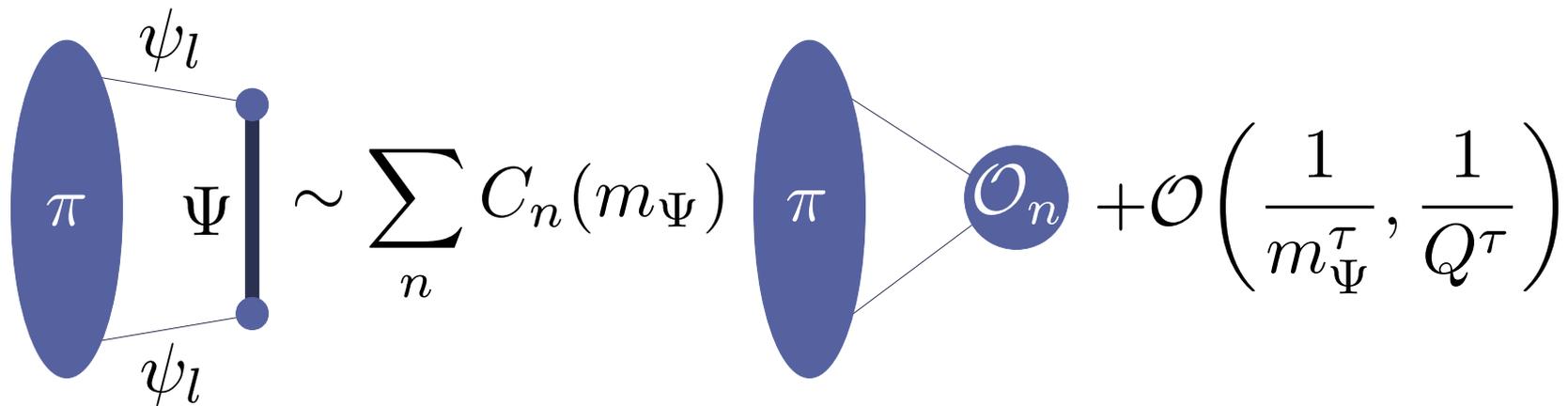


# Fourth Mellin moment of pion LCDA using the HOPE method



$$\pi \text{ (with } \psi_l \text{ lines and } \Psi \text{)} \sim \sum_n C_n(m_\Psi) \pi \text{ (with } \mathcal{O}_n \text{)} + \mathcal{O}\left(\frac{1}{m_\Psi^\tau}, \frac{1}{Q^\tau}\right)$$

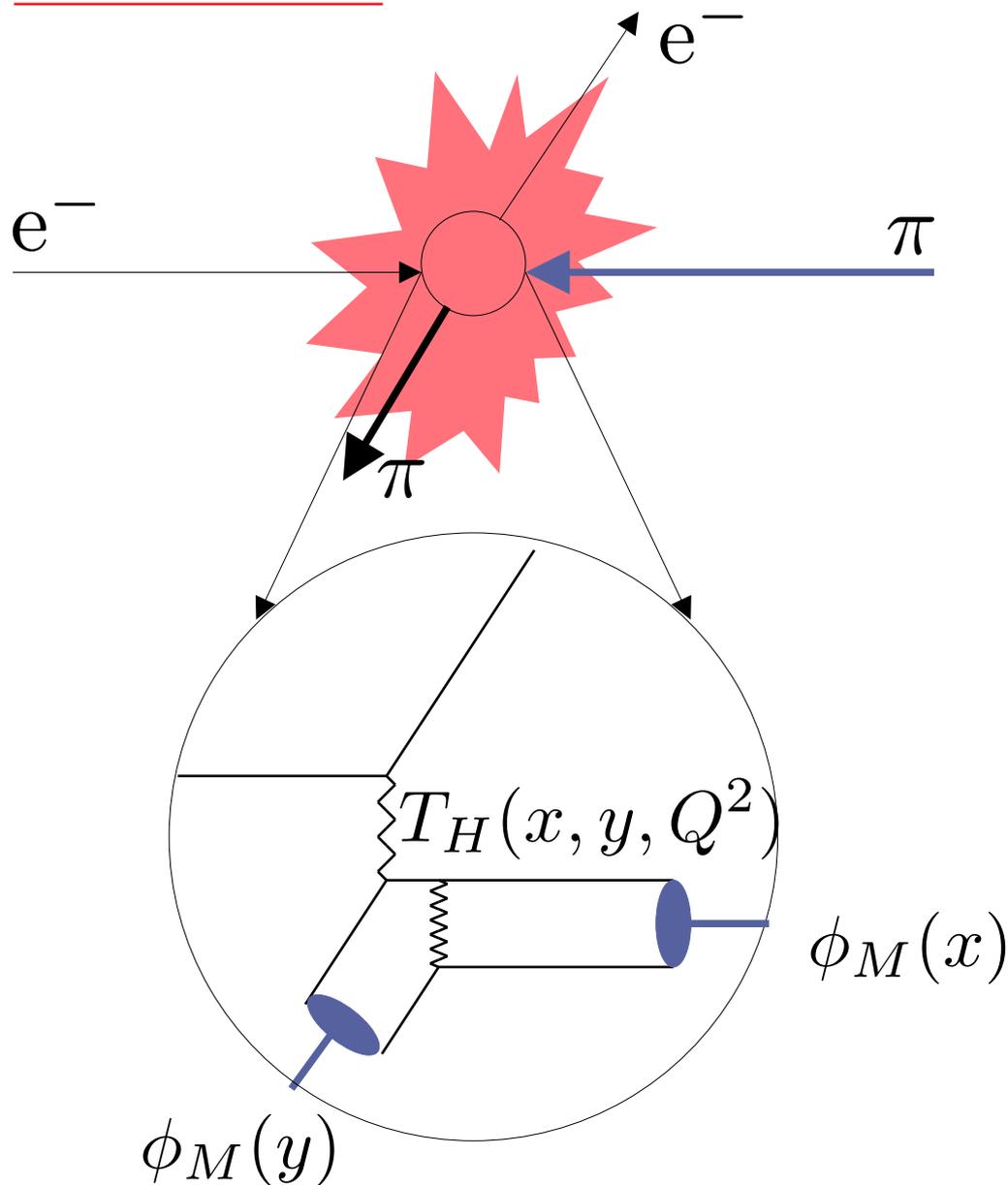
**Robert Perry**

arXiv:2509.04799 [hep-lat]

Alex Chang, William Detmold, Matias Gutierrez Escobari, Anthony V. Grebe, Issaku Kanamori, C.-J. David Lin Yong Zhao

10 October, 2025

# Exclusive processes in QCD



Exclusive processes may be  
factorized Phys.Rev.Lett. 43 (1979) 246, Phys.Rev.D 22 (1980)  
2157, Phys.Lett.B 94 (1980) 245-250

Convolution of pion wave  
function and short-distance hard  
kernel,  $T_H(x, y, Q^2)$

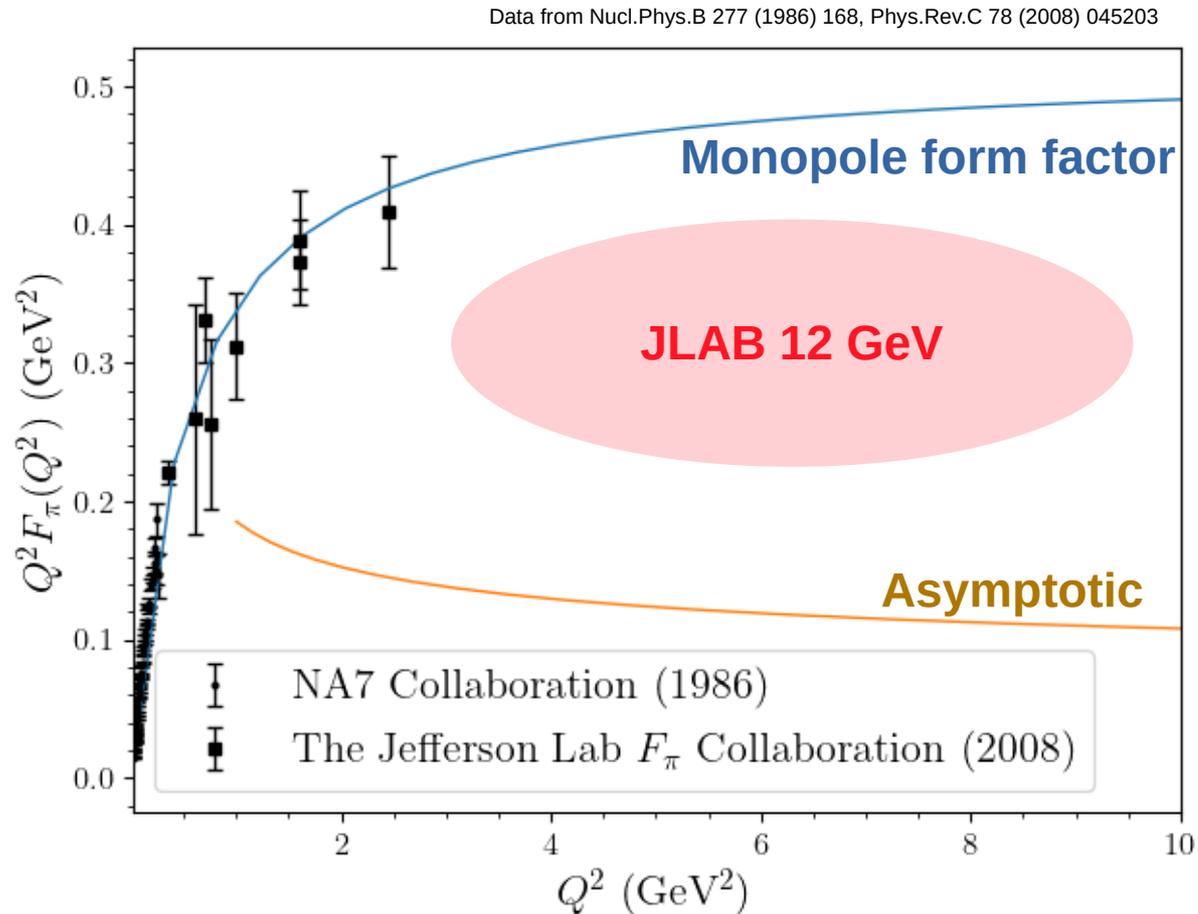
Large- $Q^2$  limit is

$$F_\pi(Q^2) = \frac{16\pi \alpha_S(Q^2)}{Q^2} f_\pi^2 \omega_\phi^2(Q^2)$$

where

$$\omega_\phi = \frac{1}{3} \int_0^1 dx \frac{\phi(x, \mu^2)}{x} \rightarrow 1$$

# Experimental status

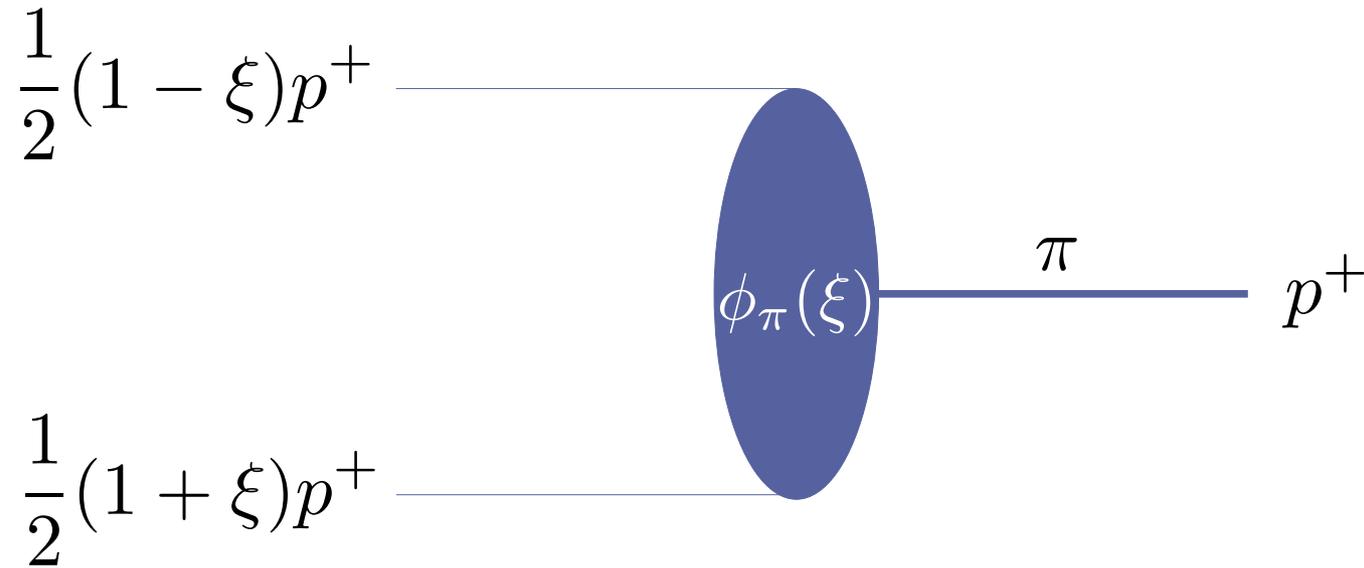


Need to know LCDA at appropriate scale.

Requires **non-perturbative** calculation

## The light-cone distribution amplitude

---



- LCDA matrix element:

$$\langle \Omega | \bar{\psi}(z_-) \gamma_\mu \gamma_5 W[z_-, -z_-] \psi(-z_-) | \pi(\mathbf{p}) \rangle = i p_\mu f_\pi \int_{-1}^1 d\xi e^{i\xi p^+ z_-} \phi_\pi(\xi, \mu^2)$$

- Probability amplitude for converting a meson into a collinear quark anti-quark pair with momentum fraction  $x$  and  $1-x$ , respectively.

# Local Operators

---

“Traditional” approach: OPE

$$\mathcal{O}_1(z)\mathcal{O}_2(0) \sim \sum_n C_n(z^2)\mathcal{O}_n(0)$$

For LCDA matrix element, twist-2 operators lead to:

$$\langle \Omega | \mathcal{O}_n | \pi(\vec{p}) \rangle = \langle \Omega | \mathcal{O}_n | \pi(\vec{p}) \rangle = f_\pi \langle \xi^{n-1} \rangle [p^{\mu_1 \dots \mu_n} - \text{traces}]$$

Coefficients related to integral (Mellin) moments of LCDA:

$$\langle \xi^n \rangle(\mu^2) = \frac{1}{2} \int_{-1}^1 d\xi \xi^n \phi_\pi(\xi, \mu^2)$$

**BUT:** lattice breaks Lorentz symmetry: non-perturbative mixing of operators!

Continuum limit challenging\*

# Introduction to HOPE

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Idea due to Detmold & Lin Phys.Rev.D 73 (2006) 014501

Compute hadronic matrix element using lattice QCD:

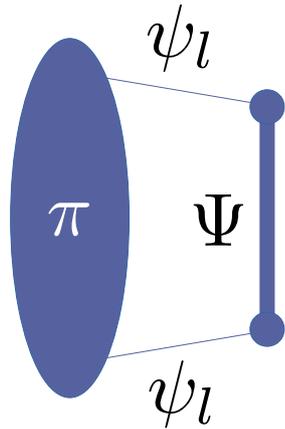
$$V^{\mu\nu}(p, q) = \int d^4z e^{iq \cdot z} \langle \Omega | T \{ J_{\Psi}^{\mu}(z/2) J_{\Psi}^{\nu}(-z/2) \} | \pi(\mathbf{p}) \rangle$$

Employ current-current operator with fictitious heavy-quark species:

$$J^{\mu} \rightarrow J_{\Psi}^{\mu} = \bar{\Psi} \Gamma^{\mu} \psi_l + \bar{\psi}_l \Gamma^{\mu} \Psi$$

- ✓ Hadronic matrix element has well-defined continuum limit
- ✓ Can compute directly using LQCD
- ✓ Heavy quark limit: Wilson line

Compute **H**eavy quark **O**perator **P**roduct **E**xpansion (**HOPE**):



$$V^{\mu\nu}(p, q) = -\frac{2if_{\pi}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}q_{\beta}}{\tilde{Q}^2} [1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \dots]$$

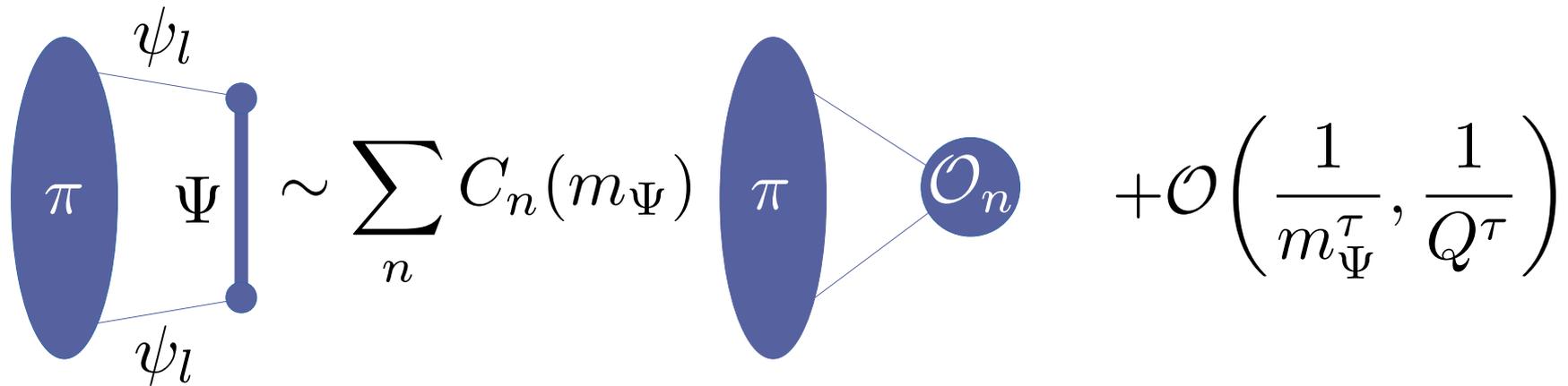
$$\tilde{Q}^2 = -q^2 + m_{\Psi}^2$$

Hard scale

$$\tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2} = p_{\mu} \frac{2q^{\mu}}{\tilde{Q}^2}$$

Small expansion parameter

Compute **H**eavy quark **O**perator **P**roduct **E**xpansion (**HOPE**):



$$\pi \Psi \sim \sum_n C_n(m_\Psi) \pi \mathcal{O}_n + \mathcal{O}\left(\frac{1}{m_\Psi^\tau}, \frac{1}{Q^\tau}\right)$$

$$V^{\mu\nu}(p, q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2} [1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \dots]$$

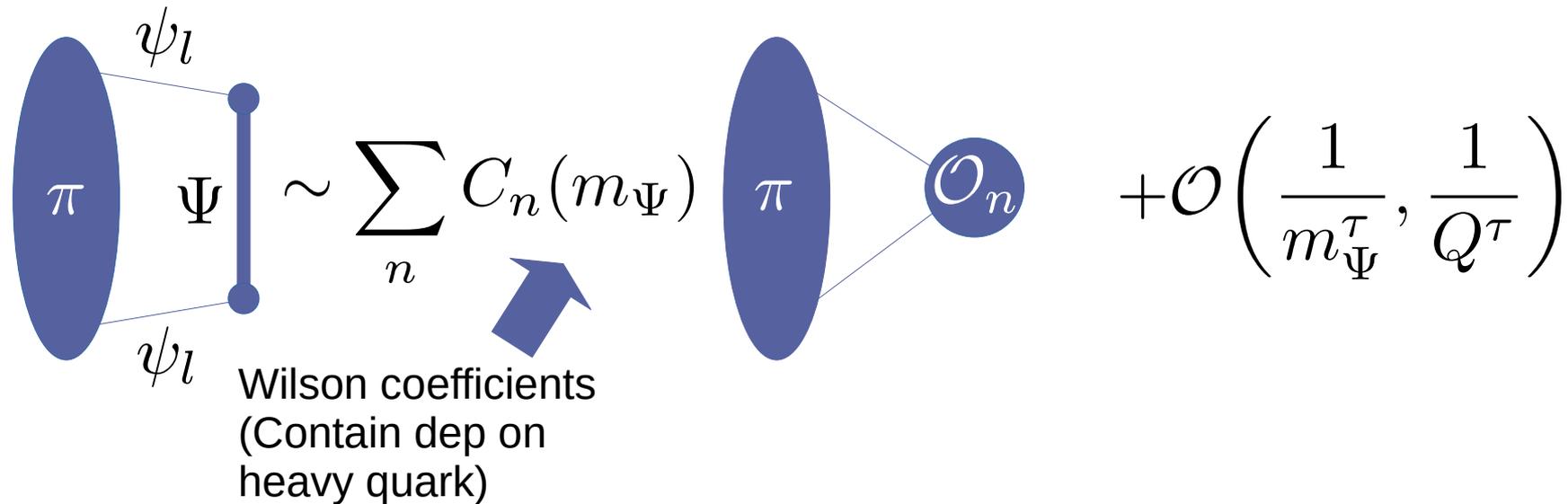
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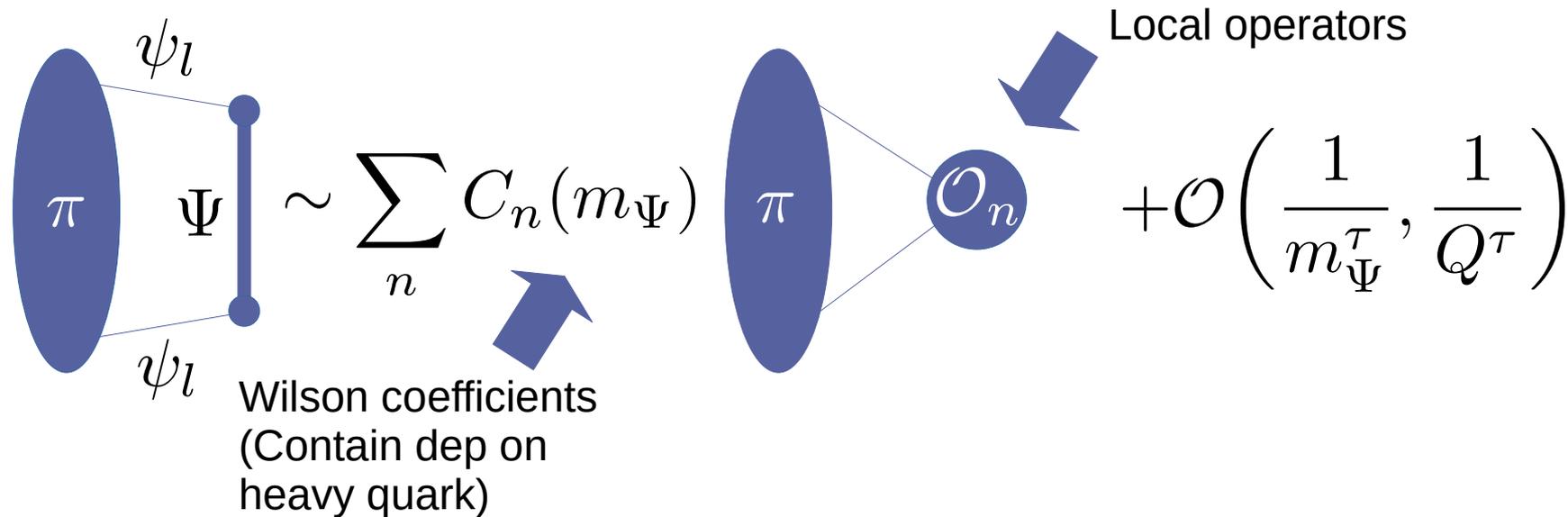
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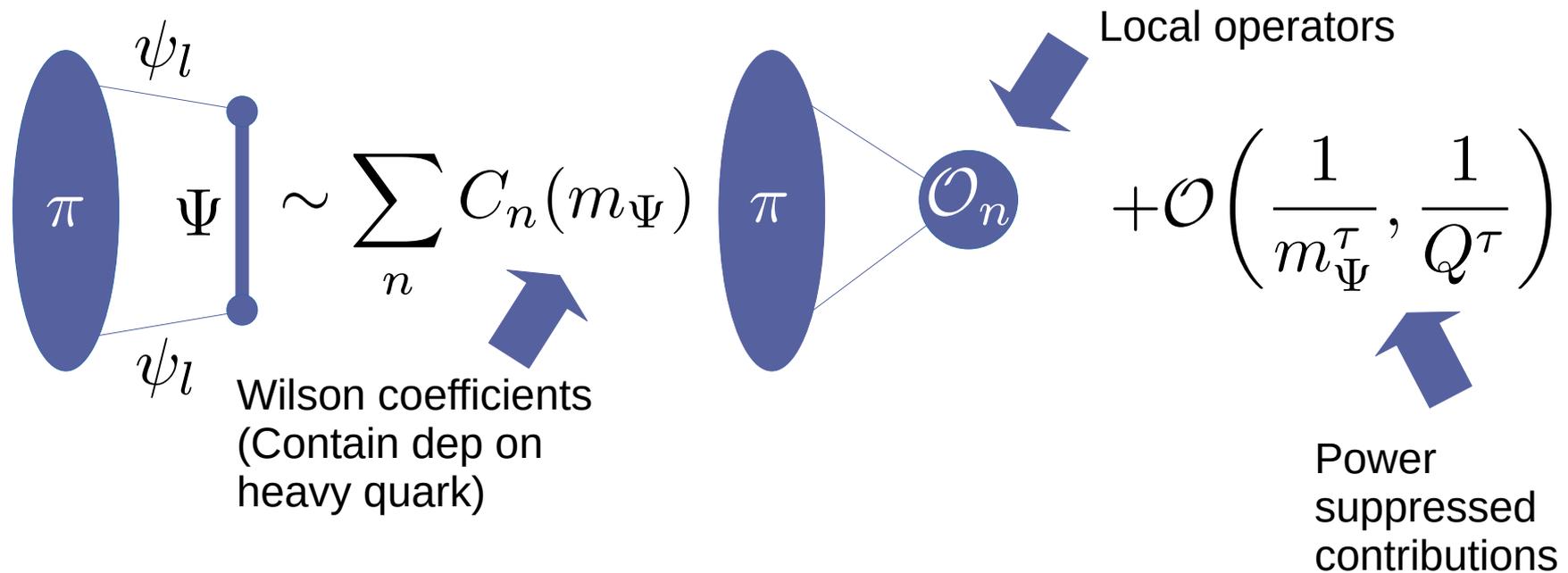
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$$V^{\mu\nu}(p, q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2} [1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \dots]$$

$$\tilde{Q}^2 = -q^2 + m_\Psi^2$$

Hard scale

$$\tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2} = p_\mu \frac{2q^\mu}{\tilde{Q}^2}$$

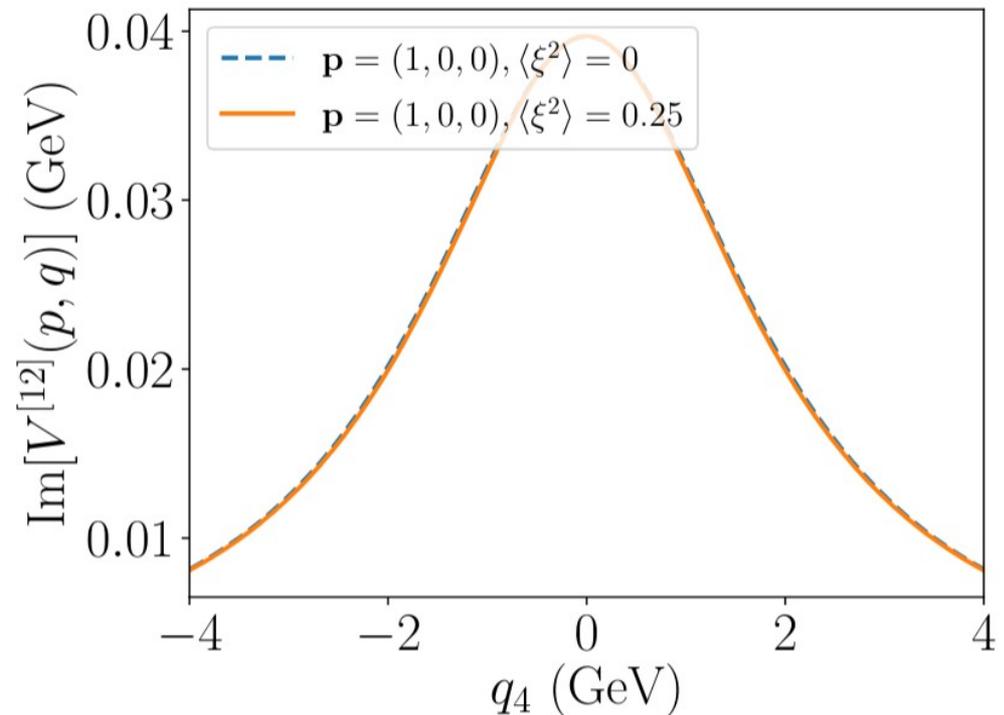
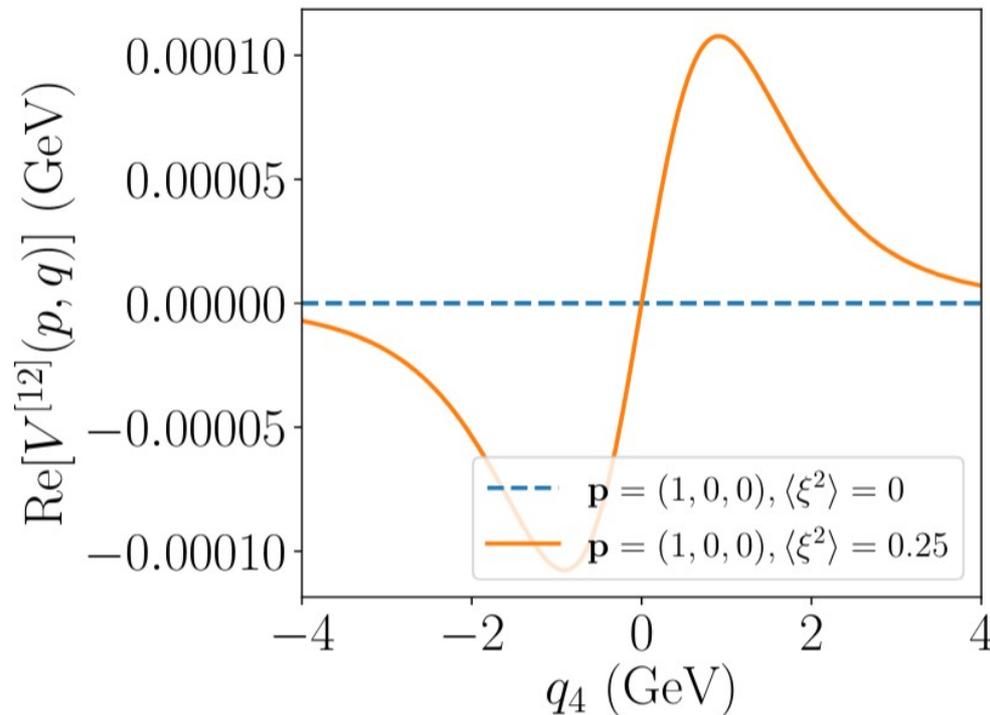
Small expansion parameter

# LaMET without ET

$$V^{\mu\nu}(p, q) = K^{\mu\nu} [1 + \langle \xi^2 \rangle \tilde{\omega}^2 + \langle \xi^4 \rangle \tilde{\omega}^4 + \dots]$$

Choose “special kinematics”

$K^{\mu\nu}$  purely imaginary. If  $\tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2}$  is complex, can isolate  $\langle \xi^2 \rangle$

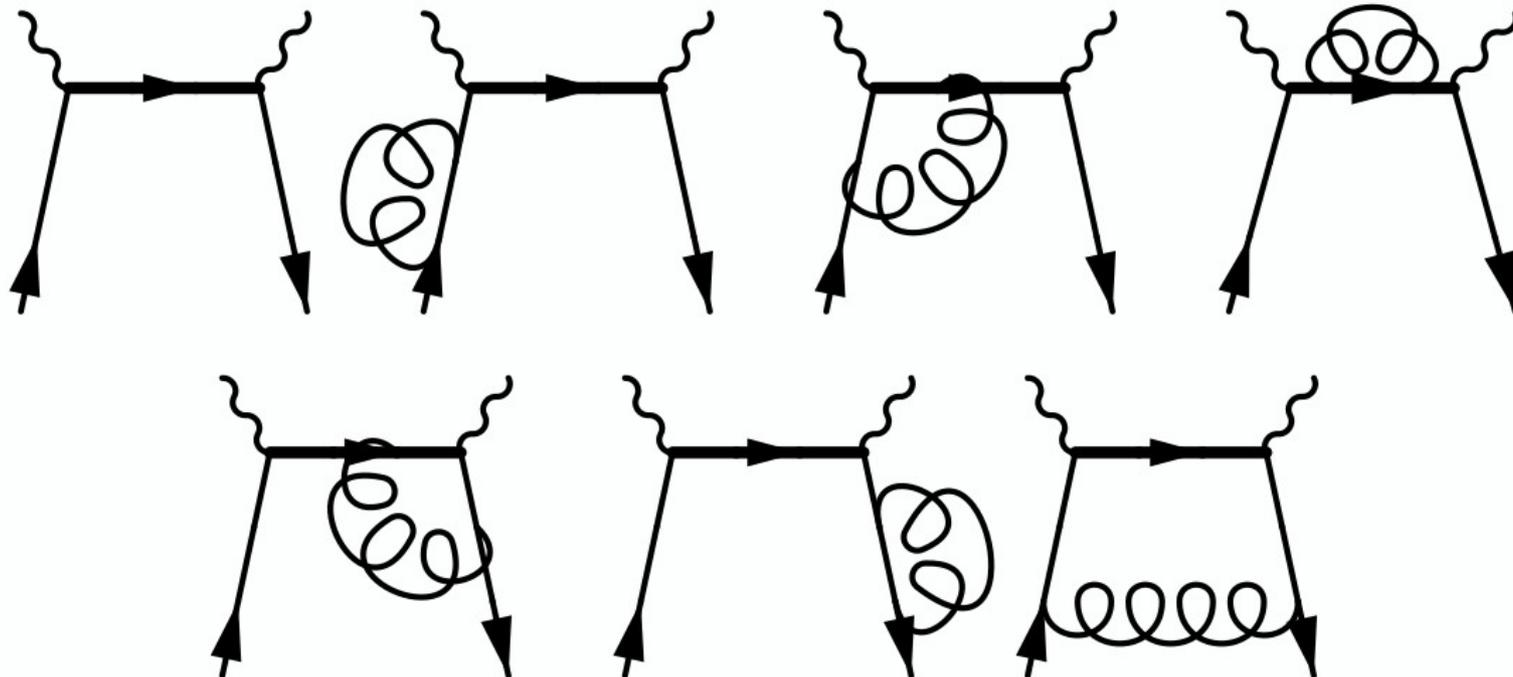


# Perturbative Matching

Compute parton-level matrix element and isolate Wilson coefficients.

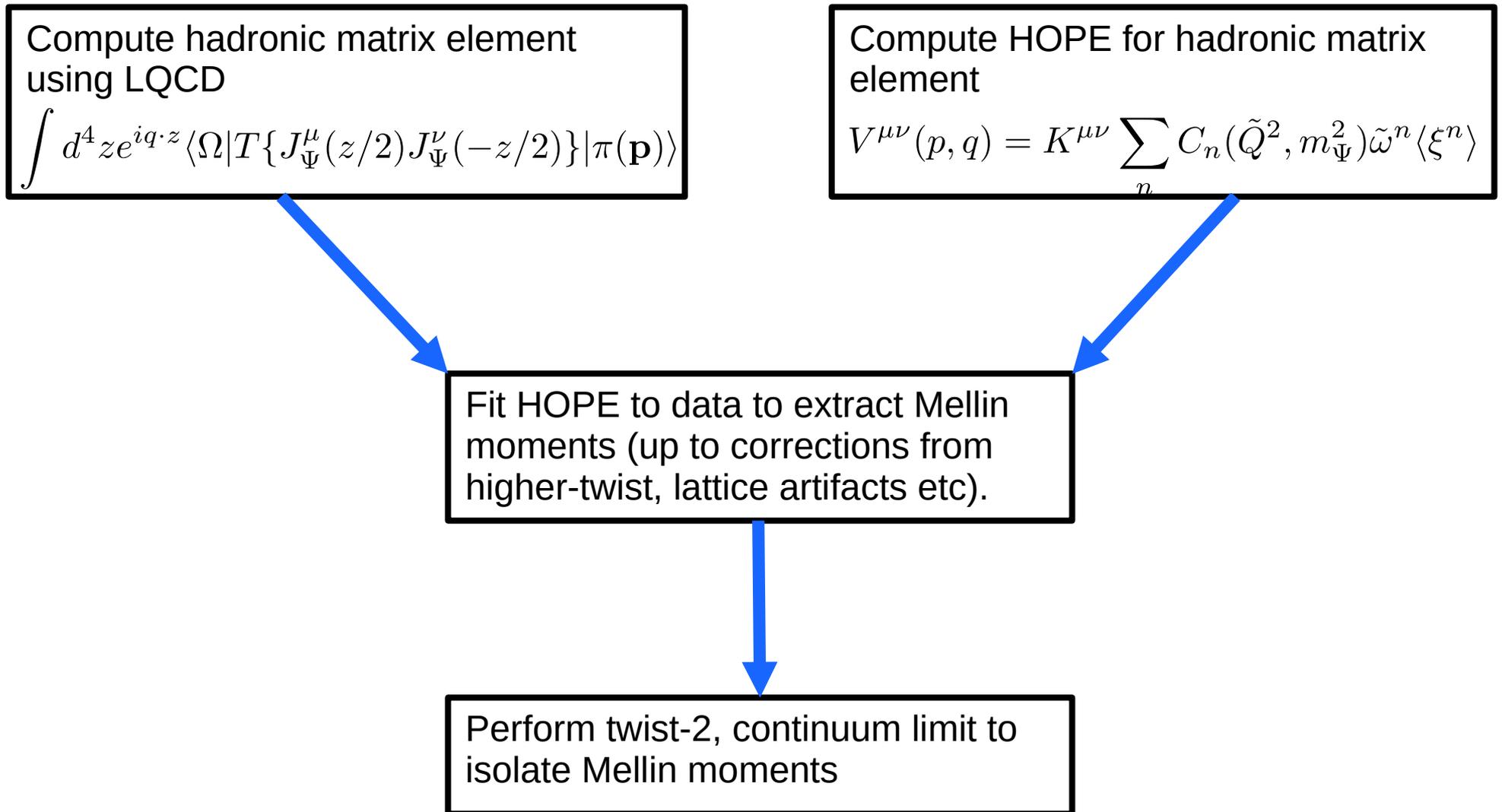
One-loop improved HOPE formula is

$$V^{\mu\nu}(p, q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2} \sum_n C_n(\tilde{Q}^2, m_\Psi^2) \tilde{\omega}^n \langle \xi^n \rangle$$



# The HOPE Method

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# The HOPE for controlled parton physics

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Phys.Rev.D 73 (2006) 014501

## 2006: First idea

PHYSICAL REVIEW D 73, 014501 (2006)

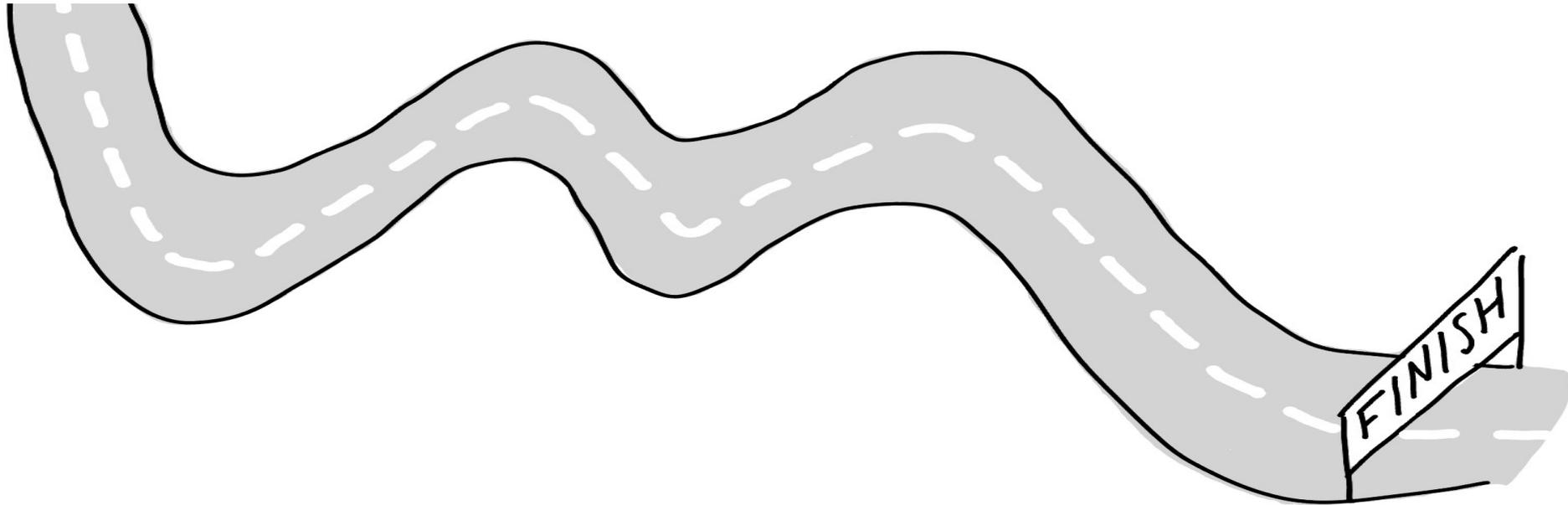
### Deep-inelastic scattering and the operator product expansion in lattice QCD

William Detmold<sup>1</sup> and C.-J. David Lin<sup>1,2</sup>

<sup>1</sup>*Department of Physics, University of Washington, Box 351560, Seattle, Washington 98195, USA*

<sup>2</sup>*Institute for Nuclear Theory, University of Washington, Box 351550, Seattle, Washington 98195, USA*

(Received 21 July 2005; published 9 January 2006)



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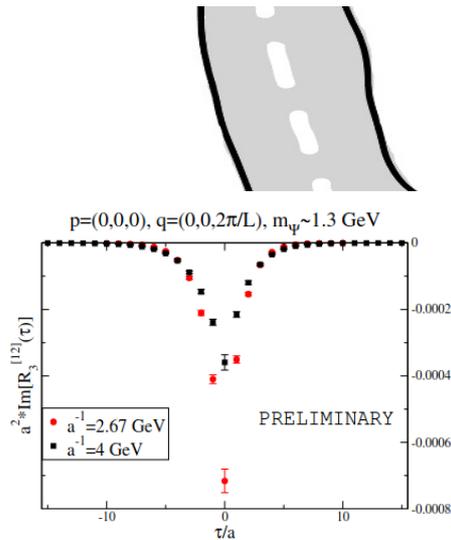
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## 2018: First numerical study

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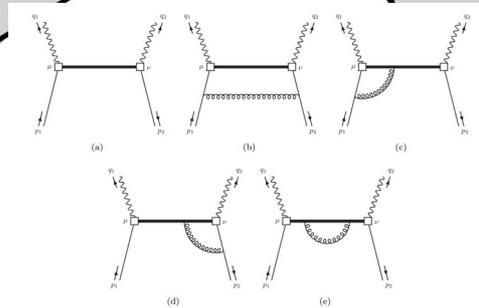
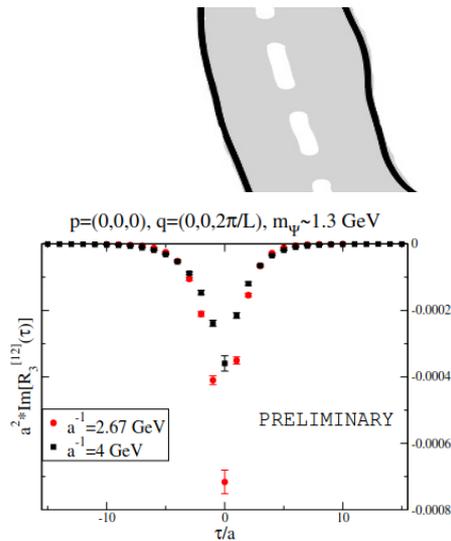
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## 2021: Formal developments

Detmold, Grebe, Kanamori, Lin, RJP, Zhao, Phys.Rev.D 104 (2021) 7, 074511

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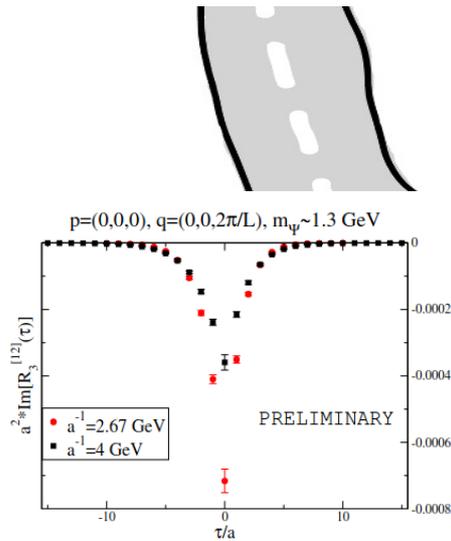
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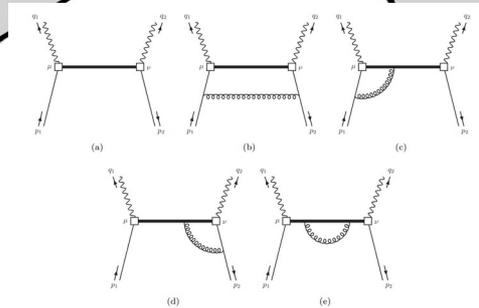
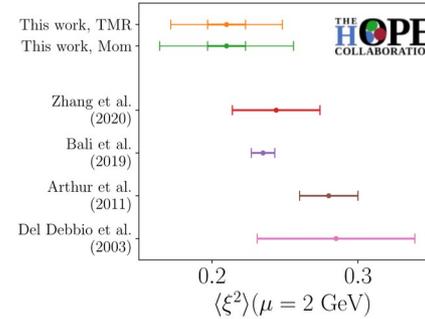
## 2018: First numerical study

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Detmold, Grebe, Kanamori, Lin, RJP, Zhao, Phys.Rev.D 104 (2021) 7, 074511

Detmold, Grebe, Kanamori, Lin, RJP, Zhao, Phys.Rev.D 105 (2022) 3, 034506

## 2021: First numerical result: Second Mellin moment



# Can we go beyond second moment?

## Lattice calculation of fourth Mellin Moment of pion LCDA

$$\pi \sim \sum_n C_n(m_\Psi) \pi \mathcal{O}_n + \mathcal{O}\left(\frac{1}{m_\Psi^\tau}, \frac{1}{Q^\tau}\right)$$

## Lattice Details

**Quenched** calculation: study continuum limit for higher moment

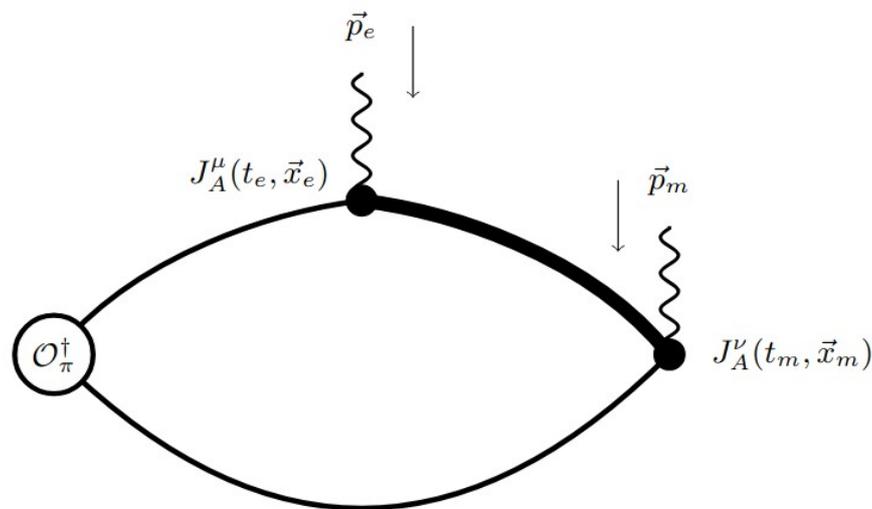
$$m_\pi \approx 550 \text{ MeV}$$

$(L/a)^3 \times T/a$	$\beta$	$a$ (fm)	$\kappa_l$	$\kappa_H$	$c_{sw}$	$N_{cfg}$	$N_{meas}$
$24^3 \times 48$	6.10050	0.0813	0.134900	0.1300	1.6842	6550	6550
				0.1250			
				0.1200			
				0.1160			
				0.1100			
$32^3 \times 64$	6.30168	0.0600	0.135154	0.1320	1.5792	7000	14000
				0.1280			
				0.1250			
				0.1184			
				0.1130			
$40^3 \times 80$	6.43306	0.0502	0.135145	0.1270	1.5292	250	10800
				0.1217			
				0.1150			
$48^3 \times 96$	6.59773	0.0407	0.135027	0.1285	1.4797	341	10000
				0.1244			
				0.1192			
				0.1150			

# Correlators and Matrix Element

Three-point correlation function:

$$C_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m) = \sum_{\vec{x}_e \in \Lambda} \sum_{\vec{x}_m \in \Lambda} e^{i\vec{p}_e \cdot \vec{x}_e} e^{i\vec{p}_m \cdot \vec{x}_m} \times \langle 0 | T \{ J_A^\mu(t_e, \vec{x}_e) J_A^\nu(t_m, \vec{x}_m) \mathcal{O}_\pi^\dagger(\vec{0}, 0) \} | 0 \rangle$$

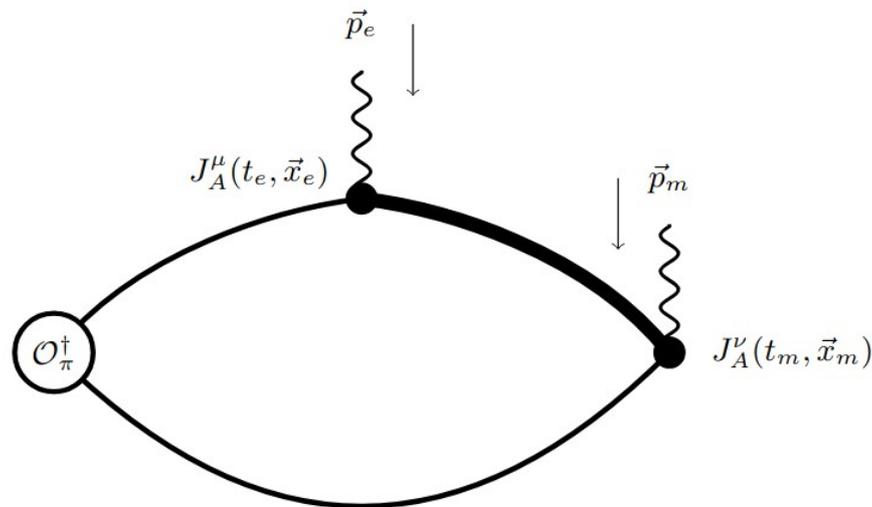


$$\lim_{t_+ \rightarrow \infty} C_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m) = R^{\mu\nu}(t_-; \vec{p}, \vec{q}) \frac{Z_\pi(\vec{p})}{2E_\pi(\vec{p})} e^{-E_\pi(\vec{p})t_+/2}$$

# Correlators and Matrix Element

Three-point correlation function:

$$C_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m) = \sum_{\vec{x}_e \in \Lambda} \sum_{\vec{x}_m \in \Lambda} e^{i\vec{p}_e \cdot \vec{x}_e} e^{i\vec{p}_m \cdot \vec{x}_m} \times \langle 0 | T \{ J_A^\mu(t_e, \vec{x}_e) J_A^\nu(t_m, \vec{x}_m) \mathcal{O}_\pi^\dagger(\vec{0}, 0) \} | 0 \rangle$$



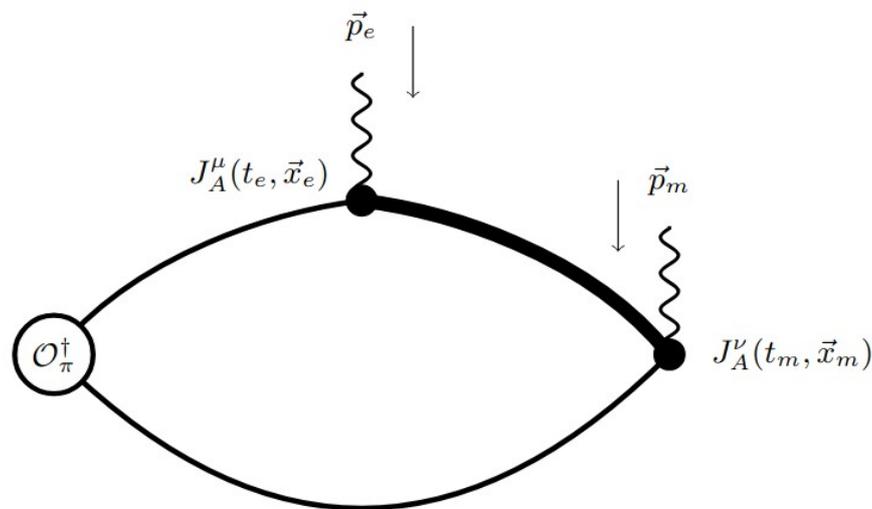
$$\lim_{t_+ \rightarrow \infty} C_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m) = \boxed{R^{\mu\nu}(t_-; \vec{p}, \vec{q})} \frac{Z_\pi(\vec{p})}{2E_\pi(\vec{p})} e^{-E_\pi(\vec{p})t_+/2}$$

Time-momentum current-current correlator

# Correlators and Matrix Element

Three-point correlation function:

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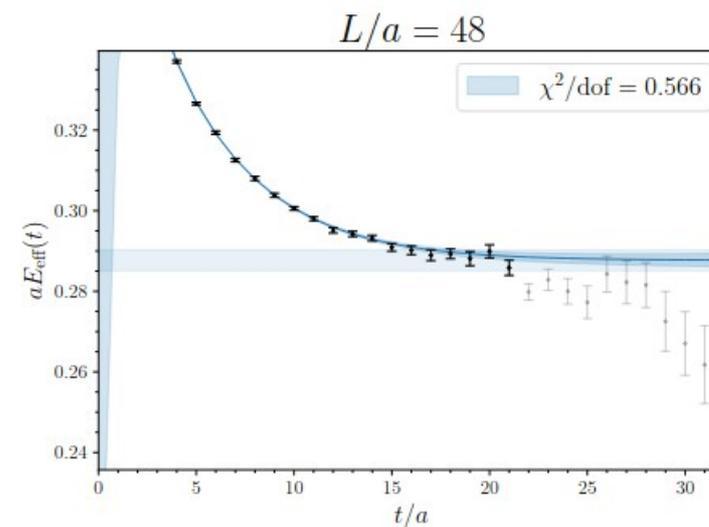
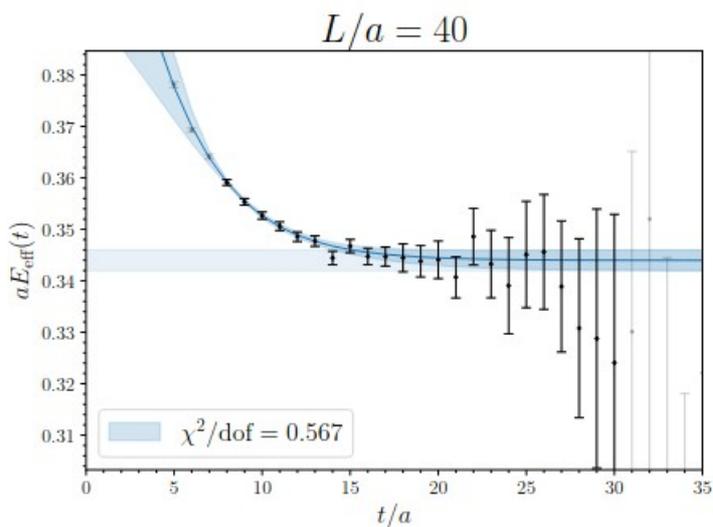
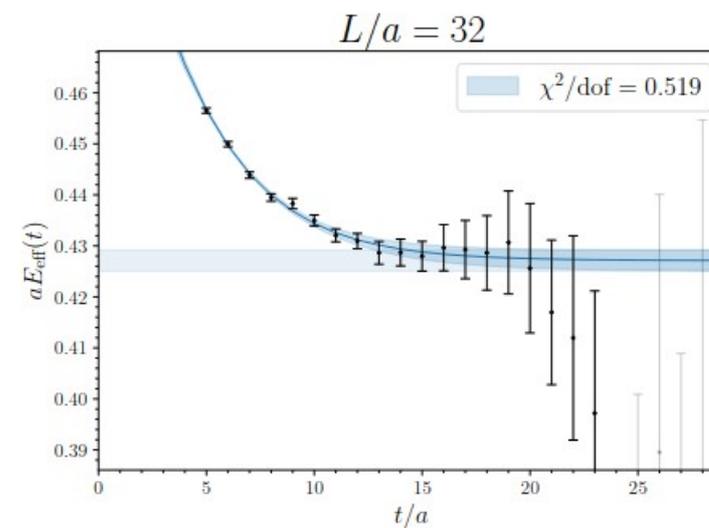
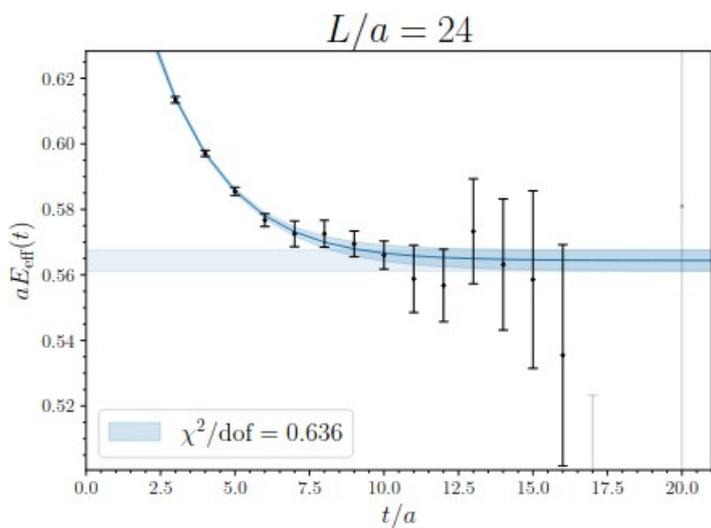
Spectroscopic factors

# Spectroscopy

2x2 GEVP:

$$\mathcal{O}_1 = \bar{\psi}\gamma_5\psi,$$

$$\mathcal{O}_2 = \bar{\psi}\gamma_4\gamma_5\psi,$$

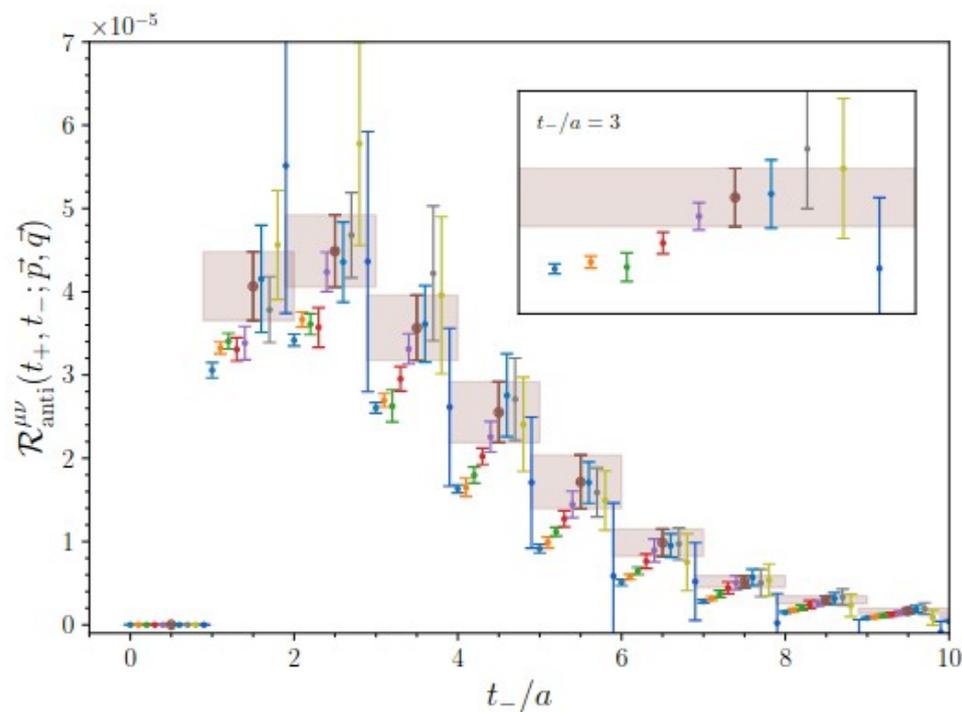
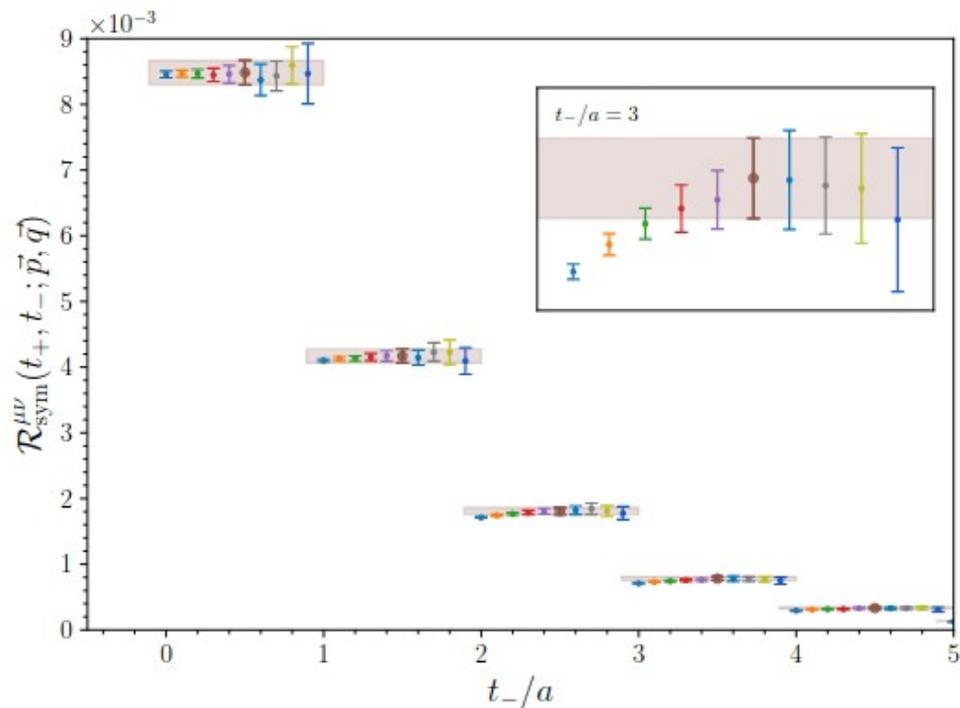


# Excited States

Construct ratio of three-point and two-point data

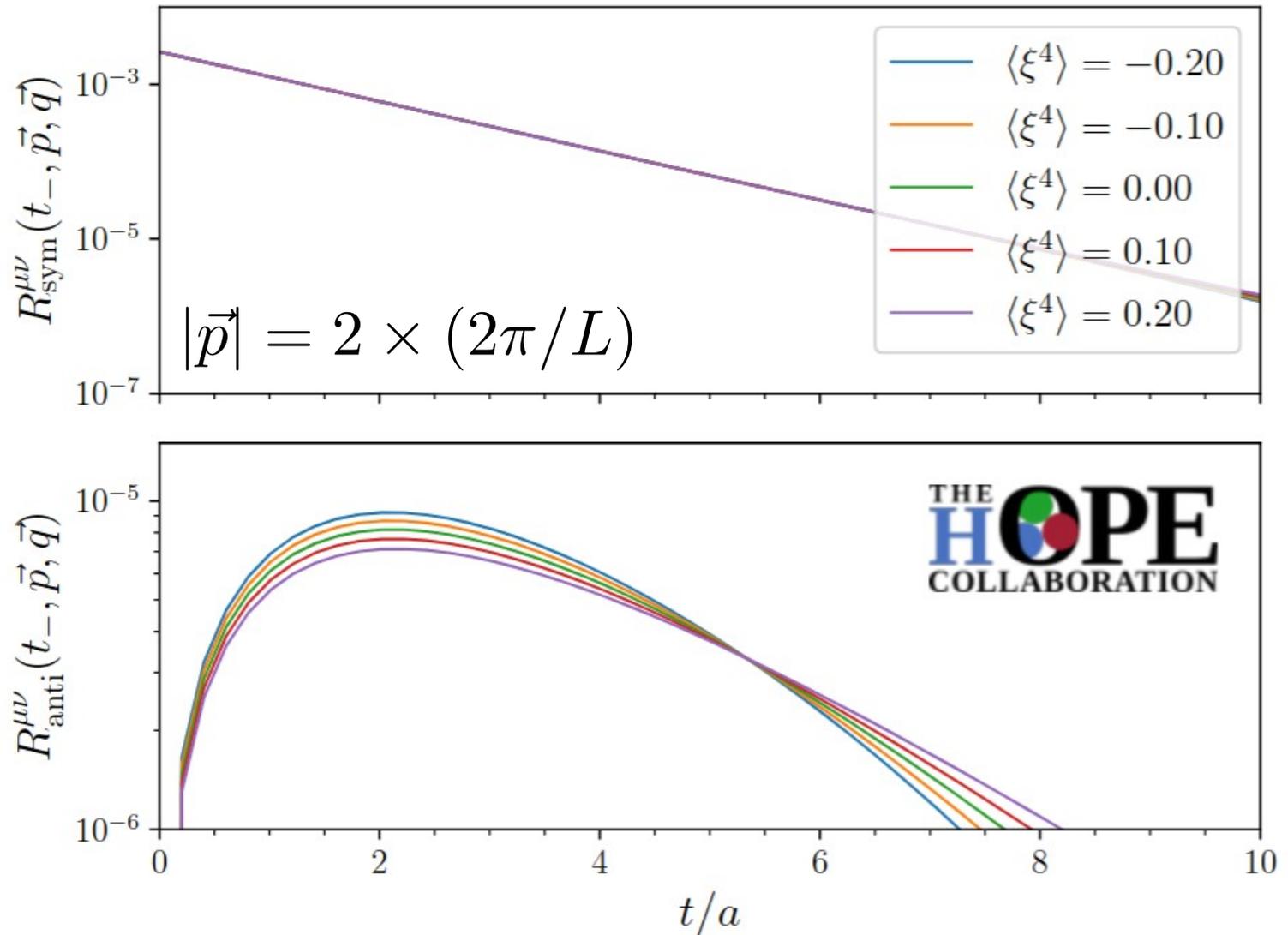
$$\mathcal{R}^{\mu\nu}(t_+, t_-; \vec{p}, \vec{q}) \propto \left[ \frac{Z_\pi(\vec{p})}{2E_\pi(\vec{p})} e^{-E_\pi(\vec{p})(t_e+t_m)/2} \right]^{-1} \times C_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m)$$

Study excited state contamination



## Sensitivity to fourth Mellin Moment

$$R^{\mu\nu}(t_-; \vec{p}, \vec{q}) = \int \frac{dq_4}{(2\pi)} e^{-iq_4 t_-} V^{\mu\nu}(q, p).$$

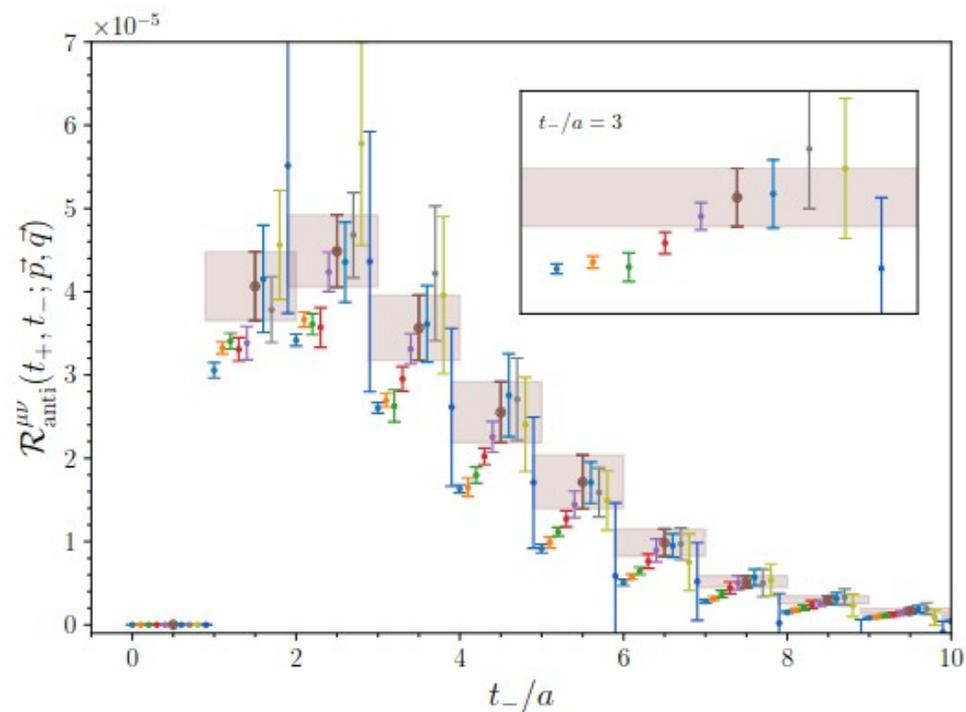
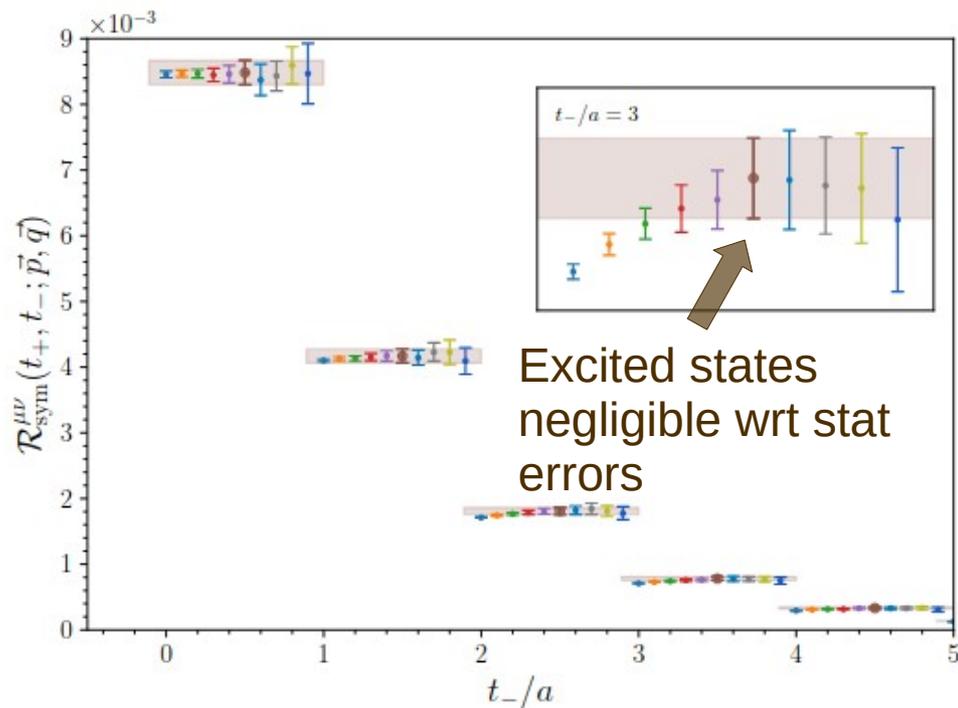


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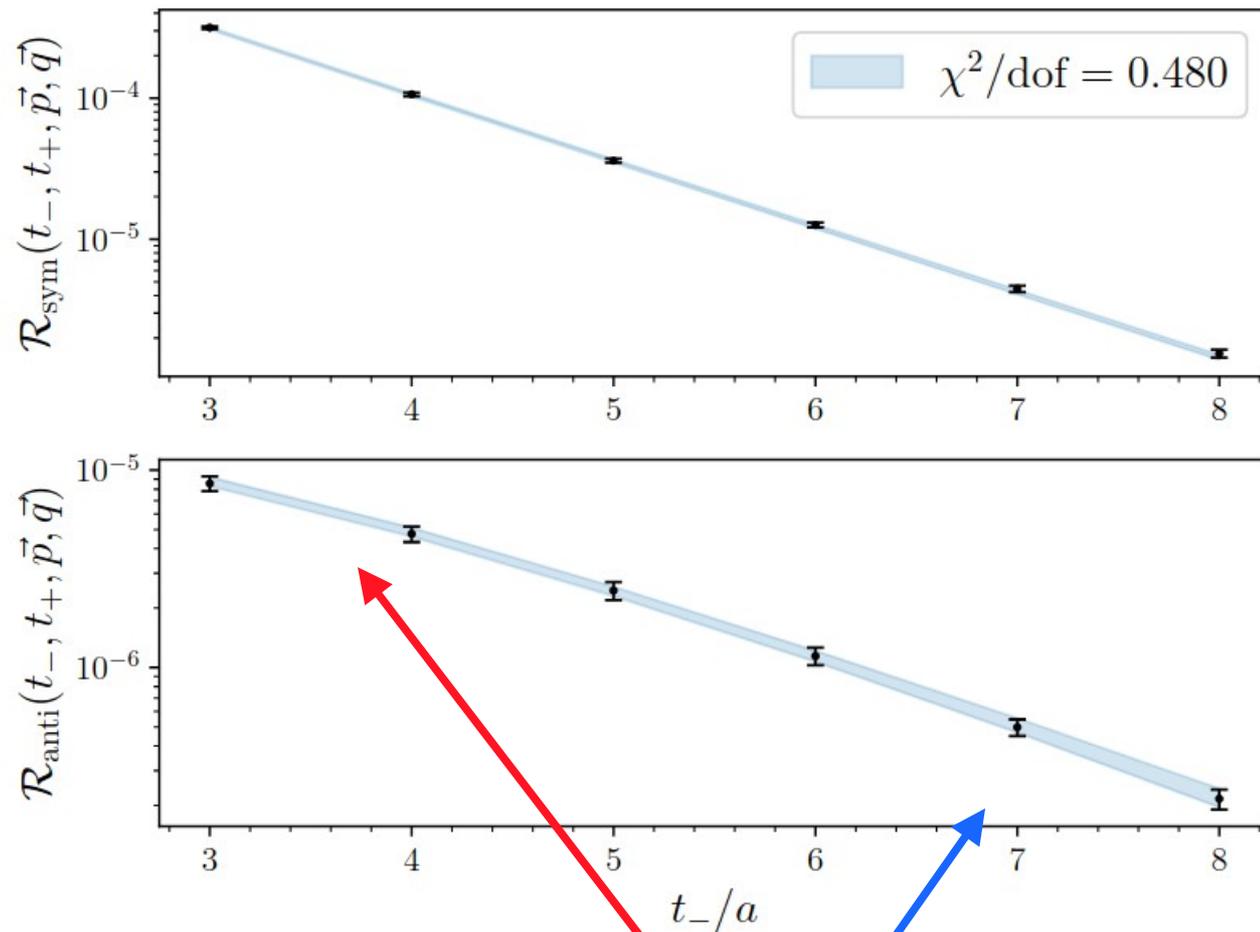
$$\mathcal{R}^{\mu\nu}(t_+, t_-; \vec{p}, \vec{q}) \propto \left[ \frac{Z_\pi(\vec{p})}{2E_\pi(\vec{p})} e^{-E_\pi(\vec{p})(t_e+t_m)/2} \right]^{-1} \times C_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m)$$

Study excited state contamination



# The HOPE Ratio

$$\mathcal{R}^{\mu\nu}(t_+, t_-; \vec{p}, \vec{q}) = \mathcal{R}_{\text{sym}}^{\mu\nu}(t_+, t_-; \vec{p}, \vec{q}) + \mathcal{R}_{\text{anti}}^{\mu\nu}(t_+, t_-; \vec{p}, \vec{q})$$

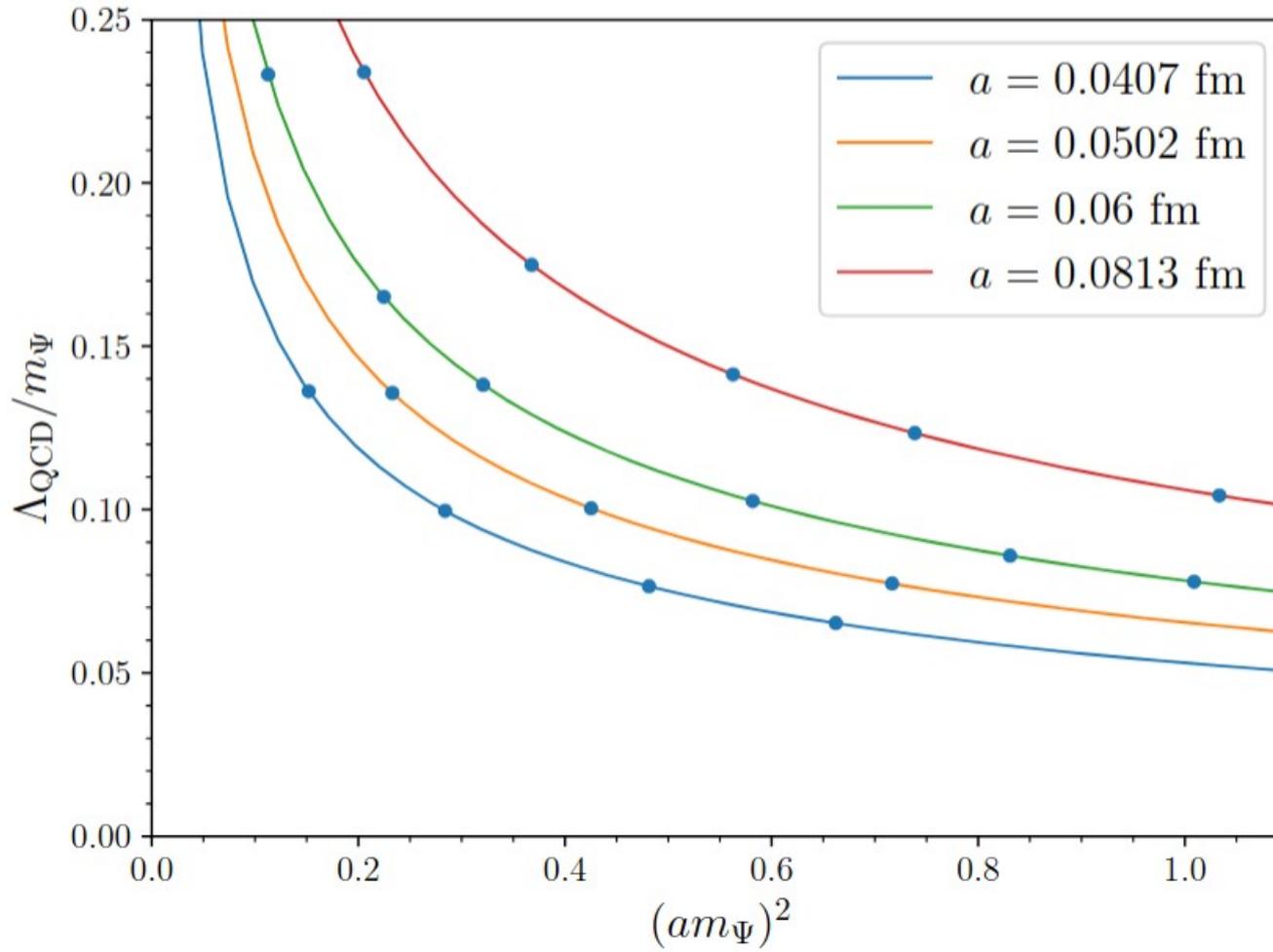


$$V^{\mu\nu}(p, q) = K^{\mu\nu} [1 + \langle \xi^2 \rangle \tilde{\omega}^2 + \langle \xi^4 \rangle \tilde{\omega}^4 + \dots]$$

# Choice of heavy quark mass

Vary heavy quark mass to study systematics

Higher-twist corrections

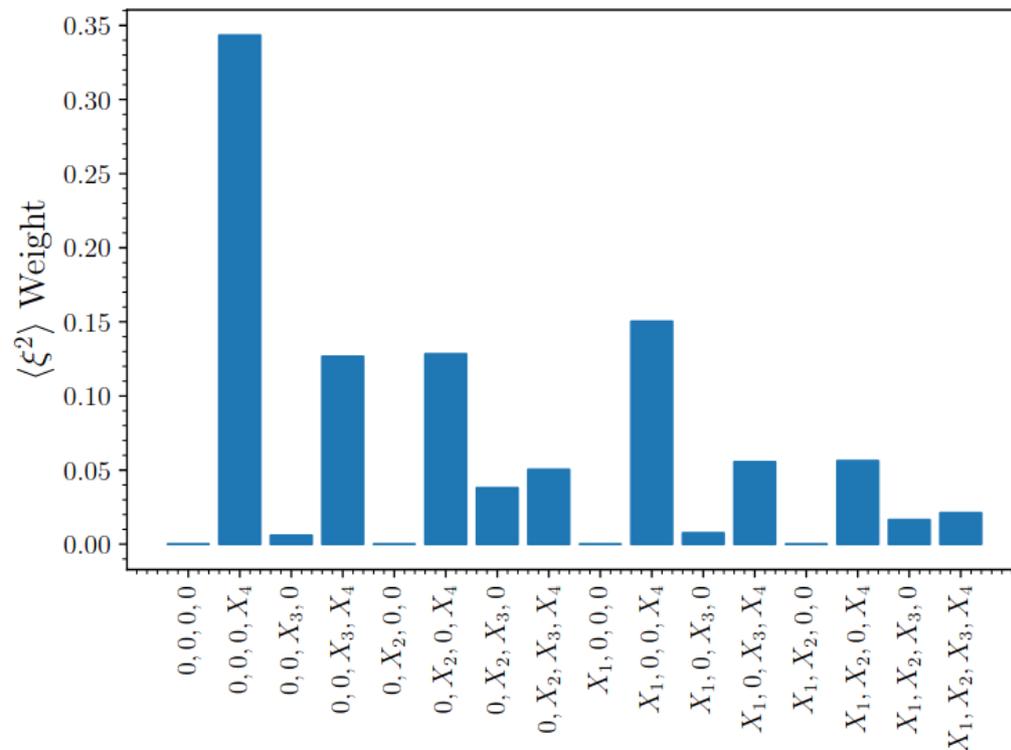
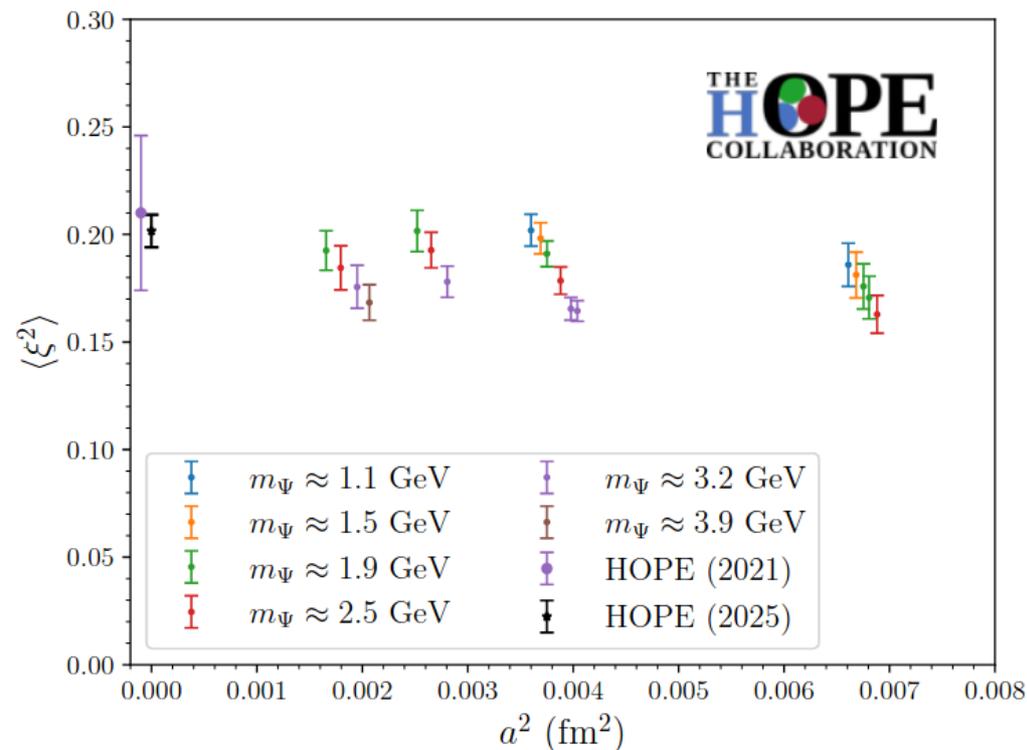


Lattice artifacts

# Continuum, Twist-2 Extrapolation

Employ Bayesian Model Averaging Phys.Rev.D 103 (2021) 114502

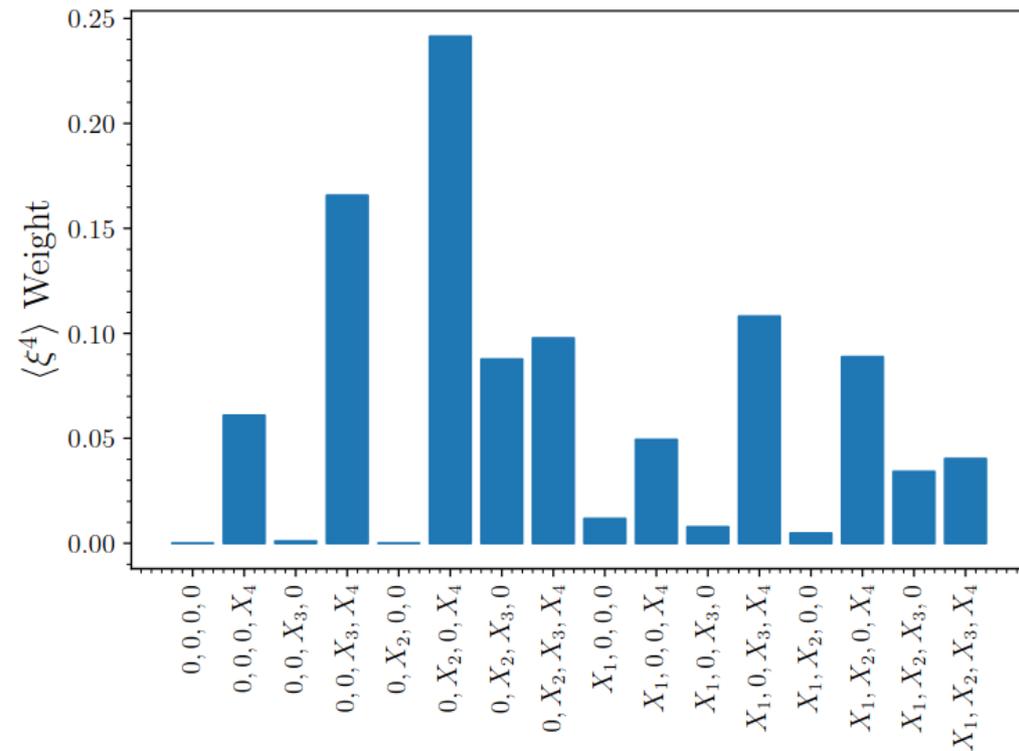
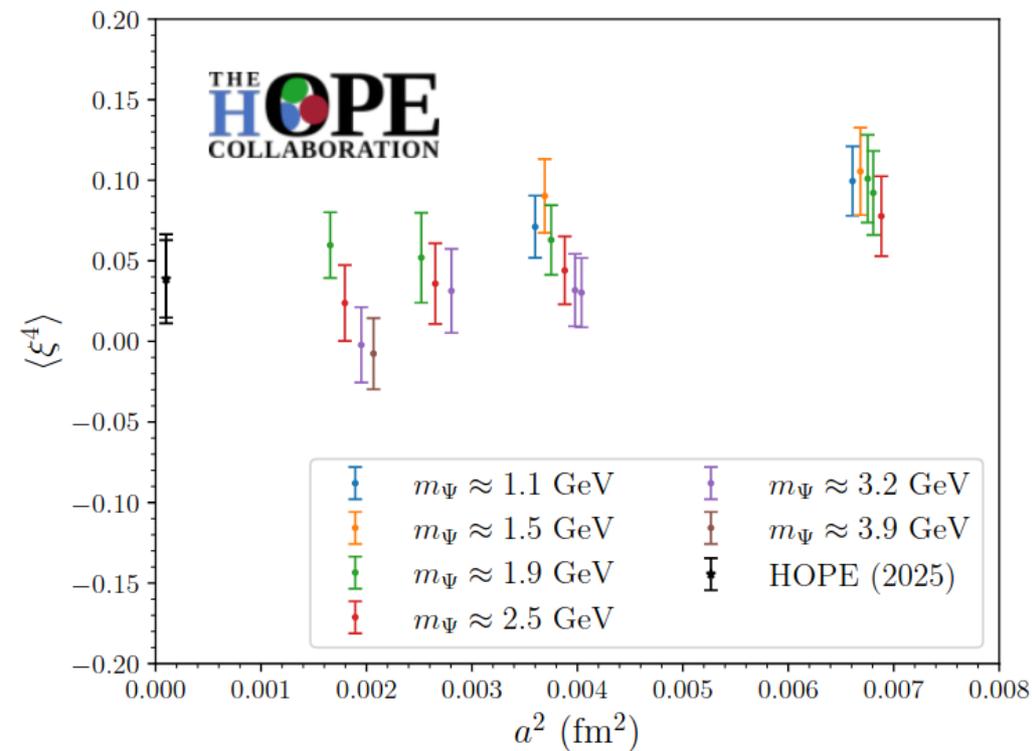
$$X(a, m_\Psi) = X_0 + \frac{X_1}{m_\Psi} + X_2 a^2 + X_3 a^2 m_\Psi + X_4 a^2 m_\Psi^2$$



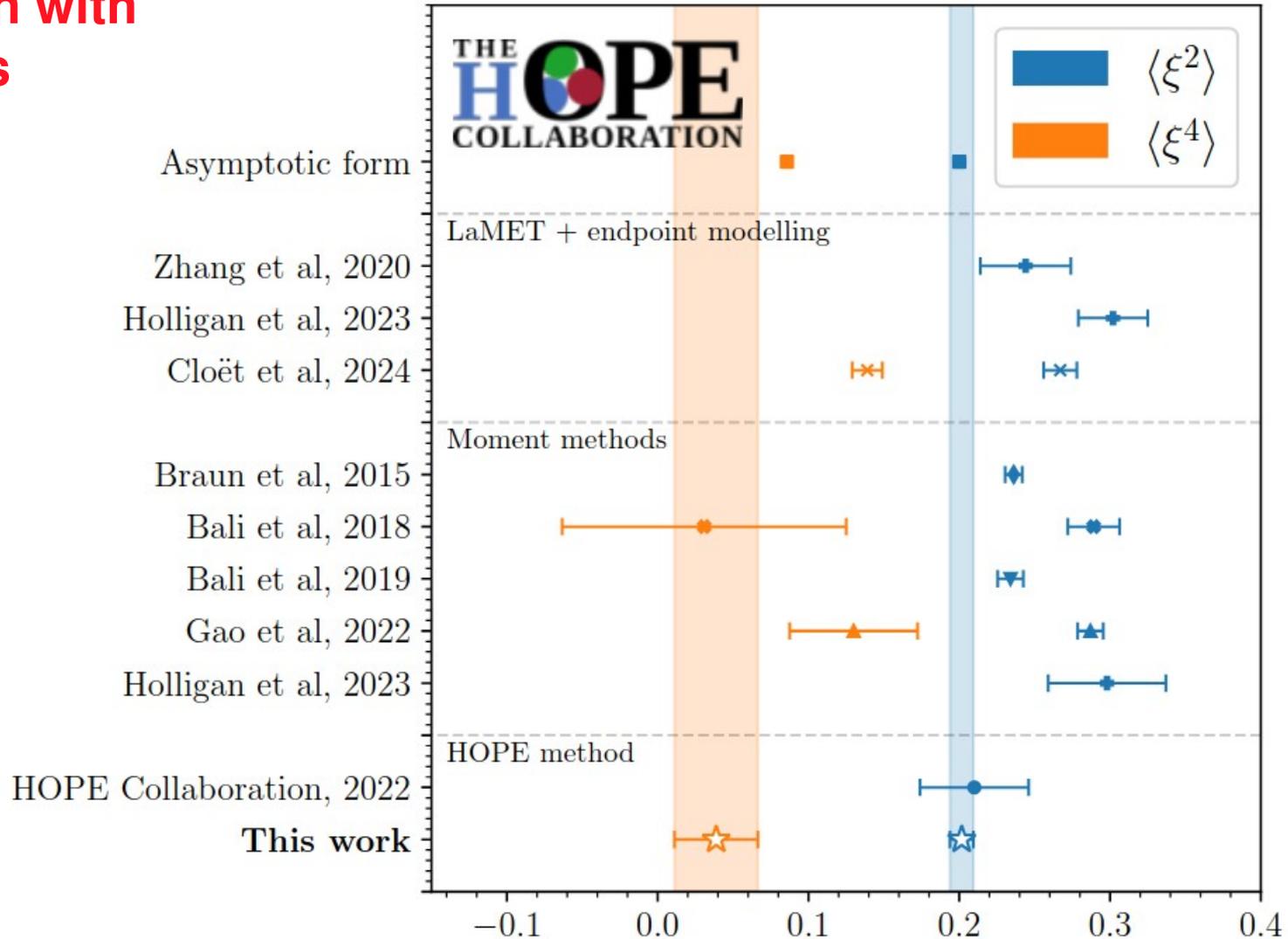
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# Comparison with other works



First (?) continuum limit determination of fourth Mellin moment of pion

Competitive with other moment methods

Systematic errors: Quenched approximation, pion mass.

# The HOPE for controlled parton physics

Phys.Rev.D 73 (2006) 014501

## 2006: First idea

PHYSICAL REVIEW D 73, 014501 (2006)

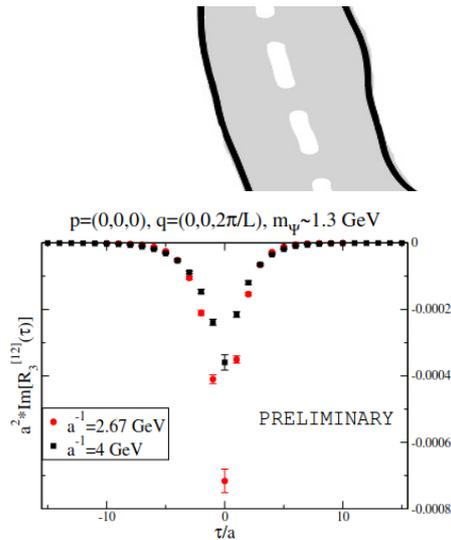
### Deep-inelastic scattering and the operator product expansion in lattice QCD

William Detmold<sup>1</sup> and C.-J. David Lin<sup>1,2</sup>

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(Received 21 July 2005; published 9 January 2006)



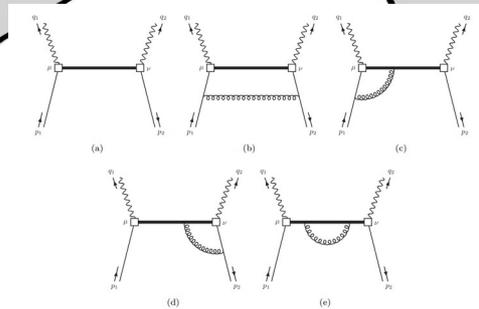
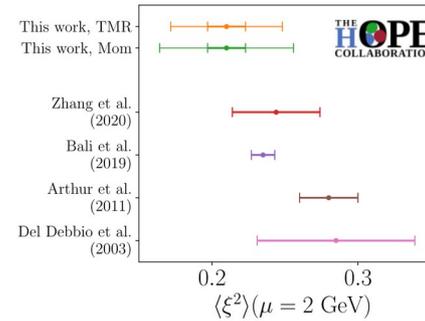
## 2018: First numerical study

## 2021: Formal developments

Detmold, Grebe, Kanamori, Lin, RJP, Zhao, Phys.Rev.D 104 (2021) 7, 074511

Detmold, Grebe, Kanamori, Lin, RJP, Zhao, Phys.Rev.D 105 (2022) 3, 034506

## 2021: First numerical result: Second Mellin moment



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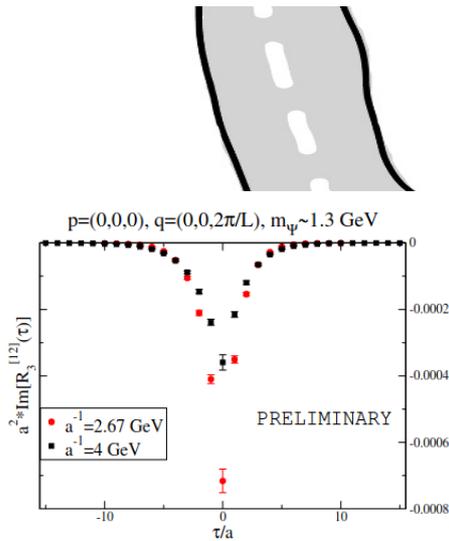
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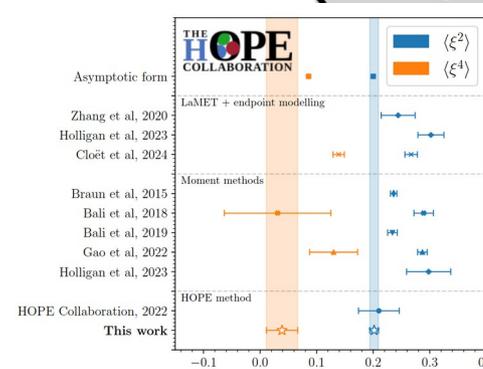
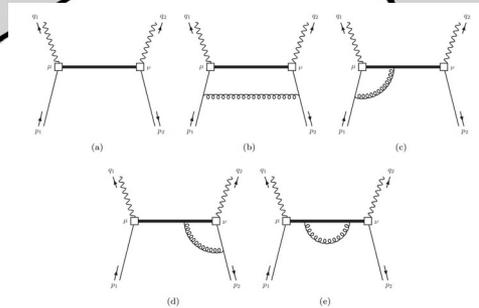
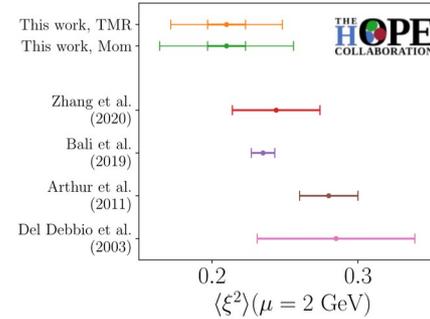
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## 2021: First numerical result: Second Mellin moment



## 2025: Continuum limit of higher moments

Detmold, Grebe, Kanamori, Lin, RJP, Zhao, arXiv:2509.04799 [hep-lat]



## Conclusions

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HOPE provides an alternate approach to parton physics

Continuum limit of higher moments possible

Determined second and fourth Mellin moment in continuum:

$$\langle \xi^2 \rangle (\mu = 2 \text{ GeV}) = 0.202(8)(9),$$

$$\langle \xi^4 \rangle (\mu = 2 \text{ GeV}) = 0.039(28)(11).$$

Up next: Dynamical calculations + PDF