# Seeking Forces in the Nucleon

## Physics Mechanisms of Momentum Current Density

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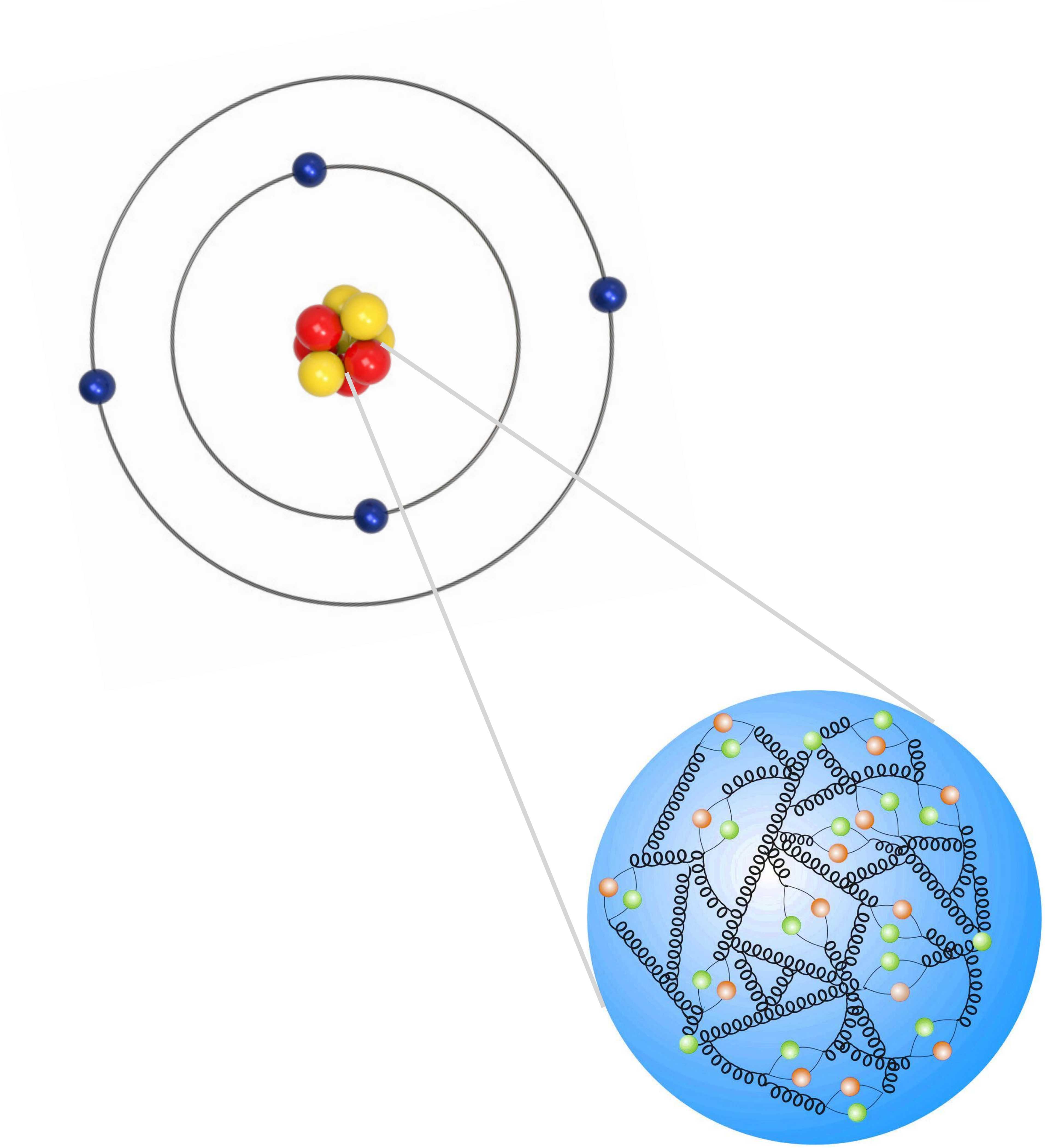
# Contents





# Introduction

# EMT and GPDs

The matrix elements of the QCD EMT can be parametrized using the gravitational form factors (GFFs),

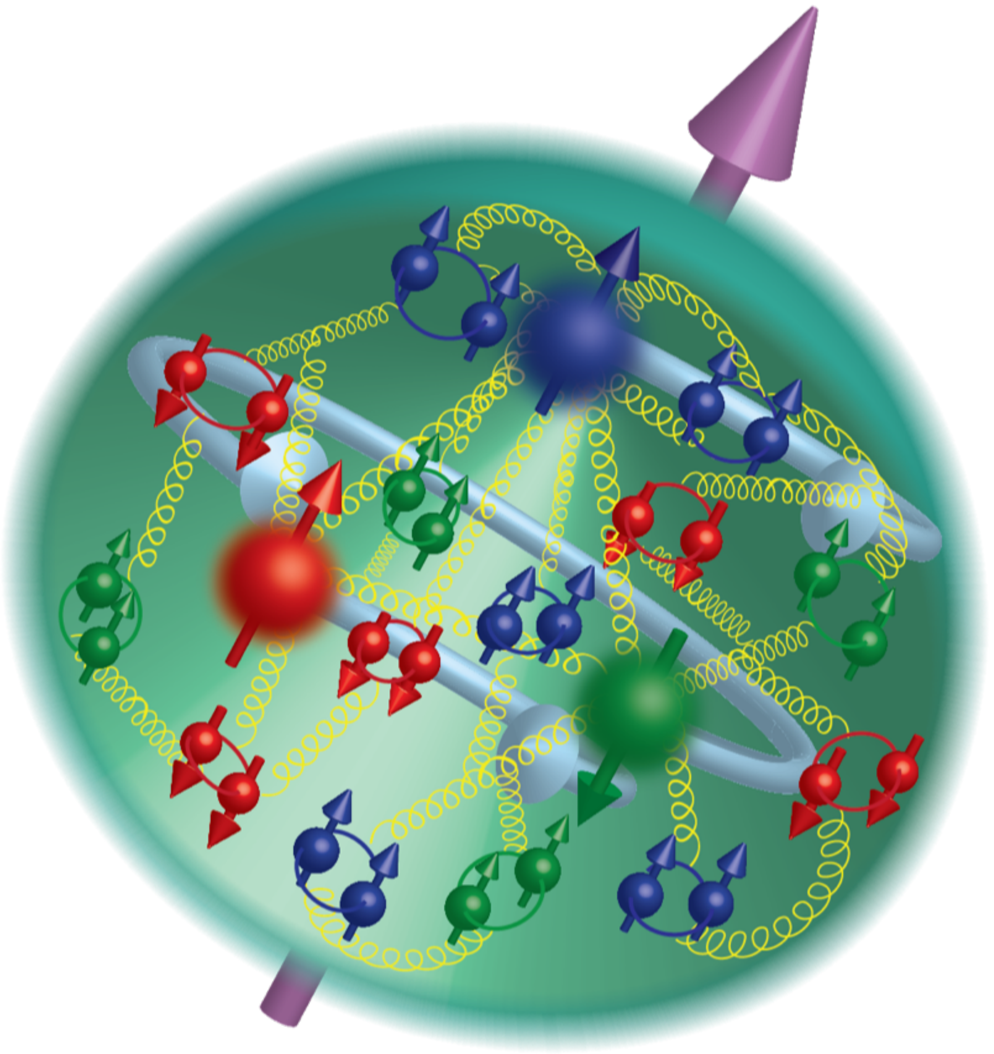
$$\left\langle P' \left| T_i^{\mu\nu} \right| P \right\rangle = \bar{U}(P') \left[ A_i\left(q^2\right) \gamma^{(\mu}\bar{P}^{\nu)} + B_i\left(q^2\right) \frac{\bar{P}^{(\mu} i\sigma^{\nu)\alpha} q_\alpha}{2M} + C_i\left(q^2\right) \frac{q^\mu q^\nu - q^2 g^{\mu\nu}}{M} + \bar{C}_i\left(q^2\right) M g^{\mu\nu} \right] U(P)$$

where $i = q, g$ are quarks and gluons.

These form factors are related to the moments of GPDs through sum rules,

$$\int_{-1}^{1} \mathrm{d}x \; xH\left(x, \xi, q^2\right) = A\left(q^2\right) + \xi^2 C\left(q^2\right)$$

$$\int_{-1}^{1} \mathrm{d}x \; xE\left(x, \xi, q^2\right) = A\left(q^2\right) - \xi^2 C\left(q^2\right)$$



These form factors encode rich information on the internal structures of the nucleon.

Ji, Phys. Rev. Lett. **78**, 610

Belitsky, et al, 10.1016/j.physrep.2005.06.002

# QCD EMT/MCD

The QCD EMT is decomposed into the traceless $\bar{T}^{\mu\nu}$ and trace $\hat{T}^{\mu\nu}$ parts, (ignore the small quark mass)

**Quark Kinetic:** $$\bar{T}^{\mu\nu}_q(\mu) = \frac{1}{2}\bar{\psi} i \overleftrightarrow{\mathscr{D}}^{(\mu}\gamma^{\nu)}\psi$$

**Gluon Tensor:** $$\bar{T}^{\mu\nu}_g(\mu) = \frac{1}{d} g^{\mu\nu}F^2 - F^{\mu\alpha}F^{\nu}{}_{\alpha}$$

**Trace Anomaly (Gluon Scalar):** $$\hat{T}^{\mu\nu}_a = \frac{1}{4} g^{\mu\nu}\frac{\beta(g)}{2g}F^2$$

The traceless parts, quarks and gluons, have a classical correspondence.

The trace anomaly comes from pure quantum effect, representing interactions.

To obtain the coordinate space distribution, we choose the Breit frame for the matrix elements of the above operators, and perform a Fourier transformation,

Jaffe, Phys. Rev. D **103**, 016017

$$\langle T^{ij}_q \rangle(\vec{r}) = +\frac{M}{4} G_{s,q}(r)\delta^{ij} + \frac{1}{M}\left(\delta^{ij}\nabla^2 - \nabla^i\nabla^j\right) C_q(r)$$
$$\langle T^{ij}_g \rangle(\vec{r}) = +\frac{M}{4} G_{s,g}(r)\delta^{ij} + \frac{1}{M}\left(\delta^{ij}\nabla^2 - \nabla^i\nabla^j\right) C_g(r)$$
$$\langle T^{ij}_a \rangle(\vec{r}) = -\frac{M}{4} G_s(r)\delta^{ij} \equiv p_a(r)\delta^{ij}$$

$G_s$: Form factor of the trace of EMT.
Also called the scalar form factor.

# Mechanical Nucleon by Polyakov

It was proposed by M. Polyakov and collaborators to study the nucleon **mechanical structure,**

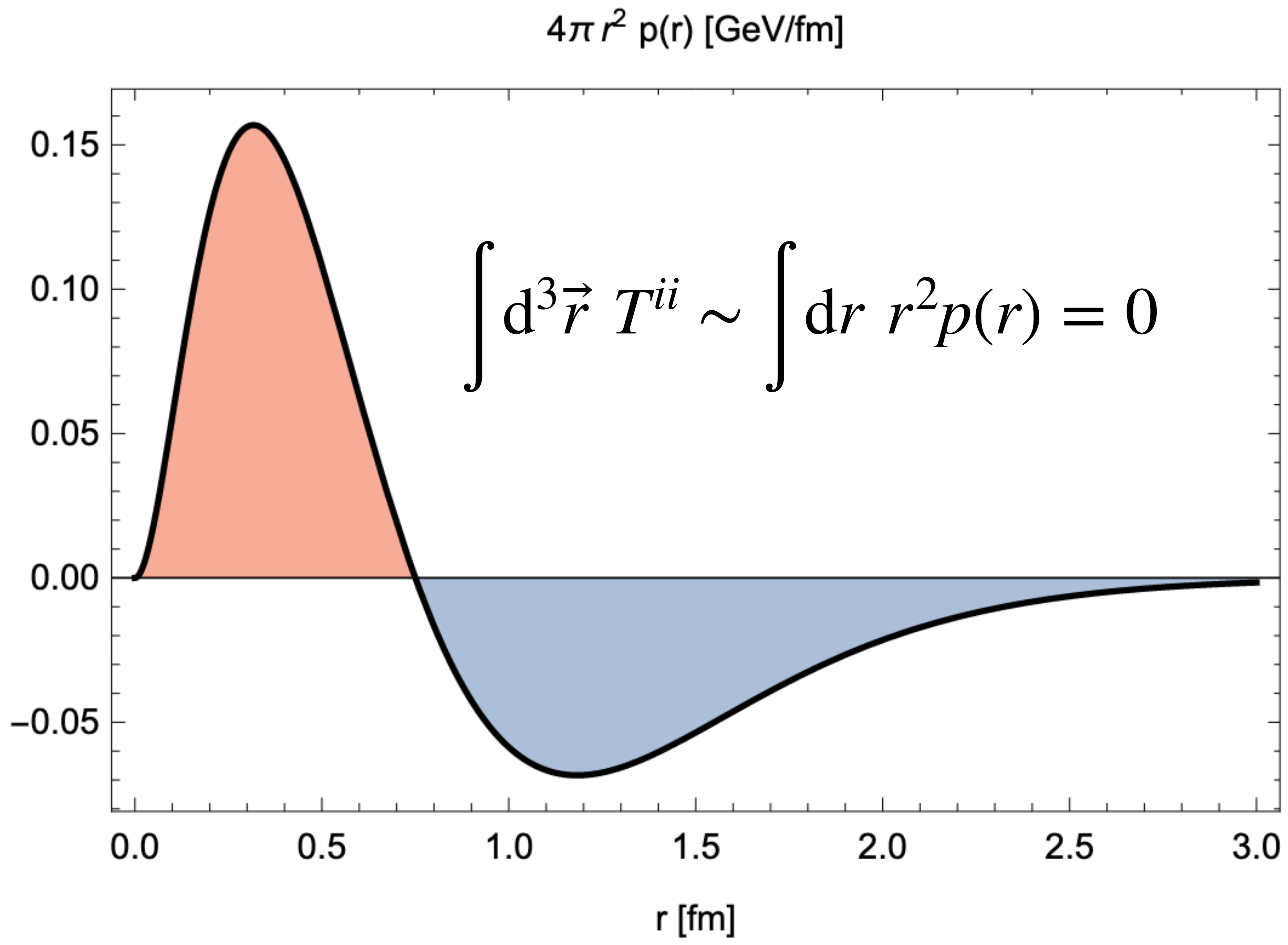
1. Nucleon is treated as a *mechanical continuum*, and its total MCD characterizes its *pressure* and *shear* distributions,

$$T^{ij}(\vec{r}) = \frac{1}{M}(\delta^{ij}\nabla^2 - \nabla^i\nabla^j)C(r) = s(r)\left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right) + p(r)\,\delta^{ij}$$

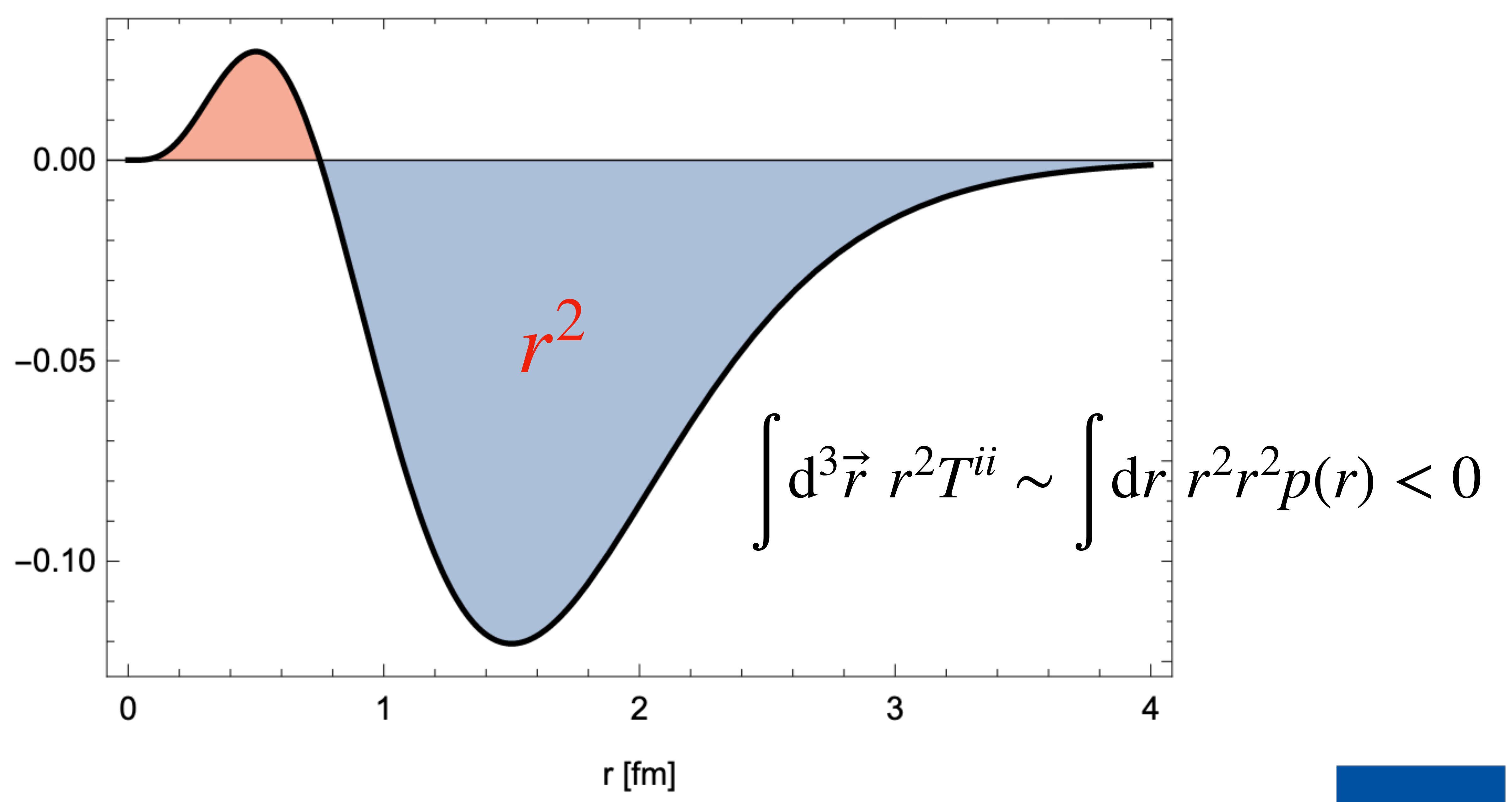
Polyakov, et al, PLB (2003)
Perevalova, et al, PRD (2016)

2. Nucleon total MCD, related to the $C/D$-form factor, satisfies two *mechanical stability conditions*,

Lorce, et al, EPJC (2018)




**von Laue condition**




**Negative D-term**

# Mechanical Nucleon by Polyakov

**Concerns,**

1. **Why is nucleon a continuous medium?**
    - *Large number of degrees of freedom?*
    - *Local thermalization?*
    - *Dynamical effective observables?*
2. **Why is total MCD a pressure/shear distribution?**
    - *Surface force?*
3. **Why do we re-introduce classical stability for quantum systems?**
    - *Quantum mechanics was introduced due to the failure of explaining the stability of H-atom in classical E&M.*
    - *Quantum stability is guaranteed by the Hermitian Hamiltonian with bounded eigenvalues.*

To understand this, we start with carefully examining the **physics mechanisms of the MCD**.

PART TWO

# Physics Mechanisms of MCD

# Momentum Flow

Momentum current density (MCD) is part of the local momentum continuity.

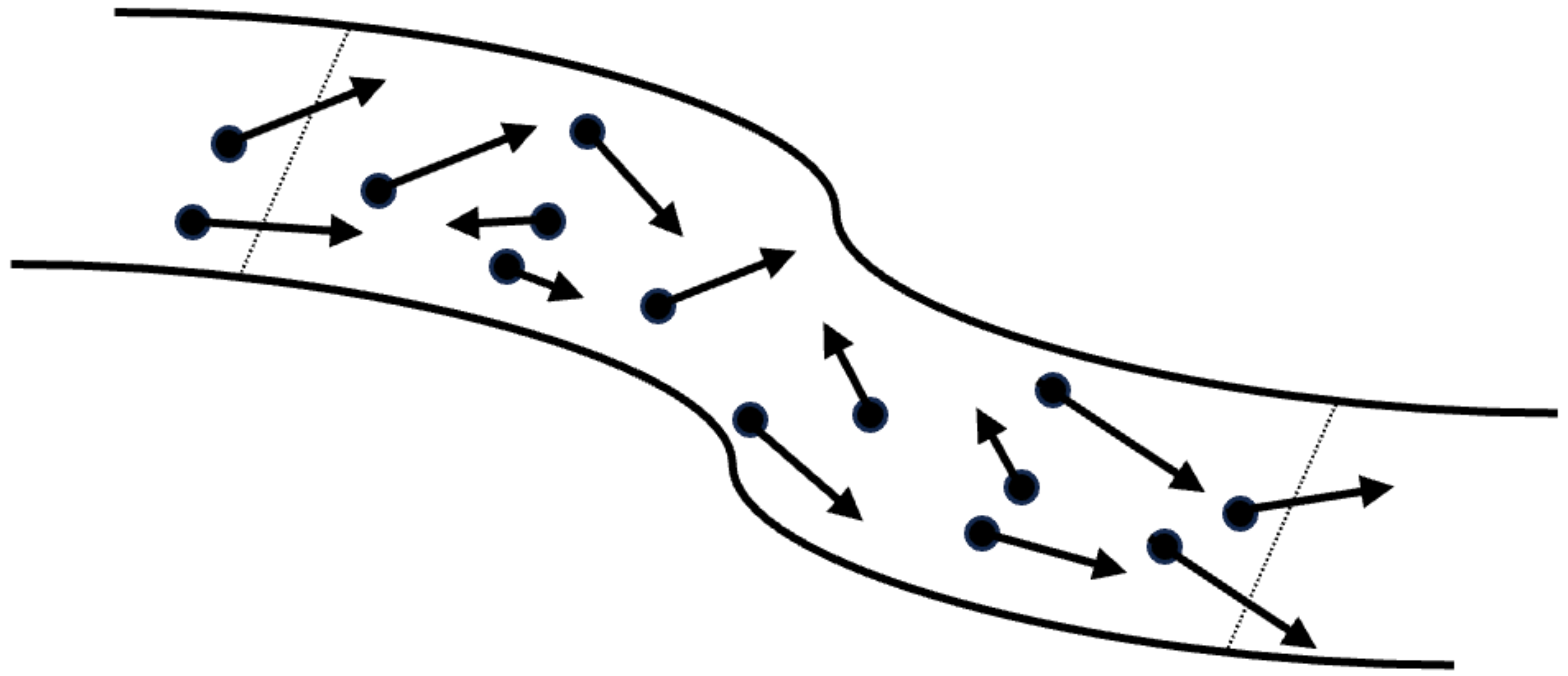
$$\partial_\mu T^{\mu j} = \partial_0 T^{0j} + \partial_i T^{ij} = 0$$

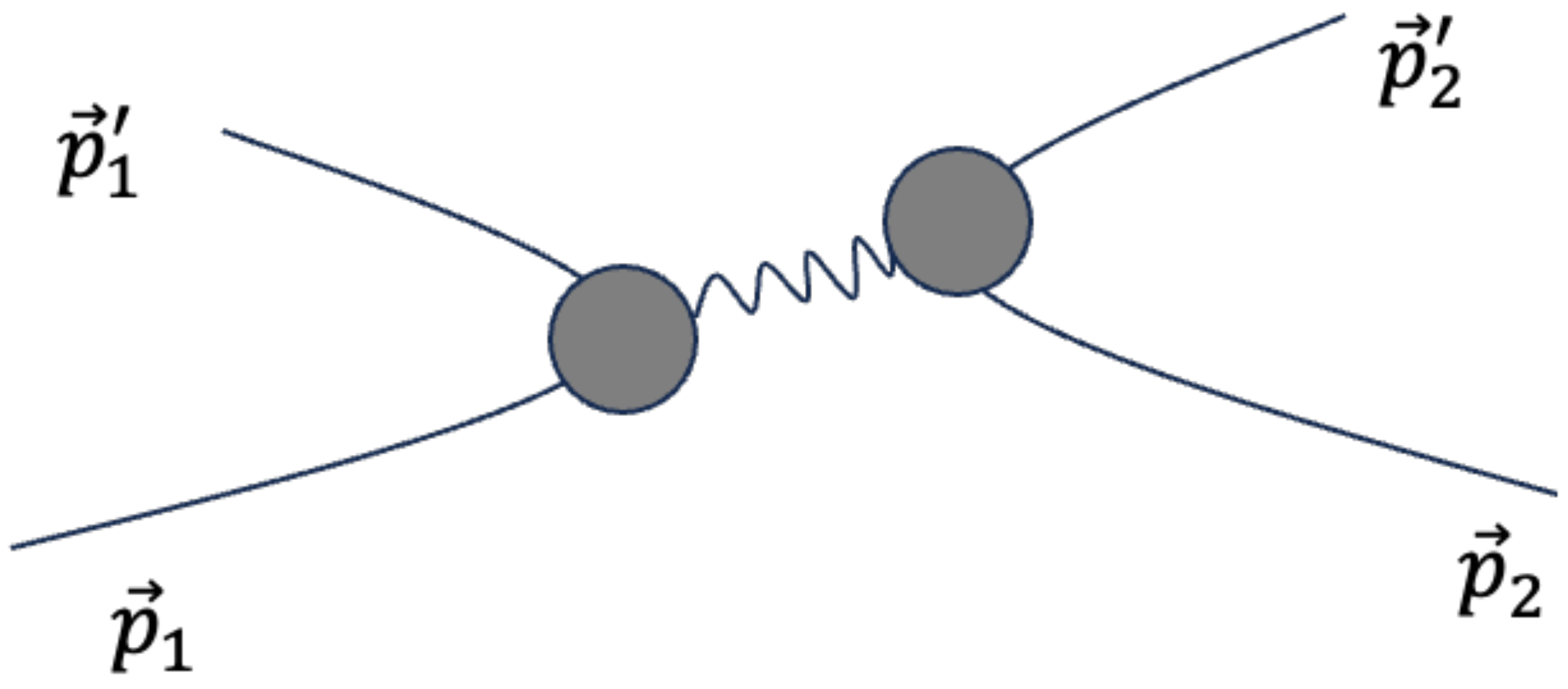
where $T^{0j}$ is the momentum density and $T^{ij}$ is the momentum current density (MCD).

- MCD, similar to the charge current $J^i \sim \rho v^i$, represents the **flow of momentum** $T^{ij} \sim v^i p^j$.
- A dynamical picture of momentum conservation through continuously flowing its density over space.

In general, the total MCD receives contributions from two different forms of momentum flow,

1. Kinetic motions of physical particles. (***Kinetic MCD***)
2. Interaction fields among particles. (***Interaction MCD***)





# Kinetic MCD

The simplest form of momentum flow, the kinetic MCD, is through the **particle transport**.

For system consisting of $N$ free particles, its momentum density and kinetic MCD,

$$T_K^{0i}(\vec{r},t) = \sum_{a=1}^{N} k_a^i(t)\delta^{(3)}\left(\vec{r}-\vec{r}_a(t)\right)$$

$$T_K^{ij}(\vec{r},t) = \sum_{a=1}^{N} v_a^i(t)k_a^j(t)\delta^{(3)}\left(\vec{r}-\vec{r}_a(t)\right) \sim \sum_a \frac{\hat{P}_a^i\hat{P}_a^j}{m}\delta^{(3)}\left(\vec{r}-\hat{\vec{r}}_a\right)$$

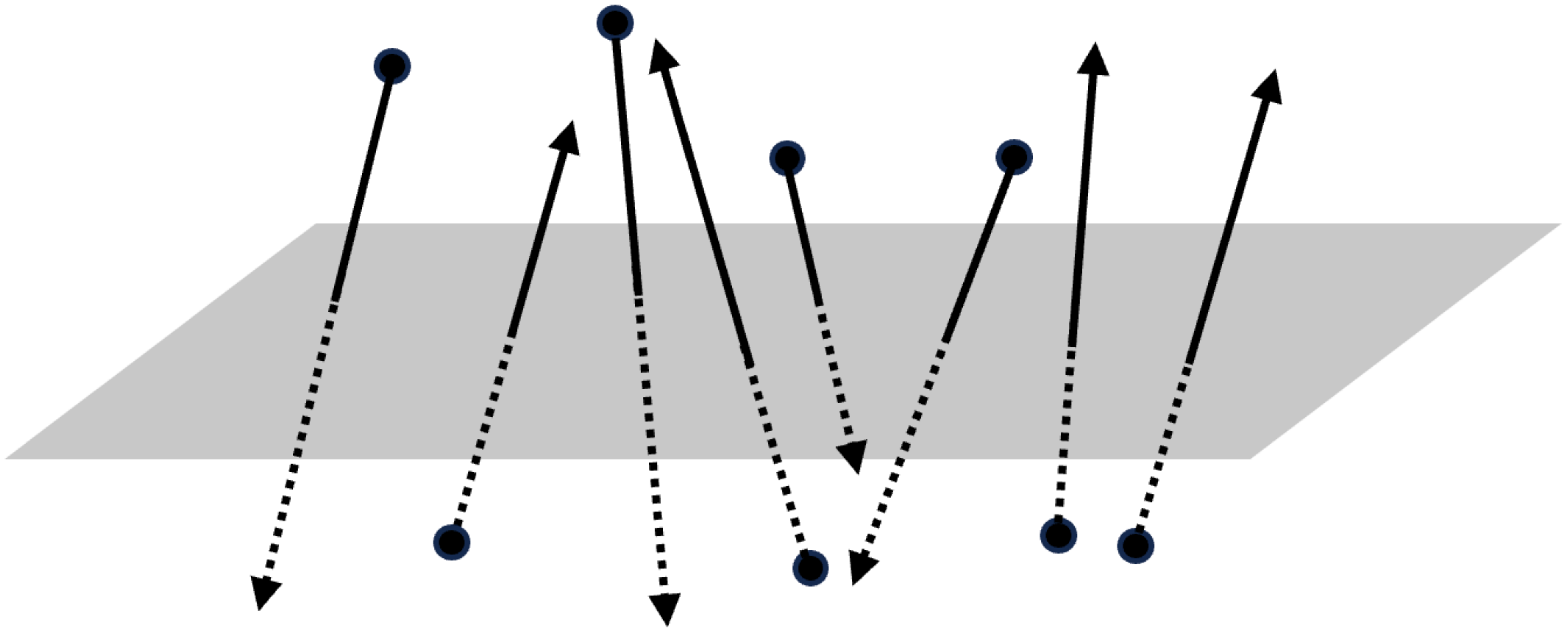
Quantum kinetic MCD

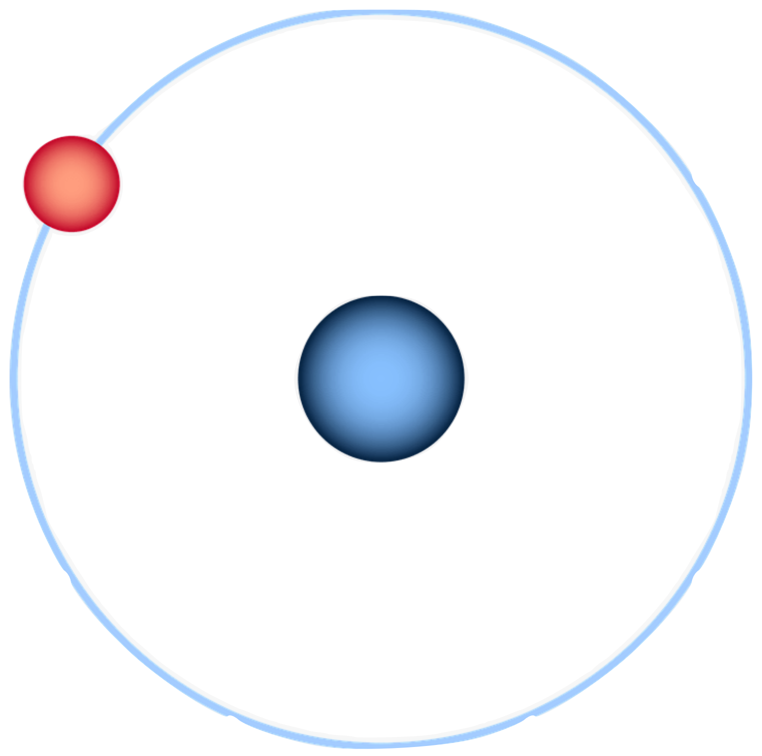
It is easy to verify that the continuity equation is satisfied.

The field theory kinetic MCD coincides with the above MCDs in the NR limit, verified for the hydrogen atom in NRQED,

$$\langle\psi_0|\hat{T}_K^{ij}|\psi_0\rangle \sim \langle\psi_0|\bar{\psi}\gamma^{(i}i\mathscr{D}^{j)}\psi|\psi_0\rangle$$

Ji, et al, Phys. Rev. D **110**, 114045



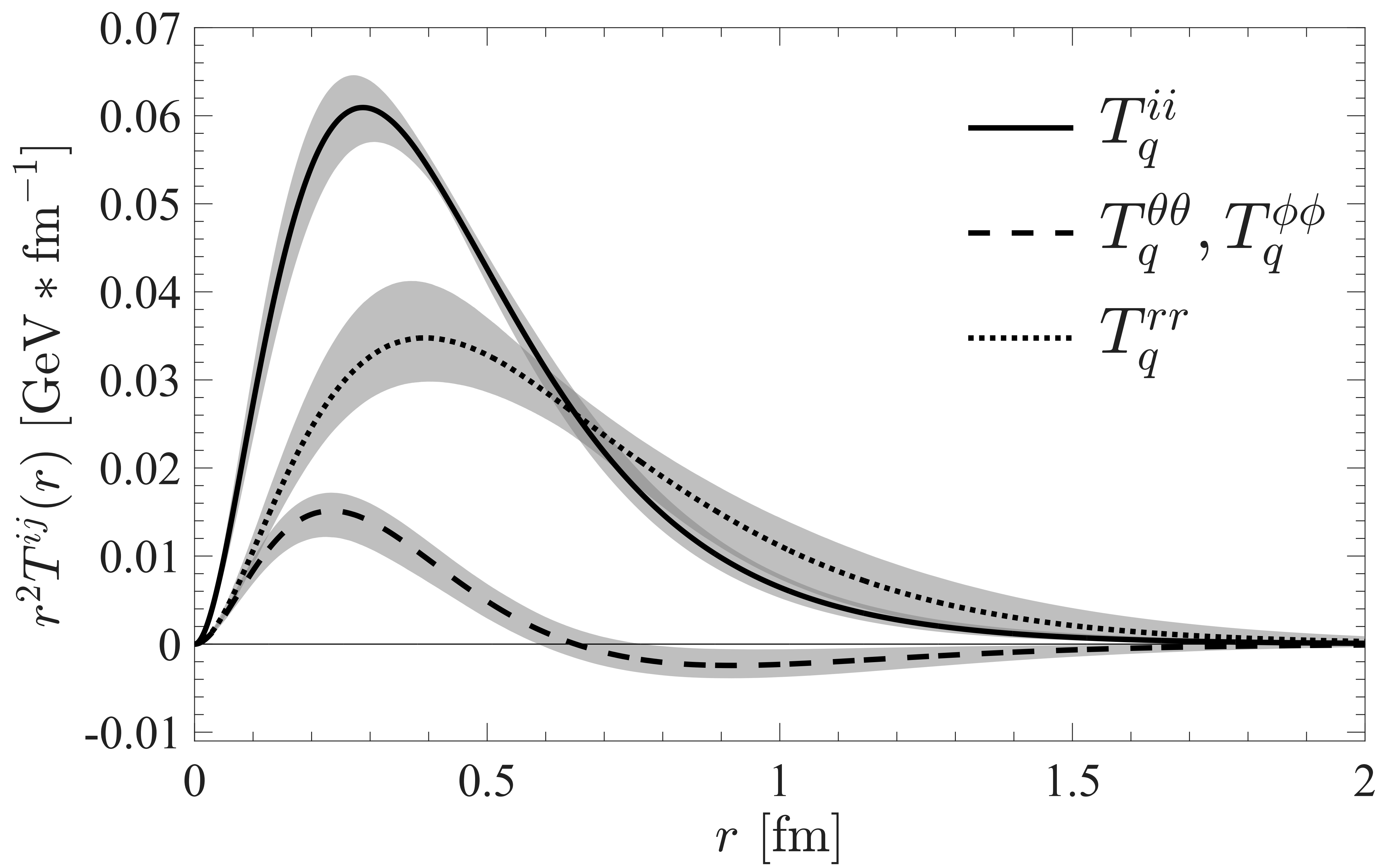


# Kinetic MCD—Quark

The traceless quark MCD is considered as from the quark kinetic motion,

$$T_q^{ij}(\mu) = \frac{1}{2}\bar{\psi} i \overleftrightarrow{\mathscr{D}}^{(i}\gamma^{j)}\psi$$

The trace of the quark MCD, or equivalently the quark kinetic energy density $\bar{T}_q^{ii}(\vec{r}) = \bar{T}_q^{00}(\vec{r}) \equiv \bar{\varepsilon}_q(\vec{r})$, is positive definite.




$$\langle \bar{T}_q^{ij} \rangle(\vec{r}) = \frac{M}{4} G_{s,q}(r)\delta^{ij} + \frac{1}{M}\left(\delta^{ij}\nabla^2 - \nabla^i\nabla^j\right) C_q(r)$$

This kinetic MCD is diagonal in spherical coordinates. The three non-vanishing components indicate both angular and radial motions.

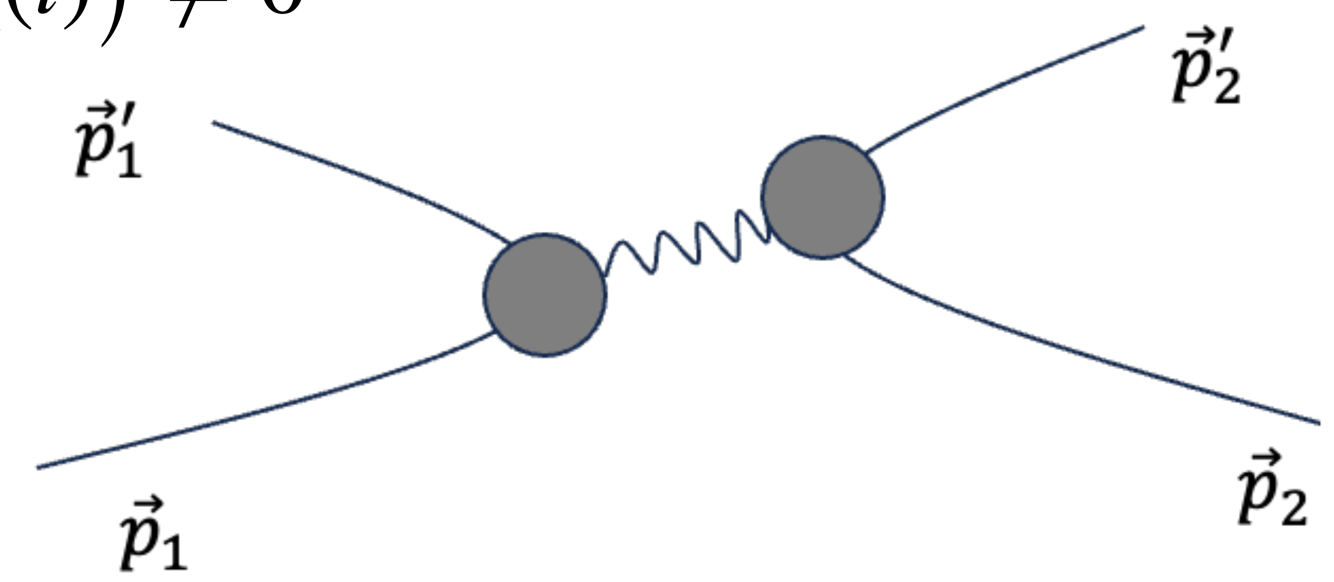
Data from Guo at al, Phys. Rev. Lett. **135**, 111902

# Interaction MCD

In the presence of interactions between particles,

$$\partial_\mu T_K^{\mu j}(\vec{r},t) = -\sum \nabla_a^j V(\vec{r}_a)\left(\vec{r}-\vec{r}_a(t)\right) = \sum F_a^j(t)\delta^{(3)}\left(\vec{r}-\vec{r}_a(t)\right) \neq 0$$

$$\nabla_i \langle \hat{T}_K^{ij}(\vec{r})\rangle = -\nabla^j V(\vec{r})\,|\psi(\vec{r})|^2 = F^j(\vec{r})\,|\psi(\vec{r})|^2 \neq 0$$




The interaction fields among particles also conduct momentum flow.

The interaction MCD is required to satisfy the continuity equation. In the NR limit (no momentum density),

$$\nabla_i T_I^{ij} = \sum \nabla_a^j V(\vec{r}_a)\delta^{(3)}\left(\vec{r}-\vec{r}_a(t)\right), \qquad \frac{\partial T_K^{0j}(\vec{r},t)}{\partial t} + \nabla_i\left(T_K^{ij}(\vec{r},t) + T_I^{ij}(\vec{r},t)\right) = 0$$

However, the momentum flow solved from the above equation is not unique,

$$T_I^{ij}(\vec{r}) = \sum_a \nabla^i\left[-\frac{\nabla_a^j V\left(\vec{r}_a\right)}{4\pi\left|\vec{r}-\vec{r}_a\right|}\right], \qquad T_I'^{ij} = T_I^{ij} + \boxed{\partial_k \chi^{[ki]j}}$$

← **Superpotential.**

**Fixed by boundary conditions, et al.**

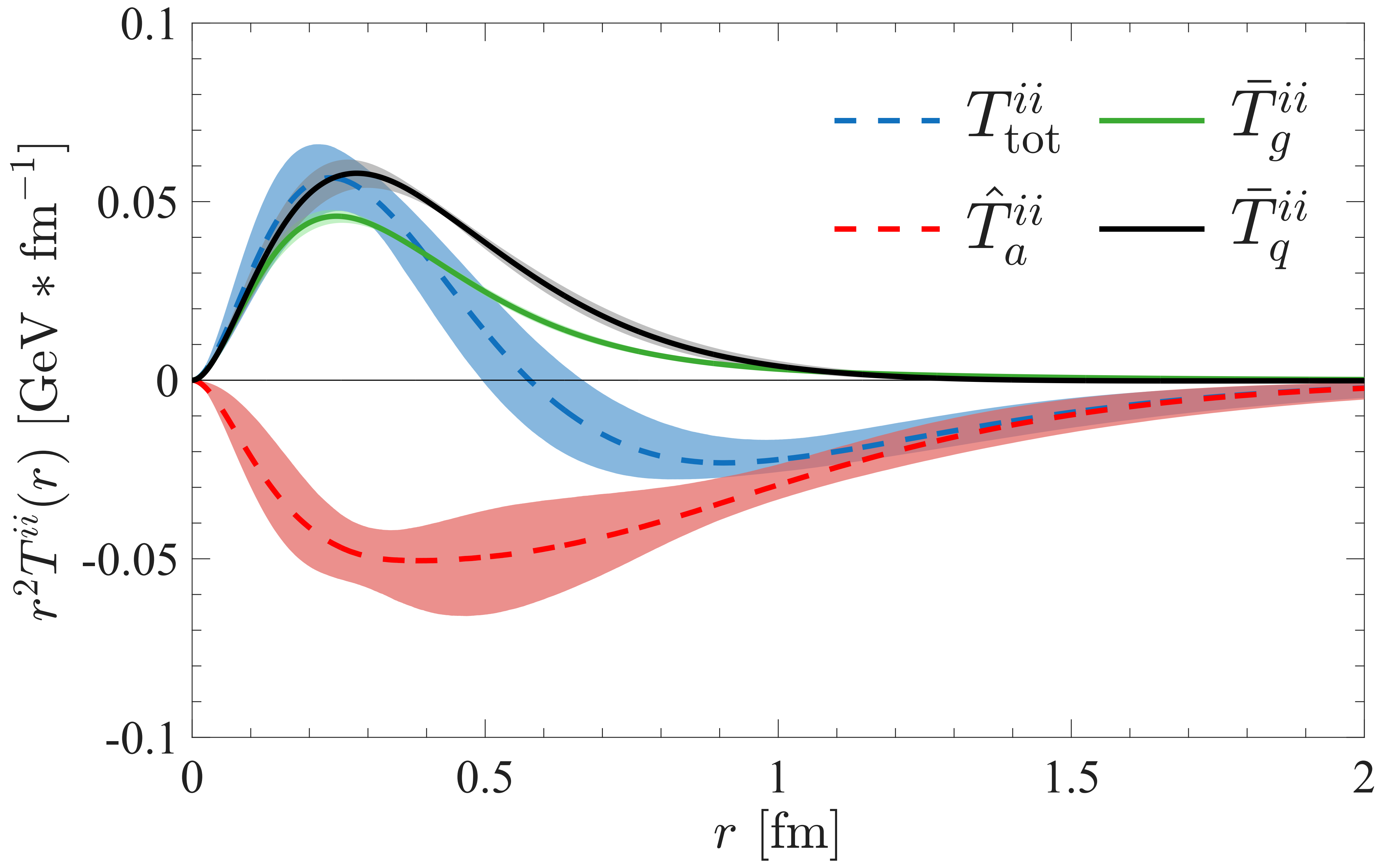
The interaction MCD behaves analogous to a **force potential**.

# Interaction MCD—Gluon Tensor

The QCD gluon tensor MCD is,

$$\bar{T}_g^{ij}(\mu) = -\frac{1}{d}\delta^{ij}F^2 - F^{i\alpha}F^j{}_\alpha$$

The gluon tensor MCD contains both Coulomb gluons (static interaction) and radiative gluons (physical particles), which cannot be separated in a gauge-invariant and frame-independent way. Therefore, these gluons are collected as the color interactions between quarks.




Data from Guo at al, Phys. Rev. Lett. **135**, 111902

$$\langle \bar{T}_g^{ij}\rangle(\vec{r}) = \frac{M}{4}G_{s,g}(r)\delta^{ij} + \frac{1}{M}\left(\delta^{ij}\nabla^2 - \nabla^i\nabla^j\right)C_g(r)$$

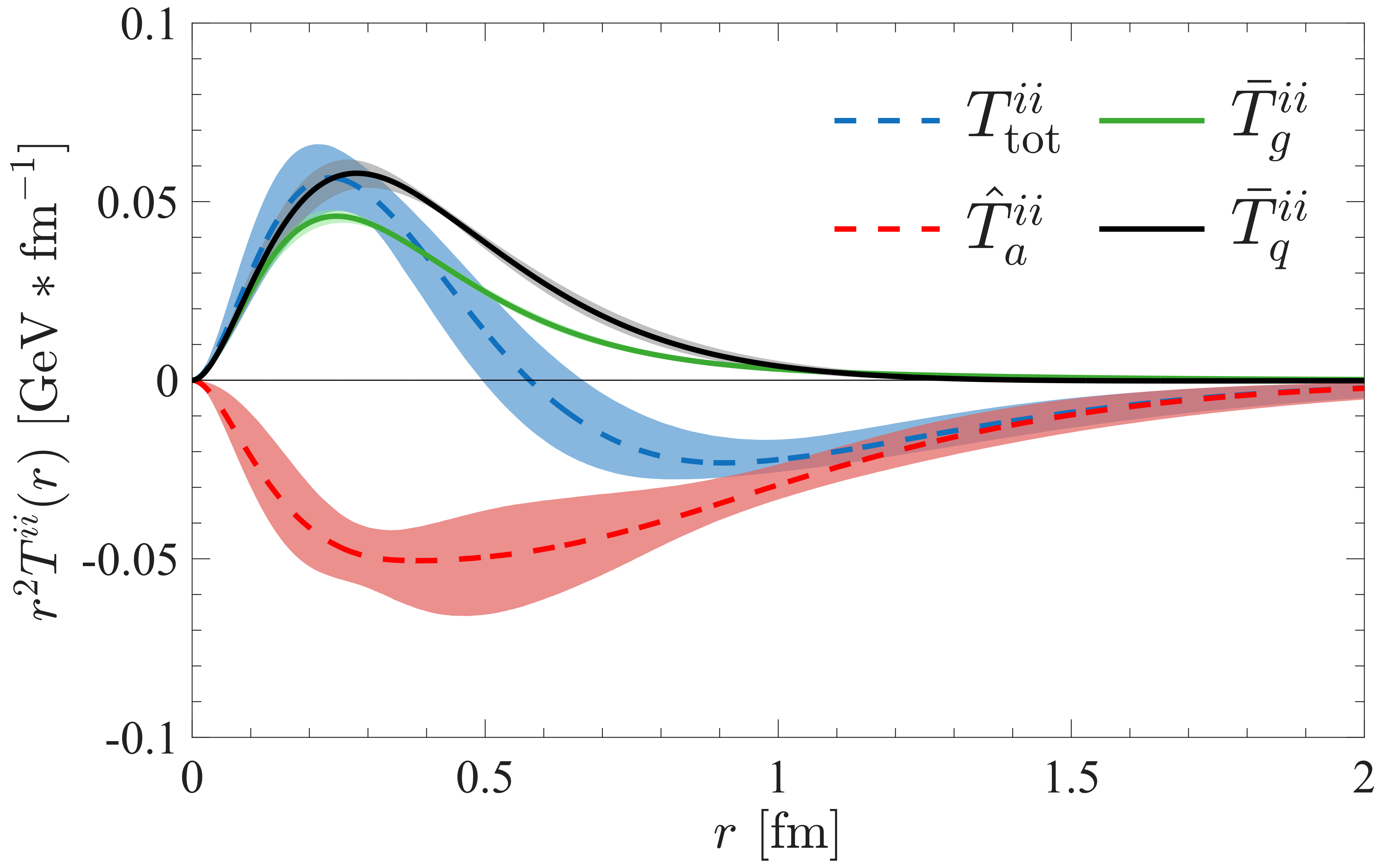
The trace of this gluon tensor MCD, or the gluon tensor energy density, $\bar{T}_g^{ii}(\vec{r}) = \bar{T}_g^{00}(\vec{r}) \equiv \bar{\varepsilon}_g(\vec{r})$, is also positive definite.

# Interaction MCD—Trace Anomaly

The most interesting part is the trace anomaly QCD gluon tensor MCD is,

$$\hat{T}_a^{ij}(\vec{r}) = -\frac{1}{4}\delta^{ij}\frac{\beta(g)}{2g}F^2$$

The anomaly part contributes to the EMT without a momentum density, and therefore represents a pure interaction effect. This MCD reflects that quarks "sweep" out the true QCD vacuum and lower the expectation of gluon condensate, similar to a bag model phenomenology.




Data from Guo at al, Phys. Rev. Lett. **135**, 111902

$$\langle \hat{T}_a^{ij}\rangle(\vec{r}) = -\frac{M}{4}G_s(r)\delta^{ij} \equiv -\hat{\varepsilon}_a(\vec{r})\delta^{ij}$$

This anomaly MCD the negative of the anomalous energy density, and is therefore negative definite, indicating that the momentum flows towards the "center".

# Total MCD

The total conserved MCD can always be written as a superpotential term,

$$T^{ij}_{\text{tot}} = T^{ij}_K + T^{ij}_I, \quad T^{ij}_I = -T^{ij}_K + \partial_k \chi^{[ki]j} \implies T^{ij}_{\text{tot}} = \partial_k \chi^{[ki]j}$$

As long as the conservation law is concerned, the total MCD is not unique.

In QCD, the total MCD is,

$$\sum_\alpha \langle T^{ij}_\alpha \rangle(\vec{r}) = \frac{1}{M}(\delta^{ij}\nabla^2 - \nabla^i \nabla^j) C(r)$$

And the $C$ form factor is simply a superpotential term, fixed by the gravitational considerations.

In terms of the decomposition, the von Laue condition satisfied by the total MCD reduces to,

$$\int \langle T^{ii}(\vec{r}) \rangle \, \mathrm{d}^3\vec{r} = \int \left[ \bar{\varepsilon}_q(\vec{r}) + \bar{\varepsilon}_g(\vec{r}) - 3\hat{\varepsilon}_a(\vec{r}) \right] \mathrm{d}^3\vec{r} = 0$$

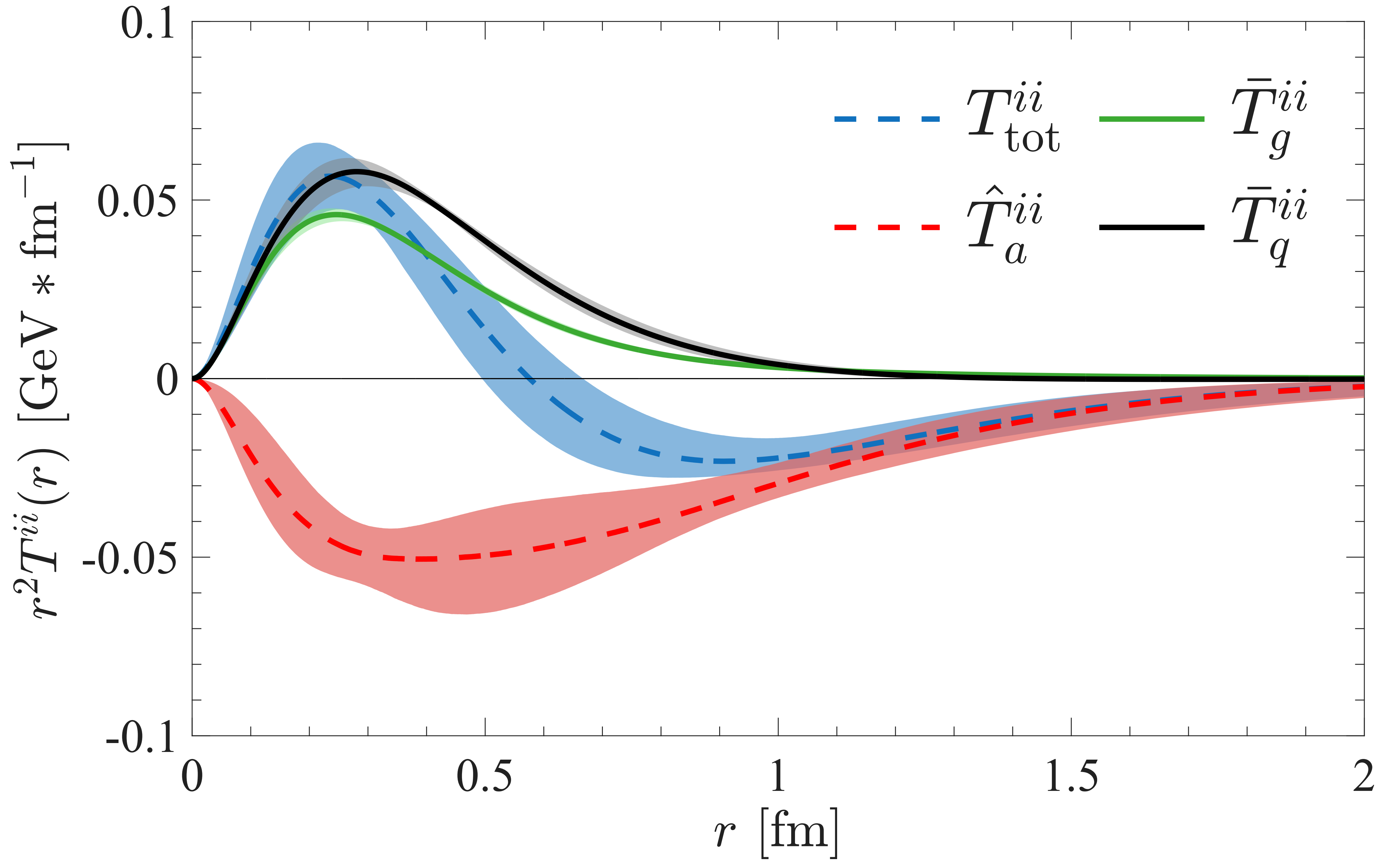
This is the Virial theorem between the scalar and tensor energy contributions,

$$E_T = 3E_S$$

# Total MCD

The total MCD, or the so-called "pressure", is,

$$\langle T^{ii}\rangle(\vec{r}) = \bar{\varepsilon}_q(\vec{r}) + \bar{\varepsilon}_g(\vec{r}) - 3\hat{\varepsilon}_a(\vec{r})$$




Data from Guo at al, Phys. Rev. Lett. **135**, 111902

The quark and gluon contributions dominate the small-$r$ behavior of the total MCD, while the trace anomaly accounts for the large-$r$ region being negative.

This has depicted a picture of the nucleon structure as a hard core of quarks and gluons near the center and an anomaly cloud surrounding them, also expected from the bag model.

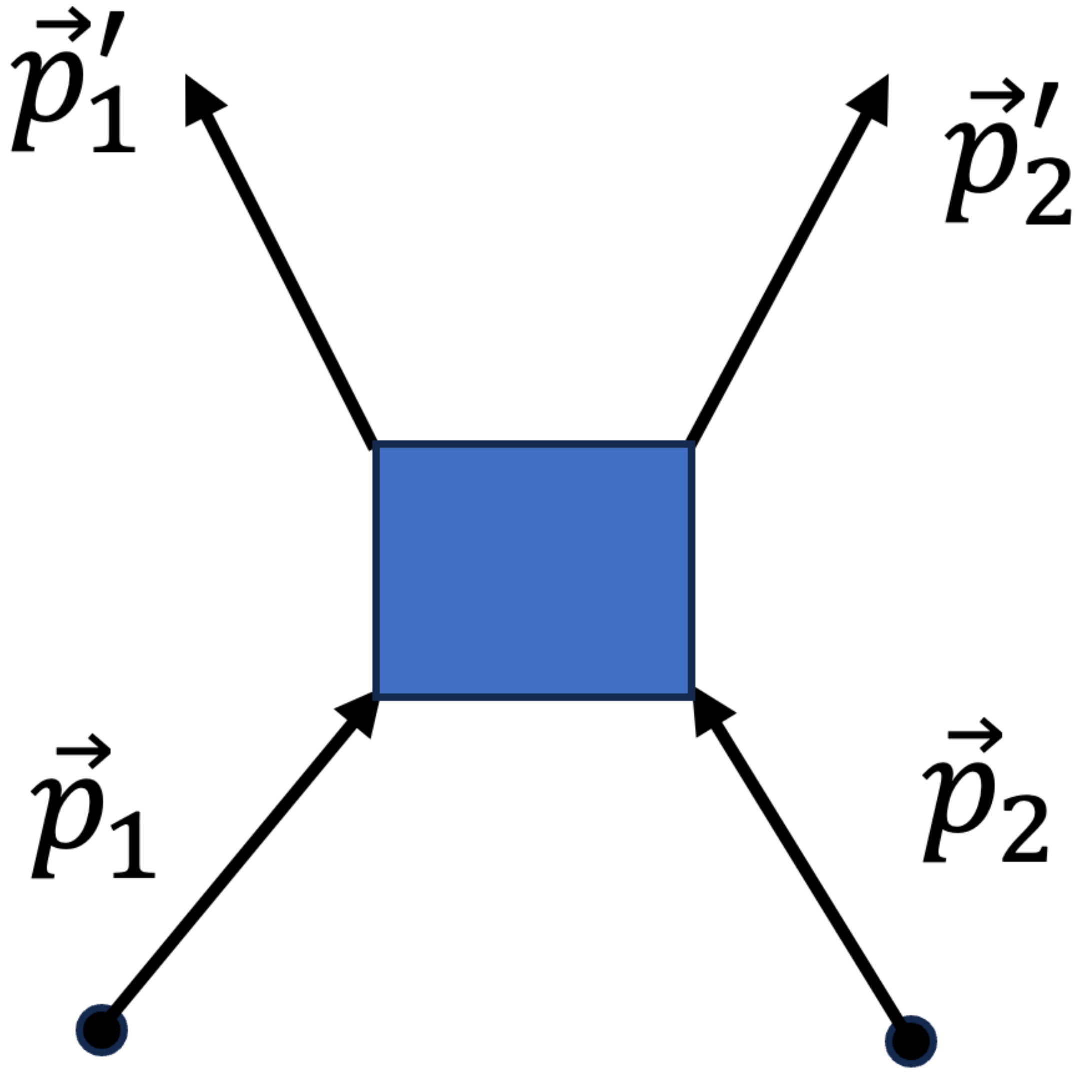
# PART THREE

# Force Densities in the Proton

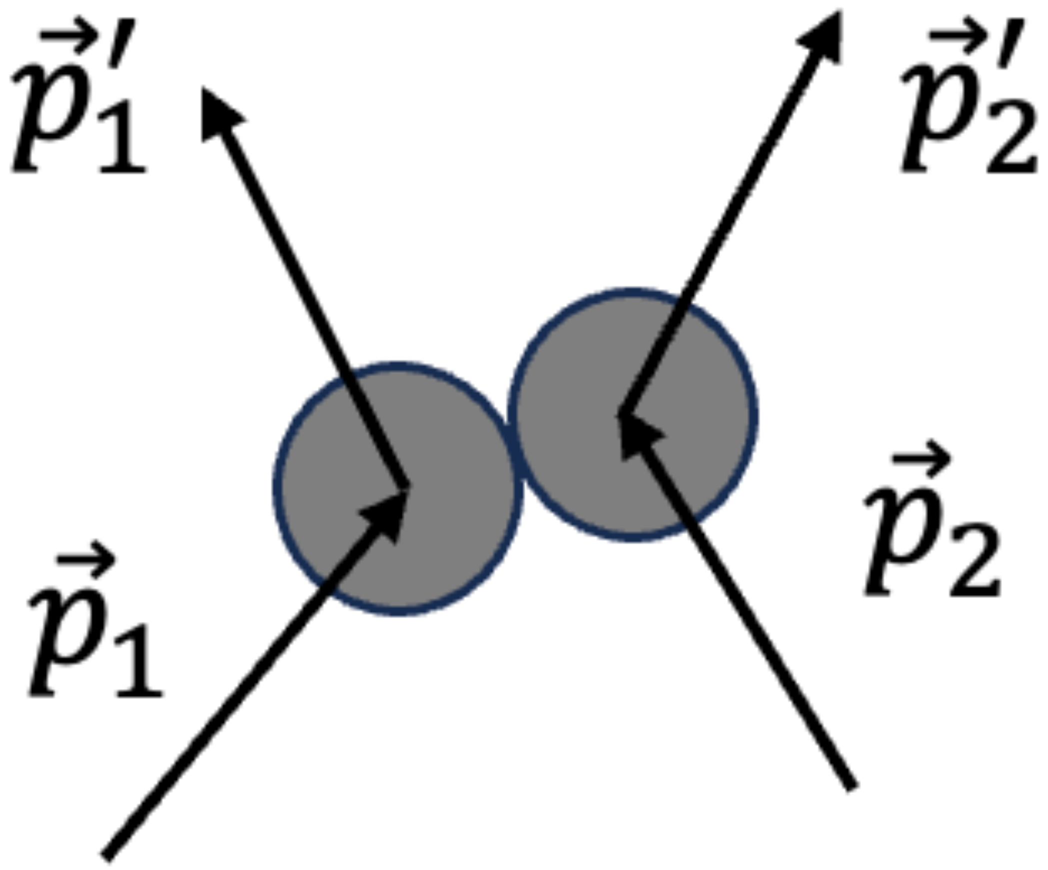
# Momentum Continuity

What is force?

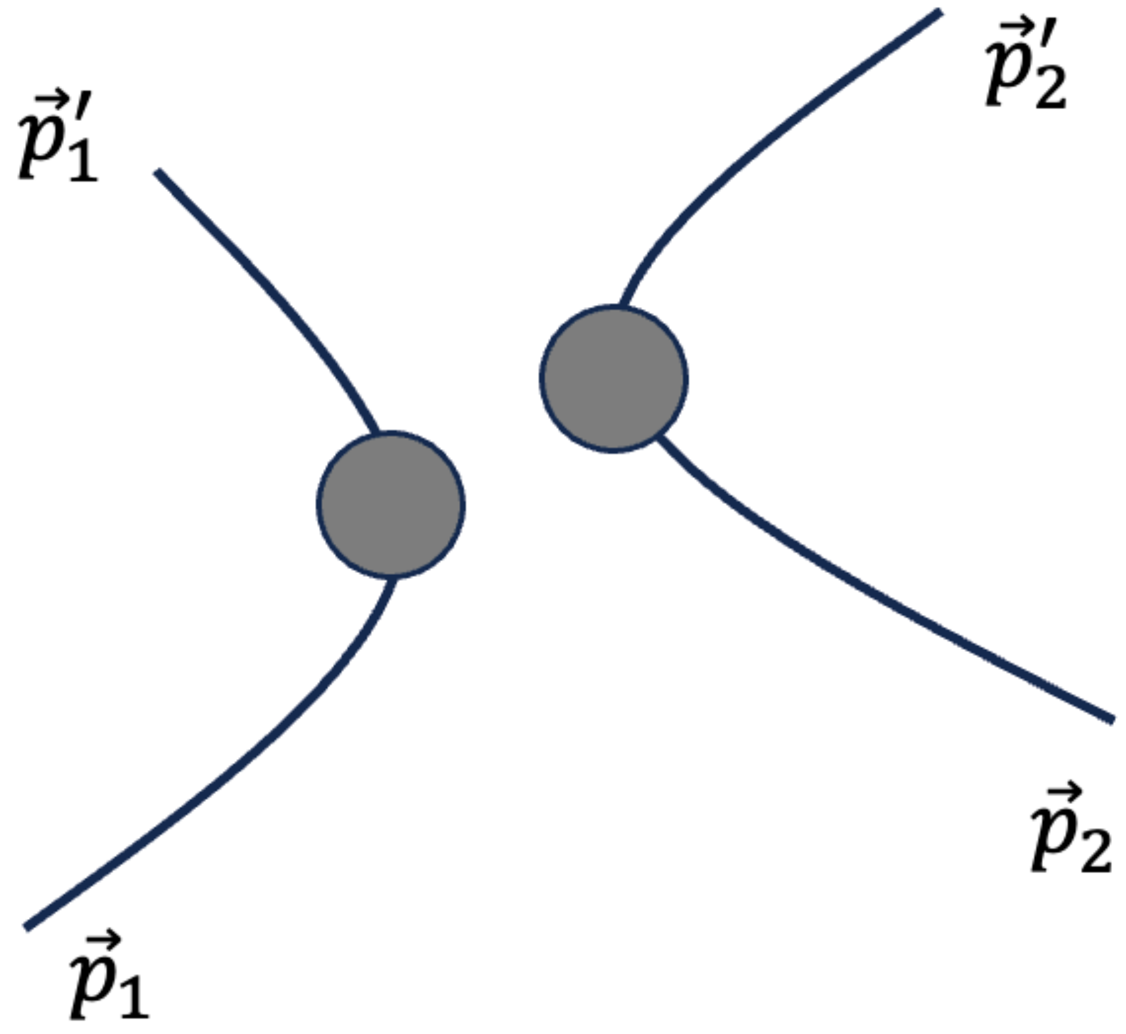
**Force comes form the momentum continuity!**




$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2'$$




Contact interaction




Long-range interaction

**Force is the momentum change!**

# Force Densities

Remember the continuity equation in the presence of interactions,

$$\partial_\mu T_K^{\mu j}(\vec{r},t) = \sum F_a^j(t)\delta^{(3)}\left(\vec{r}-\vec{r}_a(t)\right) \equiv \mathscr{F}^j(\vec{r})$$

$$\nabla_i \langle \hat{T}_K^{ij}(\vec{r})\rangle = F^j(\vec{r})\,|\psi_n(\vec{r})|^2 \equiv \mathscr{F}^j(\vec{r})$$

**Force Density!**

The divergence of kinetic EMT gives exactly the force density acting on the system! Define the force density on quarks as,

$$\mathscr{F}_q^j(\vec{r}) \equiv \partial_i \langle \bar{T}_q^{ij}\rangle(\vec{r}) = \frac{M}{4}\nabla^j G_{s,q}(r) \equiv \mathscr{F}_g^j(\vec{r}) + \mathscr{F}_a^j(\vec{r})$$

which can be traced to contributions from the gluon tensor and anomaly MCDs,

$$\mathscr{F}_g^j = -\,\partial_i \langle \bar{T}_g^{ij}\rangle(\vec{r}) = \frac{M}{4}\nabla^j G_{s,g}(r)$$

$$\mathscr{F}_a^j = -\,\partial_i \langle \hat{T}_a^{ij}\rangle(\vec{r}) = \frac{M}{4}\nabla^j G_s(r)$$

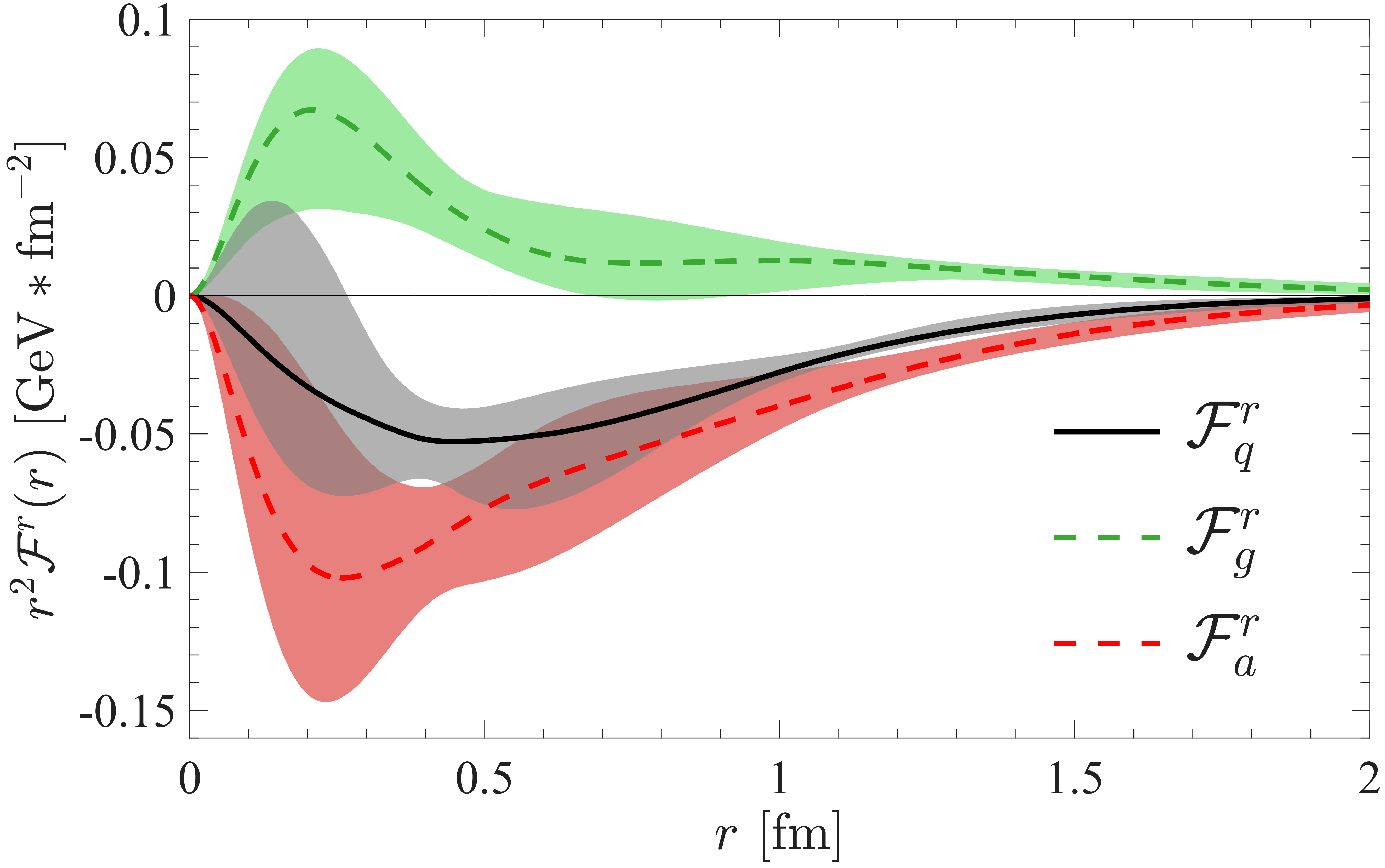
The scalar form factor $G_s$ contributes to the force, behaving analogous to a pressure potential.

The force densities are exclusively along the radial direction due to the spherical symmetry of the static nucleon.

# QCD MCD and Force Density

The average force from anomaly is of the same magnitude as the string tension, which is roughly $1\mathrm{GeV/fm}$.

$$\int \mathrm{d}^3\vec{r}\ \mathscr{F}_a(\vec{r}) = -\ 1.06^{+0.12}_{-0.11}\ \mathrm{GeV/fm}$$

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$$\mathscr{F}^j_g = -\ \partial_i \langle \bar{T}^{ij}_g \rangle(\vec{r}) = \frac{M}{4}\nabla^j G_{s,g}(r)$$

$$\mathscr{F}^j_a = -\ \partial_i \langle \hat{T}^{ij}_a \rangle(\vec{r}) = \frac{M}{4}\nabla^j G_s(r)$$

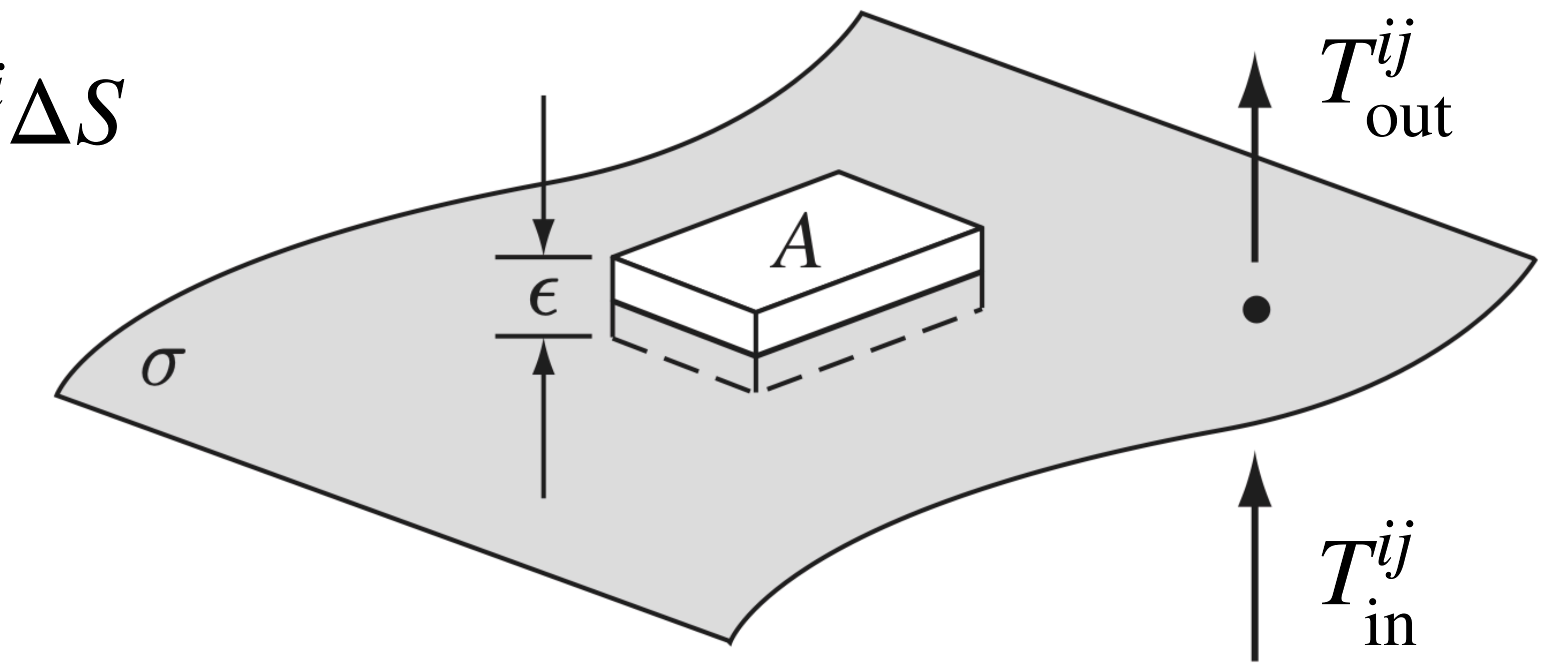
- $\mathscr{F}_g$: Forces from the gluon tensor, including the static gluons and radiative gluons. This positive force indicates the dominance of gluon radiations.
- $\mathscr{F}_a$: Forces from the trace anomaly. This attractive force is the origin of confining force acting on the quarks.

# Surface Forces: Discontinuities

The surface forces are finite forces acting on 2D surfaces, which can also be extracted from the 3D force density.

Such surface forces can be well-defined when there are **discontinuities** in the MCD,

$$\Delta F^j = \left[ -\int_{\Delta\mathscr{V}} \partial_i T_b^{ij} \mathrm{d}\mathscr{V} \right] = \left[ \left( T_b^{ij} \right)_{\mathrm{in}} - \left( T_b^{ij} \right)_{\mathrm{out}} \right] n^i \Delta S$$




where $b$ represent a part of the total MCD. The 2D pressure,

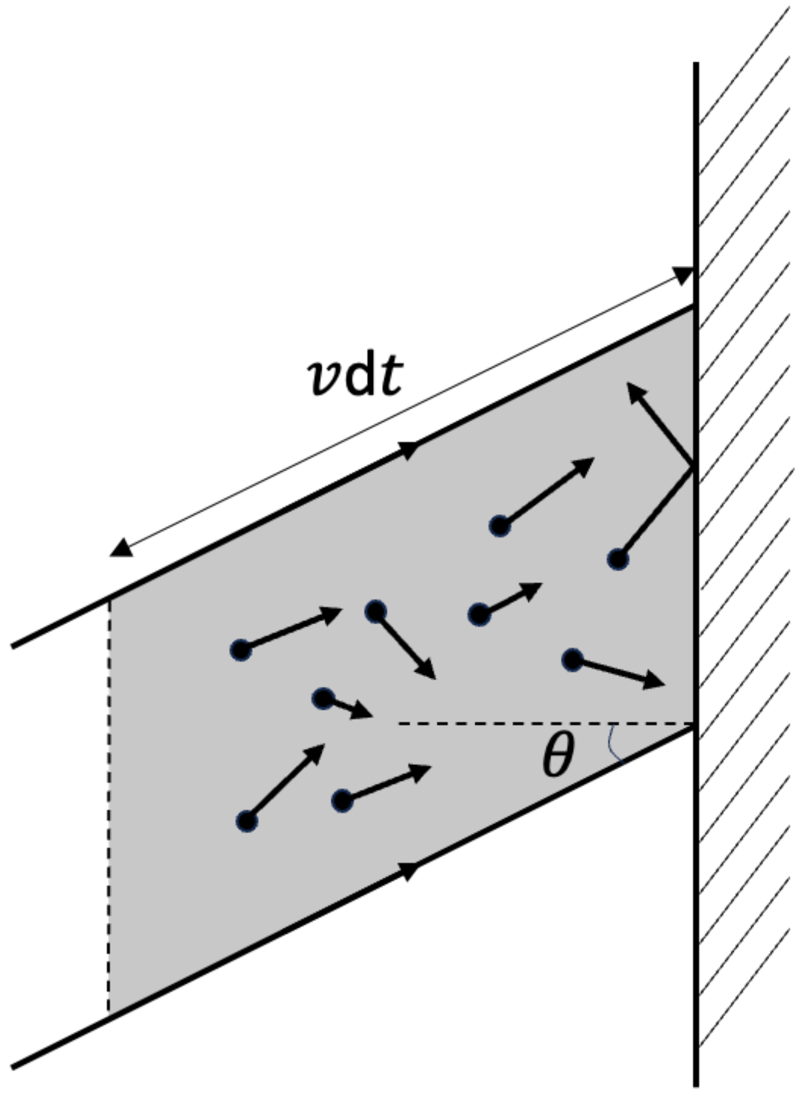
$$p \equiv \frac{\Delta F^j n^j}{\Delta S} = \left[ \left( T_b^{ij} \right)_{\mathrm{in}} - \left( T_b^{ij} \right)_{\mathrm{out}} \right] n^i n^j$$

The **surface forces arises from the discontinuities** in the (kinetic or interaction) MCD.

For example, in the ideal gas,

$$\langle T_K^{ij} \rangle (\vec{r}) = p_K \delta^{ij} \theta_V$$

No pressure within the gas!



# Surface Forces: Contact Interactions

In liquids and solids, there are contact interactions. One solution to the interaction MCD is the stress tensor,

$$T^{ij}_{I\text{–liquid}} = (p_I \delta^{ij} - \sigma^{ij})\theta_V, \quad T^{ij}_{I\text{–solids}} = -\sigma^{ij}\theta_V$$

where $p_I$ is the interaction pressure and $\sigma^{ij}$ is the stress tensor,

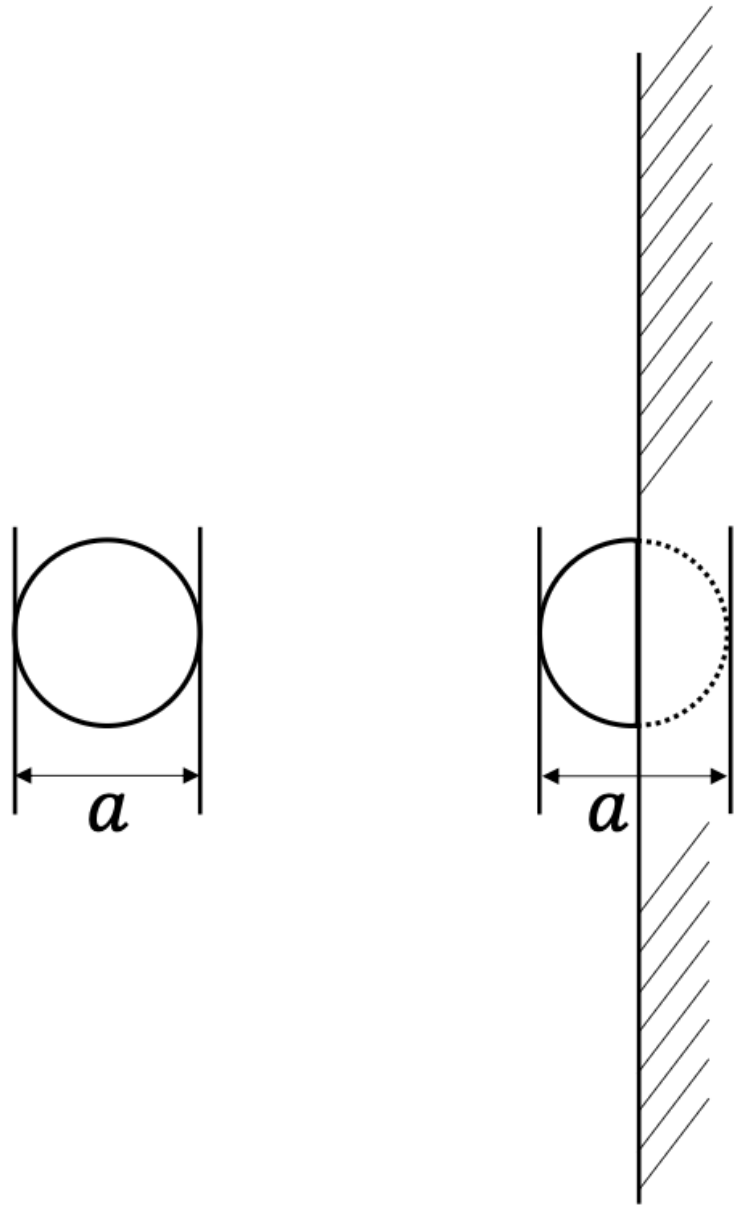
**Hooke's law**

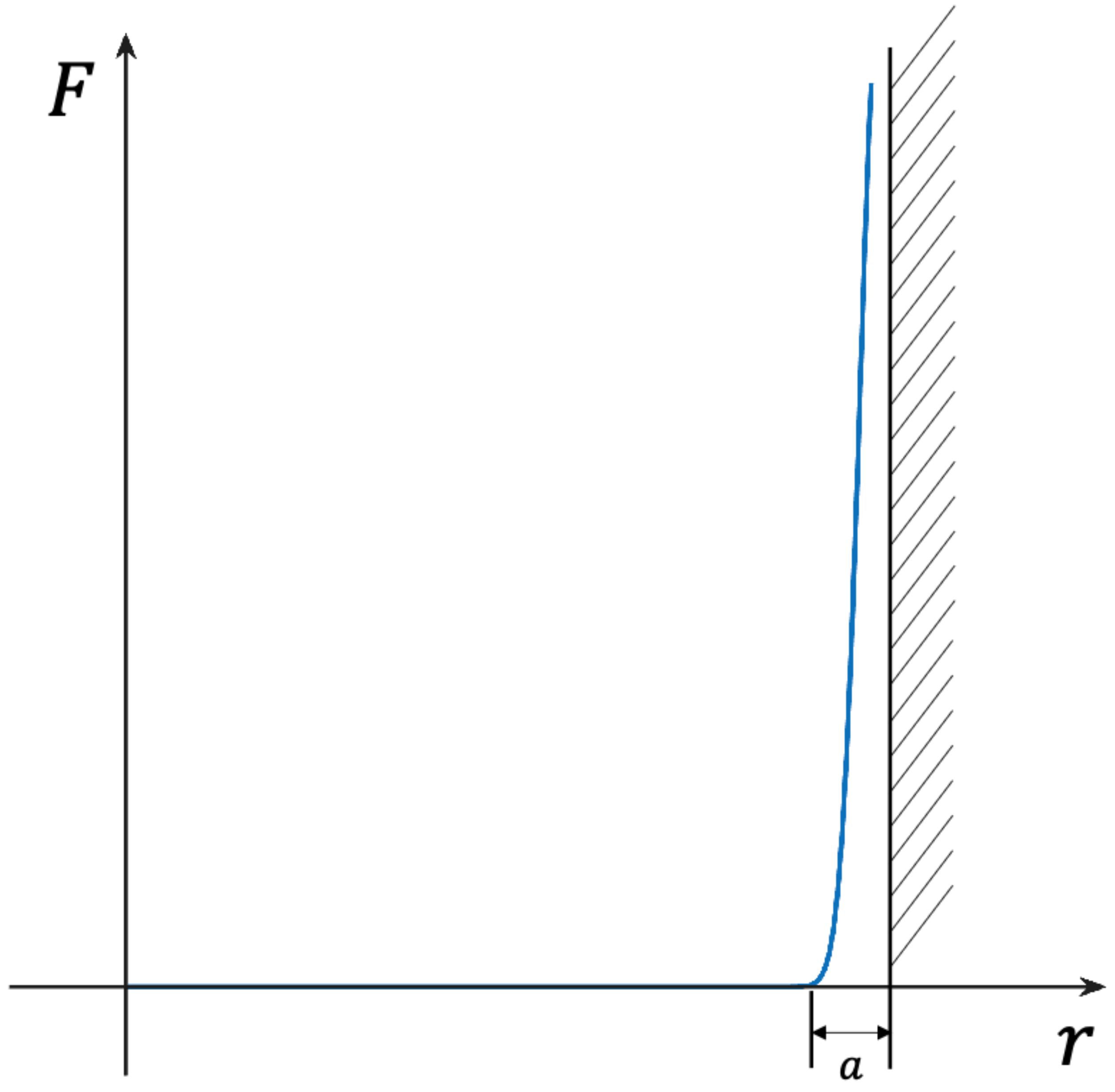
$$\sigma_{ij,\text{liquid}} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v_k\right) + \zeta \partial_k v_k \delta_{ij}, \qquad \sigma_{ij,\text{solid}}(\vec{r}) = K\delta_{ij}u_{ll}(\vec{r}) + 2\mu\left[u_{ij}(\vec{r}) - \frac{1}{3}\delta_{ij}u_{ll}(\vec{r})\right]$$

In the van de Waals gas,

$$T^{ij}_{\text{int}} = -\frac{N^2 kT}{2V^2}\int \mathrm{d}^3\vec{x}\left[e^{-V(x)/kT} - 1\right]\delta^{ij}\theta_V \equiv p_V \delta^{ij}\theta_V$$

This is the contact force within the gas.



# Contact Force vs. Discontinuities

Why do these solutions of contact interaction MCD can become the surface forces?

In fact, these surface forces arises from the **hidden discontinuities.**

Imagine a fictitious plane separating two sides of the system.
Although the total interaction MCD of these two sides is continuous,

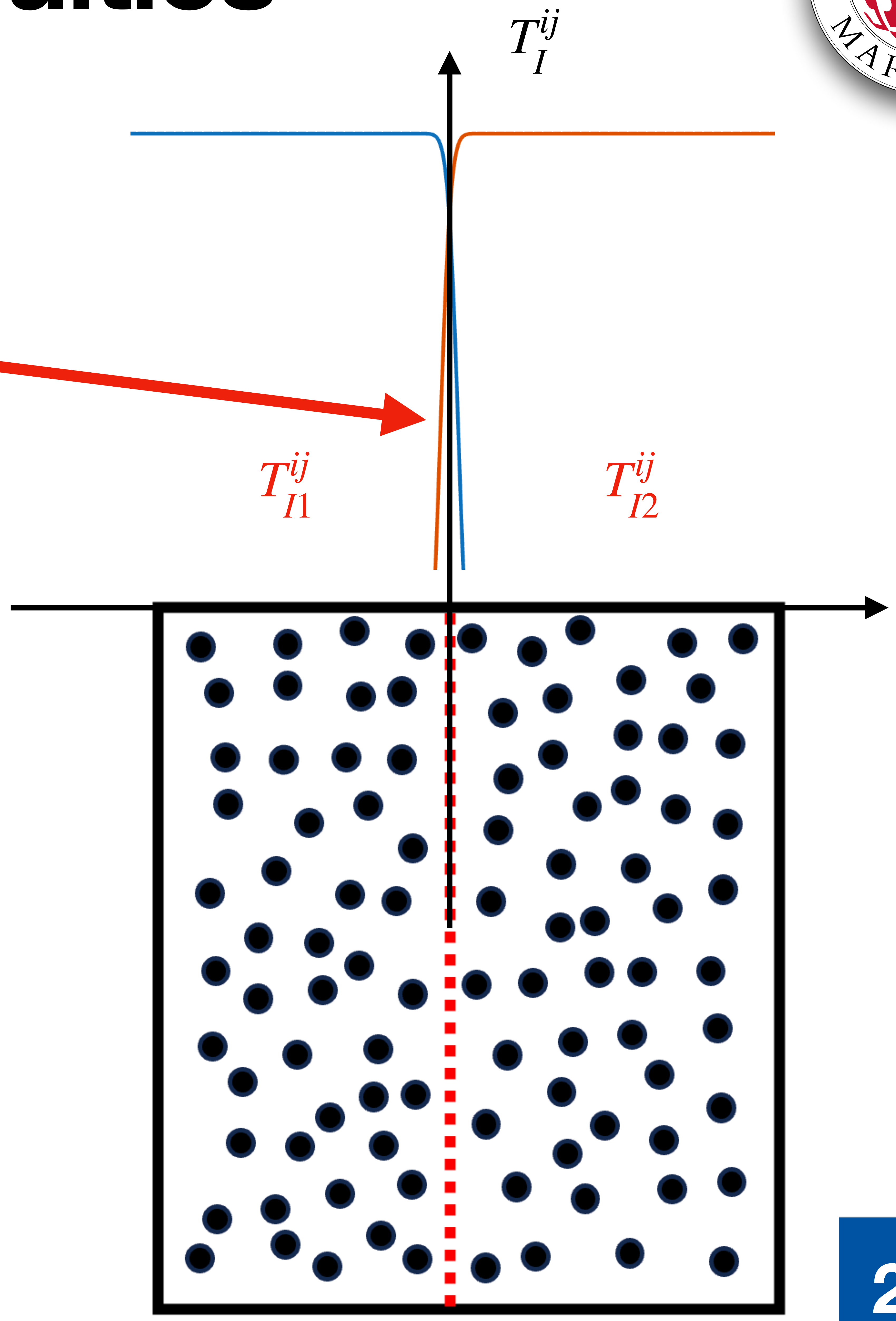
$$T_I^{ij} = T_{I1}^{ij} + T_{I2}^{ij}$$

The individual interaction MCDs have discontinuities on this plane,

$$T_{I1}^{ij}, T_{I2}^{ij}$$

These discontinuities become the surface forces as previously mentioned.

What will happen when one has **long-range interactions**?



# Long-range Force vs. Discontinuities

No matter how one separates one part of the system, isolates a small volume, the interaction MCD of long-range forces is always continuous,

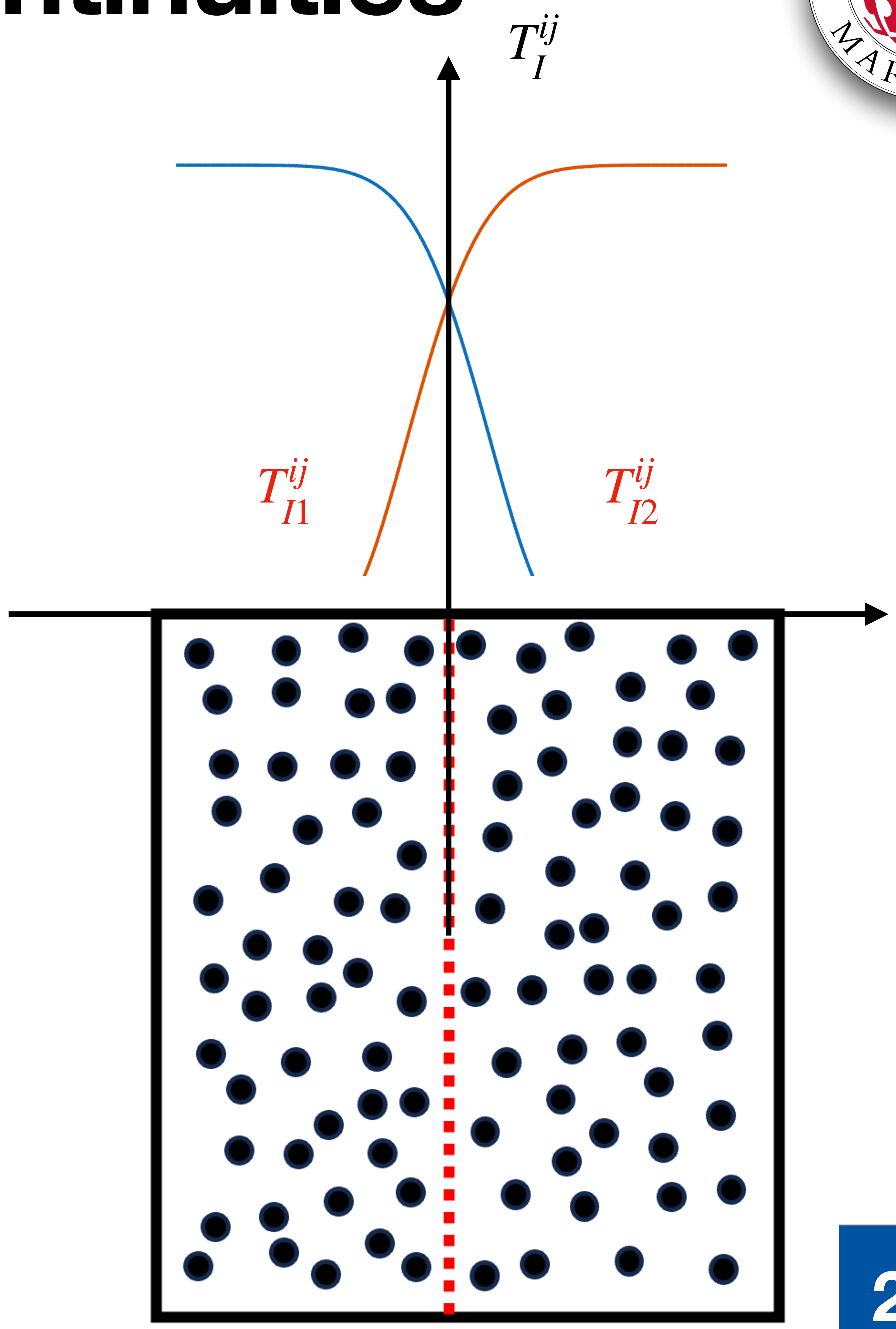
$$T^{ij}_{I1}, T^{ij}_{I2}$$

There are no **hidden discontinuities** in the interaction MCD.

Therefore, the long-range force interaction MCD can never become the surface forces within the system.

**What about nucleon?**

**The color forces among quarks are clearly long-range compared to the nucleon size! Therefore they cannot become surface forces!**



# PART FOUR

## Summary

# Summary

We have studied the underlying physical mechanisms of each component of the QCD momentum flow,

1. Quarks carry the momentum current through the ``collective'' motion both in the radial and angular directions.
2. Gluon tensor flows momentum both as momentum carriers and as a mediator of forces among quarks
3. Gluon scalar (trace anomaly) contributes through the pure interacting effect, resulting in a negative ``pressure potential'' due to a Casimir-like effect, the change of the QCD vacuum.

The total MCD shows the momentum conservation through the balance of the in and out flows of the momentum density.

The divergences of the QCD MCD give the force densities,

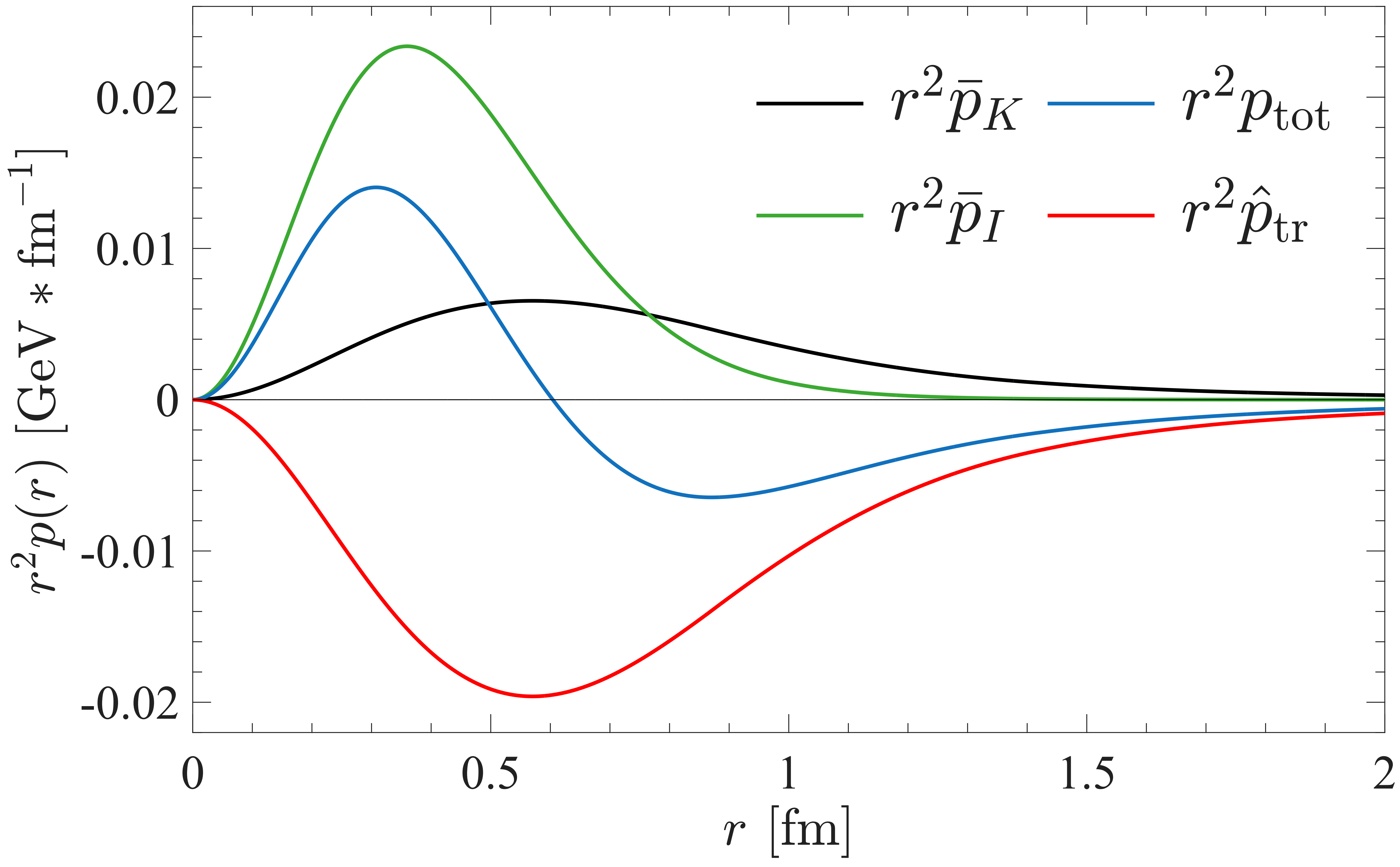
1. The total force acting on quarks is found to be strongly confining.
2. The anomaly provides a large attractive force with an average magnitude of 1 GeV/fm, similar to the QCD string tension.

# THANKS

# Skyrme Model

Lagrangian,

$$\mathscr{L} = \frac{1}{16} f_\pi^2 \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2}\mathrm{Tr}\left(\left[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U\right]\left[U^\dagger\partial^\mu U, U^\dagger\partial^\nu U\right]\right)$$



# Interaction MCD

This ambiguity can also be seen from Noether's theorem. A direct calculation gives the canonical EMT,

$$\partial_\mu T^{\mu\nu} \equiv \partial_\mu \left[ \sum_a \frac{\partial \mathscr{L}}{\partial_\mu \Phi_a} \partial^\nu \Phi_a - g^{\mu\nu} \mathscr{L} \right] = 0$$

For gauge theories like QED and QCD, this canonical EMT is neither symmetric nor gauge invariant. This is resolved by introducing a superpotential term, called the Belinfante-Rosenfeld improvement,

$$T'^{\mu\nu} = T^{\mu\nu} + \partial_\rho \chi^{[\rho\mu]\nu}[\Phi]$$

This observation also arises from the fact that Noether's theorem does not uniquely determine the conserved current. There is one particular version of EMT that is believed to couple to gravity,

$$T^{\mu\nu}[\psi, g] \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_M[\psi, g]}{\delta g_{\mu\nu}}$$

For QED/QCD, the above is the same as the Belinfante-Rosenfeld improved EMT. However, from the viewpoint of momentum continuity, there is no compelling reason to prefer one over the other.
This tells us that the interaction MCD itself cannot carry any mechanical meanings.