

From quark to gluon Collins–Soper kernels with lattice QCD



and bT -dependent LaMET matching

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in collaboration with

Yang Fu², Phiala Shanahan², Michael Wagman³, and Yong Zhao¹

[2402.06725]

+ 2 ongoing

Related talks:

1. Jin-Xin Tan, Thu 11:00 am
2. Yang Fu, Thu 1:55 pm
3. Wayne Morris, Thu 2:15 pm

XIIIth Meeting on Lattice Parton Physics from Large Momentum Effective Theory
(LaMET 2025)

Center for Frontiers in Nuclear Science, Stony Brook University
October 8–10, 2025



Talk Outline

1. *Review: TMDs and the Collins-Soper (CS) kernel*
2. *5-year progress in quark CS kernel (our group)*
3. *Takeaways for the gluon CS kernel*
4. *Takeaways for LaMET formalism*

Review:

TMDs and the Collins-Soper (CS) kernel

- TMD Factorization
- TMD evolution
- The CS kernel

TMD Factorization: Drell-Yan example

1. Collinear (**Y term**): $\Lambda_{\text{QCD}} \ll q_T \sim Q$

$$\frac{d\sigma}{dQ^2 dY} = \sigma_0 \sum_{q,\bar{q}} H_{q\bar{q}}[Q/\mu] f_{q/p_a}(x_a, \mu) f_{\bar{q}/p_b}(x_b, \mu) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{Q^2}\right)$$

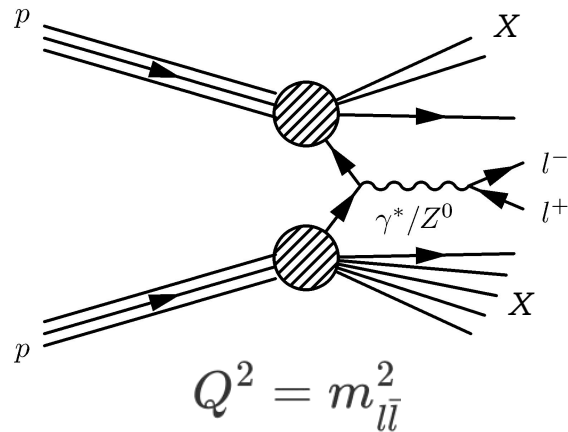
2. TMD (**W term**): $q_T \sim \Lambda_{\text{QCD}} \ll Q$

$$\frac{d\sigma^W}{dQ^2 dY d^2\mathbf{q}_T} = \sigma_0 \sum_{q,\bar{q}} H_{q\bar{q}}[Q/\mu] \underbrace{\int_0^\infty d^2\mathbf{b}_T e^{i\mathbf{b}_T \cdot \mathbf{q}_T}}_{\text{Fourier transform (large } b_T \text{ is nonperturbative)}} f_{q/p_a}(x_a, \mathbf{b}_T, \mu, \zeta_a) f_{\bar{q}/p_b}(x_b, \mathbf{b}_T, \mu, \zeta_b) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{Q^2}, \frac{q_T^2}{Q^2}\right)$$

$$\zeta_a \zeta_b = Q^4$$

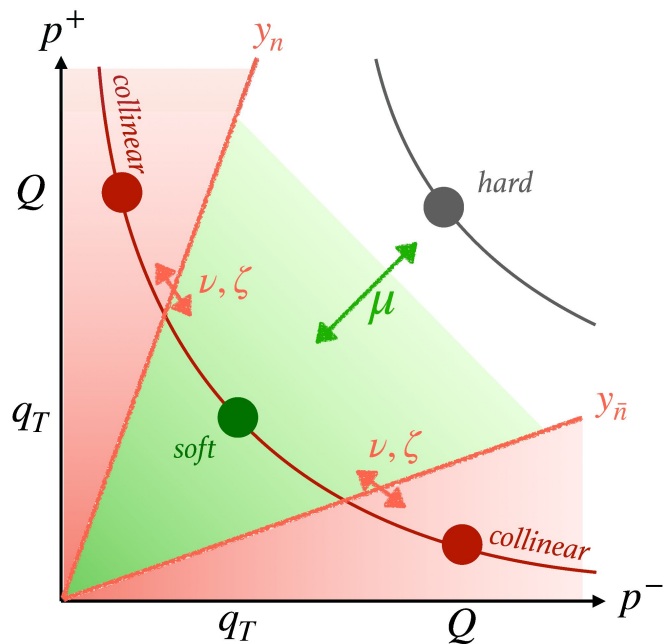
Collins-Soper scale ζ

- related to rapidity Y
- proportional to hadron momentum P



**Holds for small q_T
(~ large b_T)**

TMD evolution



RG equations of TMDs:

$$\mu \frac{d}{d\mu} \ln f_{i/h}^{\overline{\text{MS}}}(x, b_T, \mu, \zeta) = [\gamma_\mu^{q/g}]^{\overline{\text{MS}}}(\mu, \zeta)$$

$$\sqrt{\zeta} \frac{\partial}{\partial \sqrt{\zeta}} \ln f_{i/h}^{\overline{\text{MS}}}(x, b_T, \mu, \zeta) = [\gamma_\zeta^{q/g}]^{\overline{\text{MS}}}(\mu, b_T)$$

RG rapidity evolution:

$$f_{p/h}(x, b_T, \mu, \zeta) = f_{p/h}(x, b_T, \mu, \zeta_0) \exp \left[\frac{1}{2} \gamma_p(b_T, \mu) \ln \frac{\zeta}{\zeta_0} \right]$$

The Collins-Soper (CS) kernel

Computed as a ratio of TMDs at different ζ : Proportional to hadron momentum P

- Before LaMET matching: $\gamma_q(b_T, \mu) = \frac{2}{\ln(\zeta_1/\zeta_2)} \ln \frac{f_{q/h}(x, b_T, \mu, \zeta_1)}{f_{q/h}(x, b_T, \mu, \zeta_2)}$ Encoded by light-like matrix elements

- After LaMET matching: $\gamma_q(b_T, \mu) = \frac{1}{\ln(P_1/P_2)} \ln \frac{C_q(xP_1^z, \mu) \tilde{f}_{q/h}(x, b_T, \mu, P_1)}{C_q(xP_2^z, \mu) \tilde{f}_{q/h}(x, b_T, \mu, P_2)} + \text{p. c.}$
LaMET matching coefficients Encoded by space-like matrix elements

Properties of CS kernel:

- Independent of hadronic state
- Differs for quarks and gluons
- Non-perturbative at large b_T for any μ

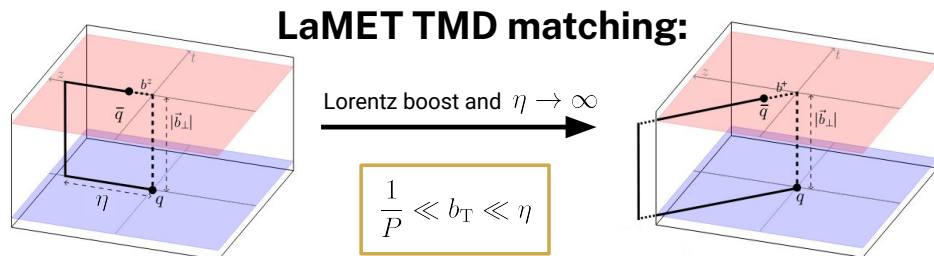
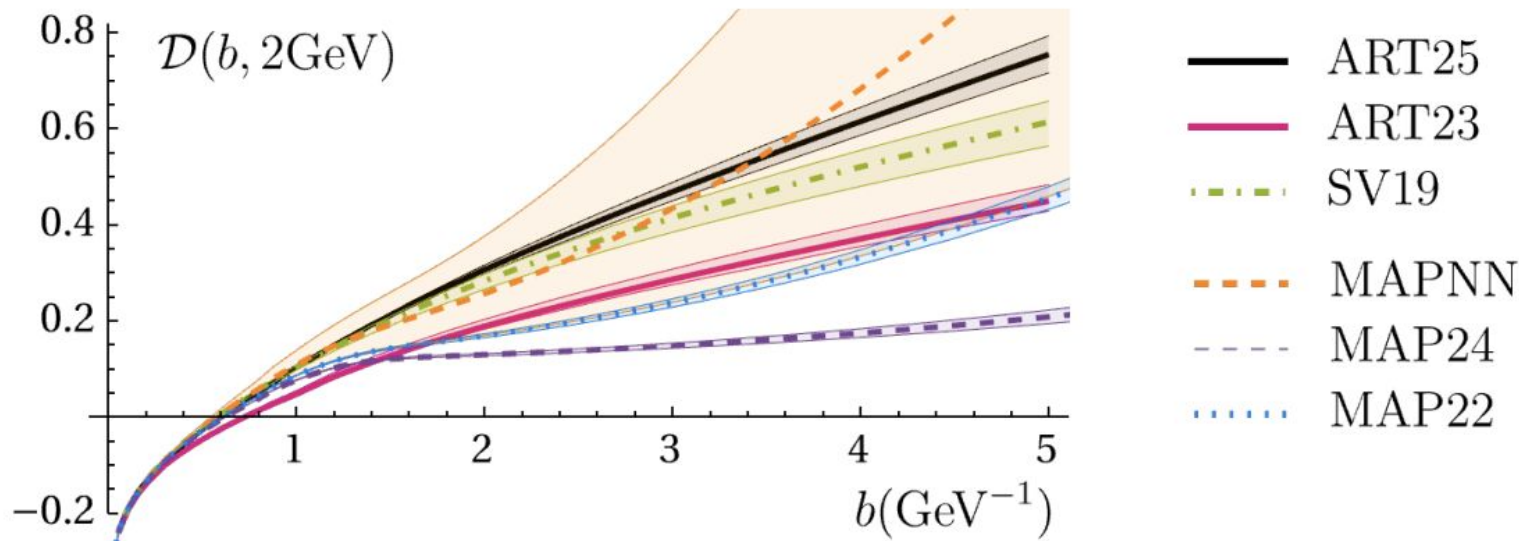


Fig. by Ebert, Stewart, Zhao, JHEP 1909 (2019)
(notation changed for consistency).

5-year progress in quark CS kernel (our group)

- Goal: joint TMD fits
- Results from lattice
- Preliminary joint fit

TMD global fits: significant uncertainty from nonperturbative modeling of CS kernel



Determination of unpolarized TMD distributions from the fit of Drell-Yan and SIDIS data at N4LL

V. Moos, I. Scimemi, A. Vladimirov, P. Zurita, PRD 112 (2025) 3, 034501 [2503.11201]

2020–2025:

Significant progress in **quark CS kernel** calculations with lattice QCD (our group)

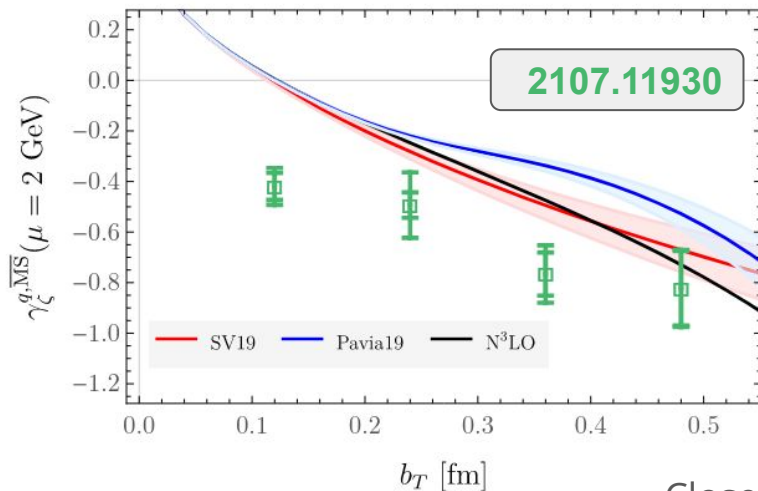
Shanahan, Wagman, Y.Zhao,
PRD 102 (2020) 1, 014511
[2003.06063]

Shanahan, Wagman, Y.Zhao
PRD 104 (2021), 2107.11930

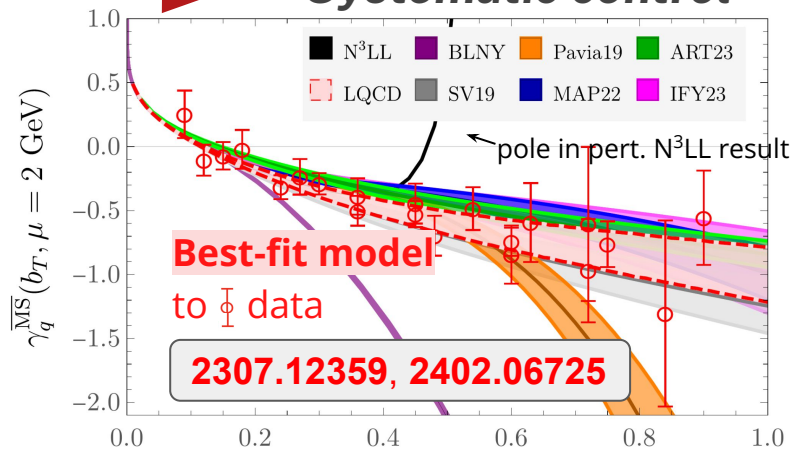
Avkhadiev, Shanahan,
Wagman, Y.Zhao,
PRD 108 (2023) 11, 114505
[2307.12359]

Avkhadiev, Shanahan,
Wagman, Y.Zhao,
PRL 132 (2024) 23, 231901
[2402.06725]

Proof of concept



Systematic control



Improvements:

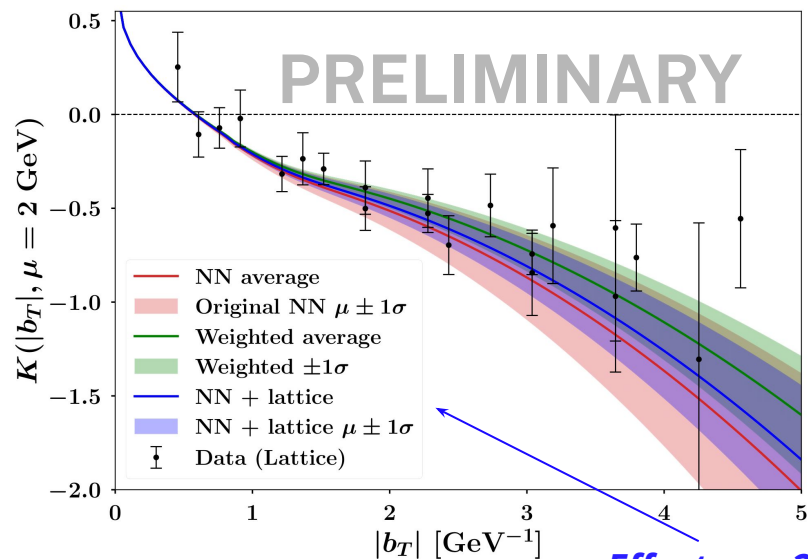
- Close-to-physical pion mass (540 MeV $\rightarrow$ $\sim$ 150 MeV)
- Increased b_T range (.48 fm $\rightarrow$ 0.90 fm)
- Quantified mixing from renormalization
- Improved LaMET matching (NLO $\rightarrow$ NNLL, renormalons, b_T corrections)
- **Continuum extrapolation (+ model fit to extrapolated data)**

Ongoing: joint fit to **lattice** + **experiment** [M.A.P. TMD]

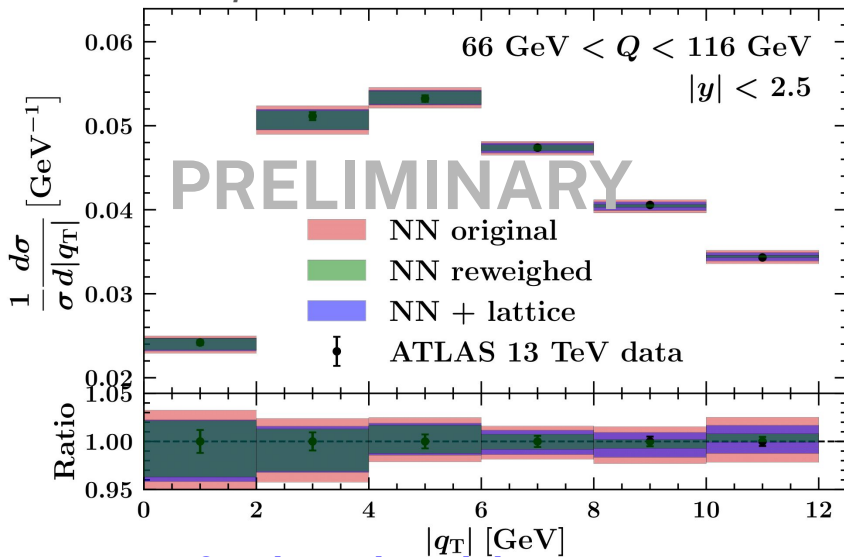
- **Experimental data [482 pts]** from full **TMD fit by MAP**
- **Lattice results [21 pts]** from the 3 ensembles
- Incorporate lattice results by (1) **reweighting** original replicas and (2) **refitting**:

A. Bacchetta, V. Bertone, C. Bissolotti, M. Cerutti, M. Radici, S. Rodini, L. Rossi,
PRL 135 (2025) 2, 021904 [2502.04166]

Impact on CS kernel:



Impact on cross-section:



Effect on g_2 parameter of CS kernel model:
shift central value by ~5%, reduce uncertainty by ~30%

Takeaways for the gluon CS kernel

- Noise from Wilson lines
 - Order of LaMET matching
 - ***Choice of TMD observable***
- } **See next talk (Yang Fu)**
+ Q&A and discussion

Calculation steps analogous for quark and gluon CS kernels:

X. Ji et. al., Phys. Lett. B 811 [1911.03840]

5. Continuum extrapolation

2. Fourier transform

1. Quasi-TMDs in position space

$$\gamma_{\zeta}^{(i)}(b_T, \mu) = \lim_{a \rightarrow 0} \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{\int_{-\infty}^{\infty} \frac{dP_1^z b^z}{2\pi} e^{-ixP_1^z b^z} \lim_{\ell \rightarrow \infty} [\tilde{\mathcal{R}}_{i/h}^{(s)\ell}(b^z, b_T, P_1^z, \mu, a)}{\int_{-\infty}^{\infty} \frac{dP_2^z b^z}{2\pi} e^{-ixP_2^z b^z} \lim_{\ell \rightarrow \infty} [\tilde{\mathcal{R}}_{i/h}^{(s)\ell}(b^z, b_T, P_2^z, \mu, a)} + \delta\gamma_{(i)}(\mu, x, P_1^z, P_2^z) + \text{p. c.}$$

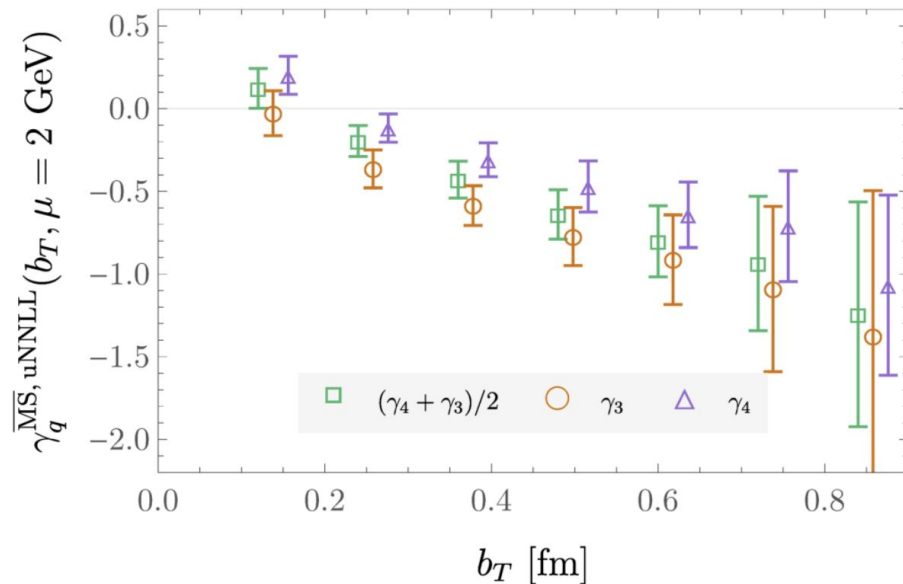
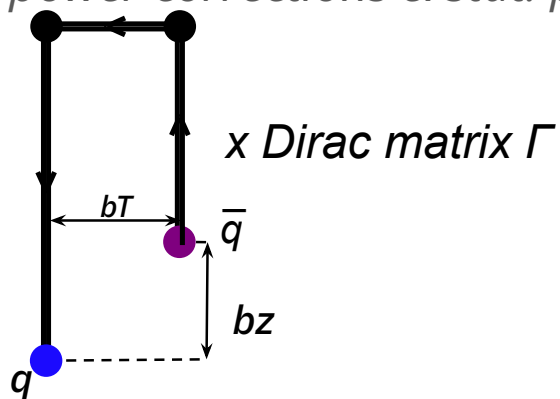
3. EFT matching

4. Power corrections

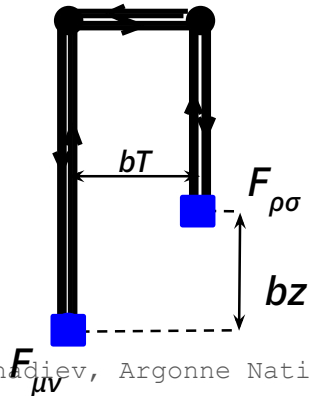
Takeaways for the gluon CS kernel calculation?

Choice of TMD observables

For quark CS kernel: operator choice affects power corrections & stat. precision



For gluon CS kernel, choices of tensor components (μ, ν, ρ, σ) in operator may differ by power corrections & stat. precision



Related talks:

1. William Good, Wed 9:00 am
2. Raza Sufian, Wed 2:15 pm
3. Yang Fu, Thu 12:30 pm

Takeaways for LaMET formalism



- Renormalon resummation
- ***bT*-dependent matching**

Renormalons in TMD matching

(numerical evidence)

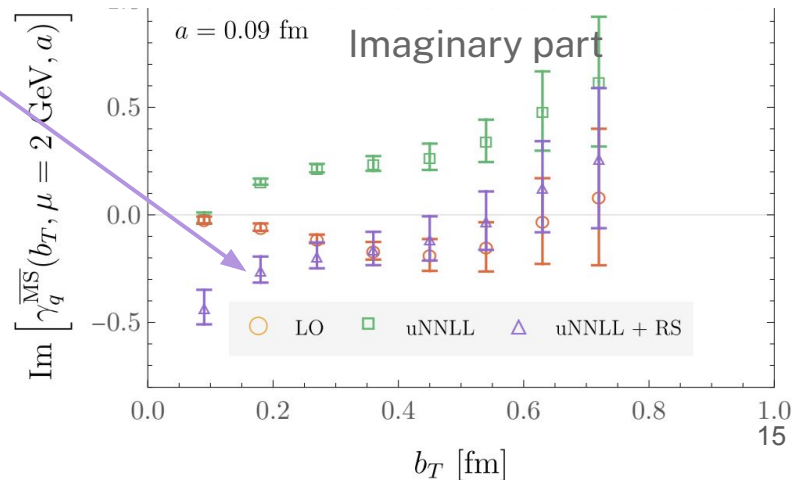
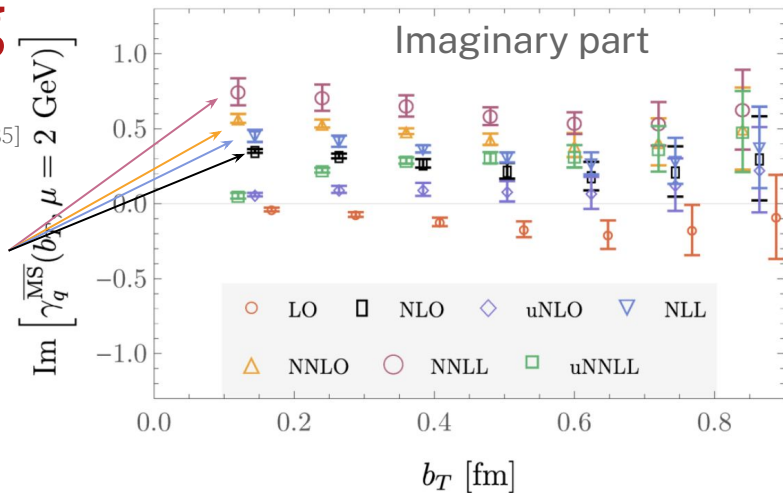
- The kernel is real-valued. M. A. Ebert et. al., JHEP 09, 037, [1901.03685]
Z.-F. Deng et. al, JHEP 09, [2207.07280]
- The kernel *estimate* with TMD WFs has an imag part, **increases w/ matching order**:
- Large b_T : IR renormalon; use **leading renormalon resummation (LRR)**: Y. Liu, Y. Su (LPC), JHEP02(204), [2311.06907]

$$C^{\text{LRR}}(p^z, \mu) = C(p^z, \mu) - R(p^z, \mu) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{p^z}\right)$$

$$R(p^z, \mu) = iN_m \frac{\mu}{p^z} \sum_{n=0}^{\infty} \beta_0^n \alpha_s^{n+1}(\mu) n!$$

= 0.552 for 4 flavors lowest-order β -function

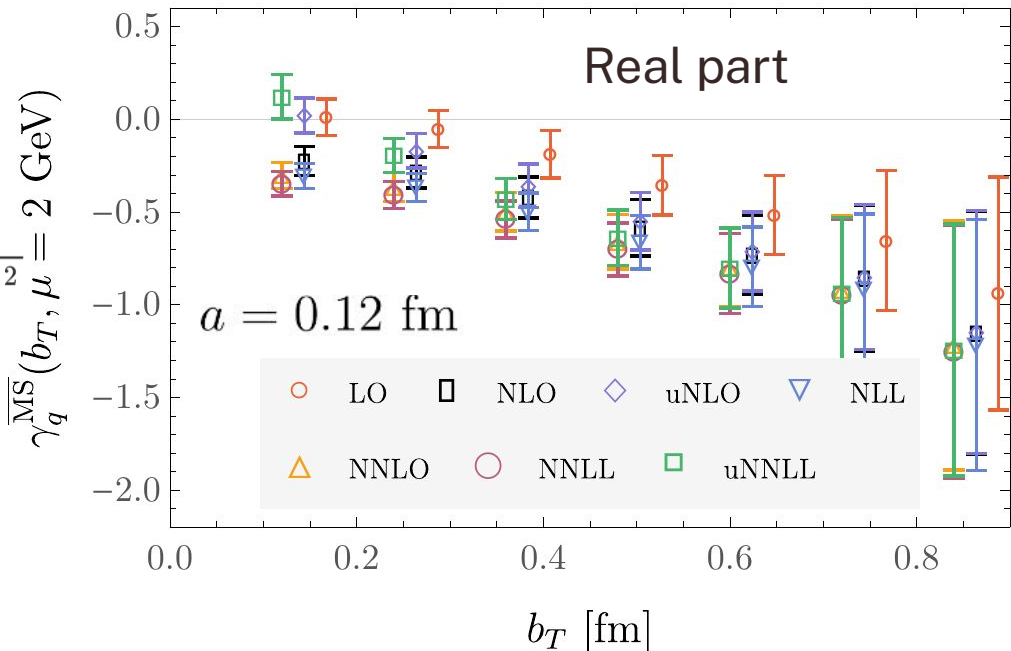
- At small b_T : due to b_T -dep. power corrections, use **unexpanded matching** (next slide)



bT -dependence of power corrections (numerical evidence) [1/3]

- TMD LaMET matching needs $\mathbf{qT} \ll \mathbf{Pz}$
- Lattice results at small bT have $\mathbf{qT} \sim \mathbf{Pz} \Rightarrow$ power corrections $\frac{1}{(xP^z b_T)^2}, \mu = 2 \text{ GeV}$
- Full matching in this regime is
 - (1) **bT -dependent** and
 - (2) **convolutional** in $\mathbf{x}$:

$$\tilde{\phi}(\mathbf{x}, b_T, \mu, P^z) = \int_0^1 d\mathbf{y} H_\phi(\mathbf{y}, \mathbf{x}, b_T, \mu, P^z) \phi(\mathbf{y}, \mu)$$



***bT*-dependence of power corrections** (numerical evidence) [2/3]

- Full ***bT*-dependent convolutional** matching: $H_\phi(y, x, \mu, P^z)$

Avkhadiev, Shanahan, Wagman, Y. Zhao

PRD 108 (2023) 11, 114505 [2307.12359]

- So far: isolate the *bT*-dependent **multiplicative** part:

D. Bollweg, X. Gao, J. He, S. Mukherjee, Y. Zhao

[2504.04625]

$$H_\phi(y, x, b_T, \mu, P^z) \xrightarrow{P^z b_T \gg 1} \delta(y - x) \left(\mathbf{C}_\phi^{(u)}(xP^z, b_T, \mu) \mathbf{C}_\phi^{(u)}((1-x)P^z, b_T, \mu) \right) \\ + \cancel{\delta H_\phi(y, x, b_T, \mu, P^z)}$$

Drop convolutional piece (for now)

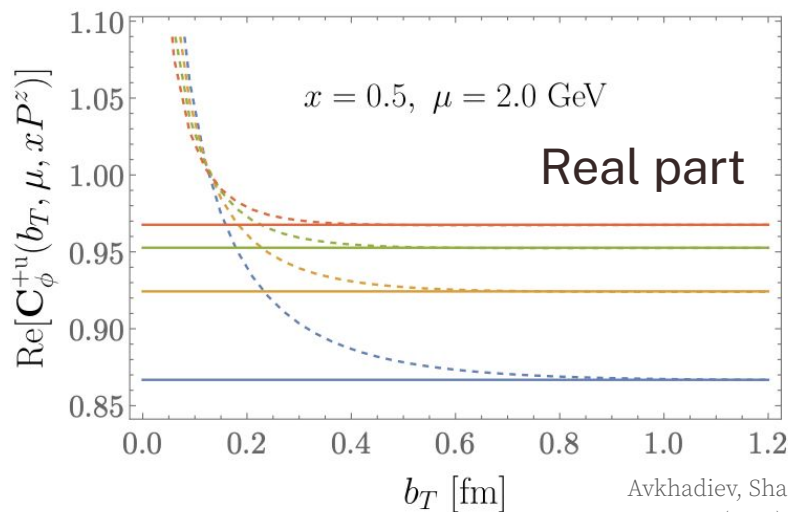
- Defines the **“unexpanded” matching** in this work:

$$\mathbf{C}_\phi^{(u)}(xP^z, b_T, \mu) = \boxed{C_\phi(xP^z, \mu)} + \delta C_\phi(xP^z, b_T, \mu)$$

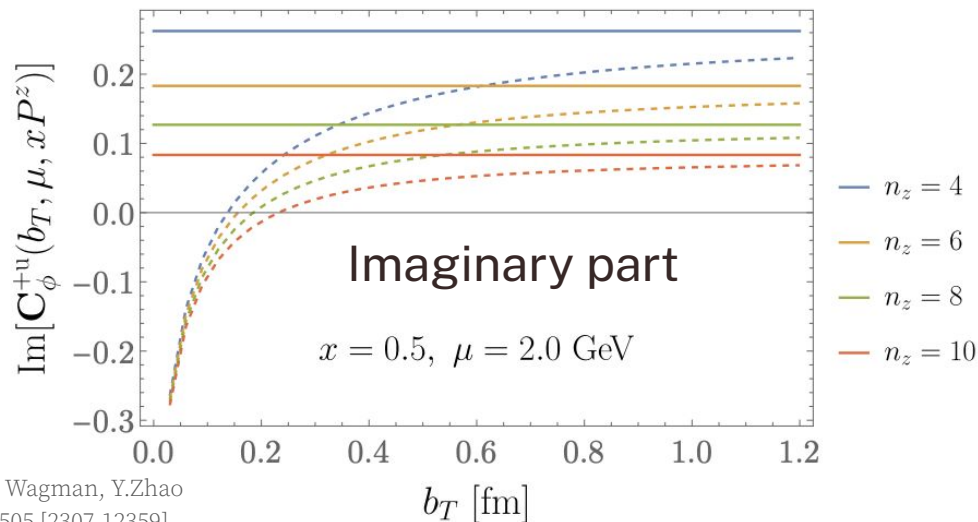
*Usual, *bT*-independent LaMET matching*

bT -dependence of power corrections (numerical evidence) [3/3]

- This “unexpanded” matching mitigates bT -dependent power corrections
[(--)**dashed** = unexpanded, (-)**solid** = bT -independent]
- Higher bT sensitivity in the imaginary part of estimate — **explains imaginary part at small bT after renormalon subtraction.**



Avkhadiev, Shanahan, Wagman, Y.Zhao
PRD 108 (2023) 11, 114505 [2307.12359]



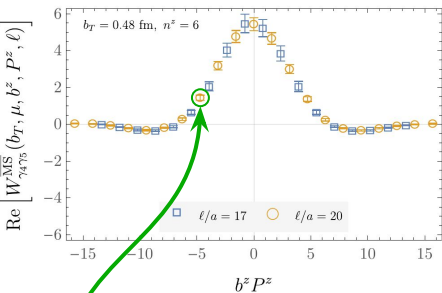
Summary and outlook

1. Nonperturbative input to quark, gluon CS kernels crucial to constrain TMDs
2. Quark CS kernel: **global joint TMD fit underway** incorporating lattice results
3. Takeaways for gluon CS kernel:
 - Challenging **statistical noise** [$\Rightarrow$ develop & apply Coulomb-Gauge (CG) methods]
 - Important **choice of TMD observable** [*noise vs. power corrections*]
 - Need for **higher-order matching** [$\Rightarrow$ higher-order CG matching]
4. Takeaways for LaMET formalism:
need bT -dependent convolutional matching.

Q&A

Steps of quark CS kernel calculation:

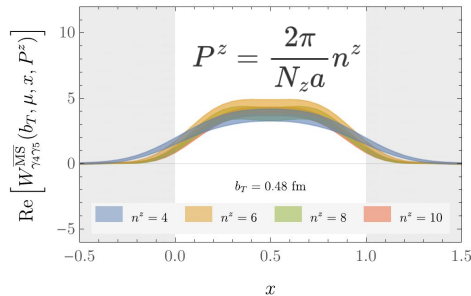
Calculate position-space MEs



$$\sum_{\Gamma'} Z_{\Gamma\Gamma'}(\mu) \lim_{\ell \rightarrow \infty} W_{\mathcal{O}}^{\Gamma'}(b^z, b_T, \ell, P_1^z)$$

Each point is a separate matrix element calculation

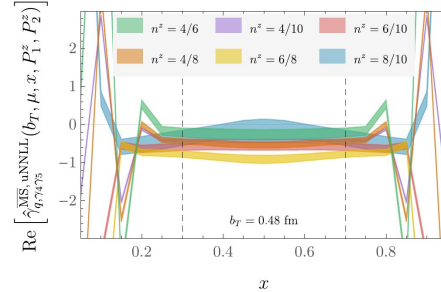
Fourier transform (FT) to momentum-space MEs



$$\int db^z e^{ib^z x P_1^z} P_1^z N_{\Gamma}(P_1^z)$$

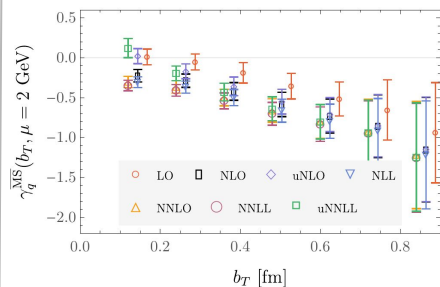
$$\sum_{\Gamma'} Z_{\Gamma\Gamma'}(\mu) \lim_{\ell \rightarrow \infty} W_{\mathcal{O}}^{\Gamma'}(b^z, b_T, \ell, P_1^z)$$

Form ratios of MEs + match + fit in x



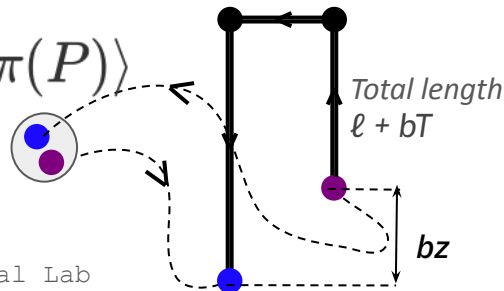
$$\gamma_q(b_T, \mu) = \lim_{a \rightarrow 0} \frac{1}{\ln(P_1^z/P_2^z)} \ln \left[\frac{\int db^z e^{ib^z x P_1^z} P_1^z N_{\Gamma}(P_1^z)}{\int db^z e^{ib^z x P_2^z} P_2^z N_{\Gamma}(P_2^z)} \right] + \delta\gamma_q(\mu, P_1^z, P_2^z) + \text{p.c.}$$

...repeat for each bT (... and for each a)



Takeaways for the gluon CS kernel calculation?

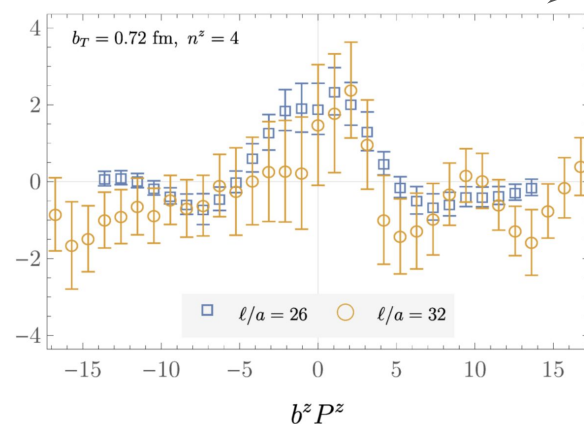
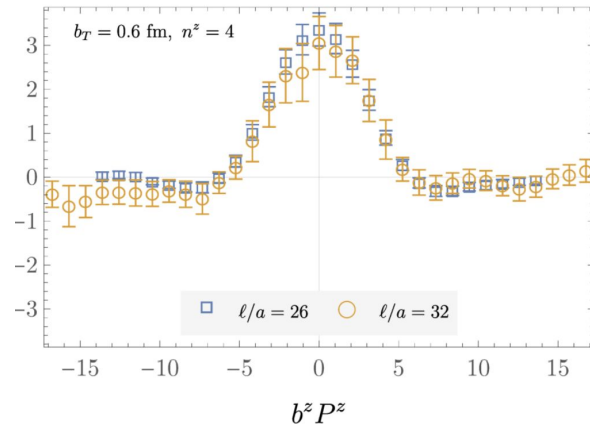
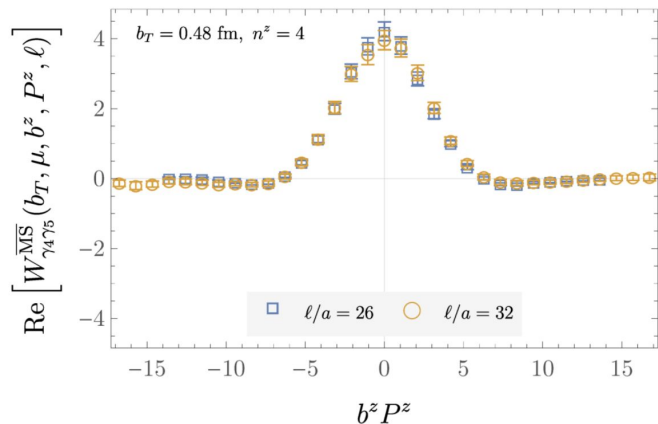
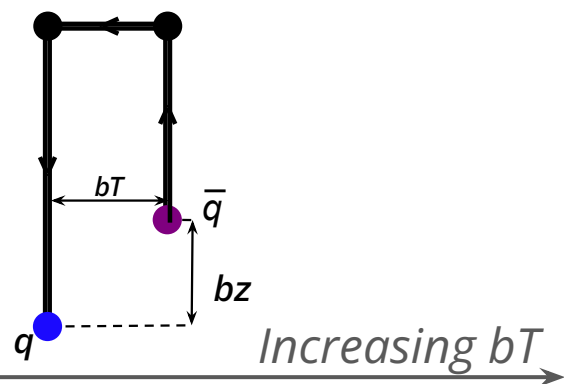
$$\tilde{\phi}_{\Gamma} = \langle 0 | O_{\Gamma} | \pi(P) \rangle$$



Statistical noise from Wilson lines

For quark CS kernel:

- Statistical noise grows with length of Wilson line.
- Compare $\ell=26$ vs $\ell=32$ with increasing bT :

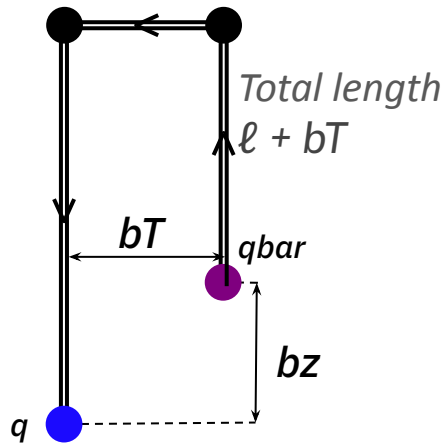


For gluon CS kernel, expect even greater challenge from statistical noise.

Coulomb-Gauge (CG) methods promising, but need to understand systematics.

Lattice QCD matrix elements (MEs) for quark CS kernel

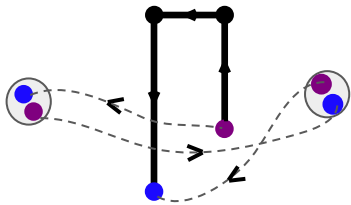
- Defined by operators $\mathcal{O}_\Gamma(b_T, b^z, y, \ell)$ with staple-shaped Wilson lines.
- Calculated for each P_z, b_T, b_z, ℓ — expensive!



2 equivalent options for MEs:

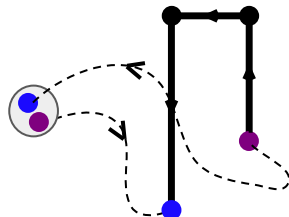
1. TMD Beam function

$$\tilde{B}_\Gamma = \langle \pi(P) | \mathcal{O}_\Gamma | \pi(P) \rangle$$



2. TMD Wavefunction (WF)

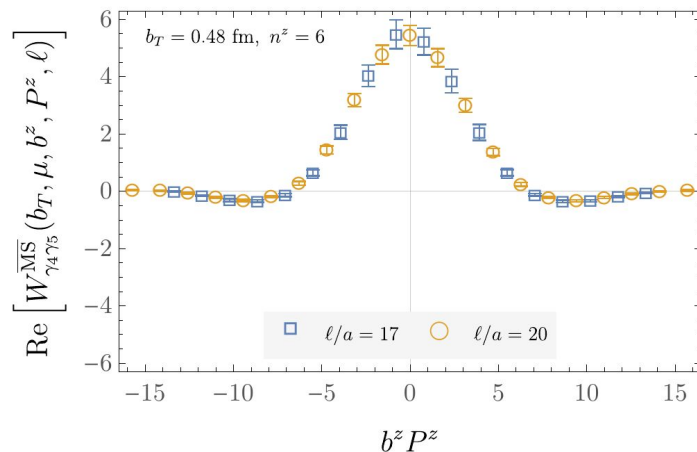
$$\tilde{\phi}_\Gamma = \langle 0 | \mathcal{O}_\Gamma | \pi(P) \rangle$$



Choose #2: less fermion propagators $\Rightarrow$ cheaper

- Linear divergences** in MEs $\sim \ell + b_T$
- Subtract the divergences in **quasi-TMD WF ratios**

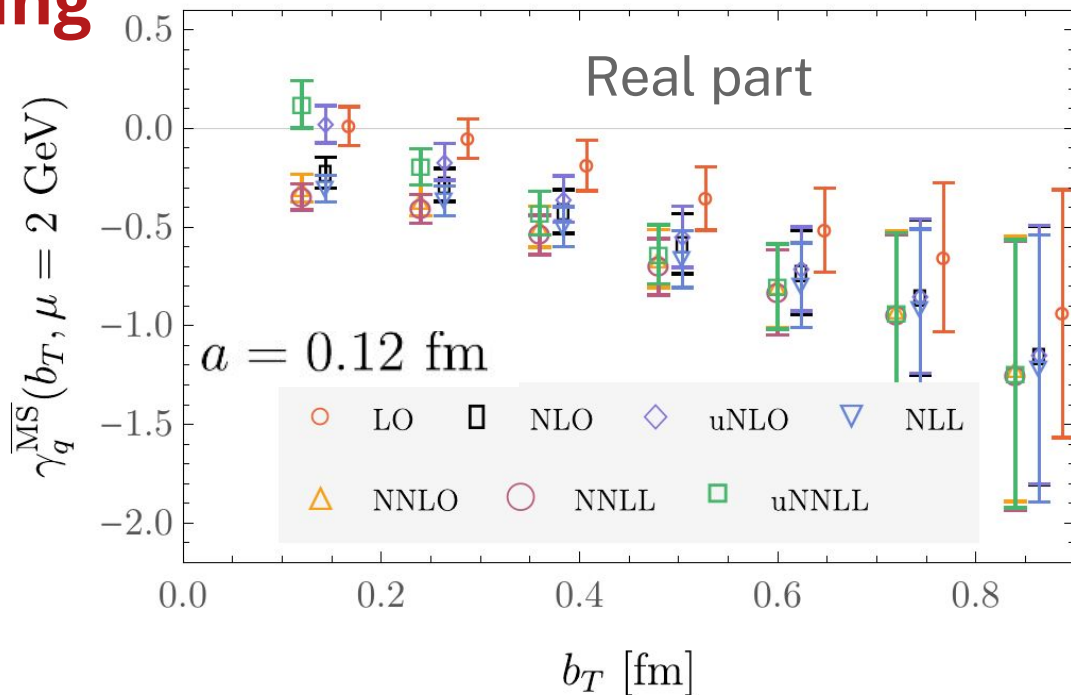
$$W_\Gamma^{(0)}(b_T, b^z, P^z, \ell) = \frac{\tilde{\phi}_\Gamma(b_T, b^z, P^z, \ell)}{\tilde{\phi}_{\gamma^4 \gamma^5}(b_T, 0, 0, \ell)}$$



Order of LaMET matching

For quark CS kernel:

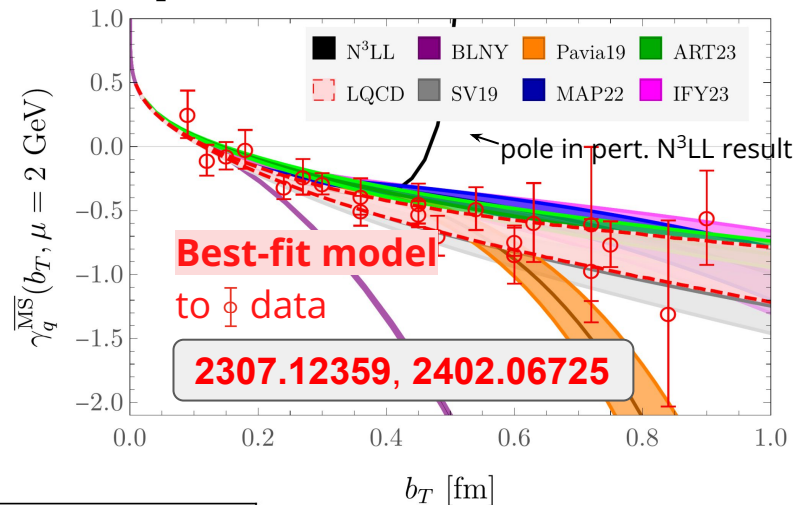
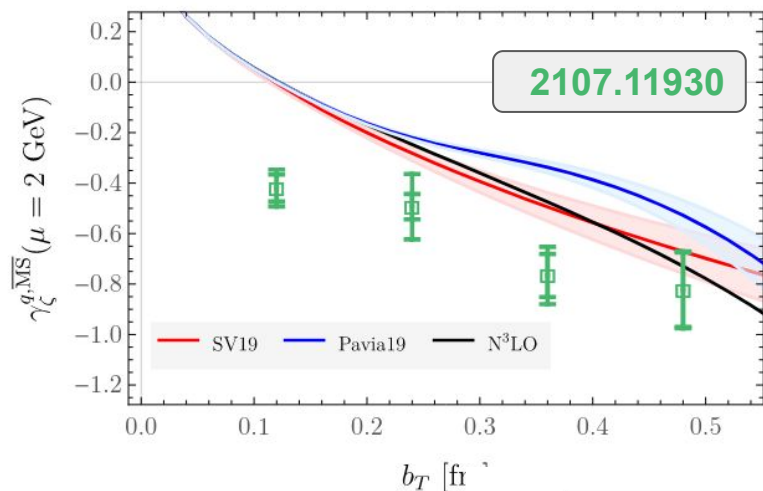
- Can access momenta up to $P_z \sim 2$ GeV.
- Results shift with **LO** $\rightarrow$ **NLO** in matching
- Need NNLO + resummations to confirm convergence within uncert.



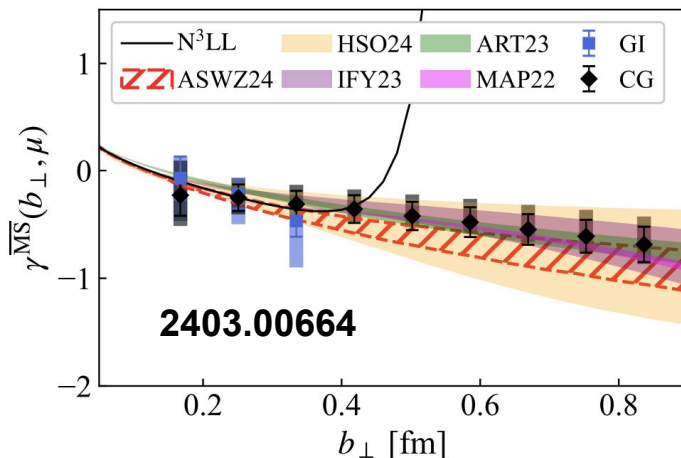
***For gluon CS kernel:** up to NLO matching calculated;
no matching yet for Coulomb-Gauge (CG) method.*

Observable	Collaboration	Power corrections	Matching	Fourier Transform	Operator Mixing	Continuum extrapolation
Quasi-Beam Functions	SWZ 20 PRD 102 (2020)	$\frac{(m_\pi^{\text{val}})^2}{(P_{\text{max}}^z)^2} = 0.219$ <i>quenched</i>	LO	Yes	✓ (RI'-MOM)	✗
	SWZ 21 PRD 104 (2021)	$\frac{(m_\pi^{\text{val}})^2}{(P_{\text{max}}^z)^2} = 0.129$	NLO	Yes	✓ (RI'-MOM)	✗
Quasi-TMD Wavefunctions (also 2403.00664) Related talks: Jin-Xin Tan, Thu 11:00 am	LPC 20 PRL 125 (2020)	$\frac{(m_\pi^{\text{val}})^2}{(P_{\text{max}}^z)^2} = 0.067$	LO	N/A	✓ (RI'-MOM)	✗
	PKU/ETMC 21 PRL 128 (2022)	$\frac{(m_\pi^{\text{val}})^2}{(P_{\text{max}}^z)^2} = 0.063$	LO	N/A	✗	✗
	LPC 22 JHEP 08 (2023)	$\frac{(m_\pi^{\text{val}})^2}{(P_{\text{max}}^z)^2} = 0.067$	NLO	Yes	✗	✗
	ASWZ 23/24 PRD 108 (2023) PRL 132 (2024)	$\frac{(m_\pi^{\text{val}})^2}{(P_{\text{max}}^z)^2} = 0.007$	NNLL	Yes	✓ (RIx-MOM)	✓ (2024)
Mellin Moments of Quasi-TMDs	SVZES 21 JHEP 08 (2021) (also 2302.06502)	$\frac{(m_\pi^{\text{val}})^2}{(P_{\text{max}}^+)^2} = 0.035$	NLO	N/A	✗	✗

Progress in systematic control and precision



Highly encouraging results for quark CS kernel with CG calculations: extends b_T range from $\sim 0.4 \text{ fm}$ to $>1 \text{ fm}$ on just 64 configurations (vs. $\sim 500 \times 3$ configurations above on the right)



Calculations of quark vs. gluon CS kernel differ by operator and matrix element

Quark CS kernel — completed

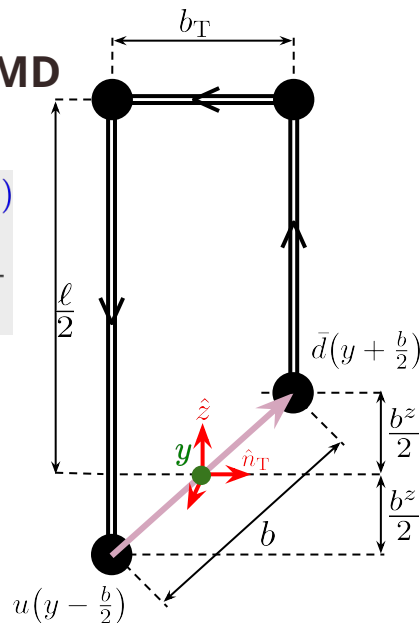
Computed using **quasi-TMD wavefunctions (WFs)**:

$$W_{\Gamma}(b_T, b^z, \ell, P^z) \quad (\Gamma \in \{\gamma^4 \gamma^5, \gamma^5\})$$

$$= \frac{\langle 0 | \mathcal{O}_{\Gamma}(b_T, b^z, \ell, P^z) | \pi(P^z) \rangle}{\langle 0 | \mathcal{O}_{\gamma^4 \gamma^5}(b_T, 0, \ell, 0) | \pi(0) \rangle}$$

$P_z=0, b_z=0$ matrix elt
to subtract divergences linear
in ℓ

Account for renormalization-
induced mixing between Γ
structures



Gluon CS kernel — ongoing

Computed using
**quasi-TMD beam
functions**:

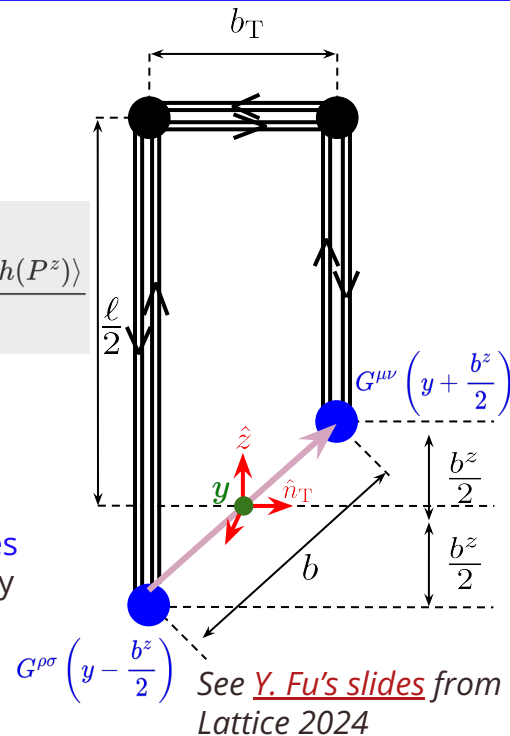
$$P^{\mu} P^{\nu} \tilde{B}_{g/h}^{\rho\sigma}(b^z, b_T, P^z, \ell)$$

$$= \frac{\langle h(P^z) | \mathcal{O}_g^{\mu\nu\rho\sigma}(b_T, b^z, 0, \ell) | h(P^z) \rangle}{\sqrt{Z_A(b_T, \ell)}}$$

Square root of adjoint
Wilson loop to subtract
divergences linear in ℓ

Can choose **Lorentz indices**
that lead to multiplicatively
renormalizable operators

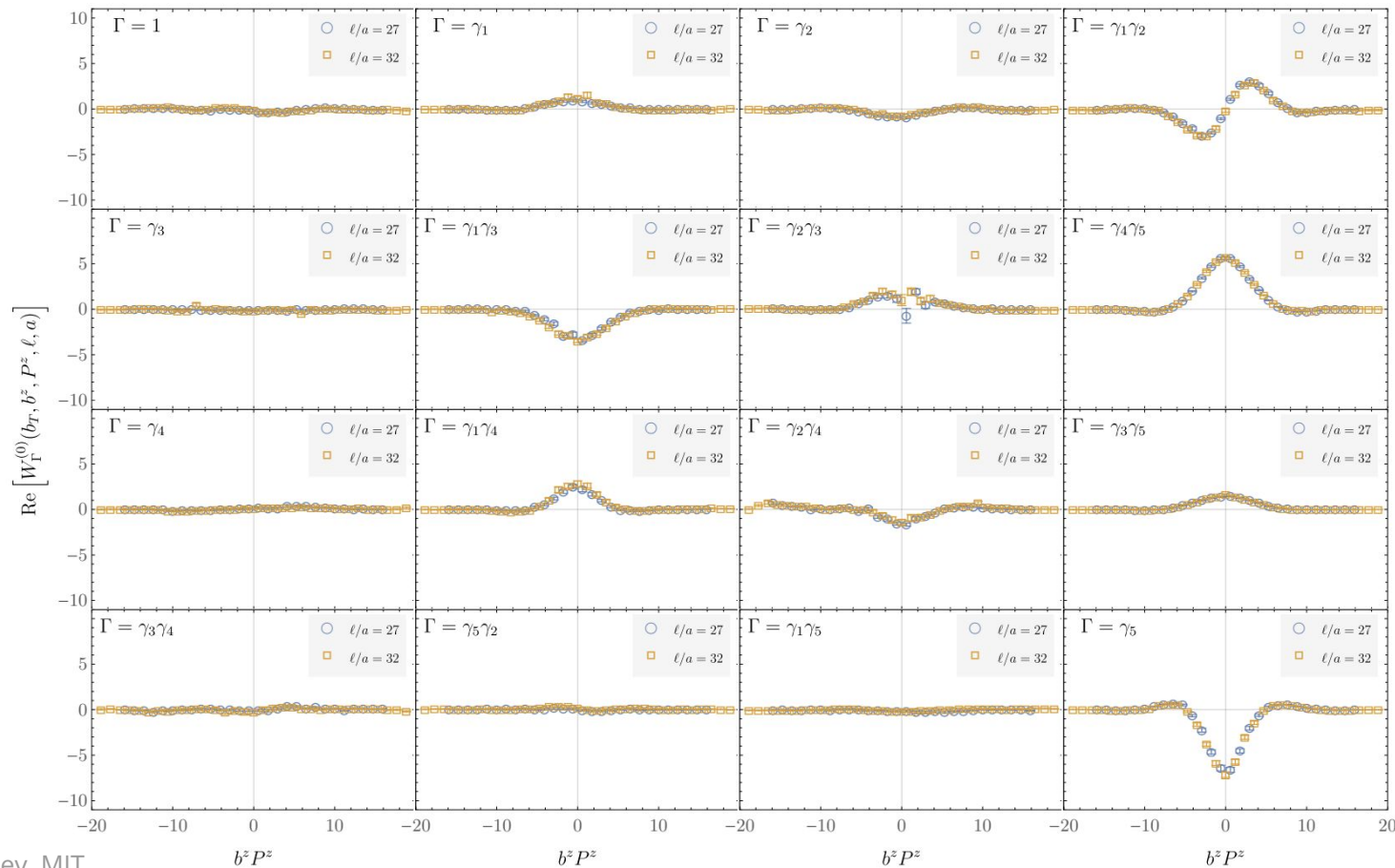
J-H. Zhang, X. Ji,
Schäfer, W. Wang, S. Zhao,
PRL 122 (2019) [1808.10824]



See Y. Fu's slides from
Lattice 2024

MEs with all 16 Dirac structures calculated

$a = 0.09$ fm, $b_T = 0.36$ fm, $P^z = 1.3$ GeV



Mixing effects quantified with RIxMOM

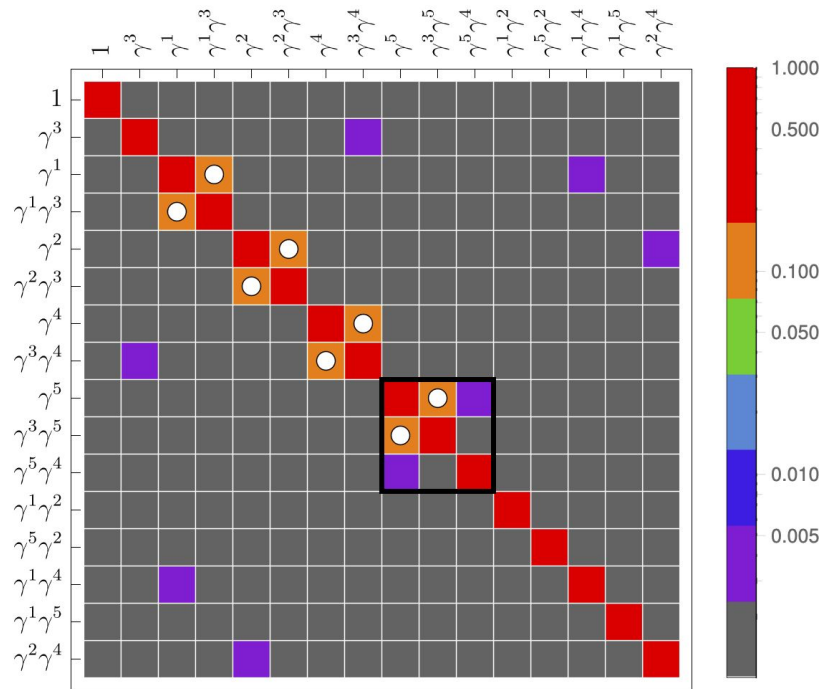
- Calculation of mixing effects in RIxMOM independent of staple geometry.

$$W_{\Gamma}^{\overline{\text{MS}}}(b_T, \mu, b^z, P^z, \ell) = \sum_{\Gamma'} Z_{\Gamma\Gamma'}^{\overline{\text{MS}}}(\mu) W_{\Gamma}^{(0)}(b_T, b^z, P^z, \ell)$$

- Full 16x16 mixing matrix computed

$$\mathcal{M}_{\Gamma\Gamma'}^{\text{RI/xMOM}}(p_R, \xi_R, a) \equiv \frac{\text{Abs}[Z_{\Gamma\Gamma'}^{\text{RI/xMOM}}(p_R, \xi_R, a)]}{\frac{1}{16} \sum_{\Gamma} \text{Abs}[Z_{\Gamma\Gamma}^{\text{RI/xMOM}}(p_R, \xi_R, a)]}$$

- Dominant mixings consistent with lattice perturbation theory at 1-loop.*



$$p_R^\mu = \frac{2\pi}{L} \times (0, 0, 10, 0), \xi = 0.24 \text{ fm}$$

X. Ji, et. al, PRL 120 (2018), [1706.08962]

*M. Constantinou et al., PRD 99 (2019), [1901.03862]

J. Green et. al, PRL 121 (2018), [1707.07152]

Y. Ji et. al., PRD 104 (2021), [2104.13345]

J. Green et. al, PRD 101 (2020), [2002.09408]

C. Alexandrou et al., [2305.11824]

Mixing reduced at finer lattice spacings, as expected

