PDFs and four-point functions on the lattice

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XIIth Meeting on Lattice Parton Physics from Large Momentum Effective Theory (LaMET 2025)

Stony Brook University

October 10, 2025

UNTERSTÜTZT VON / SUPPORTED BY

Collaborators for past, recent and ongoing four-point function calculation efforts:

RQCD:
Andreas Schäfer, Daniel Reitinger, Gunnar Bali, Markus Diehl (DESY)

χ**QCD**:
Keh-Fei Liu, Jian Liang, Bigeng Wang, Terry Draper, Raza Sabbir Sufian

Introduction

Investigation of the structure of hadrons remains an important research field, in particular for in the context of EIC physics. Only *ab-initio* approach given by **lattice QCD:**

- In the past, Moments of parton distribution functions (PDFs), generalized parton distributions (GPDs) and form factors have been studied.
- Nowadays, research is extended to more advanced structure functions, e.g. transverse momentum dependent parton distributions (TMDs) *Musch et al '12* or double parton distributions (DPDs) *RQCD '18-'24*.
- Moreover, the x-dependence of PDFs, GPDs,... has become accessible on the lattice through novel frameworks (LaMET, ITD, LCS, see e.g. *Eur. Phys. J. A **57**, no.2, 77 (2021)* for an overview).

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Four-point functions provide access to a brought variety of quantities related to the hadron structure, in particular PDFs

- Hadronic tensor: parameterized by structure functions F_i, which can be factorized in terms of PDFs: $F = \sum_a c_a \otimes f_a$
- **This talk:** Two-current correlations *Braun et al, Phys. Rev. D* ***51**, 6036-6051 (1995)* as "lattice cross sections" *Ma, Qiu, Phys. Rev. Lett.* ***120**, no.2, 022003 (2018)*
  - First calculations for the pion with tree-level and NLO matching coefficients *Sufian et al, Phys. Rev. D* ***99**, no.7, 074507 (2019), Phys. Rev. D* ***102**, no.5, 054508 (2020)*; $m_\pi = 416$ MeV, $a = 0.127$ fm, $n_f = 2+1$ clover fermions.
  - **This talk:** Calculation for the nucleon (tree-level) *CZ, Schäfer, Phys. Rev. D* ***110**, no.7, 074503 (2024)*, $m_\pi = 355$ MeV, $a = 0.085$ fm, $n_f = 2+1$ clover fermions

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Content

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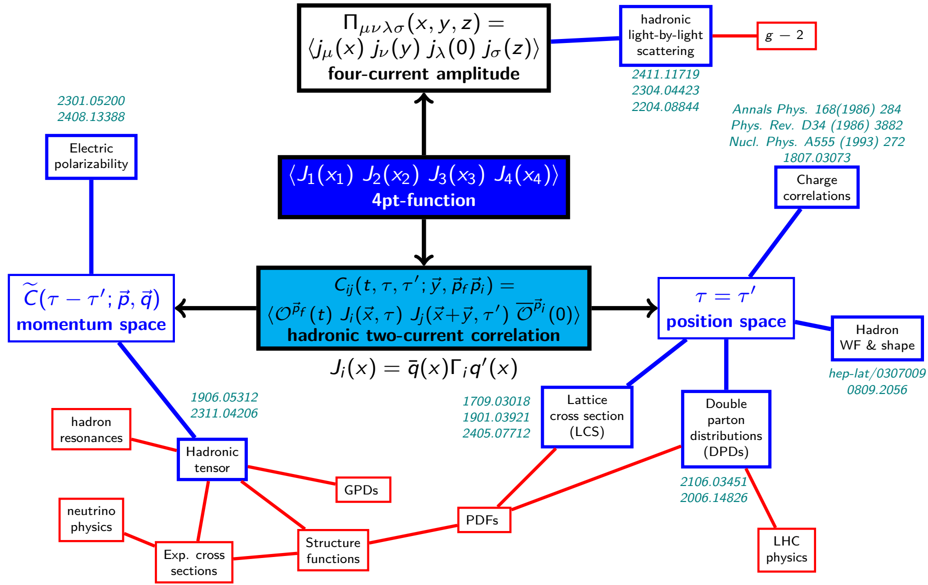
Four-point functions and two-current correlations



Good "lattice cross sections"

Terminology introduced by Ma, Qiu *Phys. Rev. D **98**, no.7, 074021 (2018)*: Any matrix element

$$\sigma(\omega = p \cdot y, y^2, \tilde{\mu}^2) = \langle p | \mathrm{T} \{ \mathcal{O}(y) \} | p \rangle$$

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- Calculable in discrete Euclidean space-time
- Well defined continuum limit (renormalizability of $\mathcal{O}(y)$)
- Collinearly factorizable in terms of PDFs and some perturbative coefficient function, e.g. in position space:

$$\sigma(\omega, y^2, \tilde{\mu}^2) = \sum_a \int_0^1 \frac{\mathrm{d}x}{x} f_a(x, \mu^2)\ K_a(x\omega, y^2, \tilde{\mu}^2, \mu^2) + \mathcal{O}(y^2)$$

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Special cases:

- Quasi-PDFs (momentum space) *Ji, Phys. Rev. Lett. 110, 262002 (2013)*
- Pseudo-PDFs *Radyushkin, Phys. Lett. B 767, 314 (2017)*

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This talk:

- Two-current correlator approach (aka *the* "lattice cross section" approach)

The two-current correlator approach

Following Ma, Qiu *Phys.Rev.Lett. 120 (2018) 2, 022003*:
Consider scalar matrix element of two quark currents, with auxiliary quark flavor Q:

$$M(\omega, y^2) := P_{\mu\nu}(p,y) \langle h(p)| \mathrm{T}\left\{(\bar{q}\Gamma_\mu Q)(0)(\bar{Q}\Gamma_\nu q)(y)\right\} |h(p)\rangle$$

$$\widetilde{M}(\omega, y^2) := P_{\mu\nu}(p,y) \langle h(p)| \mathrm{T}\left\{(\bar{Q}\Gamma_\mu q)(0)(\bar{q}\Gamma_\nu Q)(y)\right\} |h(p)\rangle$$

with projector $P_{\mu\nu}(p,y)$ and $\omega = py$

$$M(\omega, y^2) = \sum_{J,a} W_{n,P}(y^2, \mu^2) y^{\mu_1} \ldots y^{\mu_J} p_{\mu_1} \ldots p_{\mu_J} \langle p| \mathcal{O}^{(J,a)}(\mu^2) |p\rangle$$

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Considering twist-2:

$$\boxed{M(\omega, y^2) = \sum_a \int_{-1}^{1} \frac{\mathrm{d}x}{x} f_a(x, \mu^2) \; K^a_{ij,P}(x\omega, y^2, x^2p^2, \mu^2) + \mathcal{O}(y^2\Lambda^2_{\mathrm{QCD}})}$$

where $K^a_{ij,P}(x\omega, y^2, x^2p^2, \mu^2)$ are suitable matching coefficients

The two-current correlator approach

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- Higher twist contributions are neglected, enter at $\mathcal{O}(y^2\Lambda^2_{\mathrm{QCD}})$.
- Usage of heavy auxiliary quarks can help to reduce noise and suppress higher twist contributions
- Formula valid for small y^2.
- Can calculate l.h.s. on the lattice for $y^0 = 0$.
- Advantage w.r.t. quasi/pseudo-PDFs: Renormalization on local operators is trivial (e.g. RI/MOM scheme).
- Formalism basically Lorentz covariant; however, need large momenta to cover a wide Ioffe time range ($\omega = p \cdot y$).
- Need to deal with inverse problem.

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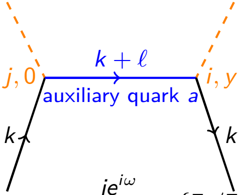
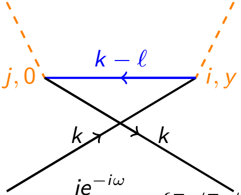
Matching Coefficients

Can be read of from PDF in a quark $q \Rightarrow M^q_{ij}(\omega, y^2, \mu^2) = K^q_{ij}(\omega, y^2, x^2, \mu^2)$

At leading order in α_s we have two diagrams for M and $\widetilde{M}$:



$$= -\frac{ie^{i\omega}}{4\pi^2 y^4} \operatorname{tr}\{\Gamma_i \not{y} \Gamma_j \not{p}\}$$



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for VV and AA: $\operatorname{tr}\{\Gamma_i \not{y} \Gamma_j \not{p}\} = 4(p^\mu y^\nu + p^\nu y^\mu - g^{\mu\nu}\omega)$

Construct projector P_D so that

$$P_{D,\mu\nu} M^{\{\mu\nu\}} = \frac{m^2}{y^4} D(\omega, y^2)$$

In order to get the valence PDF, we calculate combination $M - \widetilde{M}$:

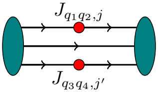
$$D_q(\omega, y^2) - \widetilde{D}_q(\omega, y^2) = -\frac{2i}{\pi^2 m^2} \int_0^1 \mathrm{d}x \; f^h_{q,v}(x) \cos(x\omega)$$

Content

Wick contractions

Worked out e.g. in *J. High Energy Phys. 09 (2021) 106*:

$$\left\langle \mathcal{O}^{\vec{p}}(t)\ J_{q_1 q_2, j}(\vec{x},\tau)\ J_{q_3 q_4, j'}(\vec{y},\tau)\ \overline{\mathcal{O}}^{\vec{p}}(0) \right\rangle \Big/ \left\langle \mathcal{O}^{\vec{p}}(t)\ \overline{\mathcal{O}}^{\vec{p}}(0) \right\rangle \xrightarrow[t \gg \tau \gg 0]{} {}_p\!\left\langle J_{q_1 q_2, j}(\vec{x},\tau)\ J_{q_3 q_4, j'}(\vec{y},\tau) \right\rangle_p$$

$C^{jj'}_{1,q_1\ldots q_4} =$


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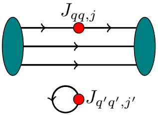

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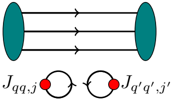
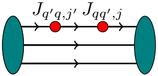


$C^{jj'}_{2,qq'} =$


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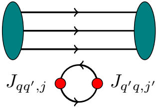

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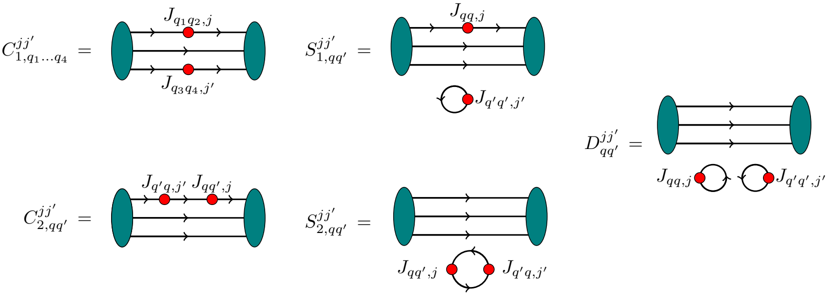

Contribution to required matrix elements

$${}_p\left\langle (\bar{q}\Gamma_j Q)(x)\ (\bar{Q}\Gamma_{j'} q)(y) \right\rangle_p = C^{jj'}_{2,qQ}(x,y) + S^{jj'}_{2,qQ}(x,y)$$

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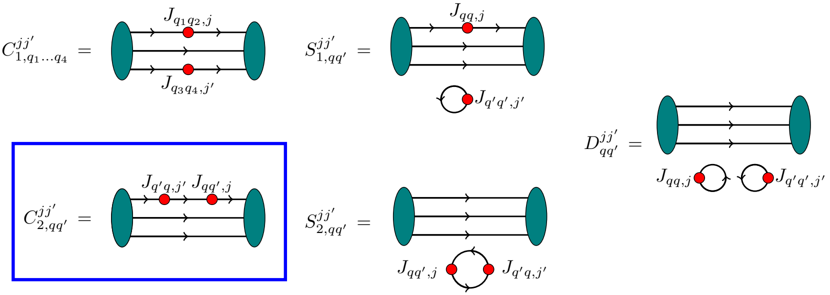

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Techniques and Strategy

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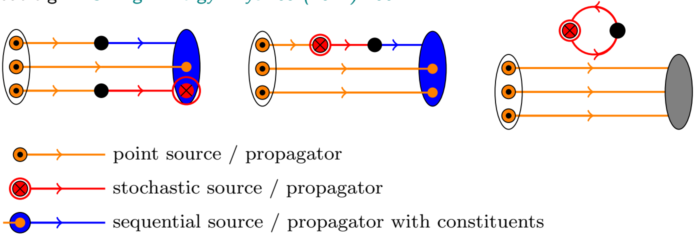

- Point source at fixed position, sequential and/or stochastic (C_1) sources at the sink; corresponding propagators have to be re-calculated for each momentum / source-sink separation
- Boosted sources (momentum smearing) *Phys.Rev.D 93 (2016) 9, 094515*
- Sequential source technique *Nucl. Phys. B316 (1989)*
- Stochastic wall sources: $\eta^{\ell}_{\alpha a \vec{x}} = (\pm 1 \pm i)/\sqrt{2}$ on requested time slice
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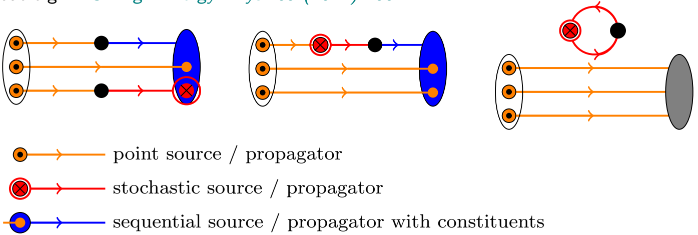

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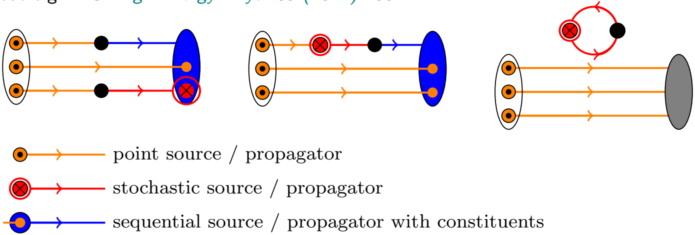

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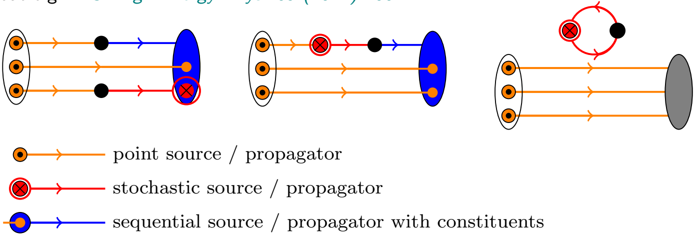

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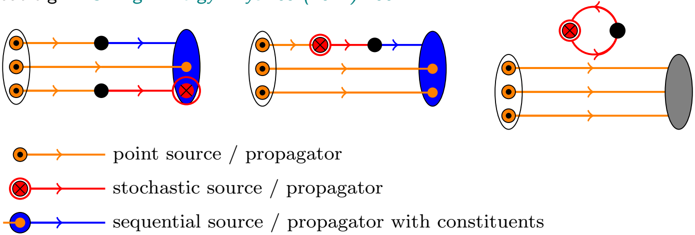

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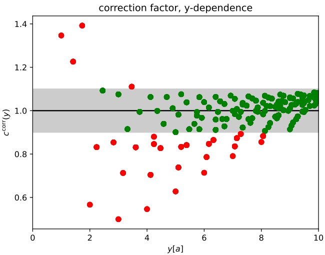
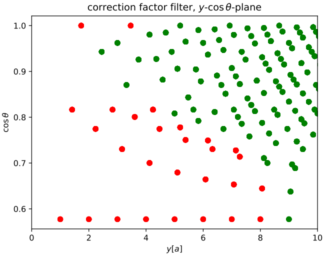
Propagator Anisotropy





Reduce effects caused by lattice propagator anisotropy (following the idea of *[Bali et al, Phys. Rev. D **98**, no.9, 094507 (2018)]*):

- Remove chiral odd contribution: Take current combination $(VV + AA)/2$
- Consider correction factor for the chiral even part of the lattice propagator:

$$c^{\rm corr}(y) = \left(\mathrm{tr}\left\{\not{y} M^{\rm free}(y)\pi^2\right\}\right)^{-1}(-m^2)K_2\left(m\sqrt{-y^2}\right)$$

- Drop all data points with $|c^{\rm corr} - 1| > 0.1$.
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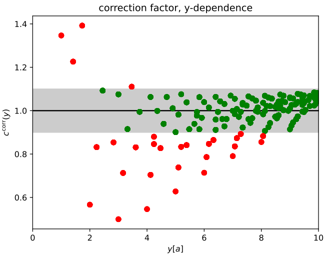

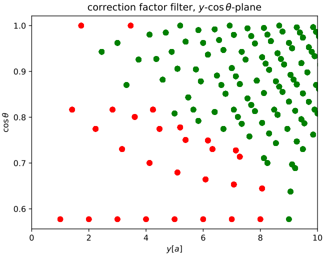

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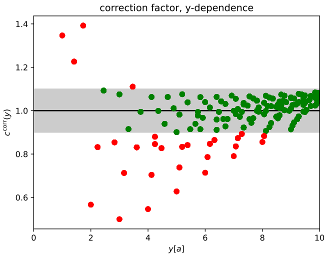

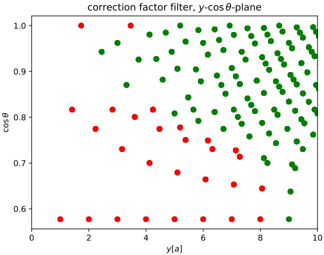

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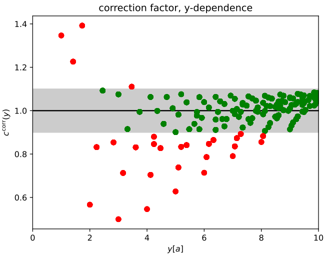

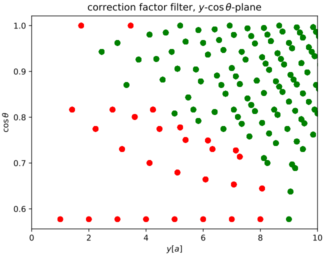

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Lattice setup

CLS ensembles

id	β	$a[\mathrm{fm}]$	$L^3 \times T$	$m_\pi[\mathrm{MeV}]$	$m_K[\mathrm{MeV}]$	$m_\pi L$	$L[\mathrm{fm}]$	# conf
H102	3.4	0.0856	$32^3 \times 128$	355	441	4.9	2.7	990

- $n_f = 2 + 1$ $\mathcal{O}(a)$-improved Sheikholeslami-Wohlert fermions.
- Lüscher-Weisz gauge action.

Setup:

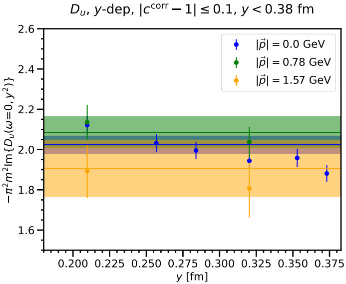
- Source-sink separation: $t/a = 10$.
- 6 momenta, $|\vec{p}|_{\mathrm{max}} \approx 1.57\ \mathrm{GeV}$.

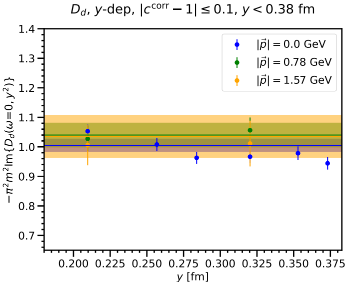
Renormalization: RI/MOM-scheme, conversion to $\overline{\mathrm{MS}}$-scheme *Bali et al, Phys. Rev. D 103 (2021) 094511*:

$$\mathcal{O}(\mu) = Z_{\mathcal{O}}(\mu)\ \mathcal{O}^{\mathrm{latt}}$$
$$\Rightarrow M_{ij}(\omega, y^2) = Z_i\ Z_j\ M_{ij}^{\mathrm{latt}}(\omega, y^2)$$

Content

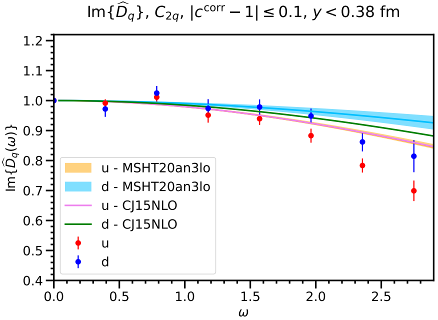
Results at $\omega = 0$





- Consistency check: $-\pi^2 m^2 \, \mathrm{Im} \left\{ D_q(\omega = 0, y^2) \right\} = n_q$
- From now on: consider $\widehat{D}(\omega) = \frac{D(\omega, y^2)}{D(0, y^2)}$, $-y^2 < 0.38 \text{ fm}$
- Average along y^2

Results for Ioffe-time dependence



- Compare with experimental data, transform to "Ioffe-time (ω) space".
- Agreement with experiment good for small ω
- Discrepancies arise for $\omega > 2$
- Extraction of PDFs "x-space"

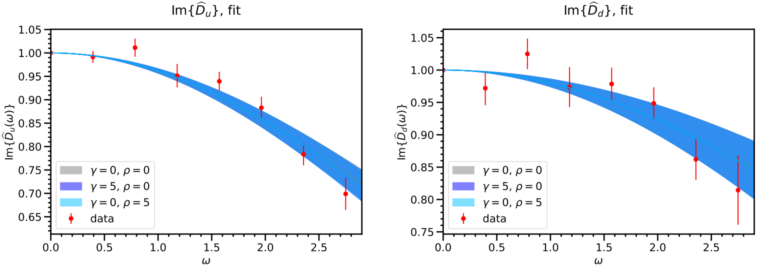
$$\mathrm{Im}\left\{D_q(\omega, y^2)\right\} = -\frac{2i}{\pi^2 m^2}\int_0^1 \mathrm{d}x\ f^h_{q,v}(x)\cos(x\omega)$$

$\Rightarrow$ Inverse problem

Extraction of the x-dependence

Ansatz:

$$f_q(x)/n_q = x^{\alpha}(1-x)^{\beta}(1+\rho\sqrt{x}+\gamma x)/\mathrm{norm}$$



Fit in ω-space:

Flavor	γ	ρ	β	χ^2/d.o.f.
u	0	0	2.56(33)	1.87
	5	0	3.86(40)	1.80
	0	5	3.49(40)	1.81
d	0	0	4.5(1.1)	1.45
	5	0	6.2(1.3)	1.43
	0	5	5.9(1.3)	1.43

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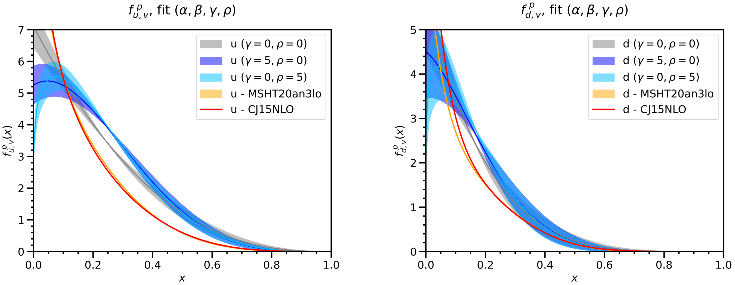

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- Two-current correlations represent so-called *good "lattice cross sections"*
- First studies available for the pion ($m_\pi = 416\ \mathrm{MeV}$) and the nucleon ($m_\pi = 355\ \mathrm{MeV}$)
- **Todo/ongoing:**
  - better deal with inverse problem
  - cover higher momenta (need kinematically enhanced interpolators *Zhang et al, Phys. Rev. D **112**, no.5, L051502 (2025)*)
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Thanks for your attention!

Backup Slides

Two-current correlation functions on the Lattice

General definition:

$$C^{\Gamma}_{4\mathrm{pt},ij}(t,\tau,\tau';\vec{y},\vec{p}_i,\vec{p}_f) := \frac{1}{V}\sum_{\vec{x}} \left\langle \Gamma\ \mathcal{O}^{\vec{p}_f}(t+t_0)\ \mathcal{J}_j(\vec{x}+\vec{y},\tau'+t_0)\ \mathcal{J}_i(\vec{x},\tau+t_0)\ \overline{\mathcal{O}}^{\vec{p}_i}(t_0) \right\rangle$$

$$\widetilde{C}^{\Gamma}_{4\mathrm{pt},ij}(t,\tau,\tau';\vec{q},\vec{p}_i,\vec{p}_f) := \left\langle \Gamma\ \mathcal{O}^{\vec{p}_f}(t+t_0)\ \widetilde{\mathcal{J}}_j(\vec{q},\tau'+t_0)\ \widetilde{\mathcal{J}}_i(-\vec{q}-\vec{\Delta},\tau+t_0)\ \overline{\mathcal{O}}^{\vec{p}_i}(t_0) \right\rangle$$

- Hadron interpolators $\mathcal{O}^{\vec{p}_f}$, $\overline{\mathcal{O}}^{\vec{p}_i}$ for initial (final) momentum $\vec{p}_i$ ($\vec{p}_f$); have $\vec{p}_i = \vec{p}_f =: \vec{p}$ ($\vec{\Delta} = \vec{0}$) in most of the considered cases; source-sink separation t.
- Polarization/parity (if applicable): Γ.
- Two currents of interest $\mathcal{J}_i$, $\mathcal{J}_j$ at timeslice τ, τ' relative to the source at t_0 (for some cases have $\tau = \tau'$).
- Momentum transfer $\vec{q}$, $-\vec{q}$ or relative spatial separation $\vec{y}$ between the two currents, respectively (always cover the entire space on the lattice!)
- Relation to two-current matrix element given by ratios, e.g.:

$$\langle p,s|\ \mathcal{J}_j(\vec{y},\tau')\mathcal{J}_i(\vec{0},\tau)\ |p,s\rangle \propto \left.\frac{C^{\Gamma}_{4\mathrm{pt},ij}(t,\tau,\tau';\vec{y},\vec{p})}{C^{\vec{p}}_{2\mathrm{pt}}(t)}\right|_{0\ll\tau,\tau'\ll t}$$

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$$\langle p,s|\ \mathcal{J}_j(\vec{y},\tau')\mathcal{J}_i(\vec{0},\tau)\ |p,s\rangle \propto \left.\frac{C^{\Gamma}_{4\mathrm{pt},ij}(t,\tau,\tau';\vec{y},\vec{p})}{C^{\vec{p}}_{2\mathrm{pt}}(t)}\right|_{0\ll\tau,\tau'\ll t}$$

Two-current correlation functions on the Lattice

General definition:

$$C^{\Gamma}_{4\mathrm{pt},ij}(t,\tau,\tau';\vec{y},\vec{p}_i,\vec{p}_f) := \frac{1}{V}\sum_{\vec{x}} \left\langle \Gamma\ \mathcal{O}^{\vec{p}_f}(t+t_0)\ \mathcal{J}_j(\vec{x}+\vec{y},\tau'+t_0)\ \mathcal{J}_i(\vec{x},\tau+t_0)\ \overline{\mathcal{O}}^{\vec{p}_i}(t_0) \right\rangle$$

$$\widetilde{C}^{\Gamma}_{4\mathrm{pt},ij}(t,\tau,\tau';\vec{q},\vec{p}_i,\vec{p}_f) := \left\langle \Gamma\ \mathcal{O}^{\vec{p}_f}(t+t_0)\ \widetilde{\mathcal{J}}_j(\vec{q},\tau'+t_0)\ \widetilde{\mathcal{J}}_i(-\vec{q}-\vec{\Delta},\tau+t_0)\ \overline{\mathcal{O}}^{\vec{p}_i}(t_0) \right\rangle$$

- Hadron interpolators $\mathcal{O}^{\vec{p}_f}$, $\overline{\mathcal{O}}^{\vec{p}_i}$ for initial (final) momentum $\vec{p}_i$ ($\vec{p}_f$); have $\vec{p}_i = \vec{p}_f =: \vec{p}$ ($\vec{\Delta} = \vec{0}$) in most of the considered cases; source-sink separation t.
- Polarization/parity (if applicable): Γ.
- Two currents of interest $\mathcal{J}_i$, $\mathcal{J}_j$ at timeslice τ, τ' relative to the source at t_0 (for some cases have $\tau = \tau'$).
- Momentum transfer $\vec{q}$, $-\vec{q}$ or relative spatial separation $\vec{y}$ between the two currents, respectively (always cover the entire space on the lattice!)
- Relation to two-current matrix element given by ratios, e.g.:

$$\langle p,s|\ \mathcal{J}_j(\vec{y},\tau')\mathcal{J}_i(\vec{0},\tau)\ |p,s\rangle \propto \left.\frac{C^{\Gamma}_{4\mathrm{pt},ij}(t,\tau,\tau';\vec{y},\vec{p})}{C^{\vec{p}}_{2\mathrm{pt}}(t)}\right|_{0\ll\tau,\tau'\ll t}$$

Matrix element data

H102, $m_\pi = 355$ MeV