Mass radius and D-term of atomic nuclei

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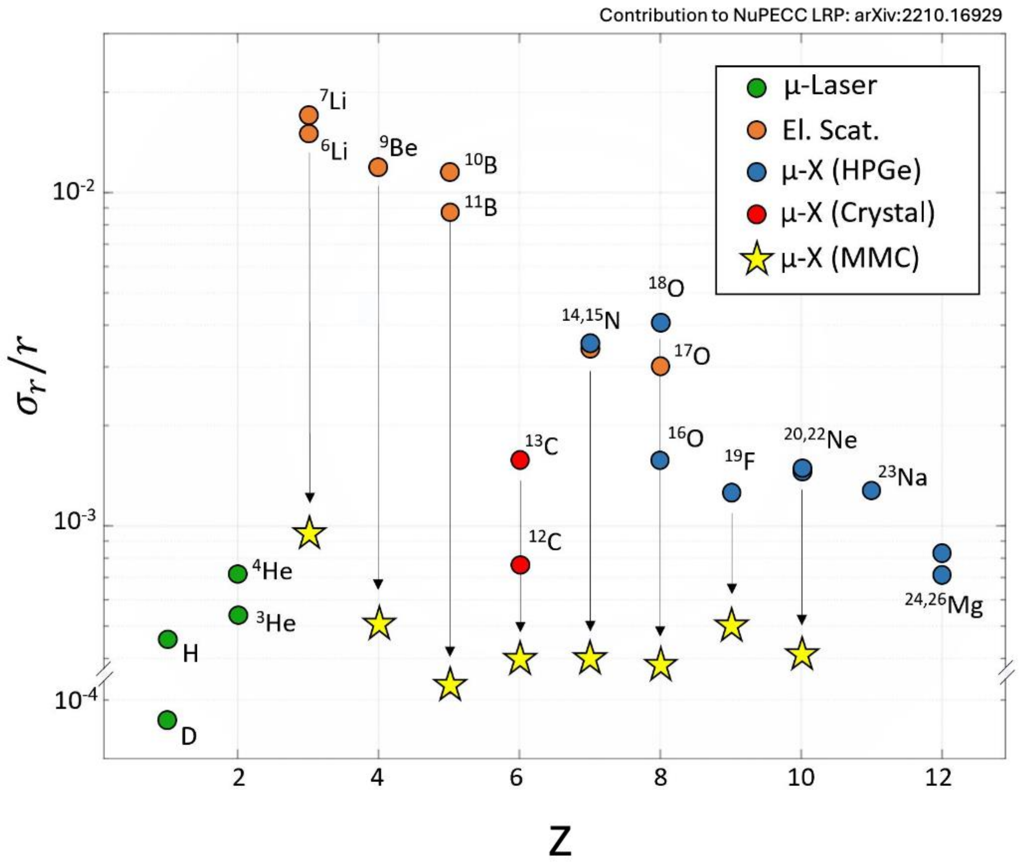
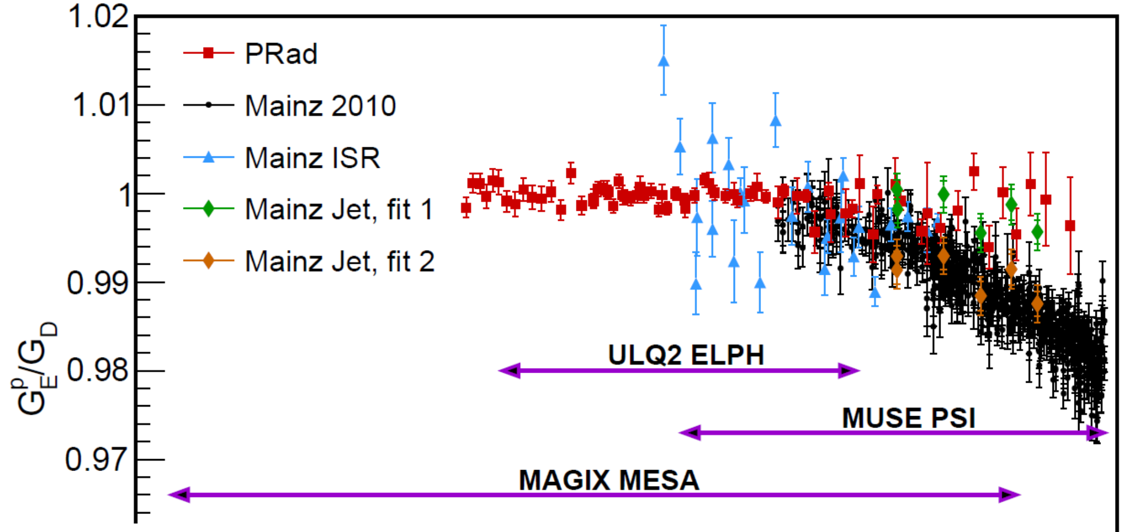
with Alberto Martin-Caro and Miguel Huidobro, 2304.05994; 2312.12984
with Tomohiro Oishi and Makoto Oka, in preparation

Proton and nuclear charge radius

70 years of experimental/theoretical effort,
(Sub-)Percent-level accuracy is the norm





PHYSICAL REVIEW LETTERS **135,** 163002 (2025)

clear Charge Radius Revisited: Closing the Fine-Structure-Anomaly Gap

Zewen Sun, Konstantin A. Beyer, Zoia A. Mandrykina, Igor A. Valuev,
Christoph H. Keitel, and Natalia S. Oreshkina*

the remaining discrepancy to theory errors which cannot be rigorously
ns charge radius with reduced model dependence to be 5.5062(17) fm. This
ichmark for charge radius extraction in heavy nuclei and paves a path for
oss the nuclear chart.

Radius zoo

2312.12984

Charge radius

$$\langle r^2\rangle_c = \frac{\int d\boldsymbol{x} x^2 \rho_c(\boldsymbol{x})}{\int d\boldsymbol{x} \rho_c(\boldsymbol{x})} = \frac{6}{G_E(0)} \frac{dG_E(t)}{dt}\bigg|_{t=0}$$

Magnetic radius

$$\langle r^2\rangle_M = \frac{6}{G_M(0)} \frac{dG_M(t)}{dt}\bigg|_{t=0}$$

Baryon number radius

$$\langle r^2\rangle_B = \frac{\int d\boldsymbol{x} x^2 \rho_B(\boldsymbol{x})}{\int d\boldsymbol{x} \rho_B(\boldsymbol{x})}$$

Mass radius

$$\langle r^2\rangle_m = \frac{\int d\boldsymbol{x}\, x^2 T^{00}(\boldsymbol{x})}{\int d\boldsymbol{x} T^{00}(\boldsymbol{x})} = 6\frac{dA(t)}{dt}\bigg|_{t=0} - \frac{3D(0)}{2M^2}$$

Scalar radius

$$\langle r^2\rangle_s = \frac{\int d\boldsymbol{x}\, x^2 T^{\mu}_{\mu}(\boldsymbol{x})}{\int d\boldsymbol{x} T^{\mu}_{\mu}(\boldsymbol{x})} = 6\frac{dA(t)}{dt}\bigg|_{t=0} - \frac{9D(0)}{2M^2}$$

Tensor radius

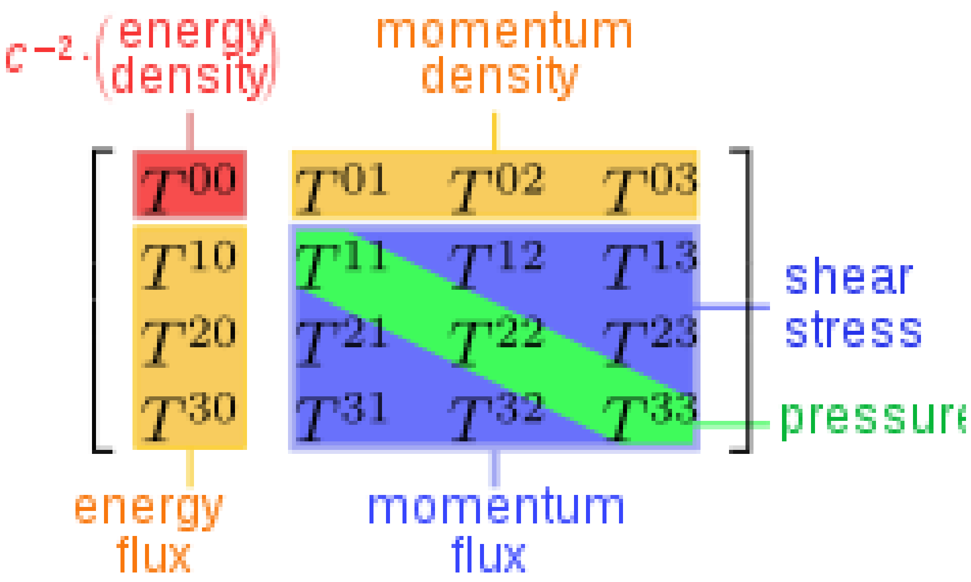
$$\langle r^2\rangle_t \equiv \frac{\int d\boldsymbol{x}\, x^2 \left(T^{00}(\boldsymbol{x}) + \frac{1}{2}T_{ii}(\boldsymbol{x})\right)}{\int d\boldsymbol{x} \left(T^{00} + \frac{1}{2}T_{ii}\right)} = 6\frac{dA(t)}{dt}\bigg|_{t=0}$$

Mechanical radius

$$\langle r^2\rangle_{mech} = \frac{\int d\boldsymbol{x} x^2 \frac{x_i x_j}{x^2} T_{ij}(\boldsymbol{x})}{\int d\boldsymbol{x} \frac{x_i x_j}{x^2} T_{ij}(\boldsymbol{x})} = \frac{6D(0)}{\int_{-\infty}^{0} dt D(t)}$$

…

Gravitational form factors



QCD energy momentum tensor

$$T^{\mu\nu} = \sum_f \bar{\psi}_f \gamma^{(\mu} iD^{\nu)} \psi_f - F^{\mu\rho} F^{\nu}{}_{\rho} + \frac{g^{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}$$

Associated form factors

$$\bar{P} = \frac{P+P'}{2}, \quad \Delta = P' - P$$

$$\langle P' | T^{\mu\nu} | P \rangle = \bar{u}(P') \left[A(t)\gamma^{(\mu}\bar{P}^{\nu)} + B(t)\frac{\bar{P}^{(\mu} i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{4M} \right] u(P)$$

$$A(0) = 1, \quad B(0) = 0 \qquad D(0) = ??$$

D-term: the last global unknown

$D(0)$ is a fundamental constant of the proton!

The value, even the sign, is unknown at the moment.

Spatial components of the energy momentum tensor
→ May be interpreted as internal `force' exerted by quarks and gluons

$$T^{ij}(\boldsymbol{r}) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right)s(r) + \delta^{ij}\, p(r) \qquad D = M\int d^3r r^2 p(r)$$

Conjecture: Stable systems must have a negative D-term $D(t=0) < 0$

D-term of atomic nuclei

Liquid drop model Polyakov (2003)

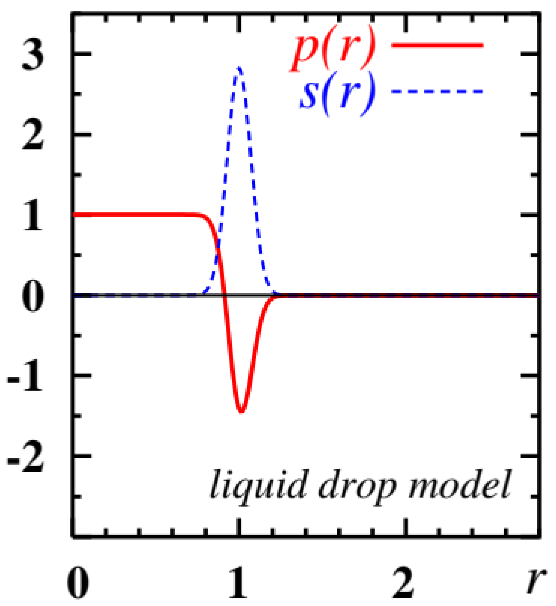
$$D=-\frac{4}{5}\frac{4\pi}{3}M\gamma R^4 \propto A^{7/3}$$

Microscopic approach Liuti, Taneja (2005)

Walecka model Guzey, Siddikov (2006)

Deuteron (light-front approach) Freese, Cosyn (2022)

Deuteron, helium (potential model) He, Zahed (2023)



Sensitive to the surface region of nucleus.

This talk:

Systematic study in the low-mass region A<10 --> Skyrme model

high-mass region A>10--> Relativistic mean field theory

The Skyrme model (1961)

One of the oldest, most well-known and phenomenologically successful models of the nucleon

$$\mathcal{L}_{\rm SK} = -\frac{f_\pi^2}{16}\mathrm{Tr}\{L_\mu L^\mu\} + \frac{1}{32e^2}\mathrm{Tr}\{[L_\mu, L_\nu]^2\} - \lambda^2\pi^4 B_\mu B^\mu + \frac{m_\pi^2 f_\pi^2}{8}\mathrm{Tr}\{U - \mathbf{1}\},$$

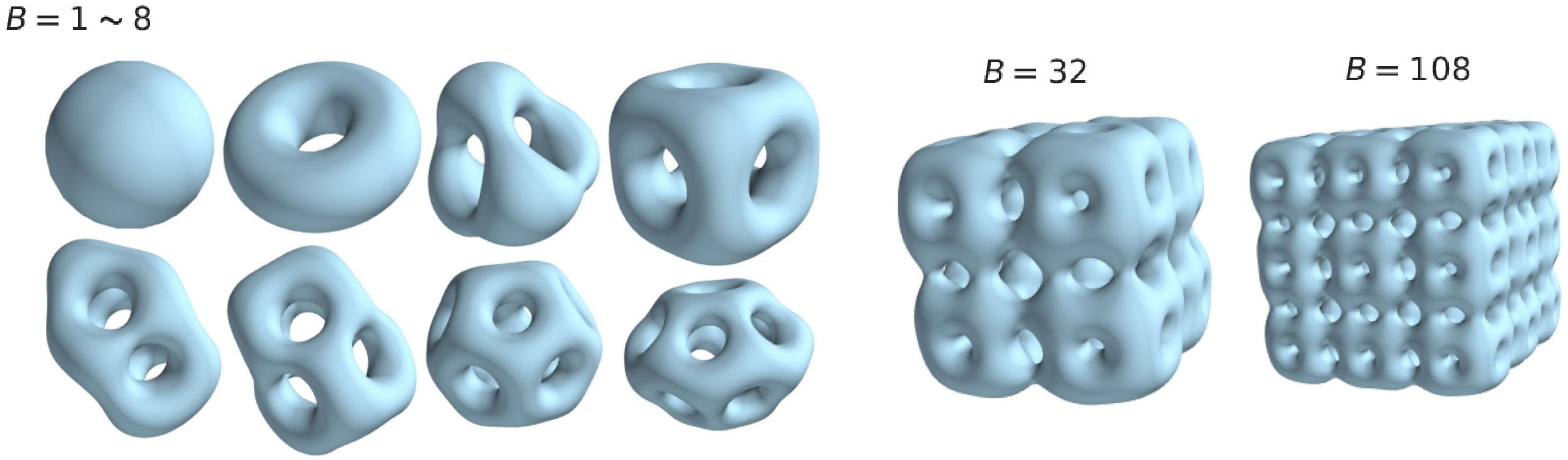
$$L_\mu = U^\dagger \partial_\mu U$$

$$B^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}}{24\pi^2}\mathrm{Tr}\{L_\nu L_\rho L_\sigma\}$$

Nucleon: Classical configuration (`hedgehog') with a definite baryon number B=1
Quantization of the collective coordinates → nucleon resonances

Gravitational form factors of the B=1 solution (nucleon) Cebulla, Goeke, Ossmann, Schweitzer (2007)

Nuclei in the Skyrme model



For more exotic solutions, see Gudnason, Halcrow (2022)

Electromagnetic form factors computed

$B=2$	deuteron	Braaten, Carson (1989)
$B=3$	triton, helium-3	Carson (1991)

Obtain classical solution that minimizes energy in each topological sector

$$U_0(\mathbf{x}) \qquad T_{ij}(U_0(\mathbf{x}))$$

Quantization

$$U(t,\boldsymbol{x}) = A(t)U_0(R_B(t)(\boldsymbol{x}-\boldsymbol{X}(t)))A^\dagger(t)$$

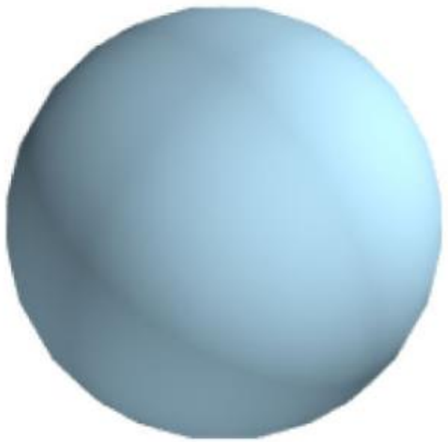
$$\langle \boldsymbol{q}/2|T_{ij}[U(R(B)(\boldsymbol{x}-\boldsymbol{X}))]|-\boldsymbol{q}/2\rangle = e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}R^T_{ia}(B)R^T_{jb}(B)\int d^3x' e^{i\boldsymbol{q}\cdot R^T(B)\boldsymbol{x}'}T^{cl}_{ab}(\boldsymbol{x}')+\mathcal{O}(I^2,J^2)$$

Neglected in this work

Partial wave decomposition

$$\exp\{i\boldsymbol{q}\cdot R^T(B)\boldsymbol{x}\} = j_0(qx) + iq_c R^T_{ck}(B)x_k\frac{3j_1(qx)}{qx} - \frac{1}{2}\left(q_cq_d - \frac{1}{3}\delta_{cd}q^2\right)R^T_{ck}(B)R^T_{dl}(B)\left(x_kx_l - \frac{1}{3}\delta_{kl}x^2\right)\frac{15j_2(qx)}{(qx)^2} + \cdots$$

B=1, nucleon



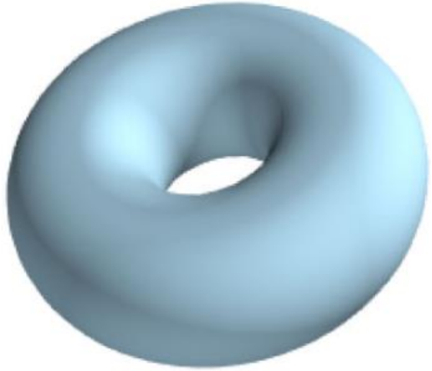
$$T_{abkl} = \int d^3\boldsymbol{x} \left(x_k x_l - \frac{1}{3}\delta_{kl} x^2 \right) \frac{15 j_2(qx)}{(qx)^2} T^{cl}_{ab}(\boldsymbol{x})$$
$$= \frac{1}{10} \left(\delta_{ak}\delta_{bl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl} \right) T_{cdcd}.$$

Classical solution spherically symmetric

R -matrices drop out. Classical formula recovered.

$$D(t) = -6M_N \int d^3x \left(x^i x^j - \frac{1}{3}\delta^{ij} x^2 \right) \frac{j_2(qx)}{(qx)^2} T^{\text{cl}}_{ij}(\boldsymbol{x})$$

B=2, deuteron



There are three D-terms for a spin-1/2 hadron.

$$\langle p'\sigma'|T_{ij}|p\sigma\rangle = \frac{1}{2}(q_i q_j - \delta_{ij}q^2)\mathcal{D}_1(t)\epsilon^*_{\sigma'}\cdot\epsilon_\sigma + \left(q_j q_k Q_{ik} + q_i q_k Q_{jk} - q^2 Q_{ij} - \delta_{ij} q_k q_l Q_{kl}\right)_{\sigma'\sigma}\mathcal{D}_2(t) + \frac{1}{2M_D^2}(q_i q_j - \delta_{ij}q^2) q_k q_l Q_{kl,\sigma'\sigma}\mathcal{D}_3(t) + \cdots$$

Only one remains after averaging over spins.

Classical configurations with quadrupole deformation Polyakov, Sun (2019)

$$T^{\mathrm{cl}}_{ij} \approx Y_2^{ij} s(x) + p(x)\delta_{ij} + 2s'(x)(Q_{ik}Y_2^{kj} + Q_{jk}Y_2^{ki} - \delta_{ij}Q_{ab}Y_2^{ab}) + p'(x)Q_{ij} - \frac{1}{M_D^2}Q^{kl}\partial_k\partial_l(p''(x)\delta^{ij} + s''(x)Y_2^{ij})$$

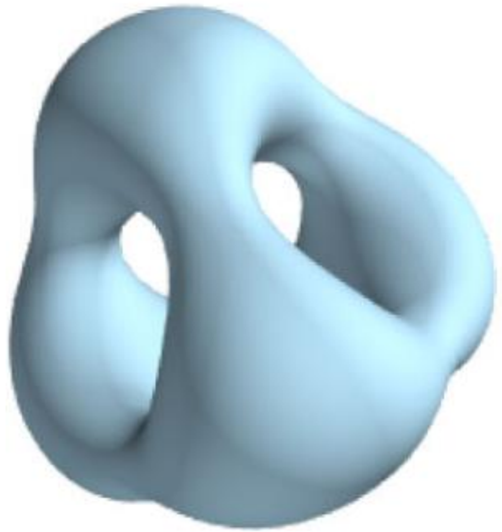
$$Y_2^{ij} = \frac{x_i x_j}{x^2} - \frac{\delta_{ij}}{3}$$

$$D(t) = -6M_N \int d^3x \left(x^i x^j - \frac{1}{3}\delta^{ij}x^2\right)\frac{j_2(qx)}{(qx)^2}T^{\mathrm{cl}}_{ij}(\boldsymbol{x})$$

The monopole part given by the same formula

All form factors computed in Cosyn, Freese, Sosa (2026)

B=3, triton and helium-3



Spin ½, only one D-term, just like the nucleon

But the classical solution is not spherically symmetric.

Yet, the solution is highly symmetric.
Discrete transformations form the tetrahedral group T_d

$$\int d^3x T^{\rm cl}_{ab}(\boldsymbol{x}) j_0(qx) \propto \delta_{ab}$$

Group theory helps! cf. Carson (1991)
Decompose tensors into the irreducible representations of T_d

$$T_1 \times T_1 = A_1 + E + T_1 + T_2$$

Rank-4 tensor associated with the D-term

$$T_{abkl} = \int d^3\boldsymbol{x} \left(x_k x_l - \frac{1}{3}\delta_{kl} x^2 \right) \frac{15 j_2(qx)}{(qx)^2} T^{\mathrm{cl}}_{ab}(\boldsymbol{x})$$

$$= \frac{1}{10}\left(\delta_{ak}\delta_{bl} + \delta_{al}\delta_{bk} - \frac{2}{3}\delta_{ab}\delta_{kl} \right) T_{cdcd} + \underline{C_{abkl}}$$

totally-symmetric, traceless

$$\langle \boldsymbol{q}/2 | T_{ij}(-R\boldsymbol{X}) | -\boldsymbol{q}/2 \rangle |_{l=2}$$

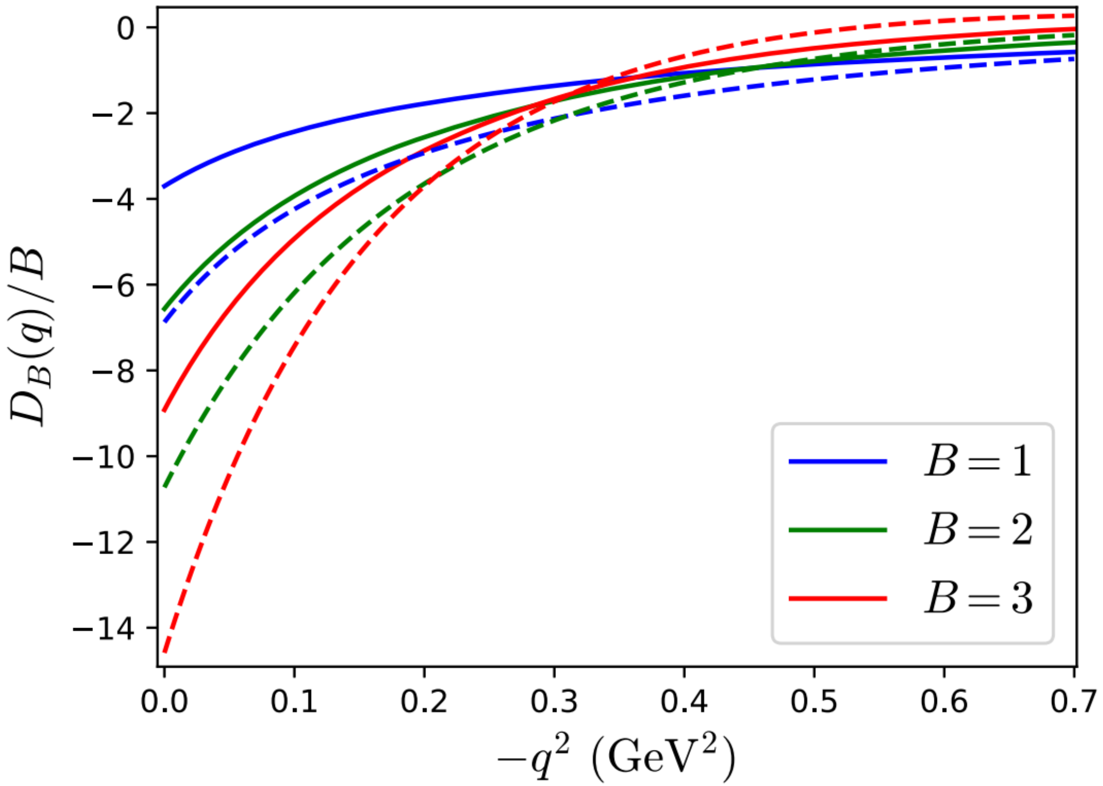
$$= -\left(q_i q_j - \frac{\delta_{ij}}{3} q^2 \right) \frac{T_{cdcd}}{10} - \frac{1}{2}\left(q_c q_d - \frac{1}{3}\delta_{cd} q^2 \right) \underline{R^T_{ia}(B) R^T_{jb}(B) R^T_{ck}(B) R^T_{dl}(B) C_{ab,kl}}$$

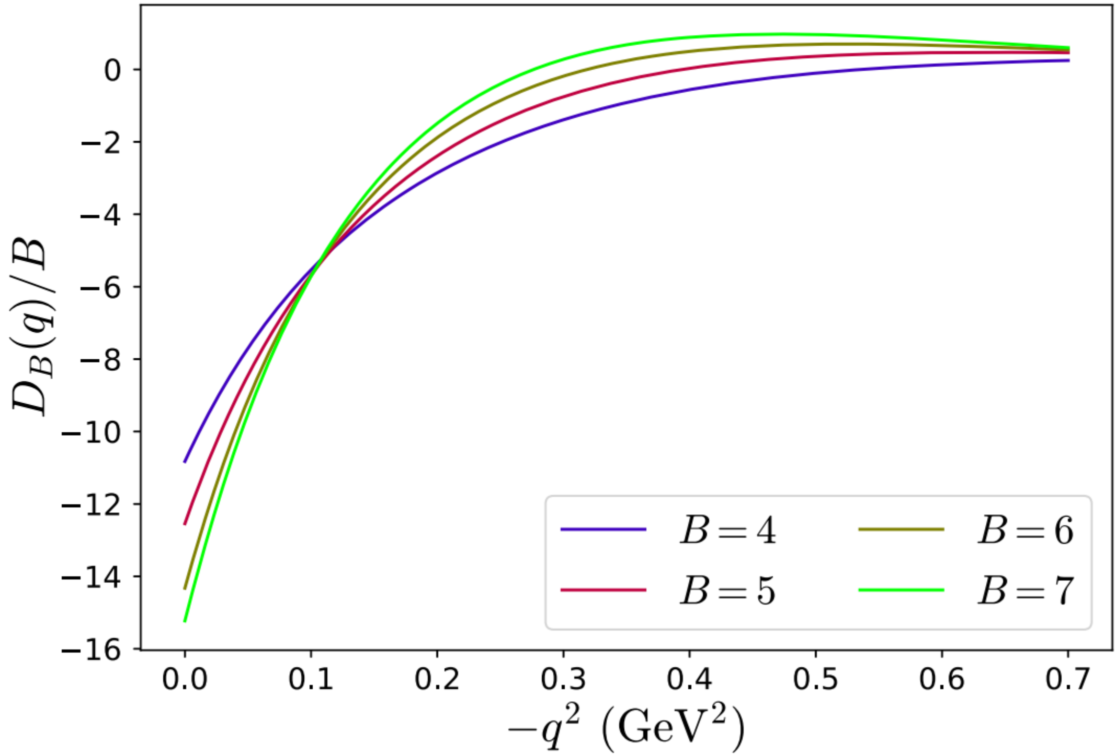
Spin-4 operator, vanishes when evaluated in spin-1/2 states

$$D(t) = -6 M_N \int d^3x \left(x^i x^j - \frac{1}{3}\delta^{ij} x^2 \right) \frac{j_2(qx)}{(qx)^2} T^{\mathrm{cl}}_{ij}(\boldsymbol{x})$$

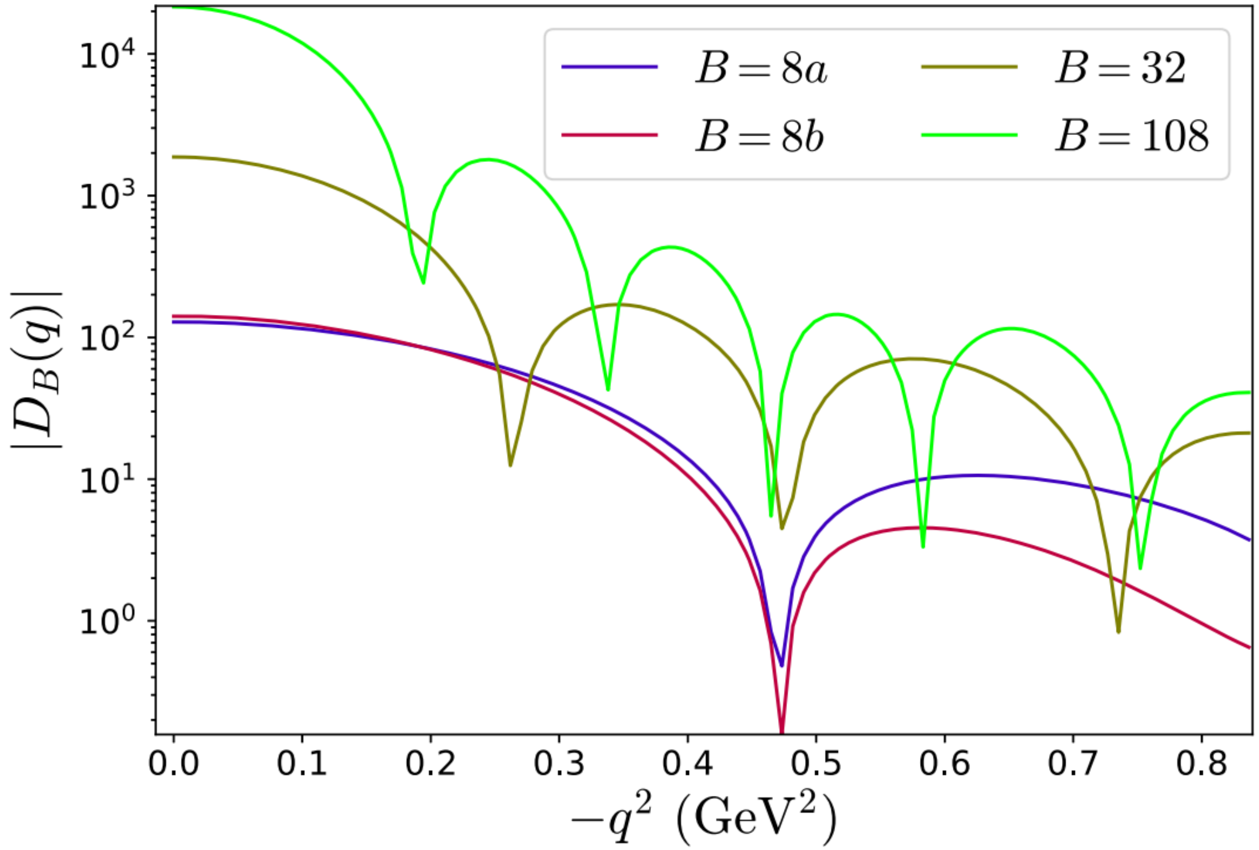
The same formula as for the B=1 solution.

Results





The dashed lines include the sextic term $\Delta\mathcal{L} \sim B_\mu B^\mu$ $\qquad B^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}}{24\pi^2}\mathrm{Tr}\{L_\nu L_\rho L_\sigma\}$



The form factor changes signs, oscillates around zero for large-B nuclei.

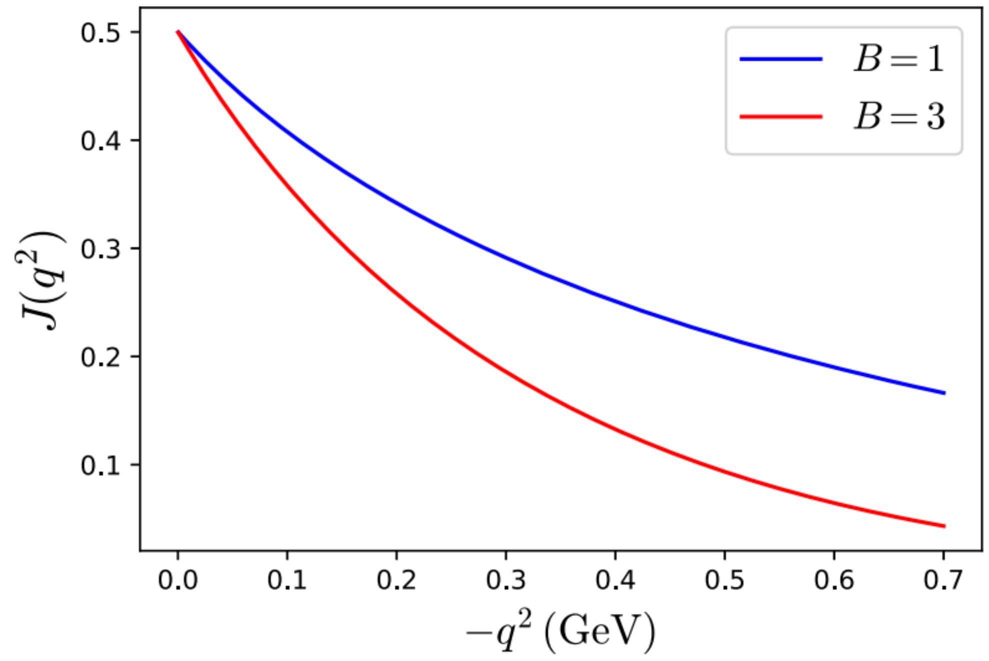
Similar to the diffractive pattern in elastic scattering off nuclei.

B	1	2	3	4	5	6	7	$8a$	$8b$	32	108
$D(0)$	−3.701	−13.126	−26.757	−43.304	−62.72	−85.95	−106.596	−128.368	−140.816	-1.874×10^3	-2.152×10^4

The value D(0) grows quickly with increasing B

^{3}He angular-momentum form factor $J(t) = \frac{1}{2}(A(t) + B(t))$

$$\frac{\langle p', s'|T_{0i}(0)|p, s\rangle}{2P^0} = -J(t) i\epsilon_{ijk} \frac{\tau^j_{s's}}{2} q^k$$



Off-diagonal components vanish for classical configuration → quantization crucial

Straightforward for B=1 Cebulla, et al.
Nontrivial for B=3

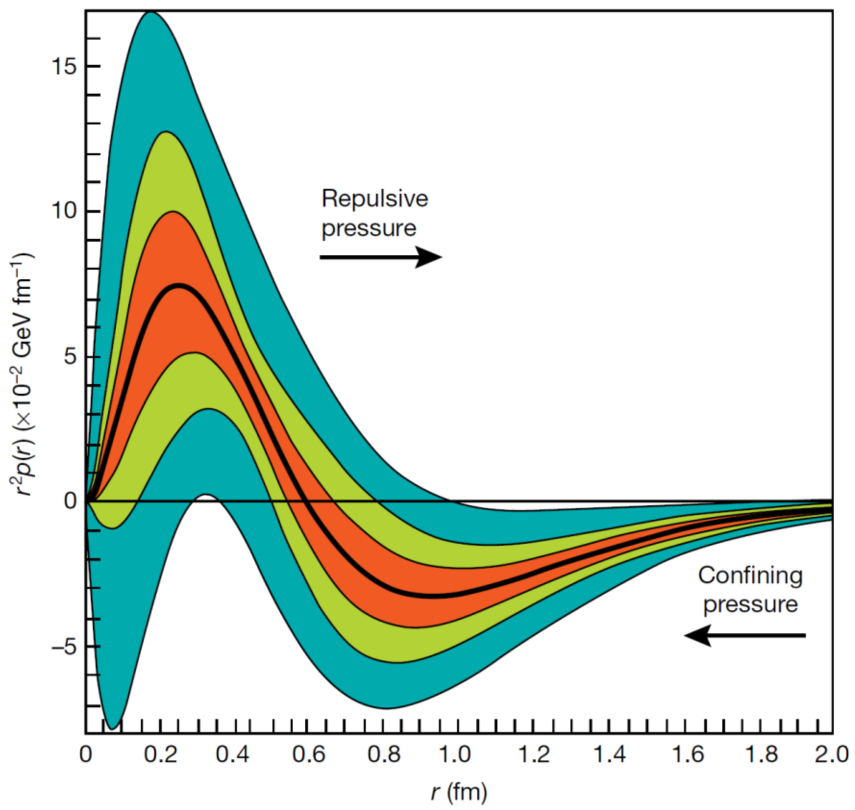
Again, group theory helps!

$$\langle \boldsymbol{q}/2|T_{0i}(-R\boldsymbol{X})| - \boldsymbol{q}/2\rangle|_{l=1} = iR^T_{ij}(B)R^T_{kl}(B)q^k \int d^3x \frac{3j_1(qx)}{qx}[I_{jm}a^m - J_{jm}b^m]x^l$$

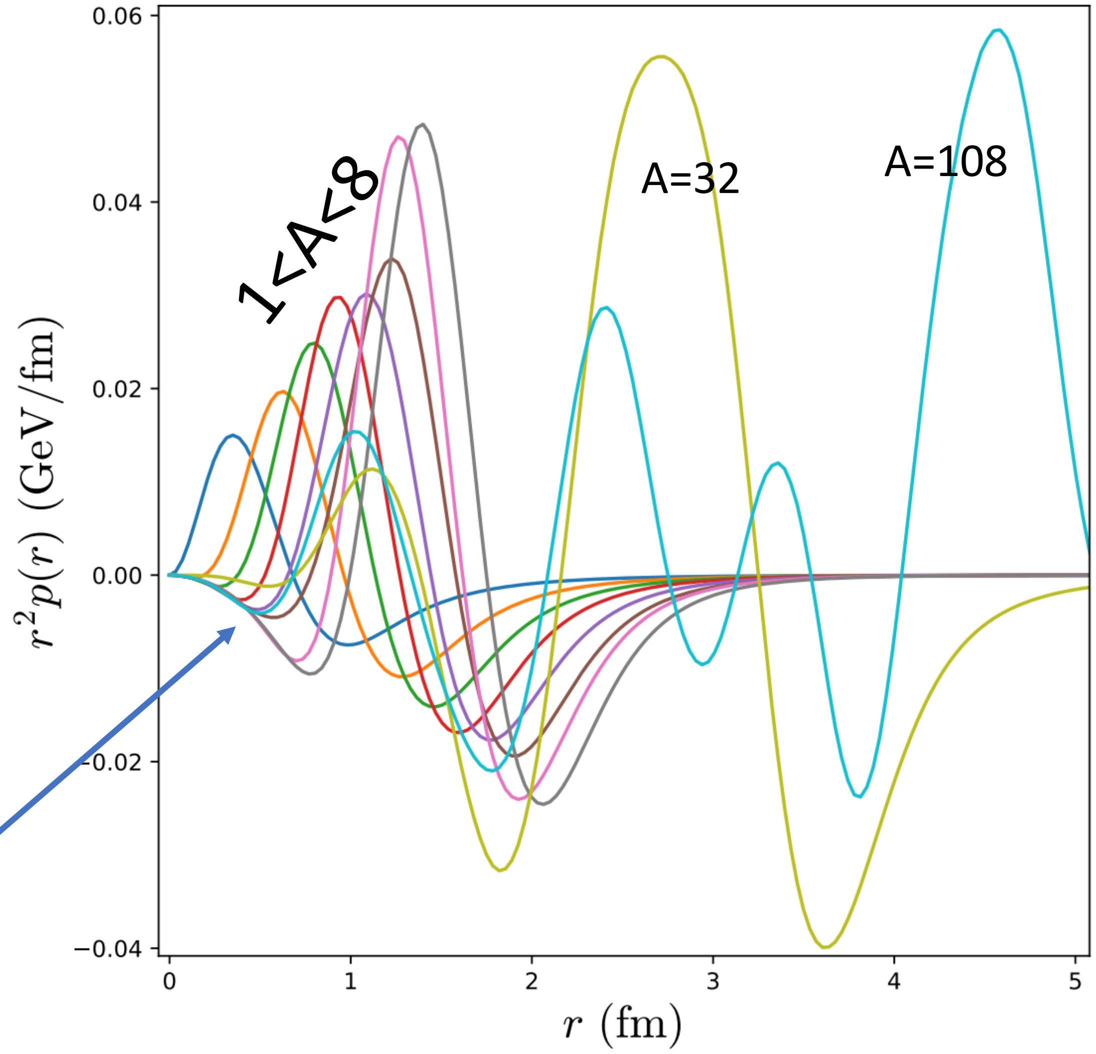
$$A_1 \in T_2 \times T_1 \times T_1$$

`Pressure' inside nucleon and nuclei

Burkert, Elouadrhiri, Girod (2018)



Martin-Caro, Huidobro, YH, 2312.12984



Negative pressure near the core for nuclei A>1
see also, Freese, Cosyn (2022), He, Zahed (2023)

Nuclear mass radius in relativistic mean field theory

YH, Oishi, Oka, in preparation

$$
\begin{aligned}
\mathcal{L} \;=\; & \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{m_\omega^2}{2}\omega_\mu\omega^\mu - g_\omega\omega_\mu\bar{\psi}\gamma^\mu\psi \\
& -\frac{1}{4}\vec{R}_{\mu\nu}\cdot\vec{R}^{\mu\nu} + \frac{m_\rho^2}{2}\vec{\rho}_\mu\cdot\vec{\rho}^\mu - g_\rho\vec{\rho}_\mu\cdot\bar{\psi}\gamma^\mu\vec{\tau}\psi \\
& +\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{m_\sigma^2}{2}\sigma^2 - g_\sigma\sigma\bar{\psi}\psi \\
& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\frac{1+\tau^3}{2}\psi.
\end{aligned}
$$

One of the most successful, time-tested models of atomic nuclei with A>16

Reproduce binding energies and charge radii with percent-level accuracy

Relativistic Lagrangian formulation --> Ideal for GFF studies.

Earlier work on Walecka model: Guzey, Siddikov (2005)

In this work we use the `DIRHB' code by Niksic, Paar, Vretenar, Ring (2014)

Dirac equation for the nucleons

$$i\partial\!\!\!/\psi - (m + g_\sigma\sigma)\psi - \left(g_\omega\omega_\mu + g_\rho\vec{\rho}_\mu\cdot\vec{\tau} + \Sigma_\mu^R + eA_\mu\frac{1+\tau^3}{2}\right)\gamma^\mu\psi = 0;$$

Poisson equation for the mesons

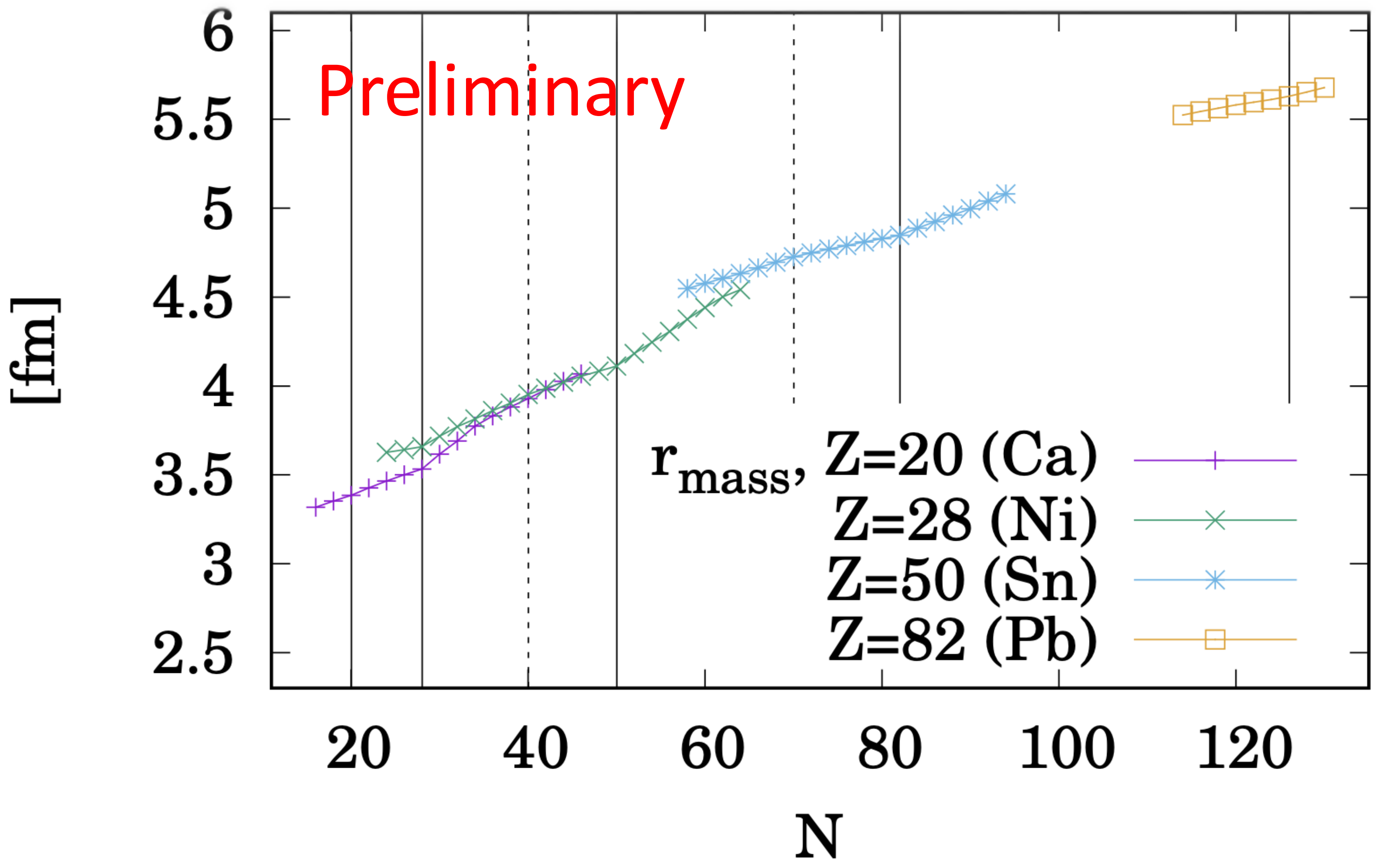
$$(\vec{\nabla}^2 - m_\sigma^2)\sigma(r) = g_\sigma\rho_s(r);$$
$$(\vec{\nabla}^2 - m_\omega^2)\omega(r) = -g_\omega\rho_v(r),$$

Shear density

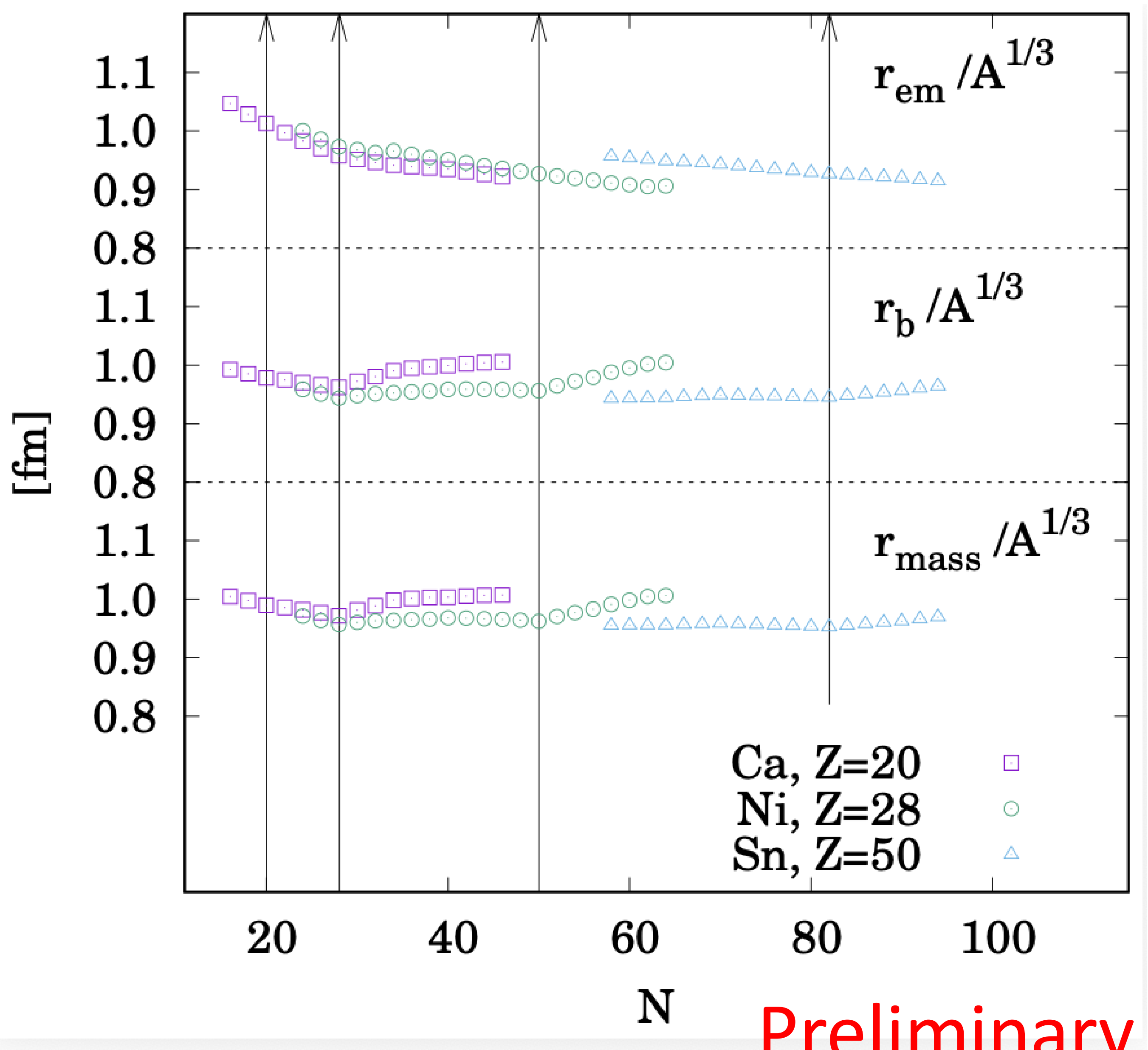
$$s(r) \equiv \frac{3}{2}\left(\frac{r^i r^j}{r^2} - \frac{\delta^{ij}}{3}\right)T_{ij} \;=\; -\frac{1}{4\pi}\sum_{a=1}^{A}\left[f_a(r)\frac{dg_a(r)}{dr} - \frac{df_a(r)}{dr}g_a(r) + \kappa_a\frac{f_a(r)g_a(r)}{r}\right] + \left(\frac{d\sigma}{dr}\right)^2 - \left(\frac{d\omega}{dr}\right)^2 - \left(\frac{d\rho}{dr}\right)^2 - \left(\frac{d\phi}{dr}\right)^2,$$

Mass radius of nuclear isotopes

YH, Oishi, Oka, in preparation



Kinks at neutron magic numbers, reflecting nuclear shell structure!

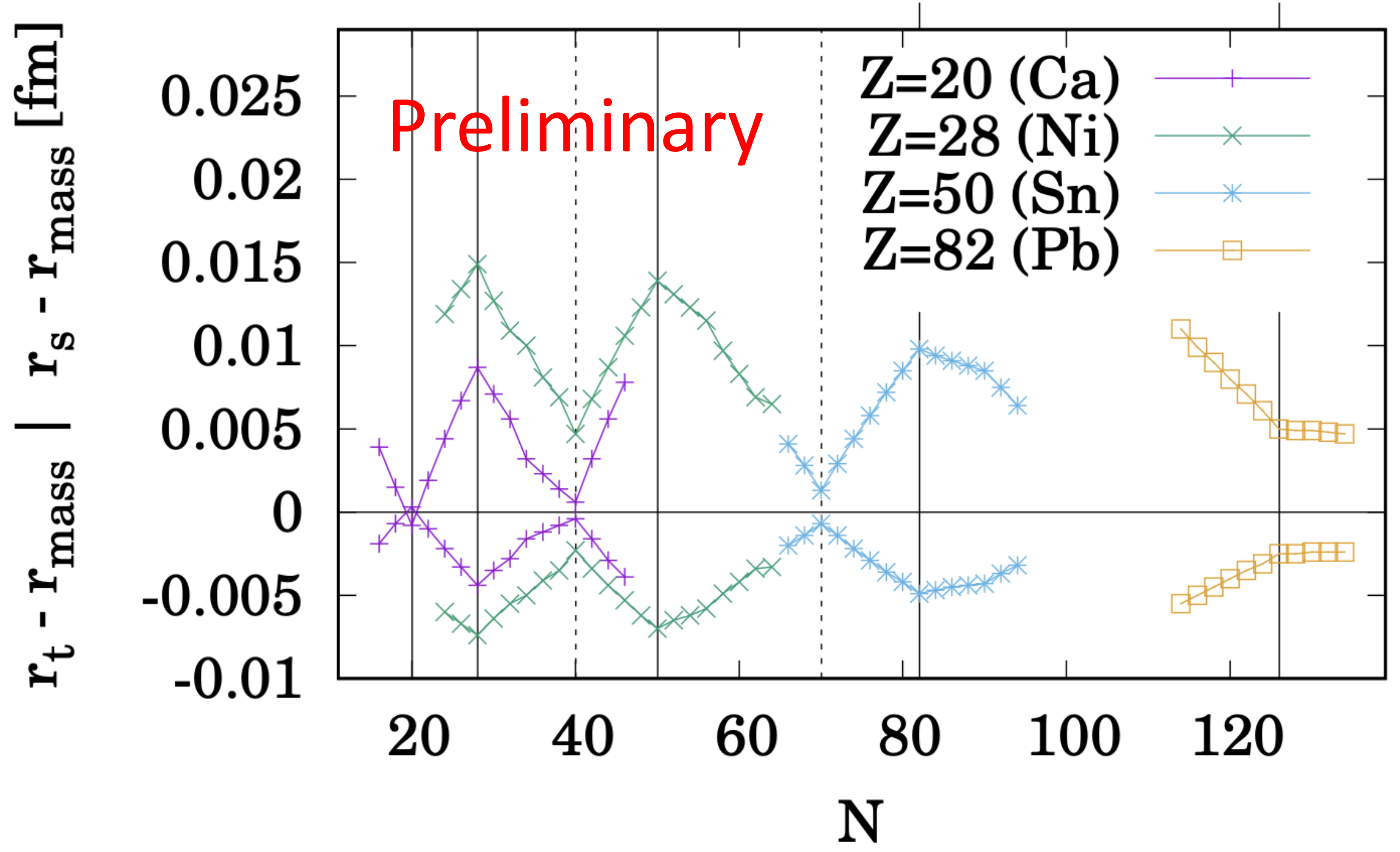


Preliminary

Scales with A^1/3 in neutron-rich region

More sensitive to nuclear shell structure than charge radius

Mass radius, tensor radius, scalar radius



$$\langle r^2 \rangle_s = \langle r^2 \rangle_m - \frac{3D(0)}{M^2}$$

$$\langle r^2 \rangle_t = \langle r^2 \rangle_m + \frac{3D}{2M^2}$$

Conclusions

- Charge radius: 70 years of history, sub-percent precision
- Radii associated with energy momentum tensor: lagging far behind, but interest growing.
- Nuclear mass radius: Terra incognita. Challenge for nuclear theory tools?