Interpolating Instant and Light-Front Dynamics

Chueng-Ryong Ji
North Carolina State University

Light Cone 2026
22nd - 26th June 2026
CFNS, Stony Brook University

Center for Frontiers
in Nuclear Science

June 22, 2026

Outline

- Conformal Symmetry Nature of Spacetime
- 1+1 Dimensional Conformal Algebra
- Interpolation between IFD and LFD
- Orientation Entanglement in Relativistic Helicity
- Critical Point Appearance between IFD and LFD
- ‘t Hooft model as a toy QCD
- Link to the Light-Front Quark Model
- Phenomenological Application of LFQM
- Conclusions and Outlook

Conformal Transformation of Spacetime

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}g'_{\alpha\beta} = \Lambda(x)g_{\mu\nu}$$

Conformal Generators

$$\begin{aligned}
P_{\mu} &= i\partial_{\mu} && \text{(translation)}\\
M_{\mu\nu} &= i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) && \text{(rotation)}\\
D &= ix_{\mu}\partial^{\mu} && \text{(dilation)}\\
\mathfrak{K}_{\mu} &= i\left(2x_{\mu}x_{\nu}\partial^{\nu} - x^{2}\partial_{\mu}\right) && \text{(SCT)}
\end{aligned}$$

In 0-space and 1-time $(0+1)$ dimension,

Möbius transformation $x_0 \to x_0' = \dfrac{\alpha x_0 + \beta}{\gamma x_0 + \delta}$, **Det** $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ **= 1**

$$P_0^{(0+1)} = i\sigma_- = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix},$$

$$D_0^{(0+1)} = \frac{i}{2}\sigma_3 = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix},$$

$$\mathfrak{K}_0^{(0+1)} = -i\sigma_+ = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix},$$

where $\sigma_\pm = \frac{\sigma_1 \pm i\sigma_2}{2}$.

V. de Alfaro, S. Fubini, and G. Furlan, Conformal invariance in quantum mechanics, Il Nuovo Cimento A (1965-1970) **34**, 569 (1976).

$$P_0^{(0+1)} = i\partial_0,$$

$$D_0^{(0+1)} = ix_0\partial_0,$$

$$\mathfrak{K}_0^{(0+1)} = ix_0x_0\partial_0.$$

Anti de-Sitter space (AdS_2)

S. J. Brodsky, G. F. de Téramond, H. G. Dosch, and J. Erlich, Light-front holographic QCD and emerging confinement, Phys. Rep. **584**, 1 (2015).

In 1-space and 0-time (1+0) dimension,

Möbius transformation $x_3 \rightarrow x_3' = \dfrac{\alpha x_3 + \beta}{\gamma x_3 + \delta}$, **Det** $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ **= 1**

C.Ji & H. Ravikumar, PRD113, 096018 (2026).

$$P_3^{(1+0)} = i\sigma_- = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix},$$

$$D_3^{(1+0)} = \frac{i}{2}\sigma_3 = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix},$$

$$\mathfrak{K}_3^{(1+0)} = +i\sigma_+ = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix},$$

$$P_3^{(1+0)} = i\frac{(a^\dagger - a)}{\sqrt{2}},$$

$$D_3^{(1+0)} = \frac{-i}{2}(a^{\dagger 2} - a^2 + 1),$$

$$\mathfrak{K}_3^{(1+0)} = \frac{-i}{2\sqrt{2}}(3a + 3a^\dagger + a^{\dagger 2}a - a^3 + a^{\dagger 3} - a^2 a^\dagger).$$

$$P_3^{(1+0)} = i\partial_3,$$

$$D_3^{(1+0)} = -ix_3\partial_3.$$

$$\mathfrak{K}_3^{(1+0)} = -ix_3x_3\partial_3,$$

$$i\sigma_- \leftrightarrow i\,\frac{(a^\dagger - a)}{\sqrt{2}},$$

$$i\sigma_+ \leftrightarrow \frac{-i}{2\sqrt{2}}(3a + 3a^\dagger + a^{\dagger 2}a - a^3 + a^{\dagger 3} - a^2 a^\dagger),$$

$$\frac{i}{2}\sigma_3 \leftrightarrow \frac{-i}{2}(a^{\dagger 2} - a^2 + 1).$$

$0 + 1$ conformal algebra

	$\mathfrak{K}_0^{(0+1)}$	$P_0^{(0+1)}$	$D_0^{(0+1)}$
$\mathfrak{K}_0^{(0+1)}$	0	$-2iD_0^{(0+1)}$	$-i\mathfrak{K}_0^{(0+1)}$
$P_0^{(0+1)}$	$2iD_0^{(0+1)}$	0	$iP_0^{(0+1)}$
$D_0^{(0+1)}$	$i\mathfrak{K}_0^{(0+1)}$	$-iP_0^{(0+1)}$	0

$1 + 0$ conformal algebra

	$\mathfrak{K}_3^{(1+0)}$	$P_3^{(1+0)}$	$D_3^{(1+0)}$
$\mathfrak{K}_3^{(1+0)}$	0	$2iD_3^{(1+0)}$	$-i\mathfrak{K}_3^{(1+0)}$
$P_3^{(1+0)}$	$-2iD_3^{(1+0)}$	0	$iP_3^{(1+0)}$
$D_3^{(1+0)}$	$i\mathfrak{K}_3^{(1+0)}$	$-iP_3^{(1+0)}$	0

CONFORMAL ALGEBRA IN (1+1) DIMENSIONS

$$P_0^{(1+1)} = i\partial_0,$$

$$\mathfrak{K}_0^{(1+1)} = i(x_0 x_0 \partial_0 - 2x_3 x_0 \partial_3 + x_3 x_3 \partial_0),$$

$$D^{(1+1)} = i x_0 \partial_0 - i x_3 \partial_3,$$

$$P_3^{(1+1)} = i\partial_3,$$

$$\mathfrak{K}_3^{(1+1)} = -i(x_3 x_3 \partial_3 - 2x_0 x_3 \partial_0 + x_0 x_0 \partial_3),$$

$$K_3^{(1+1)} = -i(x_3 \partial_0 - x_0 \partial_3).$$

$$P_0^{(1+1)} = P_0^{(0+1)},$$

$$D^{(1+1)} = D_0^{(0+1)} + D_3^{(1+0)},$$

$$P_3^{(1+1)} = P_3^{(1+0)}.$$

$$K_3^{(1+1)} = -i(x_3\partial_0 - x_0\partial_3).$$

$$\mathfrak{K}_0^{(1+1)} = \mathfrak{K}_0^{(0+1)} - x_3 K_3^{(1+1)} - x_3 x_0 P_3^{(1+0)},$$

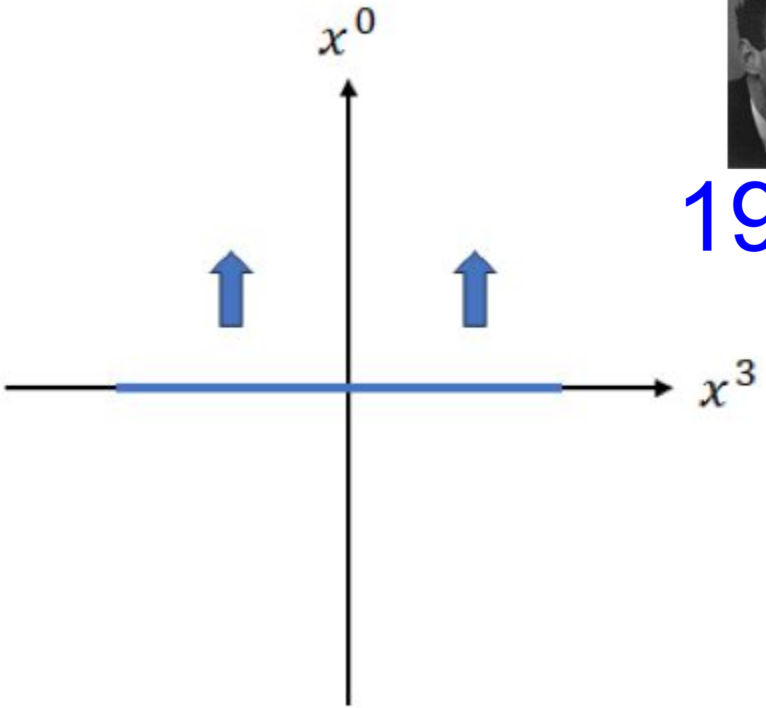
$$\mathfrak{K}_3^{(1+1)} = \mathfrak{K}_3^{(1+0)} - x_0 K_3^{(1+1)} + x_3 x_0 P_0^{(0+1)},$$

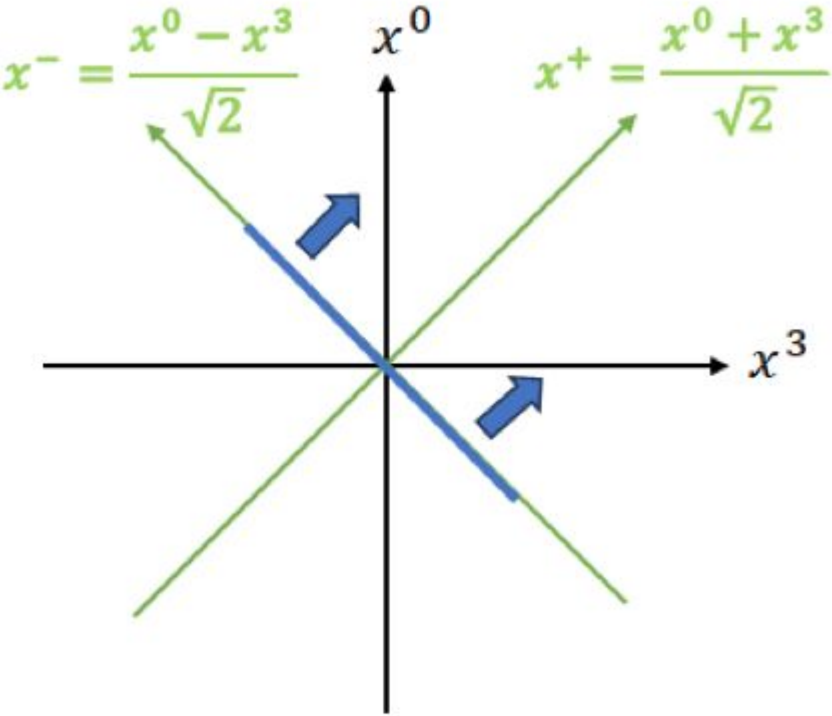
$1+1$ conformal algebra.

	$\mathfrak{K}_0$	P_0	D	$\mathfrak{K}_3$	P_3	K_3
$\mathfrak{K}_0$	0	$-2iD$	$-i\mathfrak{K}_0$	0	$-2iK_3$	$i\mathfrak{K}_3$
P_0	$2iD$	0	iP_0	$-2iK_3$	0	iP_3
D	$i\mathfrak{K}_0$	$-iP_0$	0	$i\mathfrak{K}_3$	$-iP_3$	0
$\mathfrak{K}_3$	0	$2iK_3$	$-i\mathfrak{K}_3$	0	$2iD$	$i\mathfrak{K}_0$
P_3	$2iK_3$	0	iP_3	$-2iD$	0	iP_0
K_3	$-i\mathfrak{K}_3$	$-iP_3$	0	$-i\mathfrak{K}_0$	$-iP_0$	0

Dirac’s Proposition for Relativistic Dynamics

1949





IFD

Instant Form Dynamics

LFD

Light-Front Dynamics

LFD

$P_\pm = \frac{P_0 \pm P_3}{\sqrt{2}}$, $\mathfrak{K}_\pm = \frac{\mathfrak{K}_0 \mp \mathfrak{K}_3}{\sqrt{2}}$, and $D_\pm = \frac{D \mp K^3}{\sqrt{2}}$

	$\mathfrak{K}_+$	P_+	D_+	$\mathfrak{K}_-$	P_-	D_-
$\mathfrak{K}_+$	0	$-2\sqrt{2}iD_+$	$-\sqrt{2}i\mathfrak{K}_+$	0	0	0
P_+	$2\sqrt{2}iD_+$	0	$\sqrt{2}iP_+$	0	0	0
D_+	$\sqrt{2}i\mathfrak{K}_+$	$-\sqrt{2}iP_+$	0	0	0	0
$\mathfrak{K}_-$	0	0	0	0	$-2\sqrt{2}iD_-$	$-\sqrt{2}i\mathfrak{K}_-$
P_-	0	0	0	$2\sqrt{2}iD_-$	0	$\sqrt{2}iP_-$
D_-	0	0	0	$\sqrt{2}i\mathfrak{K}_-$	$-\sqrt{2}iP_-$	0

$$P_\mu = i\partial_\mu \quad \text{(translation)}$$
$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \quad \text{(rotation)}$$
$$D = ix_\mu\partial^\mu \quad \text{(dilation)}$$
$$\mathfrak{K}_\mu = i(2x_\mu x_\nu\partial^\nu - x^2\partial_\mu) \quad \text{(SCT)}$$

C.Ji & H. Ravikumar, PRD113, 096018 (2026)

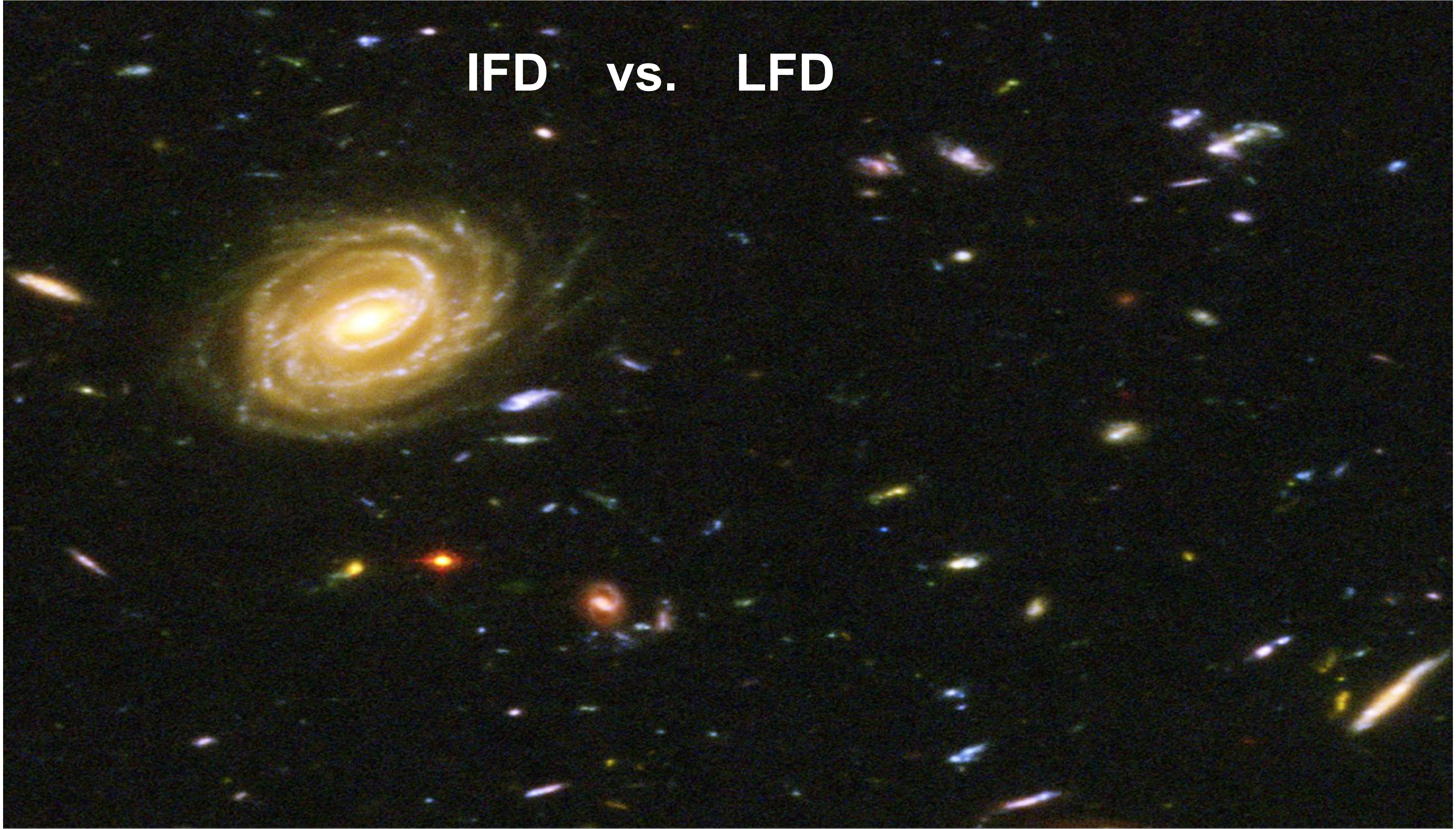
1 + 1 conformal algebra in IFD

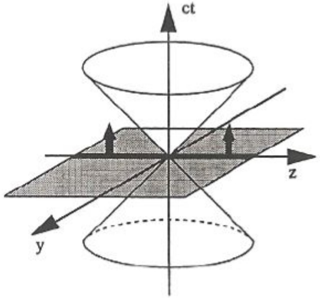
	$\mathfrak{K}_0$	P_0	D	$\mathfrak{K}_3$	P_3	K_3
$\mathfrak{K}_0$	0	$-2iD$	$-i\mathfrak{K}_0$	0	$-2iK_3$	$i\mathfrak{K}_3$
P_0	$2iD$	0	iP_0	$-2iK_3$	0	iP_3
D	$i\mathfrak{K}_0$	$-iP_0$	0	$i\mathfrak{K}_3$	$-iP_3$	0
$\mathfrak{K}_3$	0	$2iK_3$	$-i\mathfrak{K}_3$	0	$2iD$	$i\mathfrak{K}_0$
P_3	$2iK_3$	0	iP_3	$-2iD$	0	iP_0
K_3	$-i\mathfrak{K}_3$	$-iP_3$	0	$-i\mathfrak{K}_0$	$-iP_0$	0

1 + 1 conformal algebra in LFD

	$\mathfrak{K}_+$	P_+	D_+	$\mathfrak{K}_-$	P_-	D_-
$\mathfrak{K}_+$	0	$-2\sqrt{2}iD_+$	$-\sqrt{2}i\mathfrak{K}_+$	0	0	0
P_+	$2\sqrt{2}iD_+$	0	$\sqrt{2}iP_+$	0	0	0
D_+	$\sqrt{2}i\mathfrak{K}_+$	$-\sqrt{2}iP_+$	0	0	0	0
$\mathfrak{K}_-$	0	0	0	0	$-2\sqrt{2}iD_-$	$-\sqrt{2}i\mathfrak{K}_-$
P_-	0	0	0	$2\sqrt{2}iD_-$	0	$\sqrt{2}iP_-$
D_-	0	0	0	$\sqrt{2}i\mathfrak{K}_-$	$-\sqrt{2}iP_-$	0

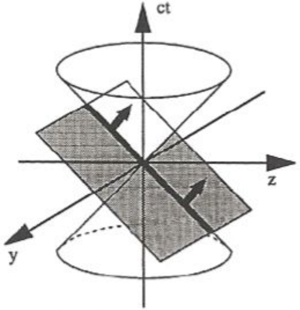
IFD vs. LFD





The instant form

Equal t		*Equal τ*
p^0	$\leftrightarrow$	$p^- = p^0 - p^3$
(p^1, p^2)	$\leftrightarrow$	$\vec{p}_\perp$
p^3	$\leftrightarrow$	$p^+ = p^0 + p^3$



The front form

Energy-Momentum Dispersion Relations

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2}$$

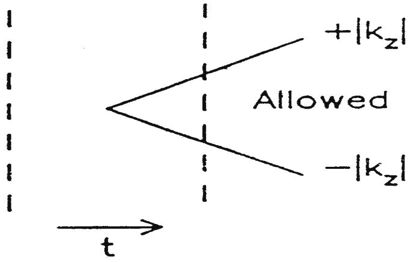
$$p^- = \frac{\vec{p}_\perp^{\,2} + m^2}{p^+}$$

Except zero-modes

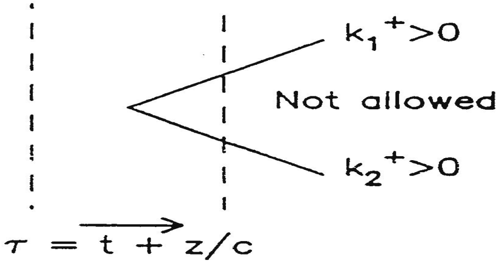
$$k_1^+ = k_2^+ = 0$$

IFD

Instant Form Dynamics



(a)



(b)

LFD

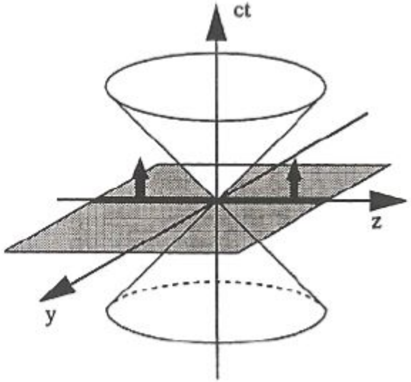
Light-Front Dynamics

Distinguished Vacuum Property

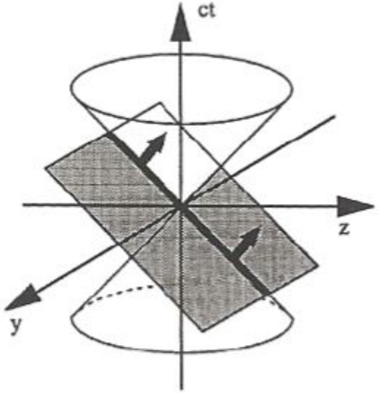
Can IFD and LFD be linked?

1949

Yes, they can!



The instant form



The front form

Traditional approach
evolved from NR dynamics

Close contact with
Euclidean space

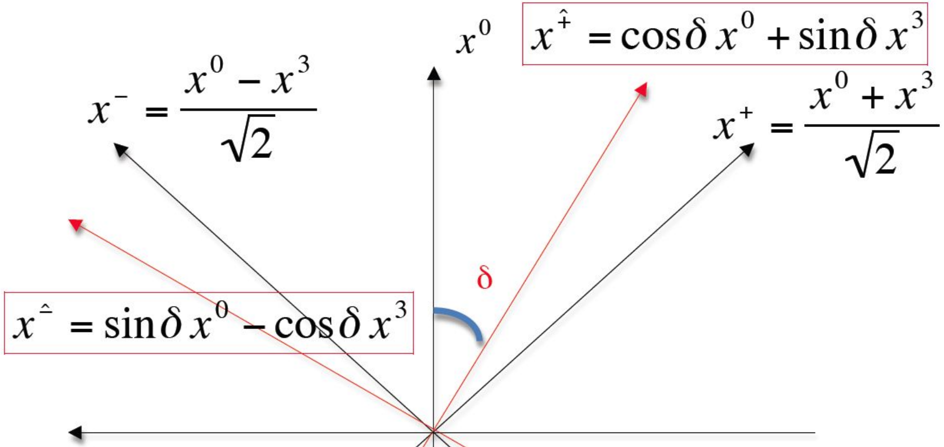
T-dept QFT, LQCD, IMF, etc.

Innovative approach
for relativistic dynamics

Strictly in Minkowski space

DIS, PDFs, DVCS, GPDs, etc.

Interpolation between IFD and LFD



$$(IFD) \quad 0 \le \delta \le \frac{\pi}{4} \quad (LFD)$$
$$1 \ge \quad C \equiv \cos(2\delta) \quad \ge 0$$

K. Hornbostel, PRD45, 3781 (1992) – RQFT
C.Ji and S.Rey, PRD53,5815(1996) – Chiral Anomaly
C.Ji and C. Mitchell, PRD64,085013 (2001) – Poincare Algebra
C.Ji and A. Suzuki, PRD87,065015 (2013) – Scattering Amps
C.Ji, Z. Li and A. Suzuki, PRD91, 065020 (2015) – EM Gauges
Z.Li, M. An and C.Ji, PRD92, 105014 (2015) – Spinors
C.Ji, Z.Li, B.Ma and A.Suzuki, PRD98, 036017(2018) – QED
B.Ma and C.Ji, PRD104, 036004(2021) – QCD_{1+1}
C.Ji and H. Ravikumar, PRD113, 096018(2026) – Conformal Symmetry

$1+1$ conformal algebra in the interpolation form

	$\mathfrak{K}_{\hat{+}}$	$P_{\hat{+}}$	$D_{\hat{+}}$	$\mathfrak{K}_{\hat{-}}$	$P_{\hat{-}}$	$D_{\hat{-}}$
$\mathfrak{K}_{\hat{+}}$	0	$-2i[(c+\mathbb{S}s)D_{\hat{+}} + (s-\mathbb{S}c)D_{\hat{-}}]$	$-i[(c+\mathbb{S}s)\mathfrak{K}_{\hat{+}} + (s-\mathbb{S}c)\mathfrak{K}_{\hat{-}}]$	0	$2i\mathbb{C}[sD_{\hat{+}} - cD_{\hat{-}}]$	$i\mathbb{C}[s\mathfrak{K}_{\hat{+}} - c\mathfrak{K}_{\hat{-}}]$
$P_{\hat{+}}$	$2i[(c+\mathbb{S}s)D_{\hat{+}} + (s-\mathbb{S}c)D_{\hat{-}}]$	0	$i[(c+\mathbb{S}s)P_{\hat{+}} + (s-\mathbb{S}c)P_{\hat{-}}]$	$-2i\mathbb{C}[sD_{\hat{+}} - cD_{\hat{-}}]$	0	$-i\mathbb{C}[sP_{\hat{+}} - cP_{\hat{-}}]$
$D_{\hat{+}}$	$i[(c+\mathbb{S}s)\mathfrak{K}_{\hat{+}} + (s-\mathbb{S}c)\mathfrak{K}_{\hat{-}}]$	$-i[(c+\mathbb{S}s)P_{\hat{+}} + (s-\mathbb{S}c)P_{\hat{-}}]$	0	$-i\mathbb{C}[s\mathfrak{K}_{\hat{+}} - c\mathfrak{K}_{\hat{-}}]$	$i\mathbb{C}[sP_{\hat{+}} - cP_{\hat{-}}]$	0
$\mathfrak{K}_{\hat{-}}$	0	$2i\mathbb{C}[sD_{\hat{+}} - cD_{\hat{-}}]$	$i\mathbb{C}[s\mathfrak{K}_{\hat{+}} - c\mathfrak{K}_{\hat{-}}]$	0	$-2i[(c-\mathbb{S}s)D_{\hat{+}} + (s+\mathbb{S}c)D_{\hat{-}}]$	$-i[(c-\mathbb{S}s)\mathfrak{K}_{\hat{+}} + (s+\mathbb{S}c)\mathfrak{K}_{\hat{-}}]$
$P_{\hat{-}}$	$-2i\mathbb{C}[sD_{\hat{+}} - cD_{\hat{-}}]$	0	$-i\mathbb{C}[sP_{\hat{+}} - cP_{\hat{-}}]$	$2i[(c-\mathbb{S}s)D_{\hat{+}} + (s+\mathbb{S}c)D_{\hat{-}}]$	0	$i[(c-\mathbb{S}s)P_{\hat{+}} + (s+\mathbb{S}c)P_{\hat{-}}]$
$D_{\hat{-}}$	$-i\mathbb{C}[s\mathfrak{K}_{\hat{+}} - c\mathfrak{K}_{\hat{-}}]$	$i\mathbb{C}[sP_{\hat{+}} - cP_{\hat{-}}]$	0	$i[(c-\mathbb{S}s)\mathfrak{K}_{\hat{+}} + (s+\mathbb{S}c)\mathfrak{K}_{\hat{-}}]$	$-i[(c-\mathbb{S}s)P_{\hat{+}} + (s+\mathbb{S}c)P_{\hat{-}}]$	0

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	P_3, D	$K_3, P_0, \mathfrak{K}_0, \mathfrak{K}_3$
$0 < \delta < \pi/4$	$P_{\hat{-}}, D = (D_{\hat{+}}c + D_{\hat{-}}s)$	$K_3 = (D_{\hat{+}}s - D_{\hat{-}}c), P_{\hat{+}}, \mathfrak{K}^{\hat{+}}, \mathfrak{K}^{\hat{-}}$
$\delta = \pi/4$	$P_-, D_+ = \frac{(D+K_3)}{\sqrt{2}}, D_- = \frac{(D-K_3)}{\sqrt{2}}, \mathfrak{K}_-$	$P_+, \mathfrak{K}_+$

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \longrightarrow g^{\hat\mu\hat\nu} = \begin{bmatrix} \cos 2\delta & 0 & 0 & \sin 2\delta \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin 2\delta & 0 & 0 & -\cos 2\delta \end{bmatrix}$$

$$\int d^3\vec{x}\, T^{0\mu} = P^{\mu} = \begin{bmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{bmatrix} \longrightarrow P^{\hat\mu} = \begin{bmatrix} P^{\hat +} = \cos\delta\, P^0 + \sin\delta\, P^3 \\ P^{\hat 1} = P^1 \\ P^{\hat 2} = P^2 \\ P^{\hat -} = \sin\delta\, P^0 - \cos\delta\, P^3 \end{bmatrix} = \int dx^{\hat -} d^2\vec{x}_{\perp} T^{\hat + \hat\mu}$$

$$\int d^3\vec{x}\, (T^{0\mu}x^{\nu} - T^{0\mu}x^{\nu}) = J^{\mu\nu} = \begin{bmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{bmatrix} \longrightarrow J^{\hat\mu\hat\nu} = \begin{bmatrix} 0 & \hat{E}^1 & \hat{E}^2 & -K^3 \\ -\hat{E}^1 & 0 & J^3 & -\hat{F}^1 \\ -\hat{E}^2 & -J^3 & 0 & -\hat{F}^2 \\ K^3 & \hat{F}^1 & \hat{F}^2 & 0 \end{bmatrix}$$

$$\hat{E}^1 = J^2 \sin\delta + K^1 \cos\delta$$

$$\hat{E}^2 = K^2 \cos\delta - J^1 \sin\delta$$

$$\hat{F}^1 = K^1 \sin\delta - J^2 \cos\delta$$

$$\hat{F}^2 = J^1 \cos\delta + K^2 \sin\delta$$

$$T^{\mu\nu} = \sum_k \frac{\partial L}{\partial(\partial_{\mu}\phi_k)} \partial^{\nu}\phi_k - g^{\mu\nu} L \quad ; \quad \partial_{\mu} T^{\mu\nu} = 0$$

Poincaré Algebra

$$[P^{\hat\mu}, P^{\hat\nu}] = 0 \qquad [P^{\hat\mu}, J^{\hat\rho\hat\sigma}] = i(g^{\hat\mu\hat\rho} P^{\hat\sigma} - g^{\hat\mu\hat\sigma} P^{\hat\rho})$$

$$[J^{\hat\mu\hat\nu}, J^{\hat\rho\hat\sigma}] = i(g^{\hat\mu\hat\sigma} J^{\hat\nu\hat\rho} + g^{\hat\nu\hat\rho} J^{\hat\mu\hat\sigma} - g^{\hat\mu\hat\rho} J^{\hat\nu\hat\sigma} - g^{\hat\nu\hat\sigma} J^{\hat\mu\hat\rho})$$

e.g. $$[P^{\hat+}, J^{\hat+\hat-}] = i(g^{\hat+\hat+} P^{\hat-} - g^{\hat+\hat-} P^{\hat+})$$

$$\downarrow$$

$$[P^{\hat+}, -K^3] = i(P^{\hat-} \cos 2\delta - P^{\hat+} \sin 2\delta)$$

$$\downarrow \quad \delta \to \pi/4$$

$$[K^3, P^+] = -iP^+$$

$$Exp(-i\omega K^3) \,|\, x^+ > \propto \,|\, x^+ >$$

One more kinematic generator appears only in the front form.

Maximum number (7) of members in the stability group.

Kinematic Operators
(Members of Stability Group)

$$Exp\left(-i\omega\hat{\aleph}^i\right) | x^{\hat{+}} > \propto | x^{\hat{+}} >$$

$$[\hat{\aleph}^i, P^{\hat{+}}] = 0$$

$$\hat{\aleph}^i = \hat{F}^i \cos 2\delta - \hat{E}^i \sin 2\delta$$

$\delta = 0$

$-J^2$ ←

J^1

$$\hat{\aleph}^1 = -J^2 \cos\delta - K^1 \sin\delta$$
$$\hat{\aleph}^2 = J^1 \cos\delta - K^2 \sin\delta$$

$\delta = \pi/4$

→ $-E^1 = -(J^2 + K^1)/\sqrt{2}$

$E^2 = (J^1 - K^2)/\sqrt{2}$

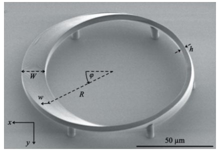
$$(J^3, P^1, P^2, P_{\hat{-}})$$

Orientation Entanglement of Quantum Mechanical Systems

- The angle should be rotated to get the same initial configuration.

$$|0,0> = \frac{1}{\sqrt{2}}(|\uparrow\downarrow - \downarrow\uparrow>)$$

- Spin- 0 → Any angle
- Spin-1/2 → 720^0
- Spin-1 → 360^0



$$|1,1> = |\uparrow\uparrow>$$

$$|1,0> = \frac{1}{\sqrt{2}}(|\uparrow\downarrow + \downarrow\uparrow>)$$

$$|1,-1> = |\downarrow\downarrow>$$

Rotation by 180^0

$$|0,0> \longrightarrow |0,0> \qquad |1,0> \longrightarrow -|1,0>$$

INTERPOLATING HELICITY

$$T_\delta = e^{i\beta_1 \mathcal{K}^{\hat{1}} + i\beta_2 \mathcal{K}^{\hat{2}}} e^{-i\beta_3 K^3}$$

$\delta \to 0$ $(\mathbb{C} \to 1)$

$(\mathbb{C} = \cos 2\delta)$

$\delta \to \pi/4$ $(\mathbb{C} \to 0)$

$$T_I = e^{-i\theta J^2} e^{-i\eta K^3}$$

$$T_l = e^{-i\beta_1 E^1} e^{-i\beta_3 K^3}$$

$$\mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta,$$

$$\mathcal{K}^{\hat{2}} = J^1 \cos\delta - K^2 \sin\delta,$$

Jacob-Wick Helicity

Light-Front Helicity

D. Dayananda & C.Ji, arXiv:2603.18208v1 [hep-th]; PRD113,116038(2016)

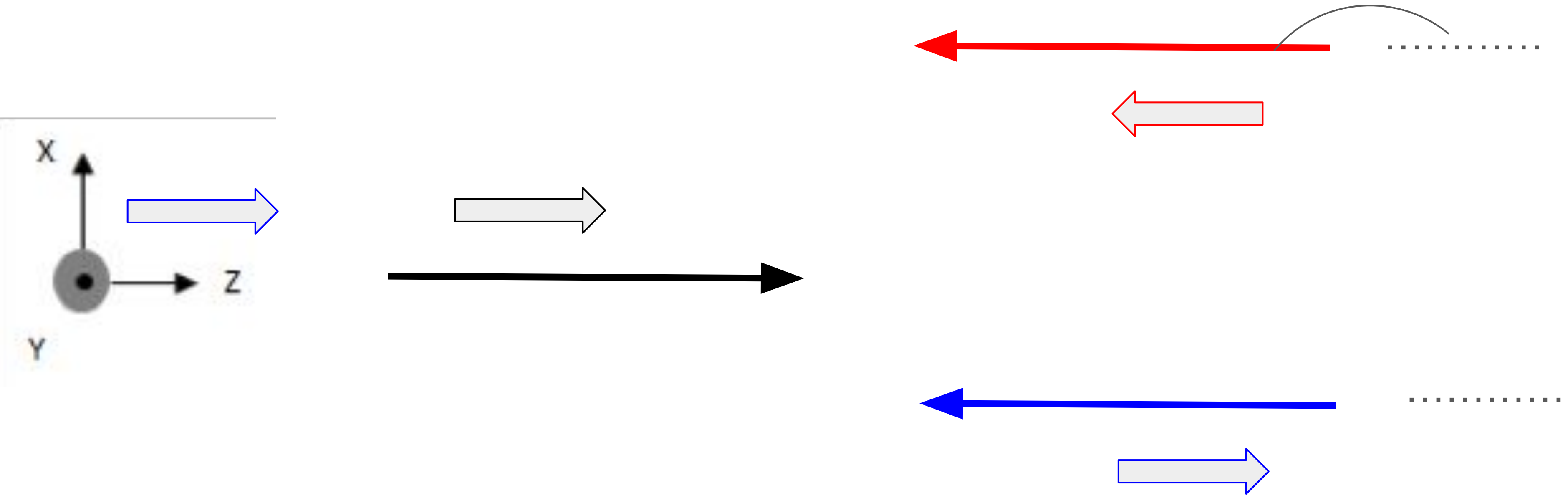
H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978).

Pauli–Lubanski pseudovector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} P_\nu M_{\alpha\beta}, \quad \hat{\mathcal{J}}_3 = W^{\hat{+}}/\mathbb{P}.$

C. Carlson and C.-R. Ji, *Phys. Rev. D* **67** (2003) 116002.

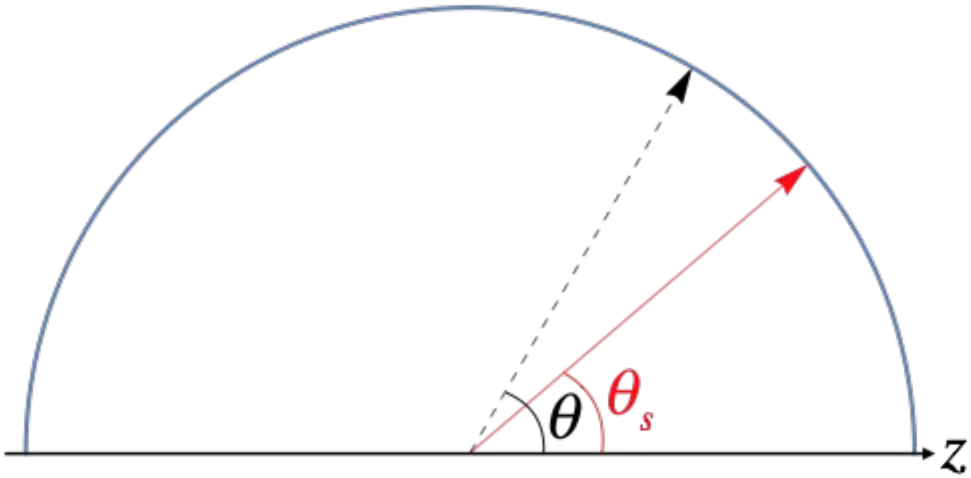
Jacob-Wick Helicity +1

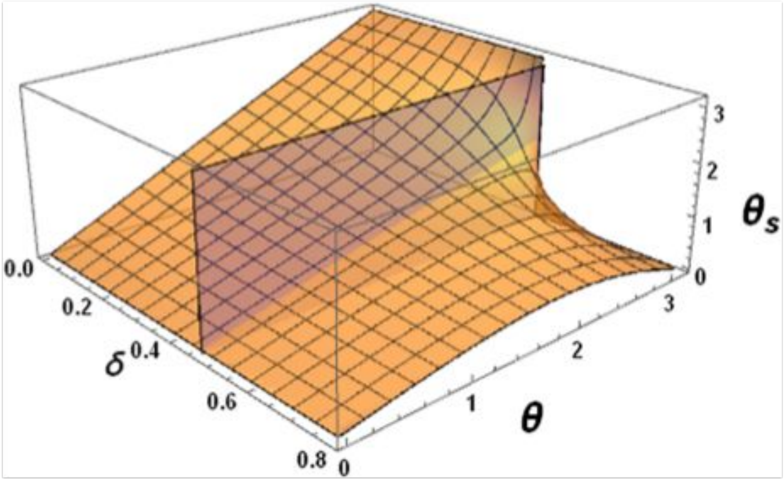
IFD: Transverse Rotation by 180^0

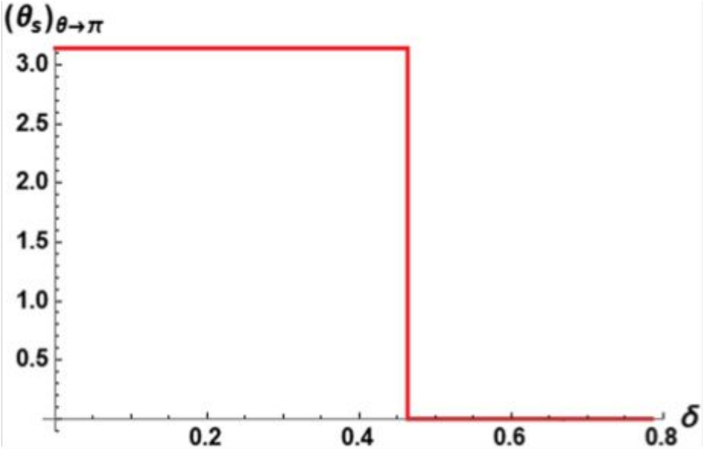


LFD: Longitudinal Boost to flip

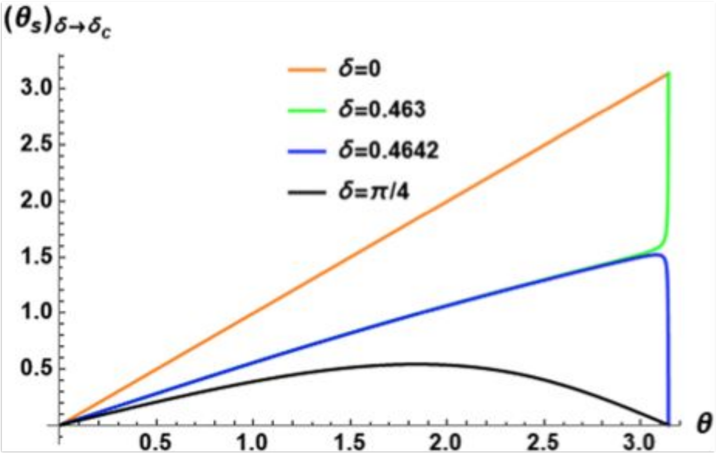
Light-Front Helicity +1







$$\delta_c = \tan^{-1}(P_v/E_0)$$



IFD and LFD are the two different formulations with two different degrees of freedom quantized at the equal time $t = x^0$ vs. at the equal light-front time $\tau \equiv (x^0 + x^3)/\sqrt{2} = x^+$, respectively. They correspond to each other with the interpolating time $x^{\hat{+}} = \cos\delta\, x^0 + \sin\delta\, x^3$.

Large N_C QCD in 1+1 dim. ('tHooft Model)

$$\mathcal{L} = -\frac{1}{4} F^a_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}a} + \bar{\psi}(i\gamma^{\hat{\mu}} D_{\hat{\mu}} - m)\psi$$

$$D_{\hat{\mu}} = \partial_{\hat{\mu}} - igA^a_{\hat{\mu}} t_a$$

$$F^a_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}} A^a_{\hat{\nu}} - \partial_{\hat{\nu}} A^a_{\hat{\mu}} + g f^{abc} A^b_{\hat{\mu}} A^c_{\hat{\nu}}$$

'tHooft Coupling $\lambda = \frac{g^2 (N_c - 1/N_c)}{4\pi}$ and mass m

$$g \to 0, N_C \to \infty; \lambda \to finite$$

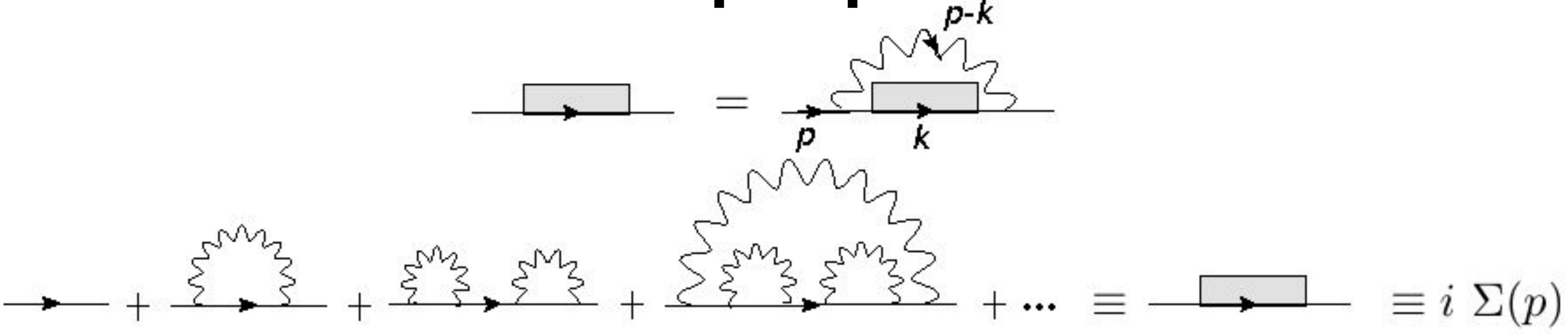
Interpolating Axial Gauge

$$A^a_{\hat{-}} = 0$$

$$A^a_1 = 0 \qquad\qquad A^+_a = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial_{\hat{-}} A^a_{\hat{+}})^2 + \bar{\psi}(i\gamma^{\hat{+}} D_{\hat{+}} + i\gamma^{\hat{-}}\partial_{\hat{-}} - m)\psi$$

Mass Gap Equation



$$\Sigma(p_{\hat{-}}) = i\frac{\lambda}{2\pi}\int \frac{dk_{\hat{-}}\,dk_{\hat{+}}}{(p_{\hat{-}} - k_{\hat{-}})^2}\gamma^{\hat{+}} \frac{1}{\not{k} - m - \Sigma(k_{\hat{-}}) + i\epsilon}\gamma^{\hat{+}}$$

Fermion Propagator

Free Propagator → **Interacting Propagator**

$$S_f(p) = \frac{1}{\not{p} - m + i\varepsilon}$$

$$S(p) = \frac{1}{\not{p} - m - \Sigma(p) + i\varepsilon} = \frac{F(p)}{\not{p} - M(p) + i\varepsilon}$$

$$\boxed{\Sigma(p) = \Sigma_s(p) + \Sigma_v(p)\not{p}}$$

$$F(p) = \left(1 - \Sigma_v(p)\right)^{-1}$$ "Wave function renormalization factor"

$$M(p) = \frac{m + \Sigma_s(p)}{1 - \Sigma_v(p)}$$ "Renormalized fermion mass function"

Energy-Momentum Dispersion Relation

Free particle | **Interacting particle**

$$E=\sqrt{p_z^2+m^2}$$

$$\frac{F(p'_{\hat{-}})E(p'_{\hat{-}})}{\sqrt{C}}=\sqrt{p'^2_{\hat{-}}+M(p'_{\hat{-}})^2}\equiv\tilde{E}(p'_{\hat{-}})$$

$$\theta_f=\tan^{-1}(p_z/m)$$

$$\theta(p'_{\hat{-}})=\theta_f(p'_{\hat{-}})+2\xi(p'_{\hat{-}})$$

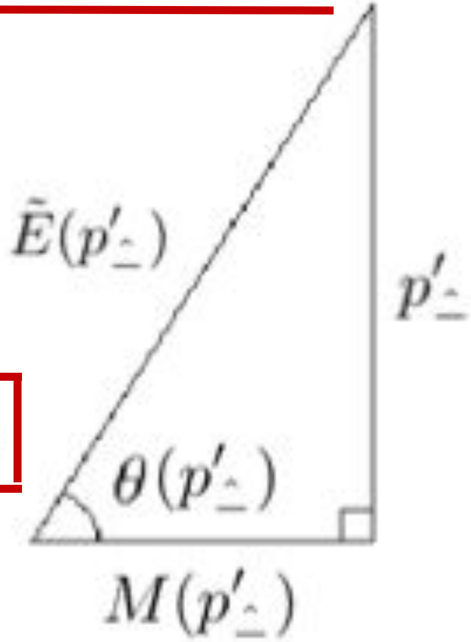
$$\beta=p_z/E$$

$$=\sin\theta_f$$

$$\begin{pmatrix} b^i(p'_{\hat{-}}) \\ d^{+i}(p'_{\hat{-}}) \end{pmatrix}=\begin{pmatrix} \cos\xi(p'_{\hat{-}}) & -\sin\xi(p'_{\hat{-}}) \\ \sin\xi(p'_{\hat{-}}) & \cos\xi(p'_{\hat{-}}) \end{pmatrix}\begin{pmatrix} b^i_f(p'_{\hat{-}}) \\ d^{+i}_f(p'_{\hat{-}}) \end{pmatrix}$$

$$=\tanh\eta$$

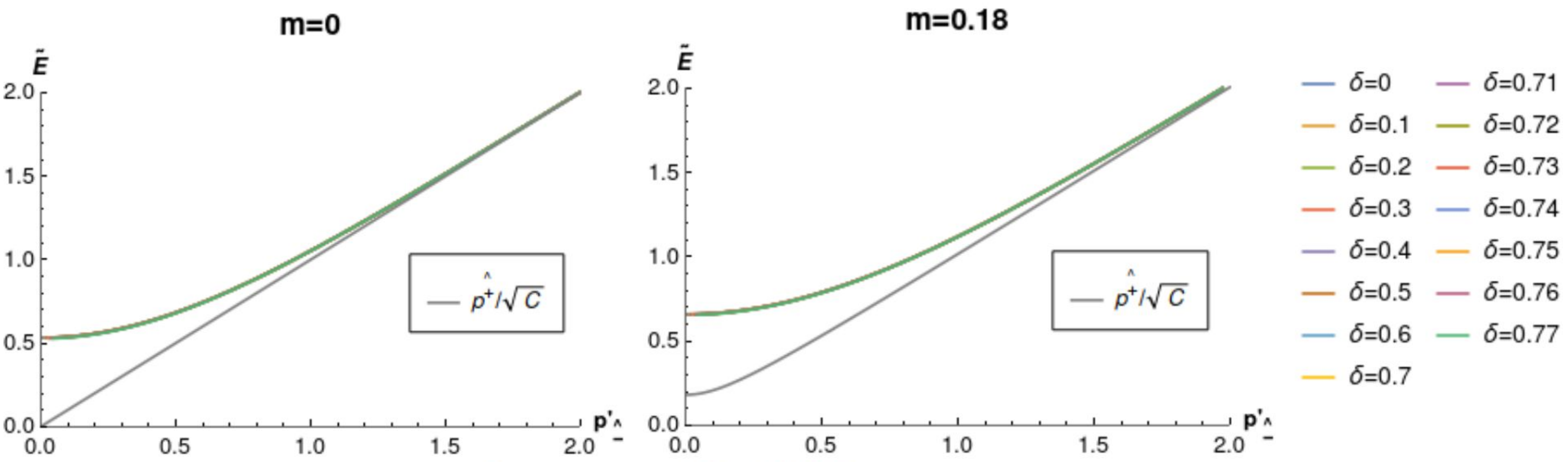
$b^i_f|0>=0,\ d^i_f|0>=0$ **vs.** $b^i|\Omega>=0,\ d^i|\Omega>=0$



Interpolation

$$(E,p_z)\Rightarrow(p^{\hat{+}}/\sqrt{C},p_{\hat{-}}/\sqrt{C}\equiv p'_{\hat{-}})$$

Mass Gap Solutions



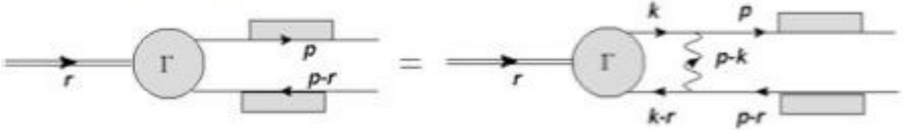
$$\tilde{E}(0) = \frac{F(0)E(0)}{\sqrt{\mathbb{C}}} = M(0)$$

m	0	0.045	0.18	0.749	1.00	2.11	4.23
$M(0)$	0.532778	0.563644	0.659112	1.10105	1.31167	2.30969	4.34358
$F(0)$	−0.495173	−0.584175	−0.987673	4.11079	2.17976	1.22134	1.05526

$$m \lesssim 0.56$$

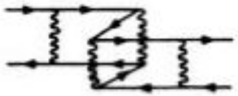
BOUND-STATE EQUATION

$$\Gamma(r,p) = \frac{i\lambda}{2\pi}\int \frac{dk_{\hat{-}}\, dk_{\hat{+}}}{(p_{\hat{-}} - k_{\hat{-}})^2} S(p)\gamma^{\hat{+}}\Gamma(r,k)\gamma^{\hat{+}} S(p-r)$$



$$\left[-r_{\hat{+}} + \frac{-\mathbb{S}p_{\hat{-}} + E(p_{\hat{-}})}{\mathbb{C}} + \frac{\mathbb{S}(p_{\hat{-}} - r_{\hat{-}}) + E(p_{\hat{-}} - r_{\hat{-}})}{\mathbb{C}}\right]\hat{\phi}_+(r_{\hat{-}}, p_{\hat{-}})$$
$$= \lambda \fint \frac{dk_{\hat{-}}}{(p_{\hat{-}} - k_{\hat{-}})^2}\left[C(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_+(r_{\hat{-}}, k_{\hat{-}}) - S(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_-(r_{\hat{-}}, k_{\hat{-}})\right],$$

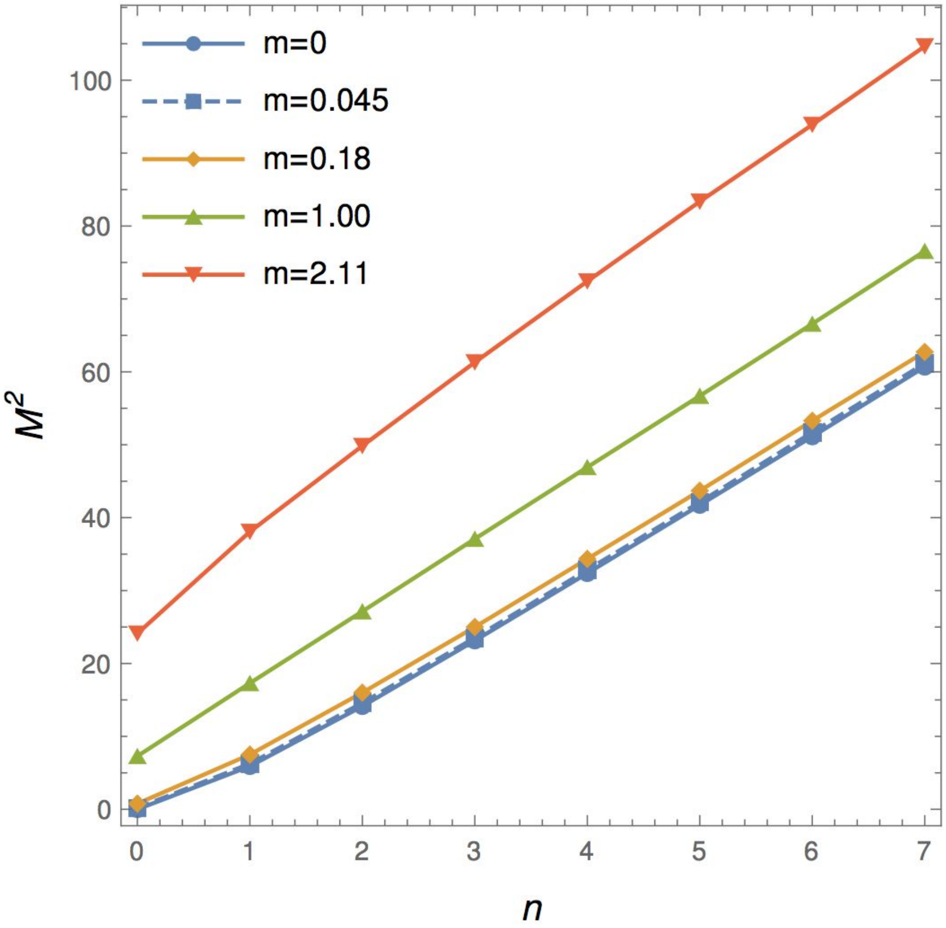
$$\left[r_{\hat{+}} + \frac{-\mathbb{S}(p_{\hat{-}} - r_{\hat{-}}) + E(p_{\hat{-}} - r_{\hat{-}})}{\mathbb{C}} + \frac{\mathbb{S}p_{\hat{-}} + E(p_{\hat{-}})}{\mathbb{C}}\right]\hat{\phi}_-(r_{\hat{-}}, p_{\hat{-}})$$
$$= \lambda \fint \frac{dk_{\hat{-}}}{(p_{\hat{-}} - k_{\hat{-}})^2}\left[C(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_-(r_{\hat{-}}, k_{\hat{-}}) - S(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_+(r_{\hat{-}}, k_{\hat{-}})\right].$$



LFD $$\left[\mathcal{M}^2 - \frac{m^2 - 2\lambda}{x} - \frac{m^2 - 2\lambda}{1-x}\right]\phi(x) = -2\lambda \fint_0^1 \frac{dy}{(x-y)^2}\phi(y)$$

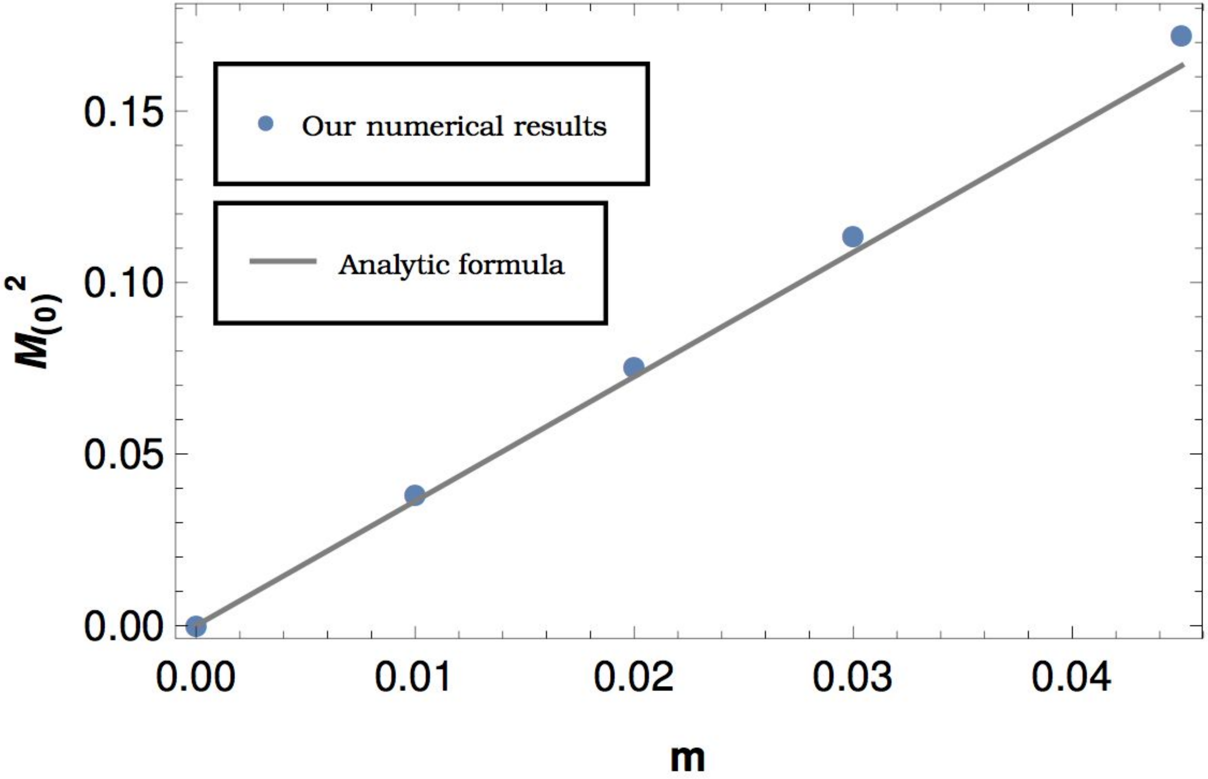
Meson Spectroscopy

- G.‘tHooft, NPB75, 461(74) - LFD



- M.Li, et al., JPG13, 915(87) - IFD (rest frame)
- Y. Jia, et al., JHEP11, 151(‘17) - IFD (moving frame)

Gell-Mann - Oaks - Renner Relation

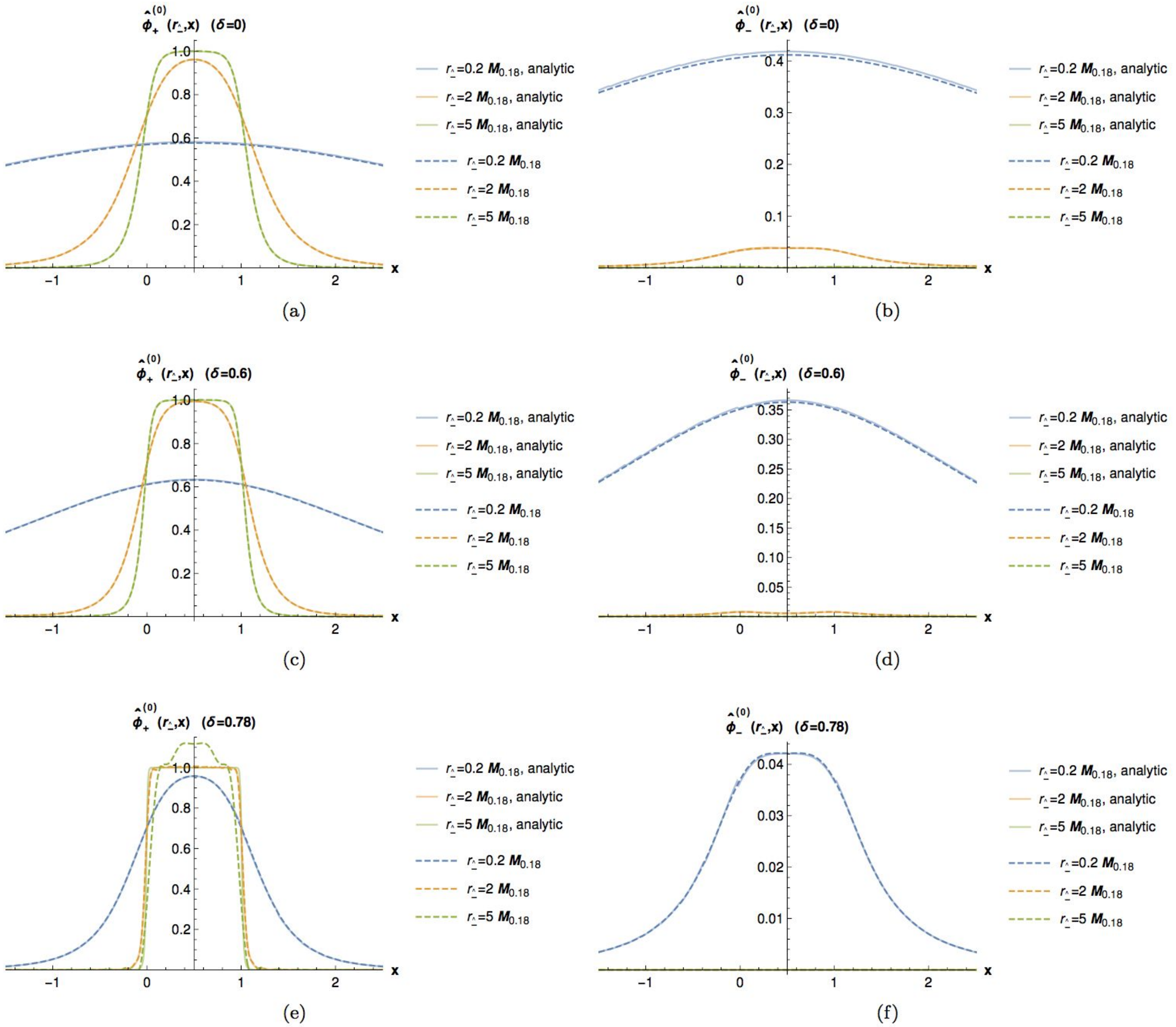


$$\mathcal{M}_\pi^2 = -\frac{4m < \bar{\psi}\psi >}{f_\pi^2} = \sqrt{\frac{8\pi^2 m^2 \lambda}{3}}$$

$$f_\pi = \sqrt{N_c/\pi}$$

Meson Ground-state Wave-function for m=0 case

$$\hat{\phi}^{(0)}_{\pm}(r_{\underline{\wedge}}, p_{\underline{\wedge}}) = \frac{1}{2}\left(\cos\frac{\theta(r_{\underline{\wedge}} - p_{\underline{\wedge}}) - \theta(p_{\underline{\wedge}})}{2} \pm \sin\frac{\theta(r_{\underline{\wedge}} - p_{\underline{\wedge}}) + \theta(p_{\underline{\wedge}})}{2}\right)$$



Parton Distribution Functions (PDFs)

$$q_n(x) = \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-}$$

$$\times \langle P_n^-, P^+ | \bar{\psi}(\xi^-)\gamma^+ \mathcal{W}[\xi^-, 0]\psi(0) | P_n^-, P^+ \rangle_C,$$

$$\mathcal{W}[\xi^-, 0] = \mathcal{P}\left[\exp\left(-ig_s \int_0^{\xi^-} d\eta^- A^+(\eta^-)\right)\right]$$

A^+=0 Gauge in LFD

Quasi-PDFs

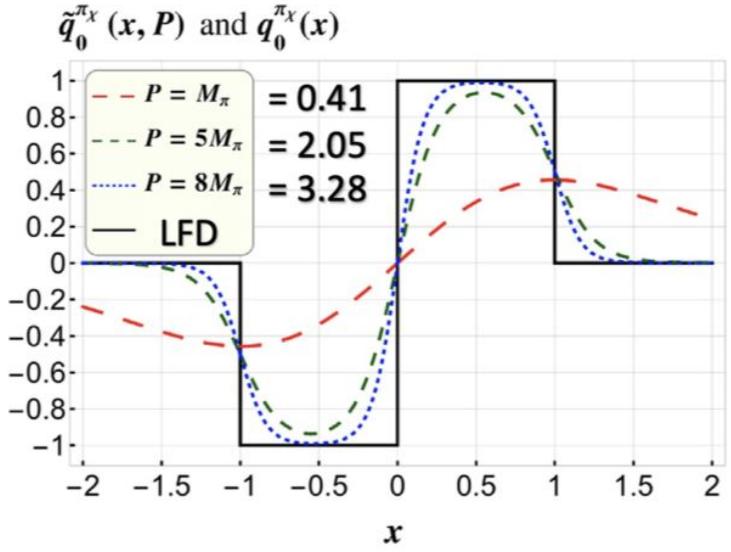
$$\tilde{q}_{(n)}(r_{\hat{-}}, x) = \int_{-\infty}^{+\infty} \frac{dx^{\hat{-}}}{4\pi} \, \mathrm{e}^{ix^{\hat{-}} r_{\hat{-}}}$$

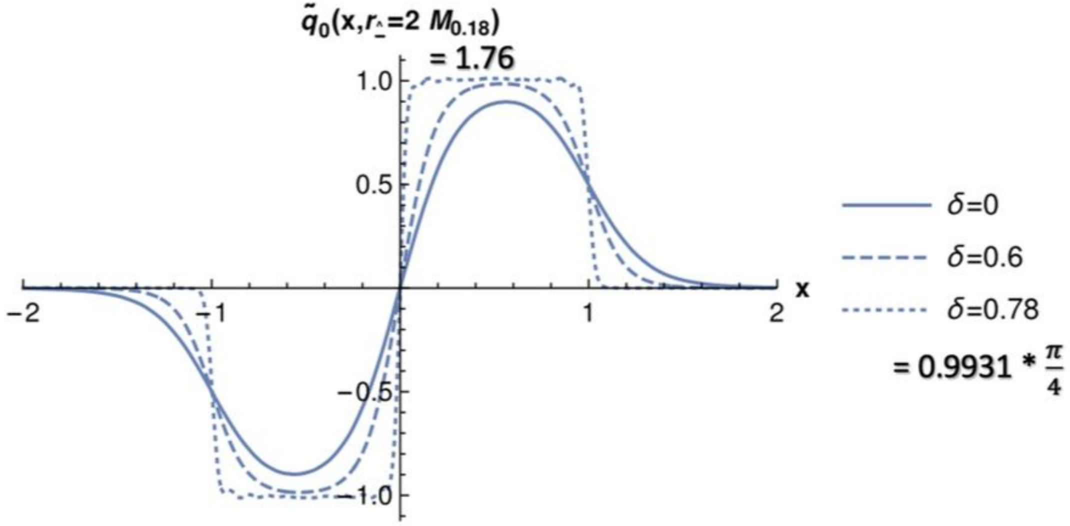
$$\times < r^{\hat{+}}_{(n)}, r_{\hat{-}} \mid \bar{\psi}(x^{\hat{-}}) \, \gamma_{\hat{-}} \, \mathcal{W}[x^{\hat{-}}, 0] \, \psi(0) \mid r^{\hat{+}}_{(n)}, r_{\hat{-}} >_C,$$

$$\mathcal{W}[x^{\hat{-}}, 0] = \mathcal{P}\left[\exp\left(-ig \int_0^{x^{\hat{-}}} dx'^{\hat{-}} A_{\hat{-}}(x'^{\hat{-}})\right)\right]$$

Interpolating dynamics

Y. Jia, et al., PRD98, 054011('18)
- IFD (quasi-PDFs)





B.Ma&C.Ji,PRD104,036004('21)
- Interpolating Dynamics

Conclusions and Outlook

- Conformal symmetry nature of the spacetime physics provides distinctively different features between the IFD and the LFD.
- Universal orientation entanglement nature distinguishes the relativistic helicities between the IFD and the LFD, i.e. Jacob-Wick helicity ≠ Light-front helicity.
- Interpolation between IFD and LFD identifies the critical point that bifurcates the two branches of helicities.
- Link between QCD and LFQM may be feasible as exemplified by the mass gap solution in the ‘t Hooft model interpolation between IFD and LFD.

Conclusions and Outlook

- **BT construction provides the self-consistency of LFQM which assures the component and frame independence of the physical observables.**

  Ref: H.M.Choi's talk today's afternoon

- **Universal entanglement nature in color dynamics can be investigated in the deuteron structure studies.**

  Ref: S.Kaur's talk Friday afternoon