Extraction of the Collins-Soper kernel on the lattice using complex directional Wilson lines

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with

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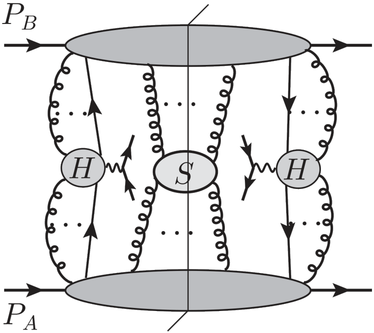
TMD factorization



Drell-Yan leading region [Collins, 2011]

- $\Lambda_{\mathrm{QCD}} \lesssim |\vec{q}_\perp| \ll Q$
- Leading region has contribution from soft momentum states
- Need to regulate rapidity divergences present in beam and soft functions
- ν, rapidity renormalization scale

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}(Q,\mu) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} B_i\left(x_a, \vec{b}_\perp, \mu, \frac{\zeta_a}{\nu^2}\right) B_j\left(x_b, \vec{b}_\perp, \mu, \frac{\zeta_b}{\nu^2}\right)$$

$$\times S_i(b_\perp, \mu, \nu)\left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$

Soft function

Bare soft function:

$$S\left(b_\perp\right) = \frac{1}{N_c}\left\langle 0\right| \mathrm{Tr}\, S_n^\dagger(\vec{b}_\perp)S_{\bar{n}}(\vec{b}_\perp)S_\perp(-\infty\bar{n};\vec{b}_\perp,\vec{0}_\perp)S_{\bar{n}}^\dagger(\vec{0}_\perp)S_n(\vec{0}_\perp)S_\perp^\dagger(-\infty n;\vec{b}_\perp,\vec{0}_\perp)\left|0\right\rangle$$

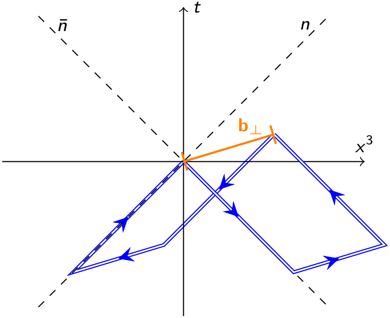
Soft Wilson line:

$$S_n\left(x\right) = P\exp\left\{-ig\int_{-\infty}^{0}\mathrm{d}s n^\mu A_\mu(x+sn)\right\}$$

Lightlike vectors:

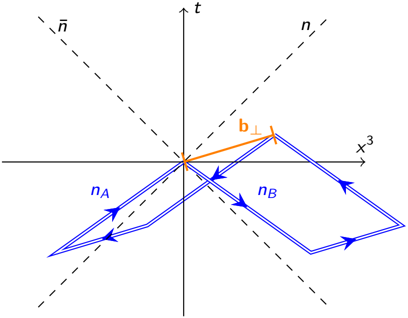
$$n = (1,0,0,1), \qquad \bar{n} = (1,0,0,-1)$$

$$n^2 = 0, \qquad \bar{n}^2 = 0, \qquad n\cdot\bar{n} = 2$$



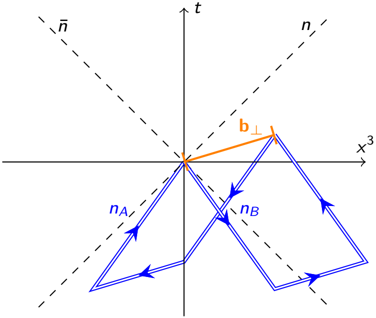
Spacelike Wilson lines:

$$n_A \equiv n - e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} - e^{+y_B}n$$

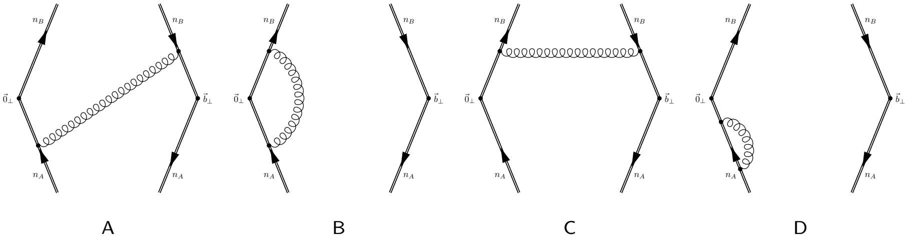


Timelike Wilson lines:

$$n_A \equiv n + e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} + e^{+y_B}n$$



One loop result in Minkowski space



One-loop result in Collins scheme:

$$S(b_\perp, \epsilon, y_A, y_B) = 1 + \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 - 2|y_A - y_B|\frac{1+e^{2(y_B-y_A)}}{1-e^{2(y_B-y_A)}}\right\} + \mathcal{O}(\alpha_s^2)$$

[Ebert, *et. al.*, 2019]

TMD evolution and Collins-Soper kernel

Evolution kernel:

$$\gamma_\mu^q(\mu, \zeta) = \frac{\mathrm{d} \log f_q \left(x, \vec{b}_\perp, \mu, \zeta \right)}{\mathrm{d} \log \mu}$$

Collins-Soper (CS) kernel:

$$\gamma_q(b_\perp, \mu) = \frac{\mathrm{d} \log f_q \left(x, \vec{b}_\perp, \mu, \zeta \right)}{\mathrm{d} \log \sqrt{\zeta}}$$

$$S_q\left(b_\perp, y_A, y_B, \mu\right) = S_I\left(b_\perp, \mu\right) e^{\gamma_q(b_\perp, \mu)(y_A - y_B)} \left(1 + \mathcal{O}\left(e^{-2(y_A - y_B)}\right)\right)$$

In the LaMET framework:

$$\sqrt{S_I\left(b_\perp,\mu\right)}\,\tilde{f}_\Gamma\left(x,b_\perp,P^z,\mu\right)$$
$$=f\left(x,b_\perp,\mu,\zeta\right)H_f\left(x,P^z,\mu\right)\exp\left[\frac{1}{2}\log\frac{(2xP^z)^2}{\zeta}\gamma_q\left(b_\perp,\mu\right)\right]$$
$$+\mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{xP^z},\frac{1}{b_\perp(xP^z)}\right)$$

[Ebert, *et. al.*, 2019], [Ji, Liu, Liu, 2020], [Ebert, *et. al*, 2022]

- Intrinsic soft function: S_I [Ji, Liu, Liu, 2020]
- quasi-TMD beam function: $\tilde{f}_\Gamma$
- TMDPDF: f
- Perturbative hard kernel: H_f
- CS kernel: γ_q

- CS kernel from ratios of quasi-TMD matrix elements

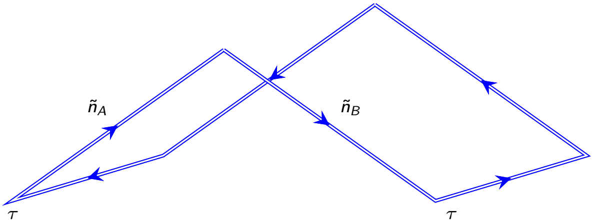
[Ji, *et. al*, 2015], [Ebert, *et. al*, 2019,], [Ji, *et. al*, 2020], [Bollweg, *et. al*, 2025]

- Soft function with timelike regulator using HQET

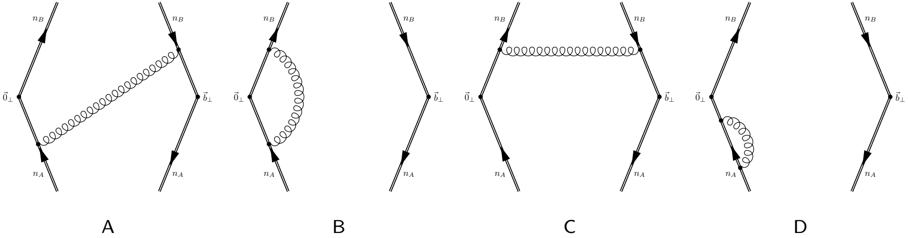
[Ji, *et. al.*, 2020], [Liu, 2022]

Euclidean space directional vectors with purely imaginary time components

$$\tilde{n}_A = \left(in_A^0, 0, 0, n_A^3\right), \quad \tilde{n}_B = \left(in_B^0, 0, 0, n_B^3\right)$$



Soft function in Euclidean space at one loop



Calculation in coordinate space at one loop:

$$S^{(1)}\left(b_\perp, \epsilon, r_a, r_b\right) = \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 + \log\left|\frac{(r_a-1)(r_b-1)}{(r_a+1)(r_b+1)}\right|\frac{r_a r_b + 1}{r_a + r_b}\right\}$$

$$|r_{a,b}| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$$

$$r_a = \frac{n_A^3}{n_A^0} = \frac{1+e^{-2y_A}}{1-e^{-2y_A}}, \quad r_b = \frac{n_B^3}{n_B^0} = \frac{1+e^{2y_B}}{1-e^{2y_B}}$$

After mapping, we obtain result from [Ebert, *et. al.*, 2019]

Auxiliary field definition of the Wilson line

Write Wilson line in terms of one dimensional 'fermions' that live along the path:

$$
\begin{aligned}
&P \exp\left\{ -ig \int_{s_i}^{s_f} \mathrm{d}s n^{\mu} A_{\mu}(y(s)) \right\} \\
&= Z_{\psi}^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \psi\bar{\psi} \exp\left\{ ig \int_{s_i}^{s_f} \mathrm{d}s \bar{\psi} i \partial_s \psi - \bar{\psi} n \cdot A \psi \right\}
\end{aligned}
$$

[Gervais, Nevau 1980], [Aref'eva 1980]

Auxiliary field propagator:

$$
in \cdot D H_n(y) = \delta(y) \xrightarrow{\text{Euclidean space}} -i\tilde{n} \cdot D_E H_{\tilde{n}}(y) = \delta(y), \qquad \tilde{n} = (in_0, \vec{n})
$$

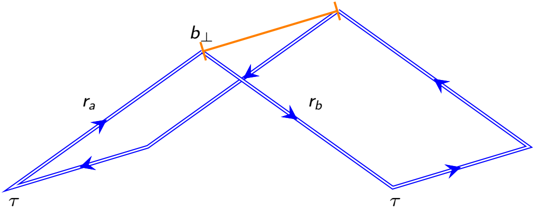
Meaningful solution only obtained with a UV cutoff

[Aglietti, *et. al.* 1992], [Aglietti, 1994]

Butterfly loop

- Large τ form of butterfly loop from UV cutoff effects:

$$\tilde{S}_{\rm bfly} = \tilde{S}\left(b_\perp, a, r_a, r_b, \tau\right) \overset{\tau\to\infty}{\sim} e^{2\pi\tau(r_a+r_b)/a} / \tau^4$$

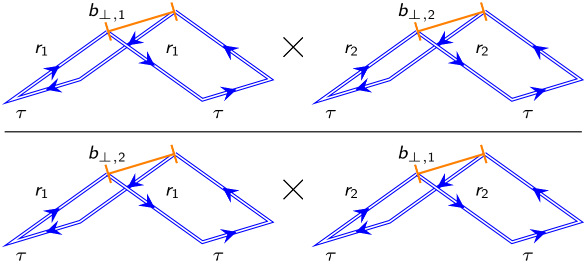


- Cutoff effects cancel in the ratio

$$R_{\rm single}\left(b_{\perp,1}, b_{\perp,2}, a, r_a, r_b, \tau\right) = \frac{\tilde{S}\left(b_{\perp,1}, a, r_a, r_b, \tau\right)}{\tilde{S}\left(b_{\perp,2}, a, r_a, r_b, \tau\right)}$$

Treatment of divergences

$$\tilde{S}_{\rm double}\left(b_{\perp,1}, b_{\perp,2}, a, r_1, r_2, L\right) = \frac{\tilde{S}\left(b_{\perp,1}, a, r_1, r_1, L\right)}{\tilde{S}\left(b_{\perp,2}, a, r_1, r_1, L\right)} \Bigg/ \frac{\tilde{S}\left(b_{\perp,1}, a, r_2, r_2, L\right)}{\tilde{S}\left(b_{\perp,2}, a, r_2, r_2, L\right)}$$
$$= \exp\left[\left(\gamma_q\left(b_{\perp,1}, a\right) - \gamma_q\left(b_{\perp,2}, a\right)\right) 2(y_1 - y_2)\right]$$



Double ratio

- For large lattice time:

$$\tilde{S}_{\text{double}}\left(b_{\perp,1}, b_{\perp,2}, a, r_1, r_2, \tau\right) = \frac{\tilde{S}_{\text{lat}}\left(b_{\perp,1}, a, r_1, r_1\right)}{\tilde{S}_{\text{lat}}\left(b_{\perp,2}, a, r_1, r_1\right)} \Bigg/ \frac{\tilde{S}_{\text{lat}}\left(b_{\perp,1}, a, r_2, r_2\right)}{\tilde{S}_{\text{lat}}\left(b_{\perp,2}, a, r_2, r_2\right)} + \mathcal{O}\left(\frac{b_1^2 - b_2^2}{\tau^2}\left(\frac{1}{r_1 - 1} - \frac{1}{r_2 - 1}\right), a^2, \frac{r_{1,2} - 1}{r_{1,2} + 1}\right)$$

- Match between lattice and continuum renormalization schemes

$$S_{\text{double}}\left(b_{\perp,1}, b_{\perp,2}, \mu, r_1, r_2\right) = C\left(r_1, r_2, \mu, a\right) \times \tilde{S}_{\text{double}}\left(b_{\perp,1}, b_{\perp,2}, a, r_1, r_2\right)$$

- We can then extract the CS kernel relative to it's value at another value of $b_\perp$:

$$\gamma_q\left(b_{\perp,1}, \mu\right) = \gamma_q\left(b_{\perp,2}, \mu\right) + \frac{\frac{1}{2}\log\left(S_{\text{double}}\left(b_{\perp,1}, b_{\perp,2}, \mu, r_1, r_2\right)\right)}{\log\left(\frac{r_1+1}{r_1-1} \Big/ \frac{r_2+1}{r_2-1}\right)}$$

- Use auxiliary field definition of the Wilson line
  [Gervais, Nevau 1980], [Aref'eva 1980], [Aglietti, *et. al.* 1992], [Aglietti, 1994], [Horgan, *et. al.*, 2009]

- Using quenched configurations [Detmold, Endres, 2018]

$L^3 \times T$	$a(fm)$	N_{config}
$24^3 \times 48$	0.081	400
$32^3 \times 64$	0.060	400
$40^3 \times 80$	0.048	250
$48^3 \times 96$	0.041	341
$64^3 \times 128^*$	0.03	200

*In progress

- Computation performed with 2048 sources per configuration

Matching procedure for double ratio

- Need at least one perturbative value of γ_q, since:

$$\gamma_q\left(b_{\perp,1},\mu\right) = \gamma_q\left(b_{\perp,2}^{\mathrm{pert}},\mu\right) + \frac{\frac{1}{2}\log\left(S_{\mathrm{double}}\left(b_{\perp,1},b_{\perp,2},\mu,r_1,r_2\right)\right)}{\log\left(\frac{r_1+1}{r_1-1}\Big/\frac{r_2+1}{r_2-1}\right)}$$

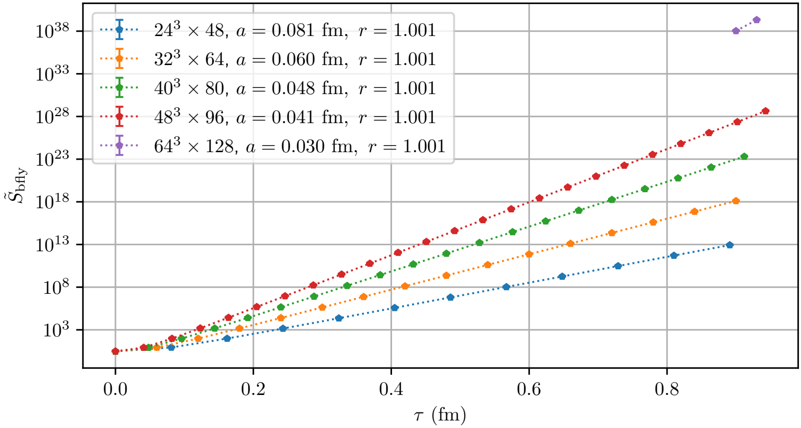
- Another perturbative value of γ_q is needed, because 'r' gets renormalized due to the breaking of $O(4)$ symmetry. Obtain renormalized rapidity factor through:

$$\log\left(\frac{r_1^{\mathrm{ren}}+1}{r_1^{\mathrm{ren}}-1}\Big/\frac{r_2^{\mathrm{ren}}+1}{r_2^{\mathrm{ren}}-1}\right) = \frac{\frac{1}{2}\log\left(\tilde{S}_{\mathrm{double}}\left(b_{\perp,1},b_{\perp,2},r_1,r_2,a\right)\right)}{\gamma_q\left(b_{\perp,1}^{\mathrm{pert}},\mu\right)-\gamma_q\left(b_{\perp,2}^{\mathrm{pert}},\mu\right)}$$

$$\left(\gamma_q\left(b_{\perp,1},\alpha_s(\mu_0)\right)-4\int_{\mu_0}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\Gamma_{\mathrm{cusp}}\left(\alpha_s(\mu')\right)\right)-\left(\gamma_q\left(b_{\perp,2},\alpha_s(\mu_0)\right)-4\int_{\mu_0}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\Gamma_{\mathrm{cusp}}\left(\alpha_s(\mu')\right)\right)$$
$$=\gamma_q\left(b_{\perp,1},\alpha_s(\mu_0)\right)-\gamma_q\left(b_{\perp,2},\alpha_s(\mu_0)\right) \quad \leftarrow \quad \text{RG invariant}$$

- Matching window: $3a \leq b_\perp \lesssim 0.2\ \mathrm{fm}$ chosen to reduce discretation error while remaining in a perturbative region of $b_\perp$

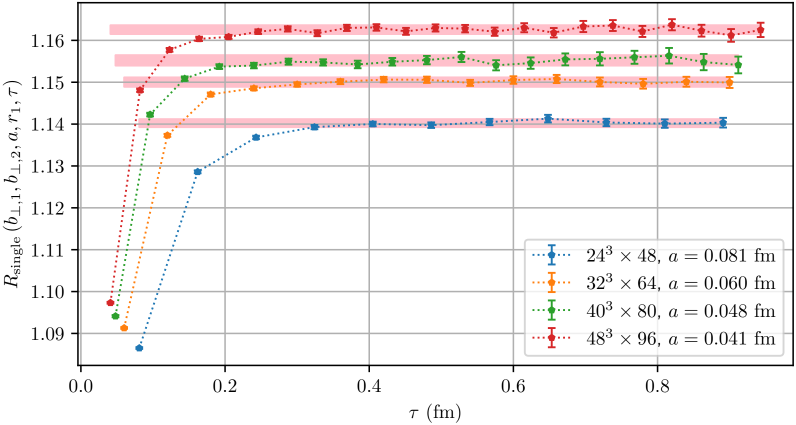
Butterfly loop, time dependence



Butterfly plot with $b_\perp = 3a$ and $r = 1.001$

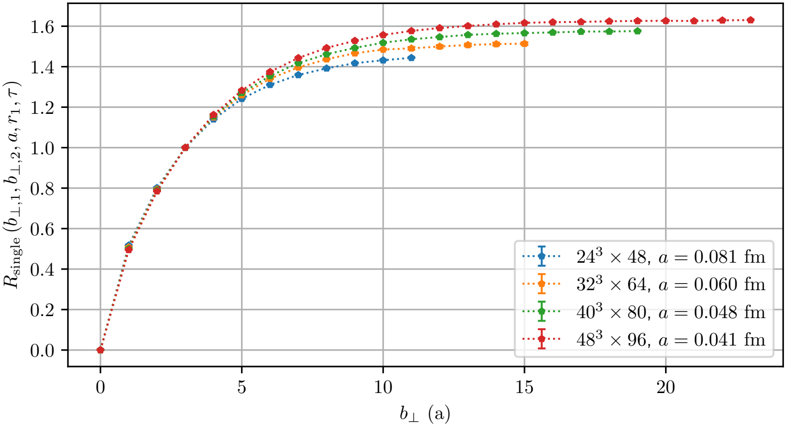
Single ratio, time dependence

$$R_{\text{single}}\left(b_{\perp,1}, b_{\perp,2}, a, r_1, \tau\right) = \frac{\tilde{S}\left(b_{\perp,1}, a, r_1, r_1, \tau\right)}{\tilde{S}\left(b_{\perp,2}, a, r_1, r_1, \tau\right)}$$



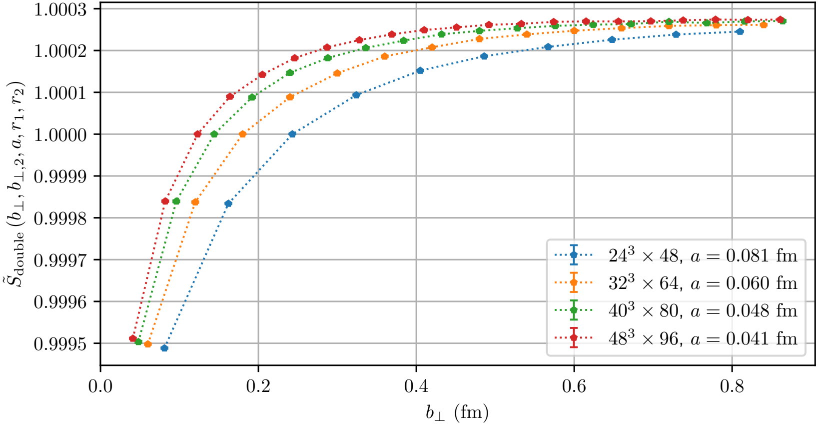
Single ratio plot with $b_{\perp,1} = 4a$, $b_{\perp,2} = 3a$, and $r_1 = 1.001$

Single ratio, $b_\perp$ dependence



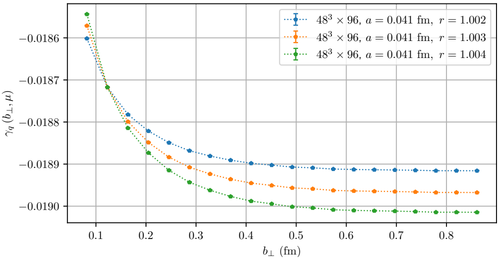
Single ratio plot with $b_{\perp,2} = 3a$, and $r_1 = 1.001$. Using values obtained from fit to τ plateau.

$$\tilde{S}_{\text{double}}\left(b_{\perp,1}, b_{\perp,2}, a, r_1, r_2\right) = \frac{R_{\text{single}}\left(b_{\perp,1}, b_{\perp,2}, a, r_1\right)}{R_{\text{single}}\left(b_{\perp,1}, b_{\perp,2}, a, r_2\right)}$$



Double ratio plot with $b_{\perp,2} = 3a$, $r_1 = 1.002$, $r_2 = 1.001$

Bare vs renormalized rapidity factor

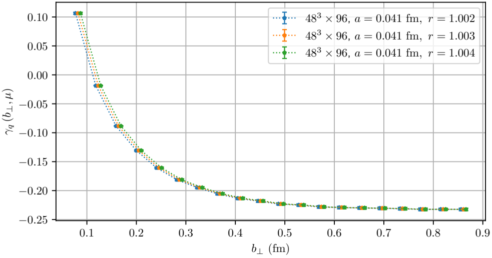


Bare rapidity:

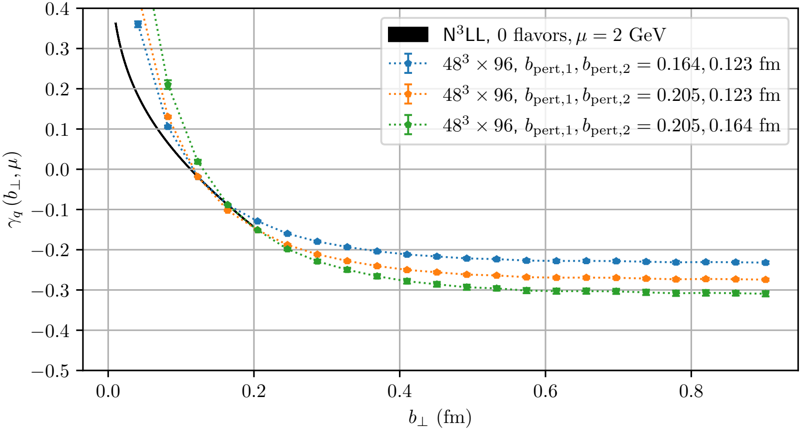
$$2(y_1 - y_2) = \log\left(\frac{r_1+1}{r_1-1}\Big/\frac{r_2+1}{r_2-1}\right)$$

With perturbative matching:

$$2(y_1^{\text{ren}} - y_2^{\text{ren}}) = \frac{\tilde{S}_{\text{double}}\left(b_{\perp,1}^{\text{pert}}, b_{\perp,2}^{\text{pert}}, r_1, r_2, a\right)}{\gamma_q\left(b_{\perp,1}^{\text{perp}}, \mu\right) - \gamma_q\left(b_{\perp,2}^{\text{pert}}, \mu\right)}$$

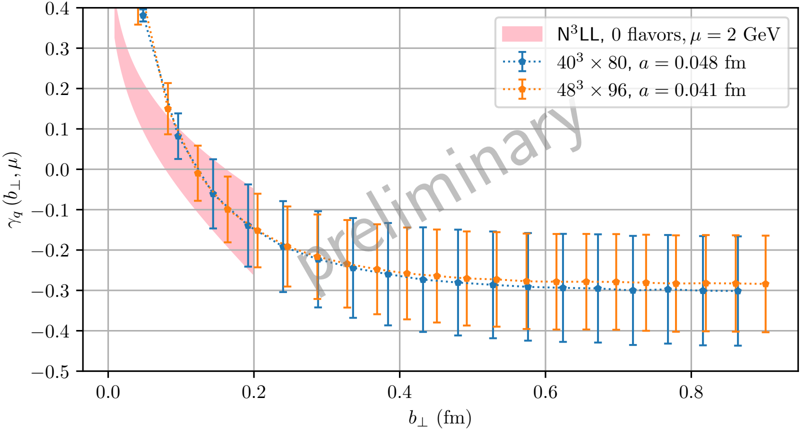


CS kernel



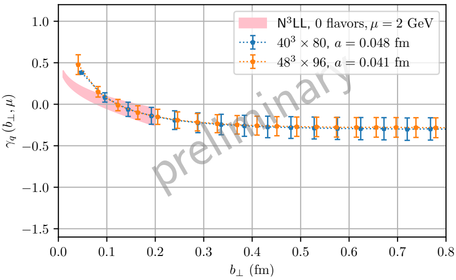
CS kernel extracted at different $b_{\rm pert,1}, b_{\rm pert,2}$ within matching window

CS kernel with systematic error

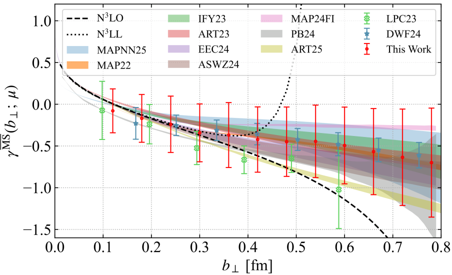


Conservative estimate of systematic error from matching points within fit window and variation of UV scale

Comparison with other results



Our result



Other recent results presented in [Bollweg, *et. al.*, 2025]

- Accounting for scale variation and variation in matching points, we give a conservative estimate of the systematic uncertainty.
- We obtain results competitive with recent lattice extractions of the Collins-Soper kernel.
- The dominant systematic error is expected to be further shrunk with data at $a = 0.03$ fm, which are now being processed.

Conclusion and outlook

- Euclidean space calculation of soft function has a direct mapping to Minkowski space result
- High precision lattice data 'butterfly' loop
- Double ratio method gives high precision computation of difference of CS kernel: $\gamma_q(b_\perp, \mu) - \gamma_q(b'_\perp, \mu)$
- Uncertainty of CS kernel extraction dominated by systematic effects in matching to perturbative computation
- Can be improved with higher precision perturbative computations or with finer lattices

Ongoing work:

- $64^3 \times 128$ lattice
- Perform analysis with interpolated data for better comparison between lattices
- A more rigorous estimate of systematic error

Thank you!