Effective Field Theory Perspective On King Non-linearity

Gil Paz

Department of Physics and Astronomy,

Wayne State University,

Detroit, Michigan, USA

U.S. DEPARTMENT OF ENERGY

Office of Science

Based on

[Assi, Carey, Jäger, Lee, Paz, Perez, Zupan, arXiv:2512.03157]

Outline

- Introduction: King linearity and non-linearity
- EFT Perspective On King Non-linearity
- Phenomenological implications
- Conclusions

Introduction:

King linearity
and
King non-linearity

Motivation

- We will discuss isotope shift: difference in spectra of isotope A and A'
- Why is this interesting?
- First experimental indication for $g_e \neq 2$ was from hyperfine structure of hydrogen and deuterium
  [Nafe, Nelson, Rabi, Phys. Rev. **71** 914 (1947)]
- This led Schwinger to calculate $g_e - 2$ in QED and show that it explained the data [Schwinger, Phys. Rev. **73** 416 (1947)]
- Isotope shift is used to look for new physics via King Non-linearity
- Isotope shift allows to go beyond the charge radius
  study nuclei structure at unprecedented precision

King linearity

- For an atomic transition i there is a difference in frequency between two isotopes A and A': $\nu_i^{AA'} = \nu_i^A - \nu_i^{A'}$

- Two main sources:

1) Mass shift $m_A \neq m_{A'}$

For hydrogen-like system

$$E_n^A = -\frac{1}{2n^2}(Z\alpha)^2 \frac{m_e m_A}{m_e + m_A} \approx -\frac{1}{2n^2} m_e (Z\alpha)^2 \left(1 - \frac{m_e}{m_A}\right)$$

King linearity

- For an atomic transition i there is a difference in frequency between two isotopes A and A': $\nu_i^{AA'} = \nu_i^A - \nu_i^{A'}$

- Two main sources:

2) Field shift: $\langle r_A^2 \rangle \neq \langle r_{A'}^2 \rangle$

- Finite nucleus $V = -Z\alpha/r + \delta V$ $\qquad \Delta E = \int d^3\boldsymbol{r}\, \phi^*(\boldsymbol{r})\delta V(\boldsymbol{r})\phi(\boldsymbol{r})$

  $\phi(\boldsymbol{r}) \sim$ atomic size $\sim 10^{-10}$ m $\qquad \delta V(\boldsymbol{r}) \sim$ nuclear size $\sim 10^{-15}$ m

  $\Rightarrow \Delta E \approx |\phi(0)|^2 \int d^3\boldsymbol{r}\, \delta V(\boldsymbol{r})$

  From Gauss's law $\nabla^2 V = -e^2\rho \Rightarrow \dfrac{4\pi\alpha}{6}\int d^3\boldsymbol{r}\, r^2\, \rho(\boldsymbol{r}) = \int d^3\boldsymbol{r}\, \delta V(\boldsymbol{r})$

  $\Delta E \approx \dfrac{4\pi\alpha}{6}|\phi(0)|^2 \int d^3\boldsymbol{r}\, r^2\, \rho(\boldsymbol{r}) \equiv \dfrac{4\pi\alpha}{6}|\phi(0)|^2 \langle r^2 \rangle$

  For hydrogen-like system

$$\Delta E_n = \frac{4\pi\alpha}{6}|\phi(0)|^2\langle r^2\rangle = \frac{4\pi\alpha}{6}\frac{(m_e Z\alpha)^3}{\pi n^3}\langle r^2\rangle$$

King linearity

- For an atomic transition i there is a difference in frequency between two isotopes A and A': $\nu_i^{AA'} = \nu_i^A - \nu_i^{A'}$

- Two main sources:

1) Mass shift $m_A \neq m_{A'}$
For hydrogen-like system

$$E_n^A = -\frac{1}{2n^2}(Z\alpha)^2\frac{m_e m_A}{m_e + m_A} \approx -\frac{1}{2n^2} m_e (Z\alpha)^2 \left(1 - \frac{m_e}{m_A}\right)$$

2) Field shift: $\langle r_A^2\rangle \neq \langle r_{A'}^2\rangle$
For hydrogen-like system

$$\Delta E_n = \frac{4\pi\alpha}{6}|\phi(0)|^2\langle r^2\rangle = \frac{4\pi\alpha}{6}\frac{(m_e Z\alpha)^3}{\pi n^3}\langle r^2\rangle$$

- Two sources with a different electronic dependence and a different nuclear dependence

King linearity

- For an atomic transition i there is a difference in frequency between two isotopes A and A': $\nu_i^{AA'} = \nu_i^A - \nu_i^{A'}$

- Two main sources:

- $m_A \neq m_{A'}$
- $\langle r_A^2 \rangle \neq \langle r_{A'}^2 \rangle$

$$\nu_i^{AA'} = \overbrace{K_i \left(\frac{1}{m_A} - \frac{1}{m_{A'}} \right)}^{\text{Mass Shift}} + \overbrace{F_i \left(\langle r_A^2 \rangle - \langle r_{A'}^2 \rangle \right)}^{\text{Field Shift}} + \cdots$$

- Factorization:

- K_i and F_i depend on atomic quantities
- $\frac{1}{m_A} - \frac{1}{m_{A'}}$ and $\langle r_A^2 \rangle - \langle r_{A'}^2 \rangle$ depend on nuclear quantities

- Two sources ⇒ linear relation between $\nu_i^{AA'}$'s ⇒ King linearity
[W. H. King, J. Opt. Soc. Am. **53** 638 (1963)]

King linearity

- Two sources ⇒ linear relation between $\nu_i^{AA'}$'s ⇒ King linearity

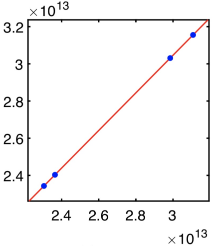


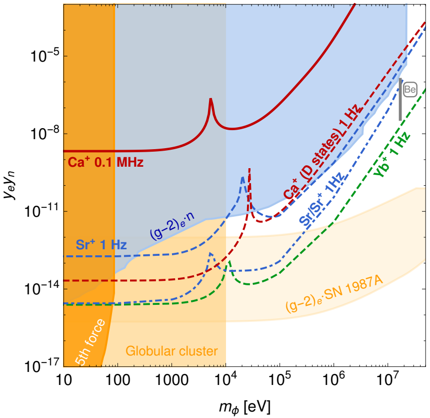
Image from [Counts et al., PRL **125** 123002 (2020)]

- New unknown interactions, e.g., light force-carrier coupling to *e* and *n* can introduce a third source, leading to non-linearity [Berengut, et al. Phys. Rev. Lett. **120**, 091801 (2018)]

King non-linearity

- New unknown interactions, e.g., light force-carrier coupling to e and n can introduce a third source, leading to non-linearity
  [Berengut, et al. Phys. Rev. Lett. **120**, 091801 (2018)]

- $V_\phi(r) = -\alpha_{\rm NP}(A-Z)e^{-m_\phi r}/r$, where $\alpha_{\rm NP} = (-1)^s y_e y_n/4\pi$



King non-linearity

- Such a non-linearity was recently found in experiments

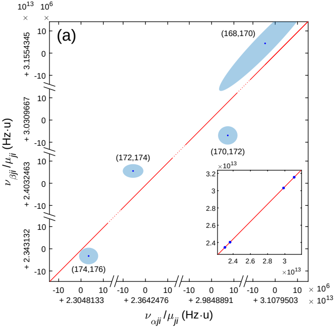


Image from [Counts et al., PRL **125** 123002 (2020)]

- Is it new physics?
- Nuclear effects such as $\langle r^4 \rangle$ or $\langle r^2 \rangle^2$ also lead to non-linearity!

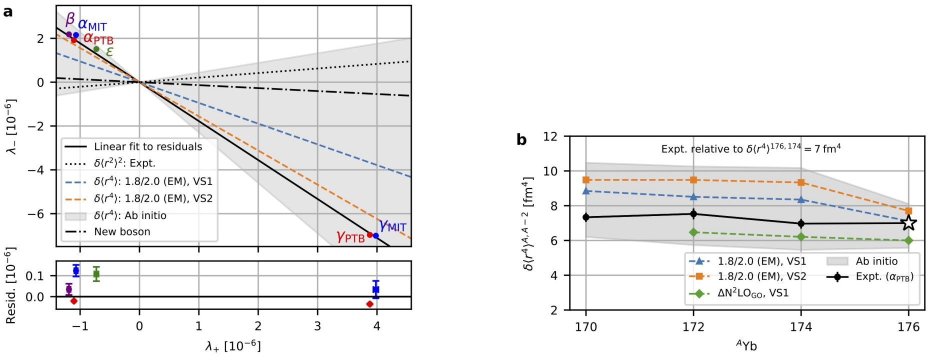
reminder: $\left[\langle r^k \rangle \equiv \int d^3r\, r^k\, \rho(r)\right]$

Experimental results

Journal	Atom	# of σ	New nuclear effects
PRL 125 123002 (2020)	Yb	3σ	$\langle r^2\rangle^2$
PRX 12 021033 (2022)	Yb	3σ	$\langle r^2\rangle^2$ and $\langle r^4\rangle$
PRL 128 163201 (2022)	Yb	41σ	$\langle r^2\rangle^2$ and $\langle r^4\rangle$
PRL 134 063002 (2025)	Yb		$\langle r^2\rangle^2$ and $\langle r^4\rangle$
PRL 134 233002 (2025)	Ca	$\sim 10^3\sigma$	mass and polarizability

- King non-linearity was observed with a large number of σ's
- Is it new physics?
- Are these all the possible nuclear effects?

King non-linearity



Images from [PRL 134 063002 (2025)]

- Are these all the possible nuclear effects?

EFT Perspective on King Non-linearity

Why EFT?

- Interested in nuclear physics effects size of $r_N \sim 10^{-15}$ m
  on atomic physics systems size of $a \sim 10^{-10}$ m

- Since $\epsilon = r_N/a \ll 1$ we can consider effects by their scaling in ϵ
- For Ca $\quad \epsilon \sim 10^{-3}$
- For Yb $\quad \epsilon \sim 10^{-2}$

- The tool for that is an effective field theory

- In hydrogen-like bound states the electron velocity is $\sim \alpha \ll 1$

- We will use Non Relativistic QED (NRQED)

Game plan

- Step 1: Match SM onto NRQED:
  Find the possible operators and their Wilson coefficients

- Step 2: Match NRQED onto Quantum Mechanics (QM)
  Calculate NRQED amplitudes and find the QM potentials

- Step 3: Use the QM potentials to find $\Delta E = h\,\Delta\nu$

Step 1: Match SM onto NRQED

SM $\to$ NRQED

- The Lagrangian can be split into
  - $\mathcal{L}_{\text{electron}}$
  - $\mathcal{L}_{\text{nucleus}}$
  - $\mathcal{L}_{\text{interaction}}$

- $\mathcal{L}_{\text{electron}}$ is known

- Consider next the $\mathcal{L}_{\text{nucleus}}$
- Define the covariant derivatives operators
  - $iD_t = i\partial_t - eZA^0$ which is P and T even
  - $i\boldsymbol{D} = i\boldsymbol{\nabla} + eZ\boldsymbol{A}$ which is P odd and T even
- For an NR nucleus χ operators of the form $\chi^\dagger iD^\mu \ldots iD^\nu \chi$ must contain and even number of $i\boldsymbol{D}$
- Let's illustrate the construction of spin-independent operators with one, two, and three covariant derivatives [GP MPLA **30**, 1550128 (2015)]

Constructing $\mathcal{L}_{\text{nucleus}}$

- With one covariant derivative we can only have $\chi^\dagger iD_t \chi$
- With two covariant derivatives we can have

- $\chi^\dagger \left(iD^j\right)\left(iD^j\right)\chi/\Lambda$
- $\chi^\dagger \left(iD_t\right)\left(iD_t\right)\chi/\Lambda$ can be redefined away by using $\chi^\dagger iD_t \chi$ and the field redefinition $\chi \to \chi - iD_t\chi/2\Lambda$

- So far we get

$$\mathcal{L}_{\text{nucleus}} = \chi^\dagger \left(iD_t + c_2 \frac{\boldsymbol{D}^2}{2M}\right)\chi$$

where M is the nucleus mass

- Lorentz invariance of full Lagrangian (a.k.a reparameterization invariance) implies $c_2 = 1$
- Equation of motion gives the Schrödinger equation

Constructing $\mathcal{L}_{\text{nucleus}}$

- With three covariant derivatives we can have

- $\chi^\dagger\left(iD_t\right)\left(iD_t\right)\left(iD_t\right)\chi$, $\chi^\dagger\left(iD_t\right)\left(iD^j\right)\left(iD^j\right)\chi$, $\chi^\dagger\left(iD^j\right)\left(iD^j\right)\left(iD_t\right)\chi$ that can be redefined away

- $\chi^\dagger\left(iD^j\right)\left(iD_t\right)\left(iD^j\right)\chi \to \frac{1}{2}\chi^\dagger iD^j[iD_t, iD^j]\chi + \frac{1}{2}\chi^\dagger[iD^j, iD_t]iD^j\chi =$
$= \chi^\dagger[iD^j, [iD_t, iD^j]\chi = \chi^\dagger(\boldsymbol{\nabla}\cdot\boldsymbol{E})\chi$

- A better way can be found in by mapping the operators onto HQET matrix elements [Gunawardna, GP JHEP **1707** 137 (2017)]

- The NRQED Lagrangian up to dimension 8 is known explicitly

NRQED Lagrangian

- The NRQED Lagrangian up to dimension 8
  [Hill, Lee, GP, Solon, PRD **87** 053017 (2013)]

$$
\begin{aligned}
\mathcal{L}_{\text{NRQED}} = \chi^\dagger \Big\{ & iD_t + \frac{\boldsymbol{D}^2}{2M} + c_F g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + i c_S g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})}{8M^2} \\
& + \frac{\boldsymbol{D}^4}{8M^3} + c_{W1} g \frac{\{\boldsymbol{D}^2, \boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} - c_{W2} g \frac{D^i \boldsymbol{\sigma}\cdot\boldsymbol{B} D^i}{4M^3} + i c_M g \frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3} \\
& + c_{p'p} g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3} + c_{A1} g^2 \frac{(\boldsymbol{B}^2-\boldsymbol{E}^2)}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} \\
& + c_{X1} g \frac{[\boldsymbol{D}^2, \boldsymbol{D}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot\boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2, [\boldsymbol{\partial}\cdot\boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} \\
& + i c_{X4} g^2 \frac{\{\boldsymbol{D}^i, [\boldsymbol{E}\times\boldsymbol{B}]^i\}}{M^4} + i c_{X5} g \frac{D^i \boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D}) D^i}{M^4} \\
& + i c_{X6} g \frac{\epsilon^{ijk}\sigma^i D^j [\boldsymbol{\partial}\cdot\boldsymbol{E}] D^k}{M^4} + c_{X7} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + c_{X8} g^2 \frac{[\boldsymbol{E}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{B}]}{M^4} \\
& + c_{X9} g^2 \frac{[\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{E}]}{M^4} + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4} \\
& + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4} + c_{X12} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}\times[\partial_t \boldsymbol{E} - \boldsymbol{\partial}\times\boldsymbol{B}]}{M^4} \Big\} \chi
\end{aligned}
$$

NRQED Lagrangian

- First simplification: King NL experiments are done with spin-zero nuclei

$$\mathcal{L}_{\rm NRQED} = \chi^\dagger \Big\{ iD_t + \frac{\boldsymbol{D}^2}{2M} + c_F g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + i c_S g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})}{8M^2}$$

$$+\frac{\boldsymbol{D}^4}{8M^3} + c_{W1} g \frac{\{\boldsymbol{D}^2, \boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} - c_{W2} g \frac{D^i \boldsymbol{\sigma}\cdot\boldsymbol{B} D^i}{4M^3} + i c_M g \frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3}$$

$$+ c_{p'p} g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3} + c_{A1} g^2 \frac{\left(\boldsymbol{B}^2 - \boldsymbol{E}^2\right)}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3}$$

$$+ c_{X1} g \frac{[\boldsymbol{D}^2, \boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2, [\boldsymbol{\partial}\cdot\boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4}$$

$$+ i c_{X4} g^2 \frac{\{\boldsymbol{D}^i, [\boldsymbol{E}\times\boldsymbol{B}]^i\}}{M^4} + i c_{X5} g \frac{D^i \boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D}) D^i}{M^4}$$

$$+ i c_{X6} g \frac{\epsilon^{ijk}\sigma^i D^j [\boldsymbol{\partial}\cdot\boldsymbol{E}] D^k}{M^4} + c_{X7} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + c_{X8} g^2 \frac{[\boldsymbol{E}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{B}]}{M^4}$$

$$+ c_{X9} g^2 \frac{[\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{E}]}{M^4} + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4}$$

$$+ c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4} + c_{X12} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}\times[\partial_t \boldsymbol{E} - \boldsymbol{\partial}\times\boldsymbol{B}]}{M^4} \Big\} \chi$$

NRQED Lagrangian

- First simplification: King NL experiments are done with spin-zero nuclei

$$
\begin{aligned}
\mathcal{L}_{\text{NRQED}} = \chi^\dagger \Big\{ & iD_t + \frac{\boldsymbol{D}^2}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} \\
& + \frac{\boldsymbol{D}^4}{8M^3} + i c_M g \frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3} \\
& + c_{A1} g^2 \frac{\left(\boldsymbol{B}^2 - \boldsymbol{E}^2\right)}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} \\
& + c_{X1} g \frac{[\boldsymbol{D}^2, \boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2, [\boldsymbol{\partial}\cdot\boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} \\
& + i c_{X4} g^2 \frac{\{\boldsymbol{D}^i, [\boldsymbol{E}\times\boldsymbol{B}]^i\}}{M^4} \Big\}\chi
\end{aligned}
$$

NRQED Lagrangian

- Second simplification: The inverse size of the nucleus, $1/r_N$ and its mass, M are separated scales:
- The inverse of proton radius, $1/r_p \simeq 1 \text{ fm}^{-1} \simeq 0.2$ GeV
  and the proton mass, $m_p \simeq 1$ GeV are close
- Not true for Ca with charge radius $1/r_N \simeq 1/(3.5 \text{ fm}) \simeq 0.06$ GeV, and mass is $M \simeq 40$ GeV
- This difference is even larger for Yb:
  $1/r_N \simeq 1/(5\,\text{fm}) \simeq 0.04$ GeV, and $M \sim 160$ GeV
- This follows from the scaling $1/r_N \sim A^{-1/3}/r_p$ and $M \sim Am_p$

- As a result we can take the $M \to \infty$ limit

NRQED Lagrangian

- Second simplification: The inverse size of the nucleus, $1/r_N$ and its mass, M are separated scales:
- As a result we can take the $M \to \infty$ limit

- Lorentz invariance implies that certain Wilson coefficients are determined by Wilson coefficients of lower dimensional operators, e..g

$$\mathcal{L}_{\text{nucleus}} = \chi^\dagger \left(iD_t + c_2 \frac{\boldsymbol{D}^2}{2M} \right) \chi$$

  Lorentz invariance implies $c_2 = 1$

- In the $M \to \infty$

$$\mathcal{L}_{\text{nucleus}} \to \chi^\dagger iD_t \chi$$

- Similarly c_M, c_{X1}, c_{X2}, and c_{X4} are related to Wilson coefficients of lower dimensional operators

NRQED Lagrangian

- Second simplification: $1/r_N \ll M$

$$\mathcal{L}_{\rm NRQED} = \chi^\dagger \Big\{ iD_t + c_D g \frac{[\boldsymbol{\partial} \cdot \boldsymbol{E}]}{8M^2} + + c_{A1} g^2 \frac{(\boldsymbol{B}^2 - \boldsymbol{E}^2)}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} + c_{X3} g \frac{[\boldsymbol{\partial}^2 \boldsymbol{\partial} \cdot \boldsymbol{E}]}{M^4} \Big\} \chi$$

- Only 5 operators remain out of the 24

NRQED Lagrangian

- NRQED Lagrangian up to dim. 8 [Hill, Lee, GP, Solon, PRD **87** 053017 (2013)]

$$
\begin{aligned}
\mathcal{L}_{\rm NRQED} = \chi^\dagger \Big\{ & iD_t + \frac{\boldsymbol{D}^2}{2M} + c_F g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + ic_S g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})}{8M^2} \\
& + \frac{\boldsymbol{D}^4}{8M^3} + c_{W1} g \frac{\{\boldsymbol{D}^2, \boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} - c_{W2} g \frac{D^i \boldsymbol{\sigma}\cdot\boldsymbol{B} D^i}{4M^3} + ic_M g \frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3} \\
& + c_{p'p} g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3} + c_{A1} g^2 \frac{\left(\boldsymbol{B}^2 - \boldsymbol{E}^2\right)}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} \\
& + c_{X1} g \frac{[\boldsymbol{D}^2, \boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2, [\boldsymbol{\partial}\cdot\boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2 \boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} \\
& + ic_{X4} g^2 \frac{\{\boldsymbol{D}^i, [\boldsymbol{E}\times\boldsymbol{B}]^i\}}{M^4} + ic_{X5} g \frac{D^i \boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D}) D^i}{M^4} \\
& + ic_{X6} g \frac{\epsilon^{ijk} \sigma^i D^j [\boldsymbol{\partial}\cdot\boldsymbol{E}] D^k}{M^4} + c_{X7} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + c_{X8} g^2 \frac{[\boldsymbol{E}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{B}]}{M^4} \\
& + c_{X9} g^2 \frac{[\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{E}]}{M^4} + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4} \\
& + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4} + c_{X12} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}\times[\partial_t \boldsymbol{E} - \boldsymbol{\partial}\times\boldsymbol{B}]}{M^4} \Big\} \chi
\end{aligned}
$$

Scalar NRQED Lagrangian in the $M \to \infty$ limit

- Changing $\chi \to S$, the scalar NRQED Lagrangian in the $M \to \infty$ limit takes the simple form

$$\mathcal{L}_S = S^\dagger \left\{ iD_t + \frac{c_{\langle r^2 \rangle}}{\Lambda_N^2} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) + \frac{c_{E^2}}{\Lambda_N^3} \boldsymbol{E}^2 + \frac{c_{B^2}}{\Lambda_N^3} \boldsymbol{B}^2 + \frac{c_{\langle r^4 \rangle}}{\Lambda_N^4} \boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \right\} S$$

- To determine the Wilson coefficients we need to perform matching

- $c_{\langle r^2 \rangle}$ and $c_{\langle r^4 \rangle}$ are determined by one-photon amplitude

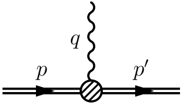
- c_{E^2} and c_{B^2} are determined by two-photon amplitude

Scalar NRQED Lagrangian in the $M \to \infty$ limit

$$\mathcal{L}_S = S^\dagger \left\{ iD_t + \frac{c_{\langle r^2\rangle}}{\Lambda_N^2}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) + \frac{c_{E^2}}{\Lambda_N^3}\boldsymbol{E}^2 + \frac{c_{B^2}}{\Lambda_N^3}\boldsymbol{B}^2 + \frac{c_{\langle r^4\rangle}}{\Lambda_N^4}\boldsymbol{\nabla}^2(\boldsymbol{\nabla}\cdot\boldsymbol{E}) \right\} S$$

- $c_{\langle r^2\rangle}$ and $c_{\langle r^4\rangle}$ are determined by one-photon amplitude
- Define the form factor via

$$\Gamma^{(3)} = -ie\,(p+p')^\mu\,F(q^2)$$



- Expand the form factor in power of $q^2 = -\boldsymbol{q}^2 + \mathcal{O}\left(1/M^2\right)$ and match onto NRQED amplitude

$$-ie\left[F(0) - F'(0)\boldsymbol{q}^2 + \frac{1}{2}F''(0)\boldsymbol{q}^4 + \cdots\right] \stackrel{!}{=} -ieZ + ie\frac{c_{\langle r^2\rangle}}{\Lambda_N^2}\boldsymbol{q}^2 - ie\frac{c_{\langle r^4\rangle}}{\Lambda_N^4}\boldsymbol{q}^4 + \cdots$$

$$\boxed{Z = F(0)} \quad \boxed{\frac{c_{\langle r^2\rangle}}{\Lambda_N^2} = F'(0) \equiv F(0)\frac{\langle r^2\rangle}{6}} \quad \boxed{\frac{c_{\langle r^4\rangle}}{\Lambda_N^4} = \frac{1}{2}F''(0) \equiv F(0)\frac{\langle r^4\rangle}{120}}$$

Scalar NRQED Lagrangian in the $M \to \infty$ limit

$$\mathcal{L}_S = S^\dagger \left\{ iD_t + \frac{c_{\langle r^2 \rangle}}{\Lambda_N^2} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) + \frac{c_{E^2}}{\Lambda_N^3} \boldsymbol{E}^2 + \frac{c_{B^2}}{\Lambda_N^3} \boldsymbol{B}^2 + \frac{c_{\langle r^4 \rangle}}{\Lambda_N^4} \boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \right\} S$$

- c_{E^2} and c_{B^2} are determined by two-photon amplitude
- From [Hill, Lee, GP, Solon, PRD **87** 053017 (2013)] in the $M \to \infty$ limit and converting to Gaussian units we have

$$\boxed{\frac{c_{E^2}}{\Lambda_N^3} = \frac{\alpha_E}{2}} \qquad \boxed{\frac{c_{B^2}}{\Lambda_N^3} = \frac{\beta_M}{2}}$$

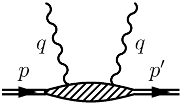
- α_E is the electric polarizability and β_M the magnetic polarizability

Scalar NRQED Lagrangian in the $M \to \infty$ limit after matching

$$\mathcal{L}_S = S^\dagger \left\{ iD_t + eZ\,\frac{\langle r^2\rangle}{6}\,(\boldsymbol{\nabla}\cdot\boldsymbol{E}) + \frac{\alpha_E}{2}\,\boldsymbol{E}^2 + \frac{\beta_M}{2}\,\boldsymbol{B}^2 + eZ\,\frac{\langle r^4\rangle}{120}\,\boldsymbol{\nabla}^2(\boldsymbol{\nabla}\cdot\boldsymbol{E}) \right\} S$$

Four-fields operators Lagrangian

- We also have four-fields operators Lagrangian
- At dimension 6 we have one possible operator: $S^\dagger S \psi_e^\dagger \psi_e$
- To find the Wilson coefficient we need the difference between full
  - and effective theory amplitude for $S + e \to S + e$ at $\mathcal{O}(e^2)$
- It arises from the two-photon interaction of the nucleus



- The imaginary part of the two-photon interaction can be split into
  - Elastic contribution depends of $F(q^2)$
  - Inelastic contribution depends of inelastic structure function

Four-fields operators Lagrangian

- The elastic piece gives a Wilson coefficient

$$\frac{\pi}{3}\,(Z\alpha)^2\, m_e\, \langle r^3\rangle_{(2)}$$

where

$$\langle r^3\rangle_{(2)} = \frac{48}{\pi}\,\frac{1}{[F(0)]^2}\int_0^\infty \frac{dQ}{Q^4}\left\{\left[F(-Q^2)\right]^2 - [F(0)]^2 + \frac{[F(0)]^2\,\langle r^2\rangle Q^2}{3}\right\}$$

- As expected, the Wilson coefficient has mass dimension -2 but it is proportional to m_e
- Up to dimension 6 we have

$$\mathcal{L}_{S\psi_e} = \frac{d_2\, m_e}{\Lambda_N^3} S^\dagger S \psi_e^\dagger \psi_e + \cdots\,,$$

where

$$\frac{d_2}{\Lambda_N^3} = \frac{\pi}{3}\,(Z\alpha)^2\,\langle r^3\rangle_{(2)} + \cdots$$

Four-fields operators Lagrangian

- At dimension 8 we have many more operators
  Lorentz invariance implies relations between Wilson coefficients
  [Hill, Lee, GP, Solon, PRD **87** 053017 (2013)]
  keeping only those which are not $1/M$ suppressed we have

$$\mathcal{L}_{S\psi_e} = \frac{d_2 m_e}{\Lambda_N^3} S^\dagger S \psi_e^\dagger \psi_e + \frac{d_3}{\Lambda_N^4} S^\dagger D_+^i S \psi_e^\dagger D_+^i \psi_e + \frac{d_6}{m_e \Lambda_N^3} S^\dagger S \psi_e^\dagger (\boldsymbol{D}^2 + \overleftarrow{\boldsymbol{D}}^2) \psi_e$$
$$+ \frac{g d_{10}}{\Lambda_N^4} S^\dagger S \; \psi_e^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{B} \psi_e + \frac{i d_{11}}{m_e \Lambda_N^3} \epsilon^{ijk} S^\dagger D_+^k S \; \psi_e^\dagger \sigma^i D_-^j \psi_e,$$

where $D_\pm = D \pm \overleftarrow{D}$

- Matching is yet to be done for $d_{>2}$
- We treat them as non-perturbative nuclear parameters

Step 2: Match NRQED onto QM

Match NRQED onto QM

- Match NRQED onto QM potentials, at leading power in $1/m_e$ and $M \to \infty$ limit by calculating NRQED amplitudes

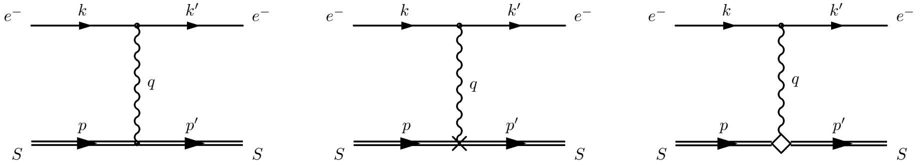
- Find three types of contributions

$$V(\boldsymbol{r}) = V_{\text{1-photon}}(\boldsymbol{r}) + V_{\text{contact}}(\boldsymbol{r}) + V_{\text{pol.}}(\boldsymbol{r})$$

- One-photon interactions

$$S^\dagger \left\{ iD_t + eZ\,\frac{\langle r^2 \rangle}{6}\,(\boldsymbol{\nabla}\cdot\boldsymbol{E}) + eZ\,\frac{\langle r^4 \rangle}{120}\,\boldsymbol{\nabla}^2(\boldsymbol{\nabla}\cdot\boldsymbol{E}) \right\} S$$

$V_{\text{1-photon}}(\boldsymbol{r})$



- Procedure: calculate $\mathcal{M}$, use $\tilde{V}(\boldsymbol{q}) = -\mathcal{M}$, take Fourier transform

$$i\mathcal{M}_Z = \frac{c}{q^2} \quad \Rightarrow \quad \tilde{V}_Z(\boldsymbol{q}) = \frac{-Ze^2}{\boldsymbol{q}^2} \quad \Rightarrow \quad V_Z(r) = -\frac{Z\alpha}{r},$$

$$i\mathcal{M}_{\langle r^2\rangle} = -\frac{c\langle r^2\rangle}{6q^2}\boldsymbol{q}^2 \quad \Rightarrow \quad \tilde{V}_{\langle r^2\rangle}(\boldsymbol{q}) = \frac{Ze^2}{6}\langle r^2\rangle \quad \Rightarrow \quad V_{\langle r^2\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{6}\langle r^2\rangle\,\delta^3(\boldsymbol{r}),$$

$$i\mathcal{M}_{\langle r^4\rangle} = \frac{c\langle r^4\rangle}{120q^2}(\boldsymbol{q}^2)^2 \quad \Rightarrow \quad \tilde{V}_{\langle r^4\rangle}(\boldsymbol{q}) = -\frac{Ze^2}{120}\langle r^4\rangle\boldsymbol{q}^2 \quad \Rightarrow \quad V_{\langle r^4\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{120}\langle r^4\rangle\,\nabla^2\left[\delta^3(\boldsymbol{r})\right]$$

- Coulomb potential, $V_Z(r)$, is solved exactly
  while $V_{\langle r^2\rangle}$ and $V_{\langle r^4\rangle}$ are treated as perturbations

$V_{\text{contact}}(\boldsymbol{r})$

- Recall that the four-fields contact interactions $M \to \infty$ limit are

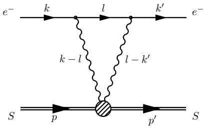
$$\mathcal{L}_{S\psi_e} = \frac{d_2 m_e}{\Lambda_N^3} S^\dagger S \psi_e^\dagger \psi_e + \frac{d_3}{\Lambda_N^4} S^\dagger D_+^i S \psi_e^\dagger D_+^i \psi_e + \frac{d_6}{m_e \Lambda_N^3} S^\dagger S \psi_e^\dagger (\boldsymbol{D}^2 + \overleftarrow{\boldsymbol{D}}^2)\psi_e$$
$$+ \frac{g d_{10}}{\Lambda_N^4} S^\dagger S \; \psi_e^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{B} \psi_e + \frac{i d_{11}}{m_e \Lambda_N^3} \epsilon^{ijk} S^\dagger D_+^k S \; \psi_e^\dagger \sigma^i D_-^j \psi_e,$$

- For the operator with the Wilson coefficient:
- d_{10} will only generate an energy shift when coupled to an external magnetic field
- d_{11} energy shift that depends on the electron spin for p levels
- d_3 use integration by parts

$$V_{d_2}(\boldsymbol{r}) = -\frac{d_2 m_e}{\Lambda_N^3} \delta^3(\boldsymbol{r}) \quad V_{d_3}(\boldsymbol{r}) = \frac{d_3}{\Lambda_N^4} \nabla^2 \left[\delta^3(\boldsymbol{r})\right]$$

$$V_{d_6}(\boldsymbol{r}) = -\frac{d_6}{m_e \Lambda_N^3} \left[\overleftarrow{\nabla}^2 \delta^3(\boldsymbol{r}) + \delta^3(\boldsymbol{r}) \nabla^2\right]$$

$V_{\text{pol.}}(\boldsymbol{r})$



- Electric (α_E) and magnetic (β_M) polarizability operators

$$S^\dagger \left\{ \frac{\alpha_E}{2} \boldsymbol{E}^2 + \frac{\beta_M}{2} \boldsymbol{B}^2 \right\} S$$

- Calculating the one-loop amplitude

$$\mathcal{M}_{\alpha_E,\beta_M} \approx \frac{\alpha}{8\pi} \left\{ (5\,\alpha_E - \beta_M)\, m_e \left[\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\,\pi\,\mu^2}{m_e^2} \right) \right] + 8\alpha_E m_e \right.$$
$$\left. + \pi^2\, i\, \alpha_E \sqrt{-\boldsymbol{q}^2 - i0} \right\} + \mathcal{O}\Big(1/m_e\Big)$$

- Constant first line $\Rightarrow \delta^3(\boldsymbol{r})$ potential, should be part of $S^\dagger S \psi_e^\dagger \psi_e$
- Second line non-analytic in $\boldsymbol{q}^2 \Rightarrow$ long-range potential

$$V_{\text{pol.}}(r) = -\frac{\alpha}{8\pi} \frac{\alpha_E}{r^4} + \mathcal{O}\left(1/m_e\right)$$

- First systematic derivation of the potential of polarizability amplitude

Step 3: Use QM potentials to find ΔE

List of potentials

- Atomic scale: a nuclear scale: r_N. Classify potentials by $\epsilon = r_N/a$ This is a rough scaling powers of Z are also important

$$\text{Order } \epsilon^2: \qquad V_{\langle r^2\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{6}\,\langle r^2\rangle\,\delta^3(\boldsymbol{r}),$$

$$\text{Order } \epsilon^3: \qquad V_{d_2}(\boldsymbol{r}) = -\frac{d_2 m_e}{\Lambda_N^3}\,\delta^3(\boldsymbol{r})$$

$$V_{\text{pol.}}(r) = -\frac{\alpha}{8\pi}\frac{\alpha_E}{r^4}$$

$$V_{d_6}(\boldsymbol{r}) = -\frac{d_6}{m_e\Lambda_N^3}\left[\overleftarrow{\nabla}^2\delta^3(\boldsymbol{r}) + \delta^3(\boldsymbol{r})\nabla^2\right]$$

$$\text{Order } \epsilon^4: \qquad V_{\langle r^4\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{120}\,\langle r^4\rangle\,\nabla^2\left[\delta^3(\boldsymbol{r})\right]$$

$$V_{d_3}(\boldsymbol{r}) = \frac{d_3}{\Lambda_N^4}\nabla^2\left[\delta^3(\boldsymbol{r})\right]$$

- Current level of experimental precision is ϵ^4

2^{nd}-order perturbation theory

- At current experimental precision for 2^{nd}-order PT only need $V_{\langle r^2\rangle} = 4\pi Z\alpha \langle r^2\rangle \delta^3(\boldsymbol{r})/6$ nonzero only for $s-$level:

$$\Delta E_n^{(2)} = \int d^3\boldsymbol{r}\, d^3\boldsymbol{r}'\phi_n(\boldsymbol{r})\, V_{\langle r^2\rangle}(\boldsymbol{r})\, \tilde{G}(\boldsymbol{r},\boldsymbol{r}',E_n)\; V_{\langle r^2\rangle}(\boldsymbol{r}')\phi_n(\boldsymbol{r}')$$
$$= \left(\frac{4\pi Z\alpha}{6}\right)^2 \langle r^2\rangle^2\, \phi_{n,0}(0)\, \tilde{G}(0,0,E_n)\, \phi_{n,0}(0),$$

where $\tilde{G}$ is the "reduced" Green's function of the system It is given by

$$\tilde{G} = G - \frac{1}{(E-E_n)}\phi_{n,0}(\boldsymbol{r})\phi_{n,0}(\boldsymbol{r}')$$

- Unlike 1^{st} order this does not depend just on perturbed energy level but the **entire** spectrum of the Hamiltonian. Very hard to calculate.

Phenomenological implications

King non-linearity

$$\nu_i^{AA'} = \overbrace{K_i \left(\frac{1}{m_A} - \frac{1}{m_{A'}}\right)}^{\text{Mass Shift}} + \overbrace{F_i \left(\langle r_A^2\rangle - \langle r_{A'}^2\rangle\right)}^{\text{Field Shift}} + \cdots$$

- Not every higher order effect leads to nonlinearity:

- Corrections of the form, e.g., $F_i \cdot (Z\alpha m_e)^2 \left(\langle r_A^2\rangle^2 - \langle r_{A'}^2\rangle^2\right)$ can be reabsorbed into the field shift via

$$\langle r^2\rangle \to \langle r^2\rangle + (Z\alpha m_e)^2 \langle r^2\rangle^2$$

- Corrections of the form, e.g., $G_i \cdot (Z\alpha)^2 \left(\langle r_A^2\rangle - \langle r_{A'}^2\rangle\right)$ can be reabsorbed into the field shift via

$$F_i \to F_i + G_i \cdot (Z\alpha)^2$$

- We need **different** atomic and nuclear quantities multiplied together
- Using the EFT results we can find the contributions to King NL

List of potentials

- Atomic scale: a nuclear scale: r_N. Classify potentials by $\epsilon = r_N/a$ This is a rough scaling powers of Z are also important

$$\text{Order } \epsilon^2: \qquad V_{\langle r^2\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{6}\langle r^2\rangle\,\delta^3(\boldsymbol{r}),$$

$$\text{Order } \epsilon^3: \qquad V_{d_2}(\boldsymbol{r}) = -\frac{d_2 m_e}{\Lambda_N^3}\,\delta^3(\boldsymbol{r})$$

$$V_{\text{pol.}}(r) = -\frac{\alpha}{8\pi}\frac{\alpha_E}{r^4}$$

$$V_{d_6}(\boldsymbol{r}) = -\frac{d_6}{m_e\Lambda_N^3}\left[\overleftarrow{\nabla}^2\delta^3(\boldsymbol{r}) + \delta^3(\boldsymbol{r})\nabla^2\right]$$

$$\text{Order } \epsilon^4: \qquad V_{\langle r^4\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{120}\langle r^4\rangle\,\nabla^2\left[\delta^3(\boldsymbol{r})\right]$$

$$V_{d_3}(\boldsymbol{r}) = \frac{d_3}{\Lambda_N^4}\nabla^2\left[\delta^3(\boldsymbol{r})\right]$$

- Current level of experimental precision is ϵ^4

⇒ Use 1st and 2nd order PT for $V_{\langle r^2\rangle}$ and 1st order for all others

List of potentials by interaction

- Use 1st and 2nd order PT for $V_{\langle r^2\rangle}$ and 1st order for all others

- $\delta^3(\boldsymbol{r}) \Rightarrow F_i\langle r^2\rangle,\, G_i^{(2)}\langle r^2\rangle^2$

$$V_{\langle r^2\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{6}\langle r^2\rangle\,\delta^3(\boldsymbol{r})$$

$$V_{d_2}(\boldsymbol{r}) = -\frac{d_2 m_e}{\Lambda_N^3}\,\delta^3(\boldsymbol{r})$$

- $\nabla^2\left[\delta^3(\boldsymbol{r})\right] \Rightarrow G_i^{(4)}\langle r^4\rangle$

$$V_{\langle r^4\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{120}\langle r^4\rangle\,\nabla^2\left[\delta^3(\boldsymbol{r})\right]$$

$$V_{d_3}(\boldsymbol{r}) = \frac{d_3}{\Lambda_N^4}\nabla^2\left[\delta^3(\boldsymbol{r})\right]$$

- $1/r^4 \Rightarrow P_i\alpha_E$

$$V_{\text{pol.}}(r) = -\frac{\alpha}{8\pi}\frac{\alpha_E}{r^4}$$

- The contribution proportional to d_6 is left for a future work

$$V_{d_6}(\boldsymbol{r}) = -\frac{d_6}{m_e\Lambda_N^3}\left[\overleftarrow{\nabla}^2\delta^3(\boldsymbol{r}) + \delta^3(\boldsymbol{r})\nabla^2\right]$$

King NL: Nuclear parameters

- Using the EFT results we can find the contributions to King NL

$$2\pi\nu_i^{AA'} = K_i\,\mu_{AA'} + F_i\left(\langle r_A^2\rangle_{\rm eff} - \langle r_{A'}^2\rangle_{\rm eff}\right) + G_i^{(2)}\left(\langle r_A^2\rangle^2 - \langle r_{A'}^2\rangle^2\right)$$
$$+\, G_i^{(4)}\left(\langle r_A^4\rangle_{\rm eff} - \langle r_{A'}^4\rangle_{\rm eff}\right) + P_i\left(\alpha_{E,A} - \alpha_{E,A'}\right) + \cdots\,,$$

- $\mu_{AA'} = 1/M_A - 1/M_{A'}$,
- We have redefined $\langle r^2\rangle$ and $\langle r^4\rangle$

$$\langle r^2\rangle_{\rm eff} = \langle r^2\rangle - \frac{3m_e}{2\pi Z\alpha}\frac{d_2}{\Lambda_N^3} + \langle r^2\rangle^2\left(\frac{2\pi Z\alpha}{3}\right)^2 \tilde{G}_Z^{n\text{-indep.}} +$$
$$+\,(m_e Z_{\rm eff}\alpha)^2\left[\frac{3}{10}\langle r^4\rangle - \frac{9}{\pi\alpha}\frac{d_3}{\Lambda_N^4}\right]$$

$$\langle r^4\rangle_{\rm eff} = \langle r^4\rangle + \frac{30}{\pi Z\alpha}\frac{d_3}{\Lambda_N^4}.$$

- The contribution proportional to d_6 is left for a future work

King NL: electronic coefficient

- Using the EFT results we can find the contributions to King NL

$$2\pi\nu_i^{AA'} = K_i\,\mu_{AA'} + F_i\left(\langle r_A^2\rangle_{\rm eff} - \langle r_{A'}^2\rangle_{\rm eff}\right) + G_i^{(2)}\left(\langle r_A^2\rangle^2 - \langle r_{A'}^2\rangle^2\right) + G_i^{(4)}\left(\langle r_A^4\rangle_{\rm eff} - \langle r_{A'}^4\rangle_{\rm eff}\right) + P_i\big(\alpha_{E,A} - \alpha_{E,A'}\big) + \cdots,$$

- K_i is the electronic coefficient multiplying the mass shift
- $F_i = F_{n,\ell} - F_{n',\ell'}$, $G_i^{(2/4)} = G_{n,\ell}^{(2/4)} - G_{n',\ell'}^{(2/4)}$, and $P_i = P_{n,\ell} - P_{n',\ell'}$

- For $V(r) = Z_{\rm eff}\,\alpha/r$ we have $\phi'_{n,0}(0) = -m_e Z_{\rm eff}\,\alpha\,\phi_{n,0}(0)$ so that

$$F_{n,\ell} = \frac{4\pi Z\alpha}{6}\,|\phi_{n,\ell}(0)|^2\;\delta_{\ell 0},$$

$$G_{n,\ell}^{(2)} = \left(\frac{4\pi Z\alpha}{6}\right)^2 \tilde{G}_{Z_{\rm eff}}^{n\text{-dep.}}(0,0,E_n)\;|\phi_{n,\ell}(0)|^2\;\delta_{\ell 0},$$

$$G_{n,\ell}^{(4)} = \begin{cases} \dfrac{4\pi Z\alpha}{20}\phi_{n,0}(0)\phi''_{n,0}(0), & \ell = 0, \\ \dfrac{4\pi Z\alpha}{80\pi}\left[R'_{n1}(0)\right]^2, & \ell = 1, \\ 0, & \ell > 1 \end{cases}$$

King NL: electronic coefficient: Hydrogen-like system

- Using the EFT results we can find the contributions to King NL

$$2\pi\nu_i^{AA'} = K_i\,\mu_{AA'} + F_i\left(\langle r_A^2\rangle_{\text{eff}} - \langle r_{A'}^2\rangle_{\text{eff}}\right) + G_i^{(2)}\left(\langle r_A^2\rangle^2 - \langle r_{A'}^2\rangle^2\right) + G_i^{(4)}\left(\langle r_A^4\rangle_{\text{eff}} - \langle r_{A'}^4\rangle_{\text{eff}}\right) + P_i\left(\alpha_{E,A} - \alpha_{E,A'}\right) + \cdots,$$

$$F_{n,\ell} = \frac{2(Z\alpha)(Z_{\text{eff}}\alpha m_e)^3}{3n^3}\,\delta_{\ell 0},$$

$$G_{n,\ell}^{(2)} = \frac{4(Z\alpha)(Z_{\text{eff}}\alpha m_e)^5}{9n^3}\left[\psi(n) - \frac{1}{n} - \ln n\right]\delta_{\ell 0},$$

$$G_{n,\ell}^{(4)} = \begin{cases} \dfrac{(Z\alpha)(Z_{\text{eff}}\alpha m_e)^5}{15n^5}, & \ell = 0, \\ -\dfrac{(Z\alpha)(Z_{\text{eff}}\alpha m_e)^5}{45n^5}, & \ell = 1, \\ 0, & \ell > 1, \end{cases}$$

$$P_{n,\ell} = \begin{cases} -\dfrac{\alpha\,\alpha_E}{8\pi}\,\dfrac{8(Z_{\text{eff}}\alpha m_e)^4}{n^3}\left\{\ln\left(\dfrac{2Z_{\text{eff}}\alpha m_e}{n}\right) + \gamma_E + C_n\right\}, & \ell = 0, \\ -\dfrac{\alpha\,\alpha_E(Z_{\text{eff}}\alpha m_e)^4}{16\pi\, n^5}\,\dfrac{3\,n^2 - \ell(\ell+1)}{\ell(\ell+1)(\ell+3/2)(\ell^2-1/4)}, & \ell \neq 0. \end{cases}$$

King non-linearity

	H-like; $A' = A+2$	$^{40,42}\mathrm{Ca}^{19+}$	$^{42,44}\mathrm{Ca}^{19+}$	$^{44,46}\mathrm{Ca}^{19+}$
$\nu^{AA'}_{\alpha=3S\to 2P}$	$4.6\times 10^2\, Z^2$ THz	1.8×10^5 THz	1.8×10^5 THz	1.8×10^5 THz
$\nu^{AA'}_{\beta=4D\to 3P}$	$1.6\times 10^2\, Z^2$ THz	6.4×10^4 THz	6.4×10^4 THz	6.4×10^4 THz
$\nu^{AA'}_{\gamma=3D\to 2S}$	$4.6\times 10^2\, Z^2$ THz	1.8×10^5 THz	1.8×10^5 THz	1.8×10^5 THz
$K_\alpha\, \mu_{AA'}/h$	$5.0\times 10^2\, Z^2/A^2$ GHz	1.2×10^2 GHz	1.1×10^2 GHz	1.0×10^2 GHz
$K_\beta/\, \mu_{AA'}/h$	$1.7\times 10^2\, Z^2/A^2$ GHz	4.4×10^1 GHz	3.9×10^1 GHz	3.6×10^1 GHz
$K_\gamma\, \mu_{AA'}/h$	$5.0\times 10^2\, Z^2/A^2$ GHz	1.2×10^2 GHz	1.1×10^2 GHz	1.0×10^2 GHz
$F_\alpha\, \langle r^2\rangle_{AA'}/h$	$8.5\times 10^{-1}\, Z^6/A^{1/3}$ Hz	2.9 GHz	2.8 GHz	2.8 GHz
$F_\beta\, \langle r^2\rangle_{AA'}/h$	0	0	0	0
$F_\gamma\, \langle r^2\rangle_{AA'}/h$	$-2.3\, Z^6/A^{1/3}$ Hz	-9.6 GHz	-9.5 GHz	-9.4 GHz
$G^{(2)}_\alpha\, \langle r^2\rangle^2_{AA'}/h$	$-1.2\times 10^{-5}\, Z^6\, A^{1/3}$ Hz	-2.6 kHz	-2.6 kHz	-2.7 kHz
$G^{(2)}_\beta\, \langle r^2\rangle^2_{AA'}/h$	0	0	0	0
$G^{(2)}_\gamma\, \langle r^2\rangle^2_{AA'}/h$	$6.0\times 10^{-5}\, Z^6\, A^{1/3}$ Hz	1.3×10^1 kHz	1.3×10^1 kHz	1.4×10^1 kHz
$G^{(4)}_\alpha\, \langle r^4\rangle_{AA'}/h$	$1.4\times 10^{-6}\, Z^6\, A^{1/3}$ Hz	3.6×10^{-1} kHz	3.6×10^{-1} kHz	3.7×10^{-1} kHz
$G^{(4)}_\beta\, \langle r^4\rangle_{AA'}/h$	$1.3\times 10^{-7}\, Z^6\, A^{1/3}$ Hz	3.3×10^{-2} kHz	3.4×10^{-2} kHz	3.5×10^{-1} kHz
$G^{(4)}_\gamma\, \langle r^4\rangle_{AA'}/h$	$-2.9\times 10^{-7}\, Z^6\, A^{1/3}$ Hz	-7.7×10^{-1} kHz	-7.8×10^{-1} kHz	-7.9×10^{-1} kHz
$P_\alpha\, \alpha_E^{AA'}/h$	$-5.8\times 10^{-1}\, Z^4\, A^{2/3}$ Hz	-1.7 MHz	-1.7 MHz	-1.8 MHz
$P_\beta\, \alpha_E^{AA'}/h$	$2.3\times 10^{-3}\, Z^4\, A^{2/3}$ Hz	6.8×10^{-2} MHz	6.9×10^{-2} MHz	7.1×10^{-2} MHz
$P_\gamma\, \alpha_E^{AA'}/h$	$2.0\, Z^4\, A^{2/3}$ Hz	5.8 MHz	5.9 MHz	6.0 MHz

Table 1. Numerical example of isotope shift frequency differences for several transitions, α: $3S \to 2P$, β: $4D \to 3P$, and γ: $3D \to 2S$, in hydrogen-like systems, taking $A' = A+2$ (2nd column) and for three isotope combinations for the case of Ca^{19+}. Different rows show either either the total frequency shift (with h being the Planck's constant), or different contributions to it as in (5.4), where we shortened $\langle r^2\rangle_{AA'} = \langle r^2\rangle_A - \langle r^2\rangle_{A'}$, etc.

Comparison to treatment of $\langle r^2\rangle^2$ in literature

- Recall that $G_i^{(2)}$, the electronic coefficient of $\langle r^2\rangle^2$ depend on the **entire** spectrum of the Hamiltonian. Very hard to calculate.

- Approximate treatments were used in the literature:
  - [Counts et al., PRL **125** 123002 (2020)]
  - [Hur et al., PRL **128** 163201 (2022)]

- For hydrogen-like systems we can test these approximations
  - These approximations are **highly** dependent on how the other moments of classical nuclear charge density are treated
  - One approximation is off by 20%, since it includes terms that can be absorbed into F_i
  - Another approximation has the right magnitude but the opposite sign

- Conclusion: current treatment of $\langle r^2\rangle^2$ term needs to be reconsidered

Conclusions and outlook

Conclusions

- Isotope shift is used to look for new physics via King Non-linearity

- Allows to go beyond the charge radius and study nuclei structure at unprecedented precision

- Such non-linearity was observed in experiments
  but its interpretation is obscured by SM effects

- Presented an EFT approach:
- Match SM onto NRQED
- Match NRQED onto QM
- Use QM potentials to find energy-level shifts

- Applied this to find a general expression for King NL
  and compare and test other more phenomenological approaches

Future directions

- Incorporate EFT approach in multi-electron effects
- Systematic discussion of renormalization
- Matching of d_6
- Extend the formalism to $1/M$
- More work to do

Thank you!

Backup

Renormalization

1^{st} order perturbation theory

- For a potential proportional to $\delta^3(\boldsymbol{r})$ ΔE is easy, e.g.

$$V_{\langle r^2\rangle}(\boldsymbol{r}) = \frac{4\pi Z\alpha}{6}\langle r^2\rangle\,\delta^3(\boldsymbol{r}) \quad\Rightarrow\quad \Delta E^{s-\text{level}}_{\langle r^2\rangle} = \frac{4\pi Z\alpha}{6}\langle r^2\rangle\,|\phi(0)|^2$$

$$V_{d_2}(\boldsymbol{r}) = -\frac{d_2 m_e}{\Lambda_N^3}\,\delta^3(\boldsymbol{r}) \quad\Rightarrow\quad \Delta E^{s-\text{level}}_{d_2} = \frac{d_2 m_e}{\Lambda_N^3}\,|\phi(0)|^2$$

- For potentials that include $\nabla^2[\delta^3(\boldsymbol{r})]$ require the integral

$$\int d^3\boldsymbol{r}\,\phi^*(\boldsymbol{r})\left[\nabla^2\delta^3(\boldsymbol{r})\right]\phi(\boldsymbol{r}) = \int d^3\boldsymbol{r}\,\nabla^2|\phi(\boldsymbol{r})|^2\,\delta^3(\boldsymbol{r}) = \nabla^2|\phi(0)|^2$$

$$= \left[\nabla^2\phi(0)^*\right]\phi(0) + 2\,\boldsymbol{\nabla}\phi(0)^*\cdot\boldsymbol{\nabla}\phi(0) + \phi^*(0)\left[\nabla^2\phi(0)\right]$$

- For s-levels this gives $r\to 0$ divergence [Lepage '97, Stryker '15]

$$\nabla^2|\phi(\boldsymbol{r})|^2\Big|_{r\to 0} = 6\phi(0)\phi''(0) + 6\left[\phi'(0)\right]^2 + \frac{4\phi(0)\phi'(0)}{r}$$

1st order perturbation theory

- For s-levels this gives $r \to 0$ divergence [Lepage '97, Stryker '15]

$$\nabla^2 |\phi(\boldsymbol{r})|^2 \Big|_{r\to 0} = 6\phi(0)\phi''(0) + 6\left[\phi'(0)\right]^2 + \frac{4\phi(0)\phi'(0)}{r}$$

- Origin of the divergent piece is clear:
  For a perturbation $\delta V(\boldsymbol{r})$ the integral $\int d^3\boldsymbol{r}\, \phi^*(\boldsymbol{r})\delta V(\boldsymbol{r})\phi(\boldsymbol{r})$ is finite
  Calculating the integral in momentum space by regions gives

$$\text{Finite} = \text{High momentum IR div.} \quad + \quad \underset{\substack{\uparrow \\ \text{EFT piece}}}{\text{Low momentum UV div.}}$$

- Renormalization is left for a future study
- For now we just note them and drop the divergent piece

Quantum Mechanics

Quantum Mechanics

- For hydrogen-like atoms corrections up to order $(Z\alpha)^6$ were calculated in [J. L. Friar, Annals Phys. 122, 151 (1979)]

$$\Delta E_n = \frac{2\pi}{3}|\phi_n(0)|^2 Z\alpha \left[\langle r^2\rangle - \frac{Z\alpha m_r}{2}\langle r^3\rangle_2 + (Z\alpha m_r)^2 F_{\rm NR} + (Z\alpha)^2 F_{\rm REL}\right]$$

$$\begin{aligned} F_{\rm NR} &= \frac{2}{3}\left[\psi(n) - \frac{1}{n} - \ln n\right]\langle r^2\rangle^2 + \frac{1}{10n^2}\langle r^4\rangle + \\ &+ \frac{2}{3}\langle r^2\rangle^2\left[2\gamma - \frac{4}{3}\right] + \frac{2}{3}\langle r^2\rangle\langle r^2\ln(2Z\alpha m_r r)\rangle \\ &+ \langle r^3\rangle\langle r\rangle + \frac{1}{9}\langle r^5\rangle\langle\frac{1}{r}\rangle + I_2^{\rm NR} + I_3^{\rm NR}\end{aligned}$$

$$\begin{aligned} F_{\rm REL} &= -\langle r^2\rangle\left[\psi(n) + 2\gamma + \frac{9}{4n^2} - \frac{1}{n} - \frac{13}{4} + \ln(2Z\alpha m_r r) - \ln n\right] \\ &- \frac{1}{3}\langle r^3\rangle\langle\frac{1}{r}\rangle + I_2^{\rm REL} + I_3^{\rm REL}\end{aligned}$$

- Only the terms in blue contribute to king non-linearity

Comparison to Friar's calculation

$$2\pi\nu_i^{AA'} = K_i\,\mu_{AA'} + F_i\left(\langle r_A^2\rangle_{\rm eff} - \langle r_{A'}^2\rangle_{\rm eff}\right) + G_i^{(2)}\left(\langle r_A^2\rangle^2 - \langle r_{A'}^2\rangle^2\right)$$
$$+\,G_i^{(4)}\left(\langle r_A^4\rangle_{\rm eff} - \langle r_{A'}^4\rangle_{\rm eff}\right) + P_i\big(\alpha_{E,A} - \alpha_{E,A'}\big) + \cdots\,,$$

- We can compare our results to the purely QM calculation of nuclear finite size effects by Friar [Friar, Annals Phys. 122, 151 (1979)]
- Friar calculated nuclear finite-size effects for *s*-levels in hydrogen-like atoms using QM PT assuming nuclear effects are described by a classical charge distribution ρ
- The conclusions are

- $G_{n,\ell}^{(2)}$ and $G_{n,\ell}^{(4)}$ agree
- $F_{n,\ell}$ agree up to relativistic corrections irrelevant to King NL
- $\langle r^2\rangle_{\rm eff}$ and $\langle r^4\rangle_{\rm eff}$ differ since Friar does not have inelastic/contact pieces but does have would-be two-loop matching coefficients
- Friar does not have polarizability