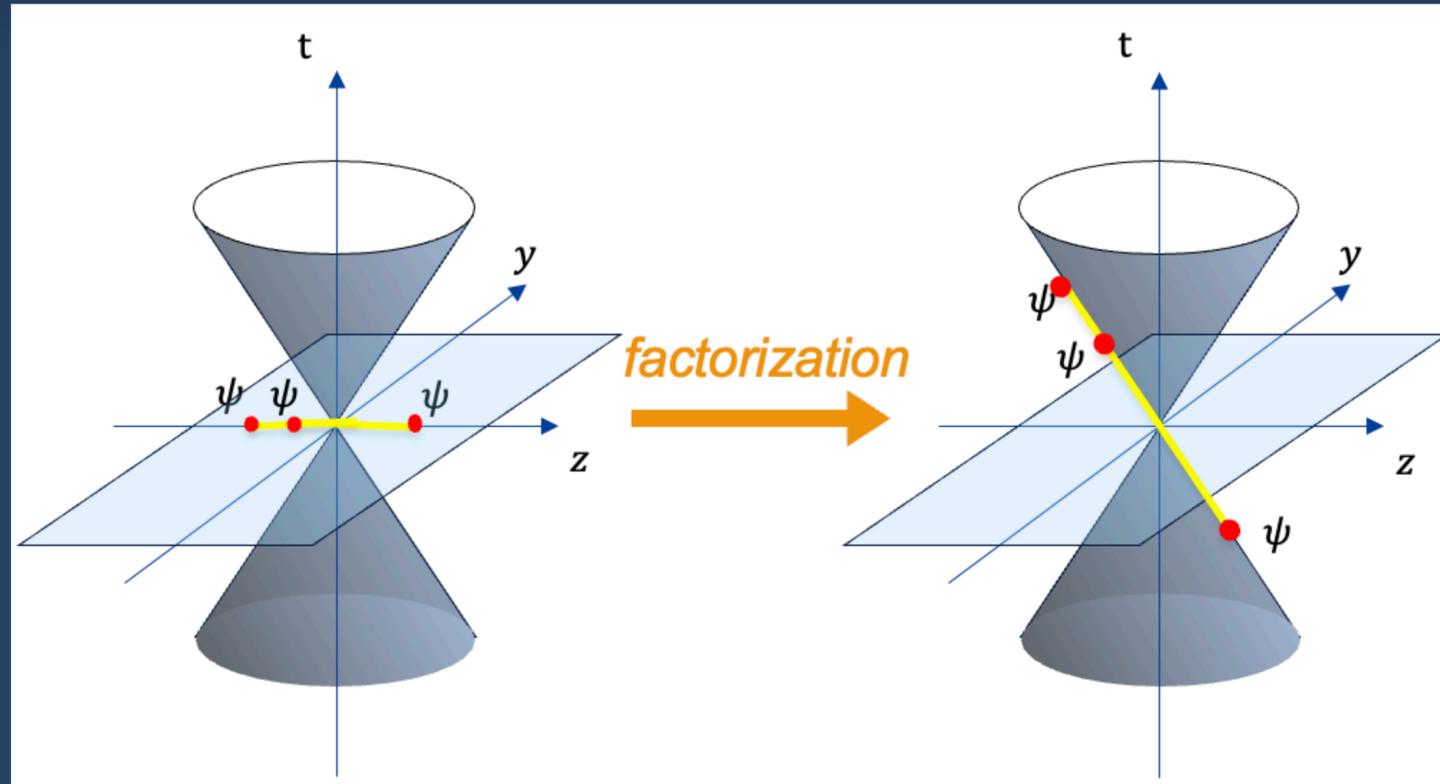
The matching is separated to two local operators

## Merits of gradient flow

- **Improved Signal-to-Noise Ratio:** It is a continuous version of stout smear.
- **Simpler renormalization for lattice QCD:** The flow time  $t$  acts as an UV regulator.
- **Matching to  $\overline{MS}$  operators:** It admit perturbative calculation and can be matched to  $\overline{MS}$  operators through factorization.
- **UV renormalon:** It illustrate the interplay between linear renormalons in lattice QCD and continuous scheme

## Our recent work: We extend the matching to Baryon Quasi-DA

*Baryon Quasi-DA is used to extract Baryon LCDA*



*LaMET framework*

*X. Ji, Sci. China Phys. Mech. Astron. 57, 1407-1412 (2014)*

*X. Ji, Phys. Rev. Lett. 110, 262002 (2013)*

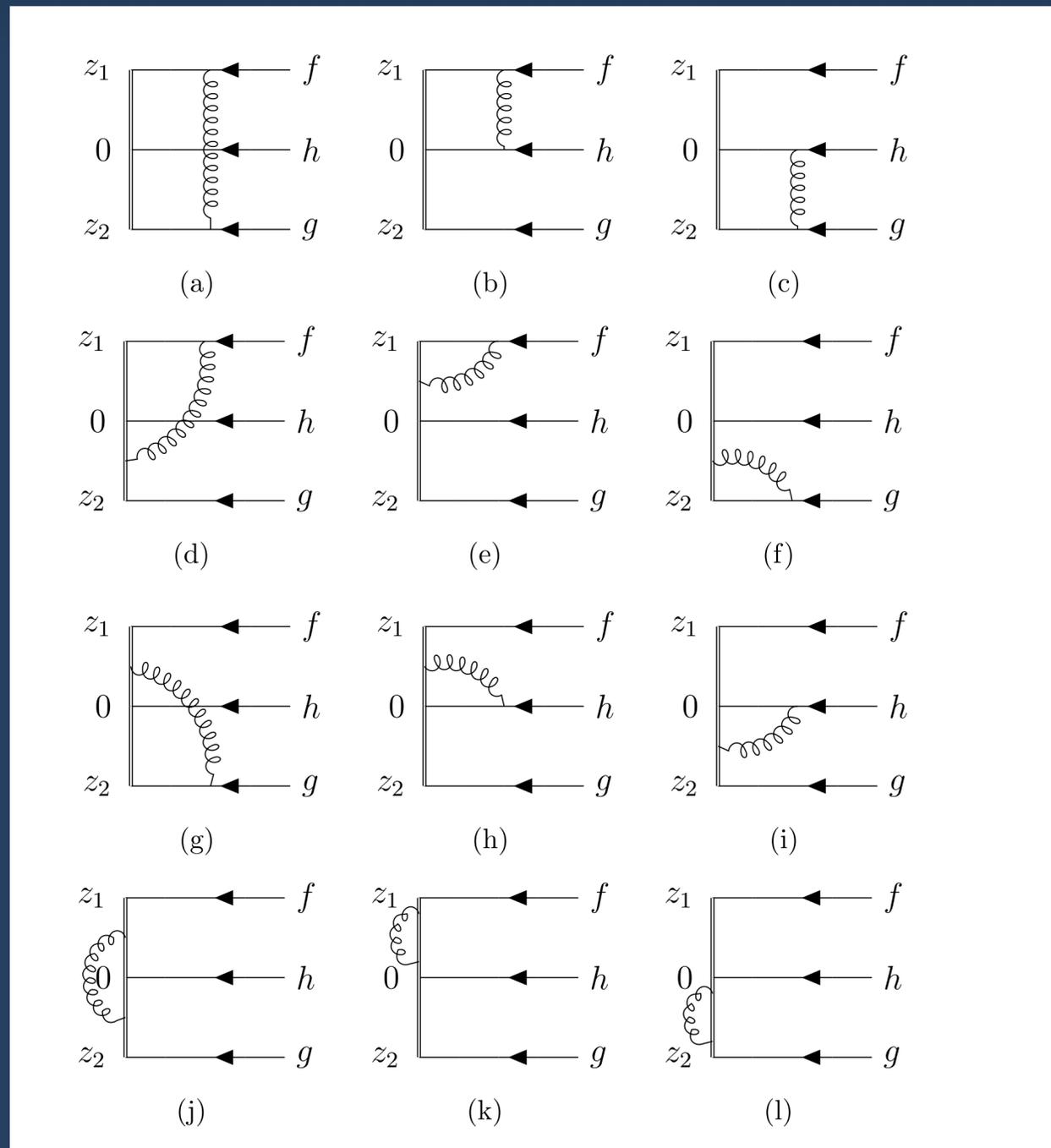
*Baryon Quasi-DA in  
coordinate space*

Matrix element definition

$$\tilde{\Phi}(z_1, z_2, \mu) = \langle 0 | \psi^T(z_1) C \gamma^z \psi(z_2) \psi(0) | P \rangle$$

## We extend the matching to Baryon Quasi-DA

With gradient flow:  $\tilde{\Phi}^{Flowed}(z_1, z_2, t) = e^{\delta m \tilde{z}} C_q(t, z_1, z_2, \mu) \tilde{\Phi}^{\overline{MS}}(z_1, z_2, \mu).$



- gluon propagator:

$$s, \nu, b \begin{array}{c} p \\ \text{-----} \\ t, \mu, a \end{array} = \delta^{ab} \frac{1}{p^2} \left( \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2} + \xi \frac{p_\mu p_\nu}{p^2} e^{-\kappa(t+s)p^2} \right)$$
- (anti)quark propagator:

$$s, \beta, j \begin{array}{c} p \\ \text{-----} \\ t, \alpha, i \end{array} = \delta_{ij} \frac{(-i\not{p} + m)_{\alpha\beta}}{p^2 + m^2} e^{-(t+s)p^2}$$
- quark-gluon vertex:

$$= g (T^a)_{ij} (\gamma_\mu)_{\alpha\beta}$$

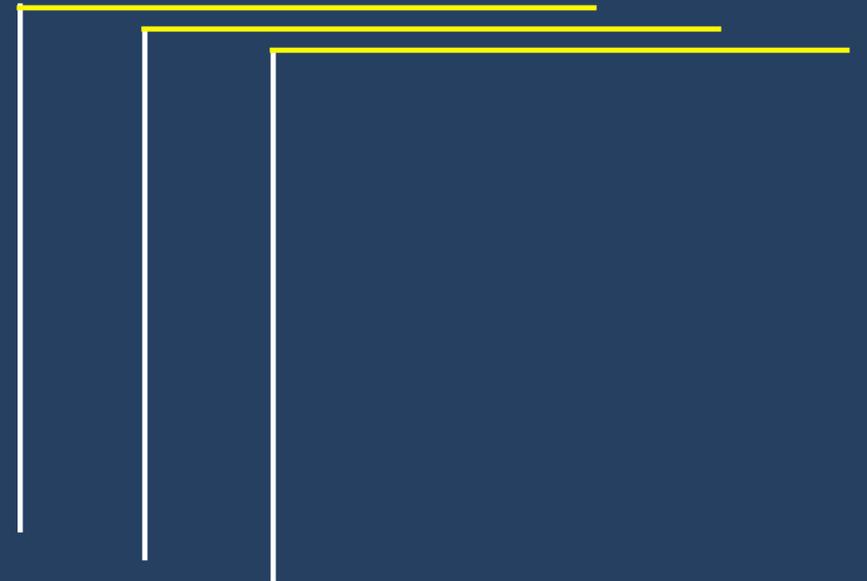
## We extend the matching to Baryon Quasi-DA

With gradient flow:  $\tilde{\Phi}^{Flowed}(z_1, z_2, t) = e^{\delta m \tilde{z}} C_q(t, z_1, z_2, \mu) \tilde{\Phi}^{\overline{\text{MS}}}(z_1, z_2, \mu)$ .

$$C_q(z_1, z_2, t, \mu) = 1 - \frac{C_F \alpha_s}{16\pi} \left[ 18 \log(2\mu^2 t e^{\gamma_E}) + 6 \log(432) + 12 + 7 \text{Ei}(-\bar{z}_1^2) + 7 \text{Ei}(-\bar{z}_2^2) + 6 \text{Ei}(-\bar{z}_{12}^2) \right. \\ \left. + 8 \left( e^{-\bar{z}_1^2} + e^{-\bar{z}_2^2} + e^{-\bar{z}_{12}^2} \right) + \frac{2 \left( e^{-\bar{z}_1^2} - 1 \right)}{\bar{z}_1^2} + \frac{2 \left( e^{-\bar{z}_2^2} - 1 \right)}{\bar{z}_2^2} + \frac{6 \left( e^{-\bar{z}_{12}^2} - 1 \right)}{\bar{z}_{12}^4} + \frac{2 \left( e^{-\bar{z}_{12}^2} + 2 \right)}{\bar{z}_{12}^2} \right] + \mathcal{O}(\alpha_s^2),$$

*In the small flow time limit, the matching kernel is free of  $z_1, z_2$*

$$\lim_{t \rightarrow 0} \mathcal{C}_q(z_1, z_2, t, \mu) = 1 - \frac{3C_F \alpha_s}{8\pi} \left[ 3 \log(2\mu^2 t e^{\gamma_E}) + 2 + \log(432) \right] + \mathcal{O}(\alpha_s^2).$$



## The 1-loop linear divergence in Gradient Flow

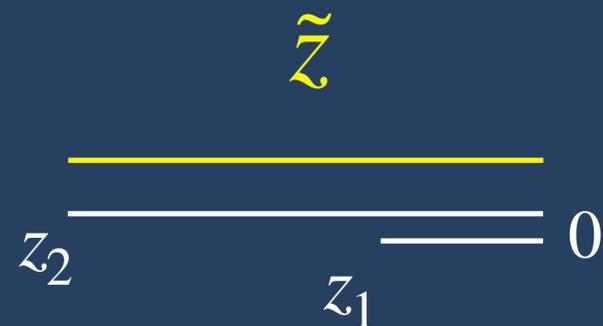
With gradient flow:

$$\delta m\tilde{z} = -\frac{C_F\alpha_s}{2\sqrt{\pi}} \left( \bar{z}_1 \operatorname{erf}(\bar{z}_1) + \bar{z}_2 \operatorname{erf}(\bar{z}_2) + \bar{z}_{12} \operatorname{erf}(\bar{z}_{12}) \right).$$

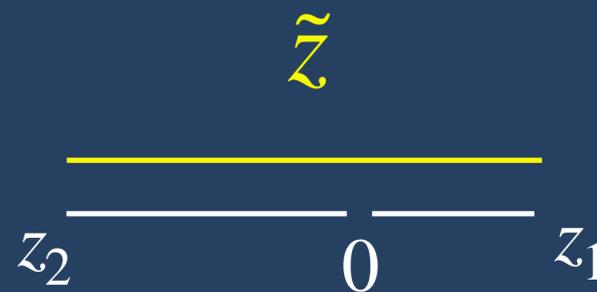
$$\lim_{t \rightarrow 0} \delta m\tilde{z} = -\frac{C_F\alpha_s}{4\pi} \frac{\sqrt{2\pi}}{\sqrt{t}} \cdot \frac{|z_1| + |z_2| + |z_1 - z_2|}{2}.$$

$$\epsilon^{ijk} W_i W_j W_k = 1$$

Case 1:



Case 2:

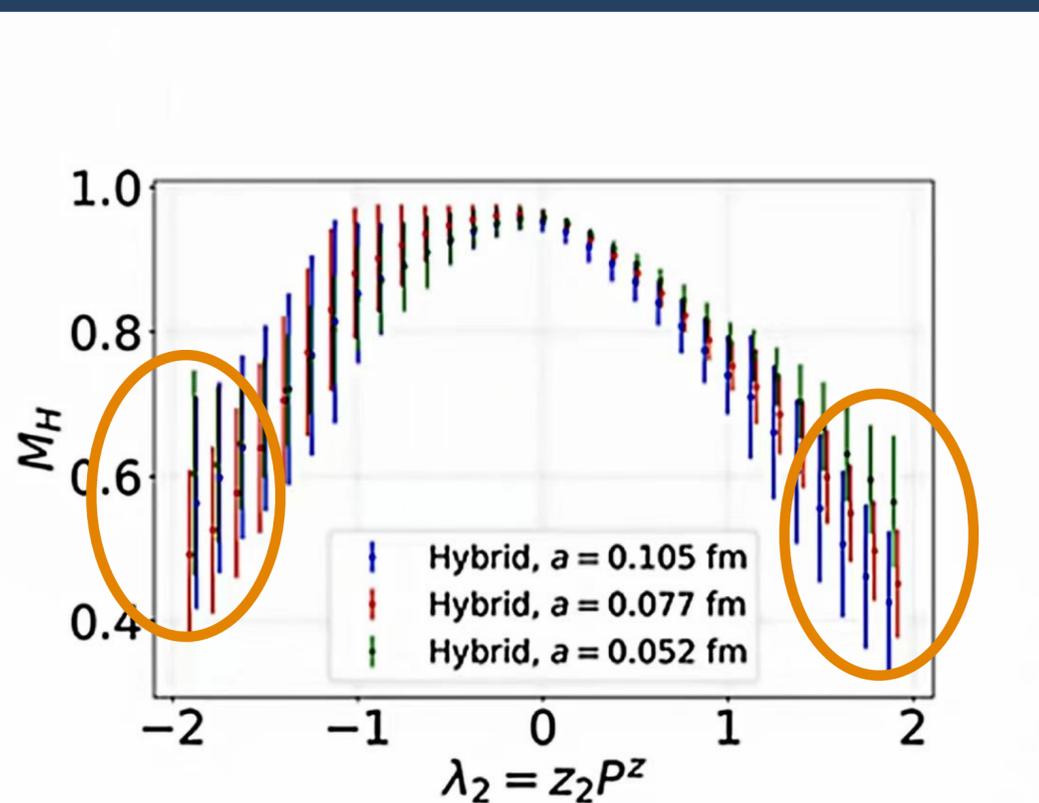


Inline with expectation of  
Wilson line effective length

## Extracting $\overline{MS}$ baryon quasi-DA is complicated

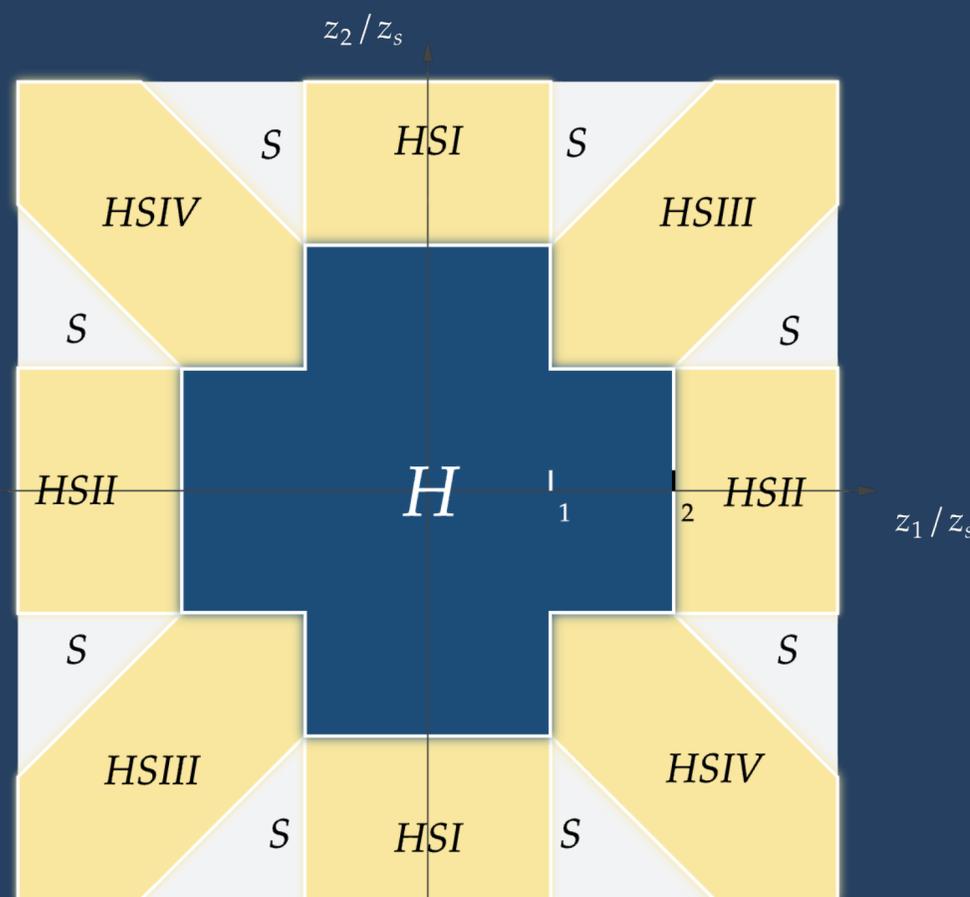
It is a 2D distribution

### Signal-to-noise problem



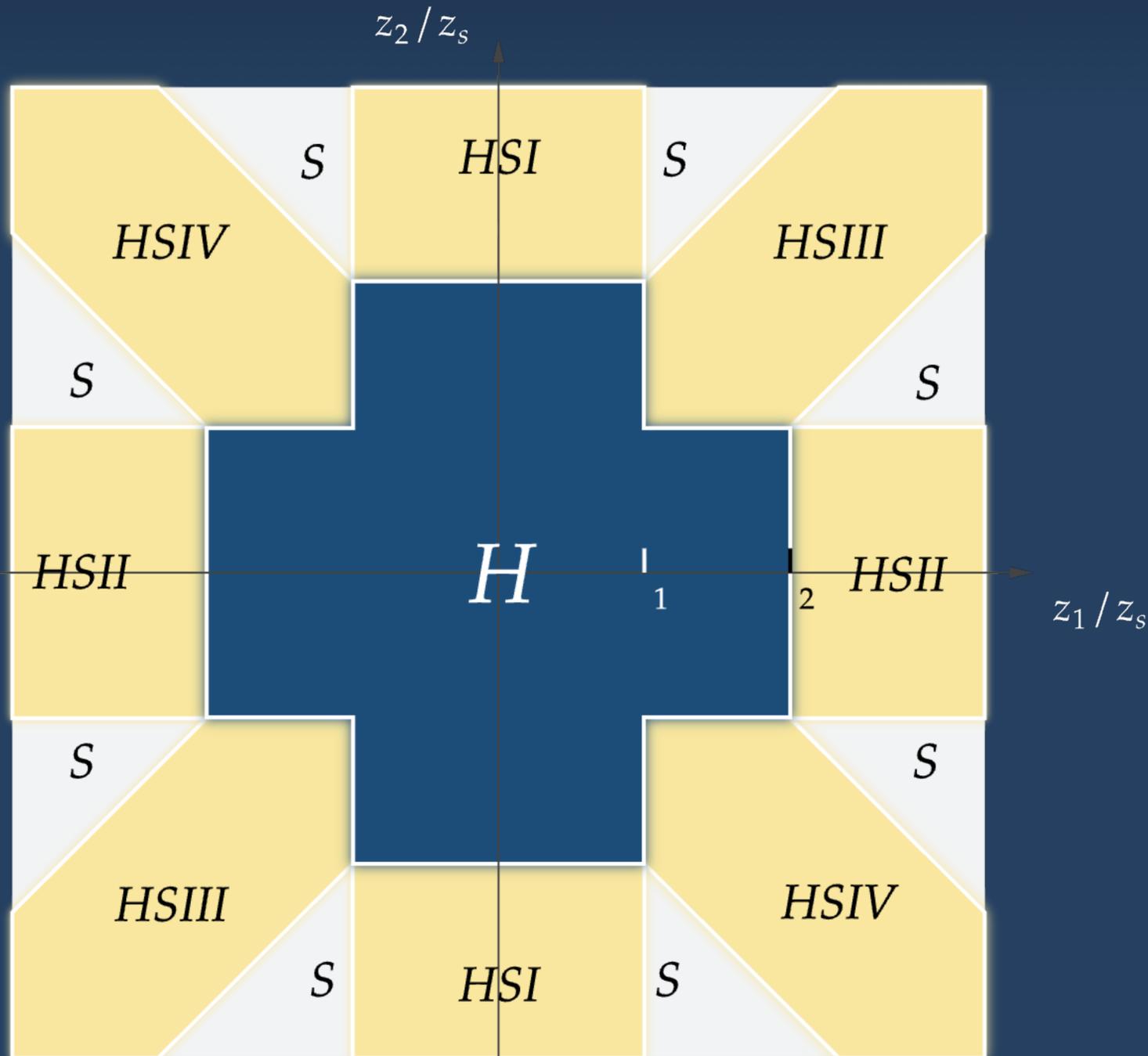
(d) Hybrid scheme result of proton at  $P = 0.5$  GeV

### Renormalization is complicated



- **Pre-step:** Convert matrix elements to  $\overline{MS}$  scheme use self-renormalization
- **Blue (hard) region:** Divide the large-momentum element **in the blue region** by the zero-momentum element **in the same blue region**.
- **Yellow (hard-soft) region:** Divide the large-momentum element **in the yellow region** by the zero-momentum element **taken on the blue-yellow boundary**.
- **Gray (soft) region:** Divide the large-momentum element **in the gray region** by the zero-momentum element **taken at the gray-yellow intersection**.

## Hybrid Renormalization with gradient flow:



We can replace self-renormalization with the matching from gradient flow.

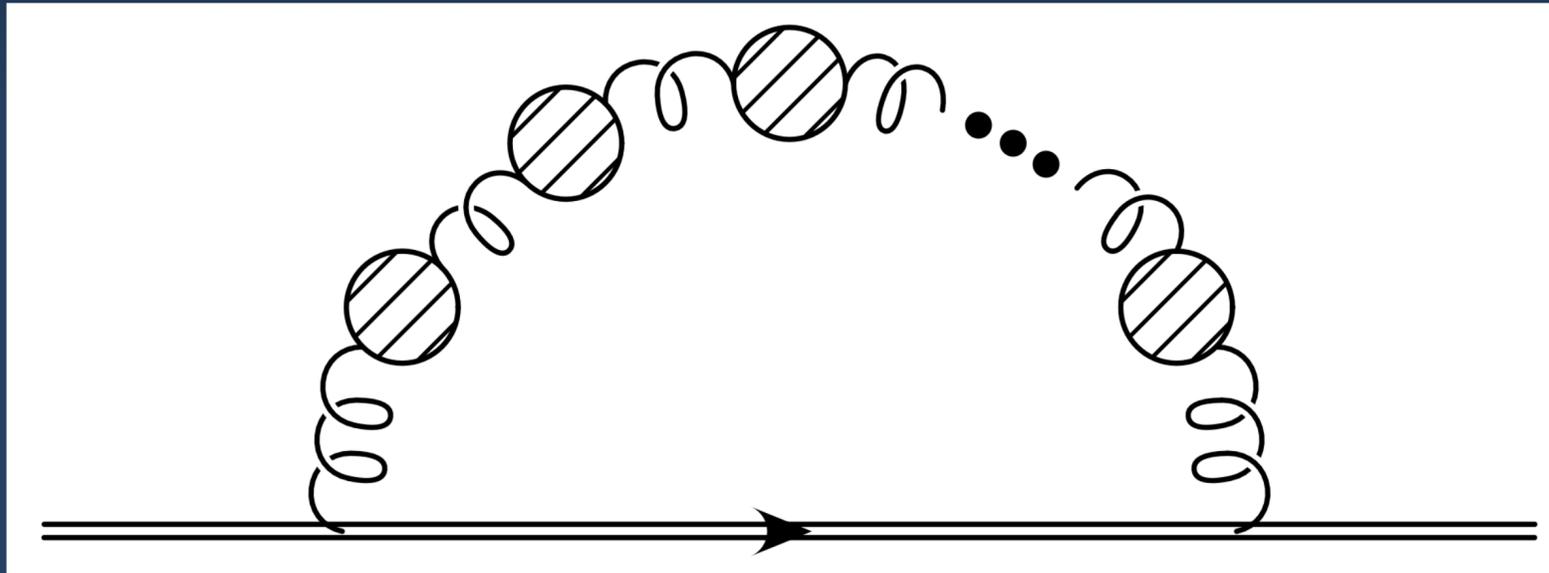
$$Z_R(z_1, z_2, a, \mu) = \exp \left[ \left( \frac{k}{a \ln[a\Lambda_{\text{QCD}}]} - m_0 \right) \tilde{z} + \frac{\gamma_0}{b_0} \ln \left[ \frac{\ln[1/(a\Lambda_{\text{QCD}})]}{\ln[\mu/\Lambda_{\overline{\text{MS}}}] } \right] + \ln \left[ 1 + \frac{d}{\ln(a\Lambda_{\text{QCD}})} \right] + f(z_1, z_2) a^2 \right].$$

$$\tilde{\Phi}^{\overline{\text{MS}}} (z_1, z_2, P^z, \mu) = \frac{\tilde{\Phi}^{\text{LATTICE}} (z_1, z_2, P^z, a)}{Z_R(z_1, z_2, a, \mu)},$$



$$\tilde{\Phi}^{\text{Flowed}} (z_1, z_2, t) = e^{\delta m \tilde{z}} C_q(t, z_1, z_2, \mu) \tilde{\Phi}^{\overline{\text{MS}}} (z_1, z_2, \mu)$$

## Renormalon in Wilson line generates uncertainty in the result



Borel transformation for self-energy diagrams:

$$\mathcal{B}[\tilde{f}_a^{\overline{\text{MS}}}] (w) = \frac{C_F 2^{1-2w} e^{5w/3} \mu^{2w} z^{2w} \Gamma(-w)}{(2w-1) \Gamma(w+1)},$$

Renormalon ambiguity is represented by the residue

$$\text{Res}_{w=1/2} \mathcal{B}[\tilde{f}_a^{\overline{\text{MS}}}] = -2e^{5/6} C_F \mu |z|$$

UV renormalons reflect the ambiguity in defining the UV divergence of Wilson lines. They must be addressed using techniques such as Renormalon Resummation ...

## Wilson Line in Gradient Flow is free from UV renormalon

$$\mathcal{B}[\tilde{f}_a^R](w, t) = C_F 2^{w-1} e^{5w/3} t^{w-1} \mu^{2w} \Gamma(-w) \\ \times \left[ w z^2 {}_2F_2 \left( \frac{1}{2}, 1 - w; \frac{3}{2}, 2; -\frac{z^2}{8t} \right) - 4t \left( L_w \left( -\frac{z^2}{8t} \right) - 1 \right) \right],$$

*J.L. Zhang, [arXiv:2507.18233 [hep-ph]].*

$$\text{Res}_{w=1/2} \mathcal{B}[\tilde{f}_a^R] = \mathbf{0}$$

When matching from Gradient flow to  $\overline{\text{MS}}$  operator, one needs to consider the  $t \rightarrow 0$  limit. This corresponds to expand the Borel transformation in  $t$ .

## The expansion of gradient flow result recreates MS result

$$\mathcal{B}[\tilde{f}_a^R](w, t) = C_F 2^w e^{5w/3} t^{w-\frac{1}{2}} \mu^{2w} \left( 2\sqrt{t}\Gamma(-w) - \sqrt{2}z\Gamma\left(\frac{1}{2}-w\right) \right) + \frac{C_F 2^{1-2w} e^{5w/3} \Gamma(-w) (\mu z)^{2w}}{(2w-1)\Gamma(w+1)}$$



Renormalon in the linear divergence



Result for MS Wilson line

**Renormalon cancellation:**

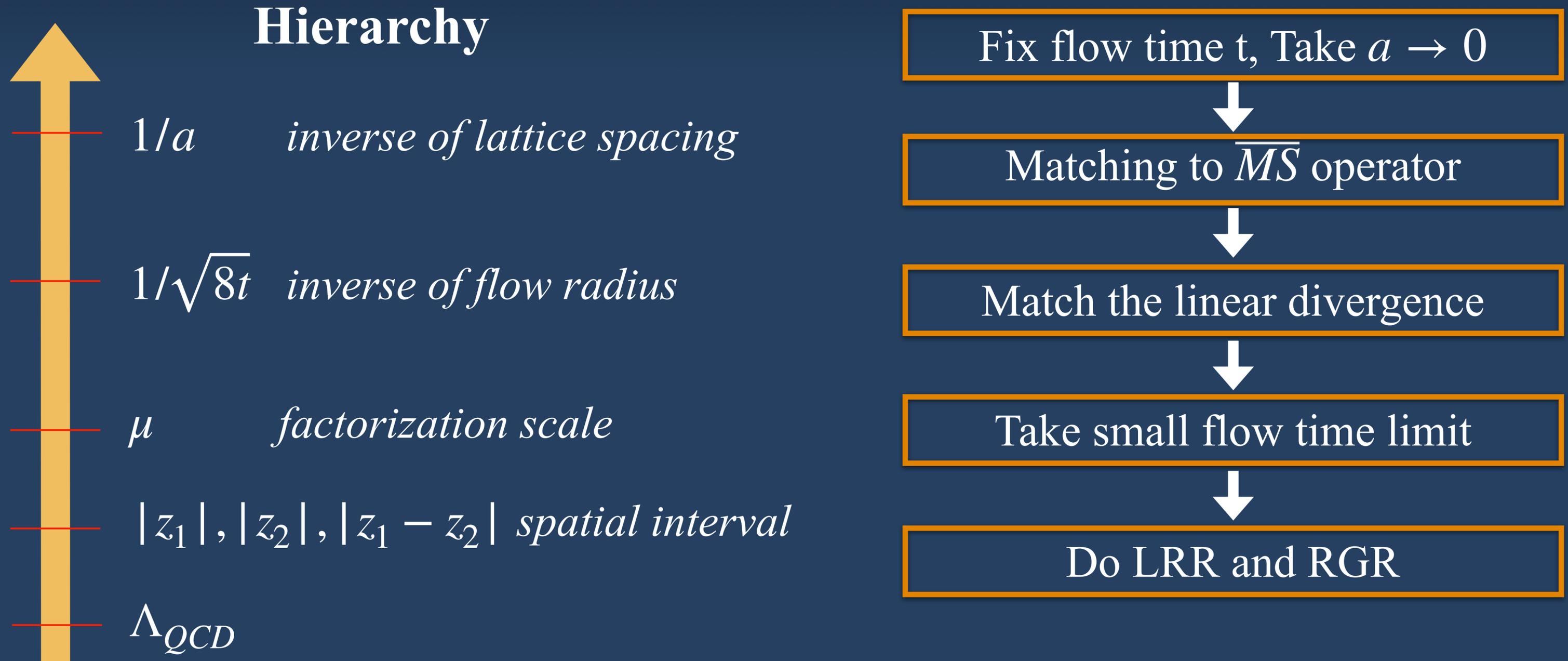
$$\text{Res}_{w=1/2} \mathcal{B}[\tilde{f}_a^{\overline{\text{MS}}}] = 2e^{5/6} C_F \mu |z|$$

$$\text{Res}_{w=1/2} \mathcal{B}[\tilde{f}_a^{\overline{\text{MS}}}] = -2e^{5/6} C_F \mu |z|$$

**One still need to do Leading Renormalon Resummation for the matching**

*Rui Zhang, Jack Holligan, Xiangdong Ji and Yushan Su, Phys. Lett. B 844, 138081 (2023)*

## A tutorial guide on Lattice Implementation



*Our work is still ongoing...*

- **The merits of Gradient flow**

Improve signal, theoretical matching, renormalon exploration

- **The implementation for quasi-DA**

Matching kernel, linear divergence and renormalon cancellation

- **A tutorial on Lattice Implementation**

***THANKS***