Regularization Prescription for the Mixing Between Nonlocal Gluon and Quark Operators

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Outline

- Mixing between nonlocal gluon and quark operators
- Existing proposals to resolve the ambiguity
- Regularization of the $1/z_{12}$ pole

Mixing between nonlocal gluon and quark operators

Factorization of gluon quasi-LF Balitsky et al.,2019 Yao et al.,2022

$$O_{g,\mathrm{s}}(z_1,z_2)=\frac{1}{\mathbf{z_{12}}}\int_0^1 d\alpha d\beta\,\theta(1-\alpha-\beta)C^{\mathrm{s}}_{gq}(\alpha,\beta,\mu^2\mathbf{z^2_{12}})O^{\mathrm{s,l.t.}}_q(z^\alpha_{12},z^\beta_{21})$$

$$\mathbb{C}^{\mathrm{s}}_{gq}=\int_0^1 d\alpha d\beta\theta(1-\alpha-\beta)\int\frac{dz_{12}}{2\pi}\frac{1}{z_{12}}e^{iz_{12}P^z(x+\xi(-\alpha+\beta)+(1-\alpha-\beta)y)}C^{\mathrm{s}}_{gq}(\alpha,\beta,\mu^2z^2_{12}),$$

$$\mathbb{H}_{g,s}(x,\xi,\Delta^2,P^z)=\int_{-1}^1 dy\,\mathbb{C}^{\mathrm{s}}_{gq}(x,y,\xi,\frac{\mu}{P^z})H^{\mathrm{s}}_q(y,\xi,\Delta^2,\mu)$$

Evolution of gluon LF Radyushkin, 1997 Blumlein et al.,1999 ⋯

$$\mu\frac{d}{d\mu}O_{g,\mathrm{s}}(z_1,z_2)=\frac{1}{\mathbf{z_{12}}}\int_0^1 du\int_0^1 dv\,\theta(1-u-v)\mathcal{K}^{\mathrm{s}}_{gq}(u,v)O^{\mathrm{s}}_q(z^u_{12},z^v_{21})$$

$$\widetilde{K}^{\mathrm{s}}_{gq}(x,y,\xi)=-i\int_a^x dx'K^{\mathrm{s}}_{gq}(x',y,\xi)\,,$$

$$\mu\frac{d}{d\mu}F_{g,\mathrm{s}}(x,\xi,\Delta^2)=\int_{-1}^1 dy\,\widetilde{K}^{\mathrm{s}}_{gq}(x,y,\xi)F^{\mathrm{s}}_q(y,\xi,\Delta^2)$$

- Due to the singularity at $z_{12}=0$, the Fourier transform of matching/evolution kernel in coordinate space is ambiguous. $f(x)=\int\frac{dz_{12}}{2\pi}e^{iz_{12}x}\frac{1}{z_{12}},\quad\frac{df(x)}{dx}=i\delta(x), f(x)=i\theta(x)=C.$
- The ambiguities on both sides are related to each other, due to the factorization requirement in momentum space: the μ dependence of LC GPDs must match the P^z dependence of quasi GPDs.

Existing proposals to resolve the ambiguity

For the evolution of LC GPDs:

$$\widetilde{K}^{\mathrm{s}}_{gq}(x,y,\xi) = -i\int_a^x dx' K^{\mathrm{s}}_{gq}(x',y,\xi)\,, \quad K^{\mathrm{s}}_{gq}(x,y,\xi) = \int_0^1 dudv\,\theta(1-u-v)\mathcal{K}^{\mathrm{s}}_{gq}(u,v)\delta((1-u-v)y+\xi(v-u)-x)\,,$$

separate it into different kinematic regions (DGLAP and ERBL), and determine lower limit a by imposing physical constraints:

ERBL region 1(with $\zeta = 1$, $X \geq Z$):

$$W^{gq,1}_{\zeta=1,X\geq Z}(Z,X) = -\int_0^Z d\tilde{X} M^{gq}_{\zeta=1,X\geq Z}(Z,X) = \frac{Z^2}{X},$$

Balitsky, Radyushkin,1997

$$W^{gq,2}_{\zeta=1,X\geq Z}(Z,X) = \int_Z^X d\tilde{X} M^{gq}_{\zeta=1,X\geq Z}(Z,X) = 1 - X + \frac{Z^2}{X},$$

Blumlein et al.,1997

DGLAP region (with $X \geq Z \geq \zeta$):

$$W^{gq,1}_{\zeta}(Z,X)|_{1\geq X\geq Z\geq\zeta} = \int_Z^X d\tilde{X} M^{gq}_{\zeta}(Z,X)|_{1\geq X\geq Z\geq\zeta} = \frac{Z}{X} + \frac{Z-\zeta}{X-\zeta} - \frac{Z(Z-\zeta)}{X(X-\zeta)},$$

Balitsky, Radyushkin,1997

$$W^{gq,2}_{\zeta}(Z,X)|_{1\geq X\geq Z\geq\zeta} = -\int_\zeta^Z d\tilde{X} M^{gq}_{\zeta}(Z,X)|_{1\geq X\geq Z\geq\zeta} = \frac{Z}{X} + \frac{Z-\zeta}{X-\zeta} - \frac{Z(Z-\zeta)}{X(X-\zeta)} + \frac{\zeta^2}{X(X-\zeta)} - \frac{\zeta}{X-\zeta},$$

The freedom to choose different a's would yield different results !

Existing proposals for handling the ambiguity

Matching Mellin moments of the kernel in coordinate and momentum space   Belitsky,Radyushkin,2005

$$\lim_{\xi\to 0}\int_0^1 dx\, x^j \mathscr{C}^{\mathrm{s}}_{gq}\left(2\frac{x}{\xi}-1, \frac{2}{\xi}-1\right) = \int_0^1 d\alpha\, C^{\mathrm{s}}_{gq}(\alpha, \mu^2 \mathbf{z}_{12}^2)\frac{-\bar{\alpha}^{j+1}}{j+1}, \quad j \geq 0\,,$$

$\mathscr{C}^{\mathrm{s}}_{gq}$ (to be solved): matching coefficient (for pseudo GPD) (Yao et al.,2022)/ evolution kernel (for LC GPD)

Evaluating the l.h.s directly:

$$\int_0^1 dx x^j \mathscr{C}^{\mathrm{s}}_{gq}(x, y, \mu^2\mathbf{z}_{12}^2) = \int_0^1 d\alpha\, C^{\mathrm{s}}_{gq}(\alpha, \mu^2\mathbf{z}_{12}^2)\Big(\frac{-(\bar{\alpha}y)^{j+1}+1}{j+1} + \frac{C(\bar{\alpha}y)}{j+1}\Big), \quad j \geq 0,$$

$C(\bar{\alpha}y)$: integration constant

- The proposal above implicitly takes $C(\bar{\alpha}y) = -1$.

- The solution is not unique, different solutions that are not equivalent at the convolution level exist.

$$\mathscr{C}^{(1)}_{gqL} = -2a_s C_F \Gamma(-\epsilon)\eta^\epsilon\Big\{D_1\theta(\tau_1-y_1) + D_2\Big[\frac{\tau_1^2\theta(\tau_1)}{y_1(y_1+y_2)} + \frac{\tau_2^2\theta(-\tau_2)}{y_2(y_1+y_2)} - \frac{(\tau_1-y_1)^2}{y_1y_2}\theta(\tau_1-y_1)\Big] + 2\epsilon D_1\Big[\frac{(-\frac{1}{2}\tau_1y_1+\tau_1y_2)\theta(\tau_1)}{y_1(y_1+y_2)} + \frac{(-\frac{1}{2}\tau_2y_2+\tau_2y_1)\theta(-\tau_2)}{y_2(y_1+y_2)} + \frac{(\tau_1-y_1)(y_1-y_2)}{y_1y_2}\theta(\tau_1-y_1)\Big]\Big\},$$

$$\mathscr{C}^{(2)}_{gqL} = -2a_s C_F \Gamma(-\epsilon)\eta^\epsilon\Big\{D_1\theta(\tau_1-y_1) + D_2\Big[\frac{\tau_1^2\theta(\tau_1)}{y_1(y_1+y_2)} + \frac{\tau_2^2\theta(-\tau_2)}{y_2(y_1+y_2)} - \frac{(\tau_1-y_1)^2}{y_1y_2}\theta(\tau_1-y_1)\Big] + 2\epsilon D_1\Big[\frac{(-\frac{1}{2}\tau_2y_1+\tau_1y_2)\theta(\tau_1)}{y_1(y_1+y_2)} + \frac{(-\frac{1}{2}\tau_1y_2+\tau_2y_1)\theta(-\tau_2)}{y_2(y_1+y_2)} + \frac{(\tau_1-y_1)(y_1-y_2)}{y_1y_2}\theta(\tau_1-y_1)\Big]\Big\},$$

Regularization of the $1/z_{12}$ pole

Local expansion of the factorization in coordinate space:

$$\tilde{h}_{g,s}(\mathbf{z}_{12}^2, \lambda) = \frac{\sqrt{2}P^z}{\mathbf{z}_{12}} \sum_{n=2}^{\infty} \frac{(-i\lambda)^{n-2}}{(n-2)!} \int_0^1 d\alpha\, \alpha^{n-2} C_{gq}^{\mathrm{coord,s}}(\alpha, \mu^2 \mathbf{z}_{12}^2) a_{n-2}^{q,s}(\mu)$$

unpolarized case:

term $\propto 1/z_{12}$ vanishes due to $a_0^{q,s} = 0$, equivalent to OPE:

$$\tilde{h}_{g,s}(\mathbf{z}_{12}^2, \lambda) = (P^z)^2 \sum_{n=2}^{\infty} \frac{(-i\lambda)^{n-2}}{(n-2)!} C_{n-2}^{gq,s}(\mu^2 \mathbf{z}_{12}^2) a_{n-1}^{q,s}(\mu)$$

free from Fourier transform of $1/z_{12}$ and $ln z_{12}^2/z_{12}$.

Izubuchi et al.,2018
Wang et al.,2019

polarized case:

the equivalence condition is

$\int_0^1 d\alpha\, C_{gq}^{\mathrm{coord,s}}(\alpha, \mu^2 \mathbf{z}_{12}^2) = 0$ (for forward case)

$\int_{\alpha\beta} C_{gq}^{\mathrm{coord,s}}(\alpha, \beta, \mu^2 \mathbf{z}_{12}^2) = 0$(for nonforward case)

At one-loop we find: $C_{gq}^{\mathrm{coord,s}}(\alpha, \mu^2 \mathbf{z}_{12}^2) = C_{gq}^{\mathrm{si,s}}(\alpha) + C_{gq}^{\mathrm{evol,s}}(\alpha) \ln\left(\frac{\mathbf{z}_{12}^2 \mu^2}{4e^{-2\gamma_E}}\right).$

$$\int_0^1 d\alpha\, C_{gq,h}^{\mathrm{evol,s}}(\alpha) = \int_{\alpha\beta} C_{gq,h}^{\mathrm{evol,s}}(\alpha, \beta) = 0,$$

$$\int_0^1 d\alpha\, C_{gq,h}^{\mathrm{si,s}}(\alpha) \neq 0, \quad \int_{\alpha\beta} C_{gq,h}^{\mathrm{si,s}}(\alpha, \beta) \neq 0.$$

Yao et al.,2022

Regularization of the $1/z_{12}$ pole

The two limits are not commutative: $P^z \to \infty / z^2 \to 0$ and $\Lambda_{UV} \to \infty$.

Izubuchi et al.,2018
Ma et al., 2022

The moments of quasi PDFs(GPDs) are divergent./The quasi-LF correlators are divergent in the local limit.

(In d dimensions)

momentum space:

$P^z \to \infty$

$$\mathbb{C}^{s}_{gq}(x, y, \frac{\mu}{P^z}, \epsilon) = a(x, y)\frac{1}{\epsilon}\left[1 - \left(\frac{4(P^z)^2}{\mu^2}\right)^{-\epsilon}\right] \quad 0\ (\varepsilon > 0)$$

$$+ d(x, y, \epsilon)|x - y|^{1-2\epsilon} + e(x, y, \epsilon)|x|^{1-2\epsilon} + f(x, y, \epsilon)|x - y|^{-2\epsilon},$$

$$\int_{-\infty}^{\infty} x\tilde{f}_{g,u}(x, \infty) = -\frac{8i a_s C_F}{3\epsilon_{\rm UV}}\langle P|O_q^{-,1}(0,0)|P\rangle,$$

$$\int_{-\infty}^{\infty} x\tilde{f}_{g,h}(x, \infty) = 0.$$

coordinate space:

$$O_g(z_{12}, 0) = \int_0^1 \left(C^{\rm si,s}_{gq}(\alpha) + C^{\rm evol,s}_{gq}(\alpha)\frac{\left(\frac{z_{12}^2\mu^2}{4e^{-2\gamma_E}}\right)^{\epsilon} - 1}{\epsilon}\right)O^{s}_q(\bar{\alpha}z_{12}, 0),$$

expanding both sides at $z_{12} = 0$

$$\langle P|O_{g,u}(0,0)|P\rangle = -\frac{8i a_s C_F}{3\epsilon_{\rm UV}}\langle P|O_q^{-,1}(0,0)|P\rangle,$$

$$0 = 0.$$

The second moments (local limit) of gluon quasi PDFs (gluon quasi-LF correlators) in d dimensions are consistent with the mixing pattern of local matrix element, the coefficients of the UV poles match exactly.

Regularization of the $1/z_{12}$ pole

The two limits are not commutative: $P^z \to \infty / z^2 \to 0$ and $\Lambda_{UV} \to \infty$.

↓

The moments of quasi PDFs(GPDs) are divergent./The quasi-LF correlators are divergent in the local limit.

(In d dimensions)

momentum space:

$$\mathbb{C}^{s}_{gq}(x, y, \frac{\mu}{P^z}, \epsilon) = a(x,y)\frac{1}{\epsilon}\left[1 - \left(\frac{4(P^z)^2}{\mu^2}\right)^{-\epsilon}\right] \xrightarrow{P^z \to \infty} 0 \ (\varepsilon > 0)$$

$$+ d(x,y,\epsilon)\left|x-y\right|^{1-2\epsilon} + e(x,y,\epsilon)\left|x\right|^{1-2\epsilon} + f(x,y,\epsilon)\left|x-y\right|^{-2\epsilon},$$

$$\int_{-\infty}^{\infty} x\tilde{f}_{g,u}(x,\infty) = -\frac{8i a_s C_F}{3\epsilon_{\rm UV}}\langle P|O_q^{-,1}(0,0)|P\rangle,$$

$$\int_{-\infty}^{\infty} x\tilde{f}_{g,h}(x,\infty) = 0.$$

coordinate space:

$$O_g(z_{12},0) = \int_0^1 \left(C^{\rm si,s}_{gq}(\alpha) + C^{\rm evol,s}_{gq}(\alpha)\frac{\left(\frac{z_{12}^2\mu^2}{4e^{-2\gamma_E}}\right)^{\epsilon} - 1}{\epsilon}\right)O_q^{s}(\bar{\alpha}z_{12},0),$$

↓ expanding both sides at $z_{12} = 0$

$$\langle P|O_{g,u}(0,0)|P\rangle = -\frac{8i a_s C_F}{3\epsilon_{\rm UV}}\langle P|O_q^{-,1}(0,0)|P\rangle,$$

$$0 = 0.$$

(On lattice) The local limits of space-like correlators are smooth, due to an effective resummation of all $\alpha_s^m ln^n z_{12}^2$ terms. (Su et al., 2022)

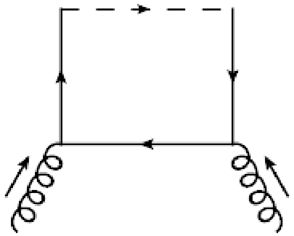
Regularization of the $1/z_{12}$ pole

Two methods of deriving the factorization of quasi PDFs/GPDs:

- Directly derive the factorization in momentum space ($\mathbb{C}_{gq}^{mom}$), (Ma et al., 2022; Wang et al., 2018)

- First derive the factorization in coordinate space,then Fourier transform it using DR.

	evolution	finite part
unpolarized	equiv.	equiv.
polarized	✓	✓



	evolution	finite part
unpolarized	✓	✓
polarized	✓	✓

FT of factorization after IBP v.s. $\mathbb{C}_{gq}^{mom}$

$$
\begin{aligned}
I_{gq} &= I_{1,gq} + I_{2,gq} \\
&= \frac{-2ia_s C_F \Gamma(-\epsilon)\eta^\epsilon}{\mathbf{z}_{12}} \Big\{ D_1\, O_q^s(z_1, z_2) + 2\epsilon\, D_1 \int_0^1 d\alpha \left[O_q^s(z_1, z_{21}^\alpha) + O_q^s(z_{12}^\alpha, z_2) \right] \\
&\qquad + 2\, D_2 \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, O_q^s(z_{12}^\alpha, z_{21}^\beta) \Big\},
\end{aligned}
$$

FT of factorization before IBP v.s. $\mathbb{C}_{gq}^{mom}$

$$
\begin{aligned}
I_{1,gq}^u = &\frac{-ia_s C_F \Gamma(-\epsilon)\eta^\epsilon}{\mathbf{z}_{12}} \bar q(p_1)\Gamma q(p_2) \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \mathrm{e}^{-i(z_{12}^\alpha \cdot p_1 + z_{21}^\beta \cdot p_2)} (1-\epsilon) \\
&\Big\{ \Big((p_1 \cdot z_{12})^2 \alpha\bar\alpha + 2i(p_1 \cdot z_{12})(2\epsilon\alpha - \bar\alpha - \epsilon) + (\alpha \leftrightarrow \beta, p_1 \leftrightarrow -p_2) \Big) \\
&+ (p_1 \cdot z_{12})(p_2 \cdot z_{12})(\alpha\beta + \bar\alpha\bar\beta) + 4\epsilon(1+\epsilon) \Big\},
\end{aligned}
$$

←

$(p_1 \cdot z_{12}) f(\alpha,\beta) \to \frac{\partial}{\partial\alpha} f(\alpha,\beta)$

Regularization of the $1/z_{12}$ pole

Two methods of deriving the factorization of quasi PDFs/GPDs:

- Directly derive the factorization in momentum space ($\mathbb{C}_{gq}^{mom}$), (Ma et al., 2022; Wang et al., 2018)
- First derive the factorization in coordinate space, then Fourier transform it to momentum space.

Besides gluon-in-quark channel, different results exist in literature for the evolution/factorization of the following parton distributions (which are free from FT of $1/z_{12}$ pole).

- unpolarized quark singlet GPDs (quark-in-gluon channel)

  (Ji,1997; Radyushkin,1997; Belitsky,Mueller,1998)

- gluon quasi PDFs and gluon quasi GPDs (gluon-in-gluon channel)

  (Wang et al.,2018; Balitsky et al.; 2020, Ma et al.,2022; Yao et al.,2022)

The results are equivalent at the convolution level, we expect if one Fourier transforms the evolution/matching kernel before applying IBP in the coordinate space, one is expected to obtain exactly the same kernel as that directly calculated in momentum space.

Regularization of the $1/z_{12}$ pole

Consider regularizing the pole at $z_{12} = 0$ with principal-value prescription (Blumlein et al.,1999):

$$\mathbb{C}_{gq}^{\mathrm{FT,PV},s}(x, y, \xi, \frac{\mu}{P^z}) = \int \frac{d\mathbf{z_{12}}}{2\pi} \mathrm{P}\left[\frac{1}{\mathbf{z_{12}}} \int_{\alpha\beta} C_{gq}^{s}(\alpha, \beta, \mu^2 \mathbf{z_{12}^2})\right],$$

- At $\mathcal{O}(\alpha_s)$, the FT of matching/evolution kernel in coordinate space gives the results same as that directly calculated in momentum space.

- For some correlators regularized using PV, taking the local limit gives inconsistent results:

$$0 = \int_0^1 d\alpha \int_0^{1-\beta} C_{gq}^{+}(\alpha, \beta, \mu^2 \mathbf{z_{12}^2})(1 - \alpha + \beta)\langle P' | O_q^{+,1}(0,0) | P \rangle$$
$$= 6 i a_s C_F \langle P' | O_q^{+,1}(0,0) | P \rangle,$$

polarized nonforward correlator, inconsistent!

Summary

- For the mixing between nonlocal gluon operators and quark operators, there is an ambiguity when relating coordinate space and momentum space results. We show such an ambiguity is due to the lack of a proper regularization of the singularity as $z_{12} \to 0$.

- We justify DR provides a simple and physically viable treatment to the coordinate→momentum space Fourier transform, yielding results consistent with that directly calculated in momentum space.

- Our work is expected to have a significant impact on lattice calculations of gluon GPDs/PDFs and singlet quark GPDs/PDFs.

Thanks for listening !

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Backup

Symmetry argument: $$\mathbb{C}^{s}_{gq} = \int_{\alpha\beta}\int \frac{dz_{12}}{2\pi} e^{iz_{12}P^z(x+\xi(-\alpha+\beta)+(1-\alpha-\beta)y}C^{s}_{gq},$$

differentiate $\mathbb{C}^{s}_{gq}$ with respect to x/y and integrate back

$$\mathbb{C}^{-}_{gq}(x,y,\xi,\frac{\mu}{P^z}) = C(x,\xi) + \int dy \int \frac{d\mathbf{z}_{12}}{2\pi} iP^z \int_{\alpha\beta} (1-\alpha-\beta) C^{-}_{gq}(\alpha,\beta,\mu^2\mathbf{z}_{12}^2)\,, \qquad \mathbb{C}^{+}_{gq}(x,y,\xi,\frac{\mu}{P^z}) = C(y,\xi) + \int dx \int \frac{d\mathbf{z}_{12}}{2\pi} iP^z \int_{\alpha\beta} C^{+}_{gq}(\alpha,\beta,\mu^2\mathbf{z}_{12}^2)\,,$$

$C(x,\xi) = 0$ for unpolarized case (odd in y) $C(y,\xi) = 0$ for polarized case (odd in x)

- Works for the evolution of twist-2 gluon GPDs.
- Doesn't work for the distributions without a definite parity over a variable, e.g:

$$\mathbb{H}^{2}_{g,h}(x,\xi,\Delta^2,P^z) = \int \frac{d\mathbf{z}_{12}}{2\pi} e^{i\mathbf{z}_{12}xP^z} \langle p' | F^{zi}(z_{12}) F^{tj}(0) \epsilon_{\perp ij} | p \rangle,$$ Zhang et al., 2019

- In comparison, DR provides a unifying treatment, regardless of whether the distributions process definite symmetries.