# Light Front Densities

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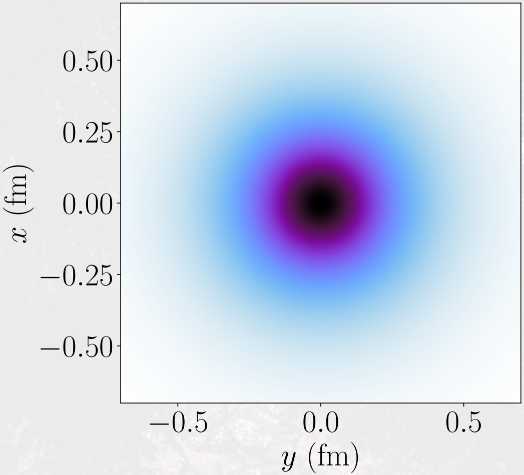
# Light front densities

- Two-dimensional densities:
  1. Over the transverse plane ($\boldsymbol{b}_\perp$)
  2. At fixed light front time (fixed $x^+$)
  3. At fixed longitudinal momentum (fixed $P^+$)
  4. Relative to center-of-momentum (center of $P^+$)
     see Ho-Yeon Won's talk (Friday) about centers!
- Given by 2D Fourier transforms of form factors
- Related to GPD moments at $\xi = 0$

Figure: proton $P^+$ density





M. Burkardt, Phys Rev D 62 (2000) 071503
M. Burkardt, Int J Mod Phys A 18 (2003) 173
G. Miller, Phys Rev Lett 99 (2007) 112001
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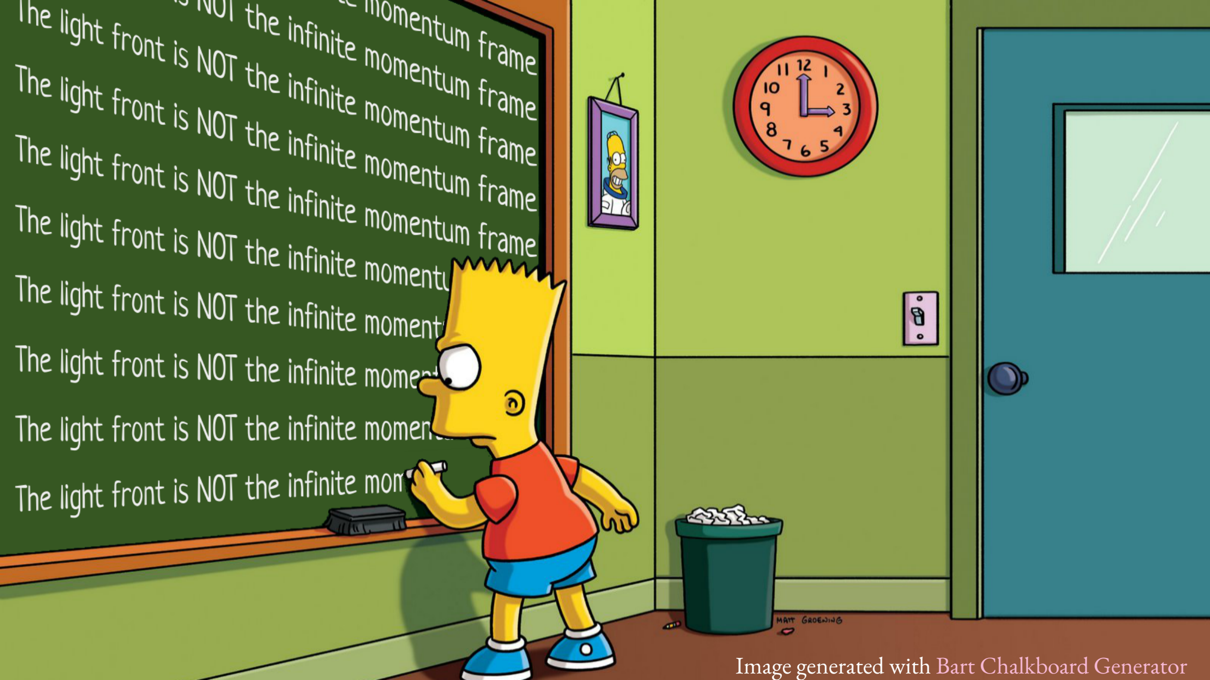


Image generated with Bart Chalkboard Generator

# Not the infinite momentum frame

- Instant form → light front is a **generalized coordinate transformation**

$$\begin{bmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

- Changes the metric—**not a Lorentz transformation**
  - Lorentz transformations are isometries of the metric

$$g_{\mu\nu}^{(\mathrm{IF})} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad g_{\mu\nu}^{(\mathrm{LF})} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Can never get to the light front by boosting

  see Chueng-Ryong Ji's talk (today) on connecting them!

AF, *A friendly introduction to the light front* (lecture series)
P. Mannheim, *A Friendly Introduction to Light-Front Quantum Field Theory* (Lecture 1 / Lecture 2)

# Galilei subgroup

- Poincaré group has a $(2+1)$D **Galilei subgroup**.
  - $x^+$ is time and $\boldsymbol{x}_\perp$ is space under this subgroup
  - $p^+ = \frac{1}{\sqrt{2}}(E_{\boldsymbol{p}} + p_z)$ is the central charge—acts like a mass, commutes with rest of subgroup
  - $x^+$ and $p^+$ are invariant under this subgroup!
- Light front time gives **fully relativistic** 2D picture that looks a lot like non-relativistic physics.

$$[B_i, B_j] = 0 \qquad [R_-, P^+] = 0$$
$$[B_i, R_-] = -i\epsilon_{3ij}B_j \qquad [R_-, P^i] = i\epsilon_{3ij}P^j$$
$$[B_i, P^+] = 0 \qquad [R_-, P^-] = 0$$
$$[B_i, P^j] = -i\delta_{ij}P^+ \qquad [P^i, P^+] = 0$$
$$[B_i, P^-] = -iP^i \qquad [P^i, P^-] = 0$$
$$[P^+, P^-] = 0$$

Kogut & Soper, Phys Rev D 1 (1970) 2901

M. Burkardt, Int J Mod Phys A 18 (2003) 173

# Barycentric separation

- Galilei subgroup permits barycentric separation:

**Barycentric coordinates**

$$P^+ = p_1^+ + p_2^+ + \dots$$

$$\boldsymbol{R}_\perp = \frac{p_1^+ \boldsymbol{r}_{1\perp} + p_2^+ \boldsymbol{r}_{2\perp} + \dots}{P^+}$$

- Total plus momentum
- Transverse center-of-$P^+$

**Internal coordinates**

$$x_n = \frac{p_n^+}{P^+}$$

$$\boldsymbol{b}_{n\perp} = \boldsymbol{r}_{n\perp} - \boldsymbol{R}_\perp$$

- Transverse position from center-of-$P^+$
- Longitudinal momentum fraction

- Spatial picture only in transverse coordinates
- Separation at fixed $x^+$

$$\Psi(p_1^+, \boldsymbol{r}_{1\perp}, p_2^+, \boldsymbol{r}_{2\perp}, \dots, x^+) = \Phi(P^+, \boldsymbol{R}_\perp, x^+)\psi(x_1, \boldsymbol{b}_{1\perp}, x_2, \boldsymbol{b}_{2\perp}, \dots, x^+)$$

see James Vary's talk (today)

- Local current conservation ...

$$\partial_\mu J^\mu(x) = \partial_+ J^+(x) + \boldsymbol{\nabla}_\perp \cdot \boldsymbol{J}^\perp(x) + \partial_- J^-(x) = 0$$

- ...with $x^-$ integrated out ...

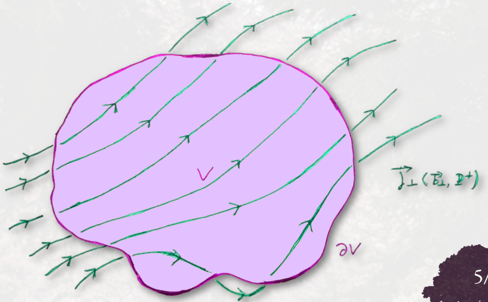
**Differential form**

$$\partial_+ J^+(\boldsymbol{x}_\perp, x^+) + \boldsymbol{\nabla}_\perp \cdot \boldsymbol{J}^\perp(\boldsymbol{x}_\perp, x^+) = 0$$

**Integral form**

$$\frac{\mathrm{d}}{\mathrm{d}x^+} \int_V \mathrm{d}^2 x_\perp \, J^+(\boldsymbol{x}_\perp, x^+) = -\oint_{\partial V} \mathrm{d}\hat{S} \cdot \boldsymbol{J}^\perp(\boldsymbol{x}_\perp, x^+)$$

- $J^+$ is a density of some stuff (e.g., charge)
- $\boldsymbol{J}^\perp$ describes its transport—**flux density**
- $J^-$ also a flux density, but lost in $P^+$ rep



# Clear and unclear components

- Expressions for **Galilean components** of densities are well-determined
  - Just take 2D Fourier transform in Drell-Yan frame
  - Galilean: $\mu \in \{+,1,2\}$

**Electromagnetic current**

$$J^\mu(\boldsymbol{b}_\perp, P^+) = \begin{bmatrix} J^+(\boldsymbol{b}_\perp, P^+) \\ J^1(\boldsymbol{b}_\perp, P^+) \\ J^2(\boldsymbol{b}_\perp, P^+) \\ J^-(\boldsymbol{b}_\perp, P^+) \end{bmatrix}$$

**Energy-momentum tensor**

$$T^{\mu\nu}(\boldsymbol{b}_\perp, P^+) = \begin{bmatrix} T^{++} & T^{+1} & T^{+2} & T^{+-} \\ T^{1+} & T^{11} & T^{12} & T^{1-} \\ T^{2+} & T^{21} & T^{22} & T^{2-} \\ T^{-+} & T^{-1} & T^{-2} & T^{--} \end{bmatrix}$$

- I will argue that **remaining components** contain artifacts of wave packet dispersion
  - Short explanation: fixed $P^+$ doesn't give fixed longitudinal velocity
- I'll also explain an attempt to remove these artifacts—and the problems we're facing

# Longitudinal velocity

- Transverse velocity is easy:

$$\boldsymbol{v}_\perp = \nabla_{(\boldsymbol{p}_\perp)} p^- = \nabla_{(\boldsymbol{p}_\perp)} \left( \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+} \right) = \frac{\boldsymbol{p}_\perp}{p^+}$$

  - Because $\boldsymbol{x}_\perp$ is in Galilei subgroup
- Longitudinal velocity is trickier:

$$\dot{x}^- = -\frac{\partial p^-}{\partial p^+} = \frac{p^-}{p^+} = \frac{m^2 + \boldsymbol{p}_\perp^2}{2(p^+)^2}$$

  - $x^-$ *not* in Galilei subgroup
  - Minus sign because of sign in metric/four-product
- Transverse boosts can **induce longitudinal motion**:

$$\begin{bmatrix} 1 & & & \\ v_x & 1 & & \\ v_y & 0 & 1 & \\ \frac{1}{2}\boldsymbol{v}_\perp^2 & v_x & v_y & 1 \end{bmatrix} \begin{bmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{bmatrix} = \begin{bmatrix} x^+ \\ x^1 + v_x x^+ \\ x^2 + v_y x^+ \\ x^- + \boldsymbol{v}_\perp \cdot \boldsymbol{x}_\perp + \frac{1}{2}\boldsymbol{v}_\perp^2 x^+ \end{bmatrix}$$

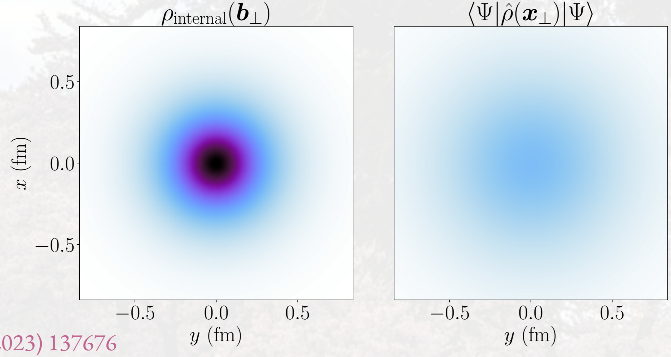

Convolution Framework

# Convolution framework

- Particles always exist in some wave packet—smears out internal structure
- *Physical* density given by quantum expectation value—schematically:

$$\langle \Psi | \hat{\rho}(\boldsymbol{x}_\perp, x^+) | \Psi \rangle = \int \frac{\mathrm{d}P^+ \,\mathrm{d}^2 R_\perp}{4\pi P^+} \left| \psi(P^+, \boldsymbol{R}_\perp, x^+) \right|^2 \rho_{\text{internal}}(\boldsymbol{x}_\perp - \boldsymbol{R}_\perp)$$

  - $|\psi|^2$ replaced by other smearing functions in general case
  - Known to work non-relativistically (but also get $P^+$ dependence on light front)




Li *et al.*, Phys Lett B 838 (2023) 137676
AF & Miller, Phys Rev D 108 (2023) 034008

- *Physical* four-current given by quantum expectation value:

$$\langle\psi|\hat{J}^\mu(x)|\psi\rangle = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2\boldsymbol{p}_\perp}{2p^+(2\pi)^3}\int\frac{\mathrm{d}p'^+\,\mathrm{d}^2\boldsymbol{p}'_\perp}{2p'^+(2\pi)^3}\langle\psi|p'\rangle\underbrace{\langle p'|\hat{J}^\mu(x)|p\rangle}_{\text{form factors appear here}}\langle p|\psi\rangle$$

  - Depends on wave packet—not entirely internal.
- Light front allows exact factorization if $x^-$ is integrated out:

$$\langle J^\mu_{2D}(\boldsymbol{x}_\perp,x^+)\rangle \equiv \int \mathrm{d}x^-\,\langle\Psi|\hat{J}^\mu(x)|\Psi\rangle = \int\frac{\mathrm{d}P^+\,\mathrm{d}^2R_\perp}{4\pi P^+}\underbrace{\Theta^\mu{}_\nu\big(\psi(P^+,\boldsymbol{R}_\perp,x^+),P^+\big)}_{\text{smearing function}}\overbrace{\mathfrak{J}^\nu(\underbrace{\boldsymbol{x}_\perp-\boldsymbol{R}_\perp}_{\text{invariant under boosts}},P^+)}^{\text{internal density}}$$

  - Move wave packet dependence into **smearing function**.
  - Call what remains the “internal” density.
  - Only possible on light front!

Li *et al.*, Phys Lett B 838 (2023) 137676
AF & Miller, Phys Rev D 107 (2023) 074036 (proof it’s only possible on light front in appendix)

# Ambiguity in convolution framework

- The factorization is not unique!

$$\langle J_{2\mathrm{D}}(\boldsymbol{x}_\perp, x^+)\rangle = \int \frac{\mathrm{d}P^+\,\mathrm{d}^2R_\perp}{4\pi P^+} \underbrace{\Theta^\mu{}_\nu(\psi(P^+, \boldsymbol{R}_\perp, x^+), P^+)}_{\text{smearing function}} \overbrace{\mathfrak{J}^\nu(\underbrace{\boldsymbol{x}_\perp - \boldsymbol{R}_\perp}_{\text{invariant under boosts}}, P^+)}^{\text{internal density}}$$

- Can shuffle terms between between smearing function & internal density.

# Ambiguity in convolution framework

- The factorization is not unique!

$$\langle J_{2D}(\boldsymbol{x}_\perp, x^+)\rangle = \int \frac{\mathrm{d}P^+\,\mathrm{d}^2R_\perp}{4\pi P^+} \underbrace{\frac{1}{2}\Theta^\mu{}_\nu\big(\psi(P^+,\boldsymbol{R}_\perp,x^+),P^+\big)}_{\text{smearing function}} \overbrace{2\mathfrak{J}^\nu(\underbrace{\boldsymbol{x}_\perp - \boldsymbol{R}_\perp}_{\text{invariant under boosts}},P^+)}^{\text{internal density}}$$

- Can shuffle terms between between smearing function & internal density.
  - Could move a constant.

# Ambiguity in convolution framework

- The factorization is not unique!

$$\langle J_{2D}(\boldsymbol{x}_\perp, x^+)\rangle = \int \frac{\mathrm{d}P^+\,\mathrm{d}^2R_\perp}{4\pi P^+} \underbrace{(\Lambda^{-1})^\mu{}_\alpha \Theta^\alpha{}_\nu\big(\psi(P^+,\boldsymbol{R}_\perp,x^+),P^+\big)}_{\text{smearing function}} \overbrace{\Lambda^\nu{}_\beta \mathfrak{J}^\beta(\underbrace{\boldsymbol{x}_\perp - \boldsymbol{R}_\perp}_{\text{invariant under boosts}},P^+)}^{\text{internal density}}$$

- Can shuffle terms between between smearing function & internal density.
  - Could move a constant.
  - Could move a Lorentz transform!
- **Internal density is a matter of convention.**
- Can we pick a reasonable convention?

# Boost scheme

$$\langle J_{2\mathrm{D}}(\boldsymbol{x}_\perp, x^+)\rangle = \int \frac{\mathrm{d}P^+\,\mathrm{d}^2R_\perp}{4\pi P^+} \overbrace{\Theta^\mu{}_\nu\big(\psi(P^+,\boldsymbol{R}_\perp,x^+),P^+\big)}^{\text{smearing function}} \overbrace{\mathfrak{J}^\nu(\boldsymbol{x}_\perp - \boldsymbol{R}_\perp, P^+)}^{\text{internal density}}$$

- **Boost scheme** — smearing function is a transverse boost by *average* velocity of kets:

$$\Theta^\mu{}_\nu = \psi^*(P^+,\boldsymbol{R}_\perp,x^+)\begin{bmatrix} 1 & & & \\ V_x & 1 & & \\ V_y & 0 & 1 & \\ \frac{1}{2}\boldsymbol{V}_\perp^2 & V_x & V_y & 1 \end{bmatrix}\psi(P^+,\boldsymbol{R}_\perp,x^+)$$

$$\boldsymbol{V}_\perp \equiv \frac{\boldsymbol{P}_\perp}{P^+}$$

$$\boldsymbol{P}_\perp \to -\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}$$

- Internal density is Fourier transform in **Drell-Yan frame**:

$$\mathfrak{J}^\nu(\boldsymbol{b}_\perp, P^+) \equiv \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\frac{\langle P^+, \frac{1}{2}\boldsymbol{\Delta}_\perp | \hat{J}^\nu(0) | P^+, -\frac{1}{2}\boldsymbol{\Delta}_\perp\rangle}{2P^+} e^{-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{b}_\perp}$$

# Spurious longitudinal current

- Boost scheme current of a spin-zero target:

$$\mathfrak{J}^+(\boldsymbol{b}_\perp) = \delta^{(2)}(\boldsymbol{b}_\perp)$$
$$\mathfrak{J}^\perp(\boldsymbol{b}_\perp) = 0$$
$$\mathfrak{J}^-(\boldsymbol{b}_\perp) = \frac{P^-}{P^+}\delta^{(2)}(\boldsymbol{b}_\perp)\Big|_{\boldsymbol{P}_\perp=0} = \frac{m^2}{2(P^+)^2}\left(1-\frac{\nabla_\perp^2}{4m^2}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$$

- Even when $P_z \to 0$—i.e., $P^+ \to \frac{m}{\sqrt{2}}$ ...

$$\mathfrak{J}^3(\boldsymbol{b}_\perp) = \frac{1}{\sqrt{2}}\left(\mathfrak{J}^+(\boldsymbol{b}_\perp) - \mathfrak{J}^-(\boldsymbol{b}_\perp)\right) \xrightarrow{P^+\to\frac{m}{\sqrt{2}}} \frac{1}{\sqrt{2}}\frac{\nabla_\perp^2}{4m^2}\delta^{(2)}(\boldsymbol{b}_\perp) \neq 0$$

- Spurious longitudinal current comes from kets in matrix element **not being at rest**

$$\mathfrak{J}^\nu(\boldsymbol{b}_\perp, P^+) \equiv \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \frac{\langle P^+, \frac{1}{2}\boldsymbol{\Delta}_\perp | \hat{J}^\nu(0) | P^+, -\frac{1}{2}\boldsymbol{\Delta}_\perp \rangle}{2P^+} \mathrm{e}^{-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{b}_\perp}$$

# Origin of the spurious longitudinal current

- Spurious longitudinal current comes from kets in matrix element **not being at rest**

$$\mathfrak{J}^\nu(\boldsymbol{b}_\perp, P^+) \equiv \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \frac{\langle P^+, \frac{1}{2}\boldsymbol{\Delta}_\perp | \hat{J}^\nu(0) | P^+, -\frac{1}{2}\boldsymbol{\Delta}_\perp \rangle}{2P^+} \mathrm{e}^{-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{b}_\perp}$$

- Each ket has a transverse velocity

$$\boldsymbol{v}_\perp = -\frac{\boldsymbol{\Delta}_\perp}{2P^+} \qquad \boldsymbol{v}'_\perp = +\frac{\boldsymbol{\Delta}_\perp}{2P^+}$$

- Associated boost matrices:

$$\Lambda^\mu{}_\nu = \begin{bmatrix} 1 & & & \\ -\frac{\Delta_x}{2P^+} & 1 & & \\ -\frac{\Delta_y}{2P^+} & 0 & 1 & \\ \frac{\boldsymbol{\Delta}_\perp^2}{8P^+} & -\frac{\Delta_x}{2P^+} & -\frac{\Delta_y}{2P^+} & 1 \end{bmatrix} \qquad \Lambda^\mu{}_\nu = \begin{bmatrix} 1 & & & \\ \frac{\Delta_x}{2P^+} & 1 & & \\ \frac{\Delta_y}{2P^+} & 0 & 1 & \\ \frac{\boldsymbol{\Delta}_\perp^2}{8P^+} & \frac{\Delta_x}{2P^+} & \frac{\Delta_y}{2P^+} & 1 \end{bmatrix}$$

- Both kets introduce longitudinal current—**artifact of transverse motion**

$$\langle J_{2\mathrm{D}}(\boldsymbol{x}_\perp, x^+)\rangle = \int \frac{\mathrm{d}P^+\,\mathrm{d}^2R_\perp}{4\pi P^+} \overbrace{\Theta^\mu{}_\nu\big(\psi(P^+,\boldsymbol{R}_\perp,x^+),P^+\big)}^{\text{smearing function}}\,\overbrace{\mathfrak{J}^\nu(\boldsymbol{x}_\perp-\boldsymbol{R}_\perp,P^+)}^{\text{internal density}}$$

- **Butter scheme** — smearing function is *average of transverse boosts* by velocities of kets:

$$\Theta^\mu{}_\nu = \psi^*(P^+,\boldsymbol{R}_\perp,x^+)\begin{bmatrix} 1 & & & \\ V_x & 1 & & \\ V_y & 0 & 1 & \\ \frac{1}{4}(\boldsymbol{v}_\perp^2+\boldsymbol{v}_\perp'^2) & V_x & V_y & 1 \end{bmatrix}\psi(P^+,\boldsymbol{R}_\perp,x^+)$$

$$\boldsymbol{v}_\perp \equiv \frac{\boldsymbol{p}_\perp}{P^+} \qquad \boldsymbol{v}'_\perp \equiv \frac{\boldsymbol{p}'_\perp}{P^+} \qquad \boldsymbol{V}_\perp \equiv \frac{\boldsymbol{P}_\perp}{P^+} \qquad \boldsymbol{P}_\perp \to -\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}$$

  - ...called butter scheme because it works smoothly (and because I like butter)

- Spin-zero internal current in the butter scheme:

$$\mathfrak{J}^+(\boldsymbol{b}_\perp) = \delta^{(2)}(\boldsymbol{b}_\perp) \qquad \mathfrak{J}^\perp(\boldsymbol{b}_\perp) = 0 \qquad \mathfrak{J}^-(\boldsymbol{b}_\perp) = \frac{m^2}{2(P^+)^2}\delta^{(2)}(\boldsymbol{b}_\perp)$$

# How to use the butter scheme

- Inverting formula for quantum expectation values gives:

$$\mathfrak{J}^{\mu}(\boldsymbol{b}_\perp, P^+) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} L^{\mu}{}_{\nu} \frac{\langle p'|\hat{J}^{\nu}(0)|p\rangle}{2P^+} \mathrm{e}^{-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{b}_\perp}$$

- $L$ here is basically an inverse of $\Theta$ (without the wave functions):

$$L = \begin{bmatrix} 1 & & & \\ -\frac{P_x}{P^+} & 1 & & \\ -\frac{P_y}{P^+} & 0 & 1 & \\ \frac{\boldsymbol{P}_\perp^2 - \frac{1}{4}\boldsymbol{\Delta}_\perp^2}{2(P^+)^2} & -\frac{P_x}{P^+} & -\frac{P_y}{P^+} & 1 \end{bmatrix}$$

- Easiest method is to build a dictionary—what does $L$ map four-vectors to?

# Dictionary for butter map

| $X^\mu$ | **Boost scheme** (Drell-Yan frame) | **Butter scheme** $(L^\mu{}_\nu X^\nu)$ |
|---|---|---|
| $P^\mu$ | $\left(P^+, 0, 0, \frac{m^2+\frac{1}{4}\boldsymbol{\Delta}_\perp^2}{2P^+}\right)$ | $\left(P^+, 0, 0, \frac{m^2}{2P^+}\right)$ |
| $\Delta^\mu$ | $(0, \Delta_x, \Delta_y, 0)$ | $(0, \Delta_x, \Delta_y, 0)$ |
| $n^\mu = (0,0,0,1)$ | $n^\mu$ | $n^\mu$ |
| $\bar{n}^\mu = (1,0,0,0)$ | $\frac{P^\mu}{P^+}$ | $\frac{P^\mu}{P^+} - \frac{\boldsymbol{\Delta}_\perp^2}{2(P^+)^2} n^\mu$ |
| $g^{\mu\nu}$ | $g^{\mu\nu}$ | $g^{\mu\nu} - \frac{\boldsymbol{\Delta}_\perp^2}{4(P^+)^2} n^\mu n^\nu$ |

# Boost scheme vs butter scheme

## Boost scheme

- Internal density boosted by *average velocity of kets*
- ☑ Matrix element in Drell-Yan frame
- ☒ Unphysical spin-zero current
- ☑ Preserves metric

## Butter scheme

- Internal density boosted by *average of boosts to ket velocities*
- ☒ No associated frame
- ☑ Sensible spin-zero current
- ☒ Does not preserve metric

- Ultimately—just different conventions
- Differ in what they ascribe to wave packet dispersion
- No particle exists without its wave packet—no way to decide which is "correct"
- But maybe one scheme is more sensible in certain cases?

- Any sensible smearing function agrees on **Galilean components**

$$\Theta^{\mu}{}_{\nu}(\psi, P^+) = \psi^*(P^+, \boldsymbol{R}_\perp, x^+) \begin{bmatrix} 1 & & & \\ \frac{P_x}{P^+} & 1 & & \\ \frac{P_y}{P^+} & 0 & 1 & \\ ? & ? & ? & ? \end{bmatrix} \psi(P^+, \boldsymbol{R}_\perp, x^+) \quad : \quad \boldsymbol{P}_\perp \to -\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}_\perp$$

- Leads to unambiguous answers for **Galilean densities**:

$$\mathfrak{J}^A(\boldsymbol{b}_\perp, P^+) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} \frac{\langle p', s' | \hat{J}^A(0) | p, s\rangle}{2P^+} e^{-i\boldsymbol{\Delta}_\perp \cdot \boldsymbol{b}_\perp} \Big|_{\boldsymbol{P}_\perp = 0} \quad : \quad A \in \{+, 1, 2\}$$

- Longitudinal barycentric separation needs to be done in momentum space
  - Spatial picture in third dimension is lost
  - Longitudinal currents drop from continuity equation
  - Schemes differ in minus components of currents—interesting to see *how*

# Spin-half Particles

# Light front helicity

- **Light front helicity** used to index spin states
- Eigenvalue of Pauli-Lubanski vector's plus component:

$$s = \frac{W^+}{P^+} = \pm\frac{1}{2} \qquad W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$$

  - Invariant under Galilei subgroup — and longitudinal boosts
  - No need for Wigner D-matrices when using light front boosts
- Appropriate spinors are **Kogut-Soper spinors**

$$u(p,\uparrow) = \frac{1}{\sqrt{\sqrt{2}p^+}} \begin{bmatrix} \sqrt{2}p^+ \\ p_x + i p_y \\ m \\ 0 \end{bmatrix} \qquad u(p,\downarrow) = \frac{1}{\sqrt{\sqrt{2}p^+}} \begin{bmatrix} 0 \\ m \\ -p_x + i p_y \\ \sqrt{2}p^+ \end{bmatrix}$$

  - What you get after boosting $m_s = \pm\frac{1}{2}$ states from rest

Kogut & Soper, Phys Rev D 1 (1970) 2901

- Spin index added to completeness relations:

$$\langle\psi|\hat{J}^\mu(x)|\psi\rangle = \sum_{s,s'}\int\frac{\mathrm{d}p^+\,\mathrm{d}^2\boldsymbol{p}_\perp}{2p^+(2\pi)^3}\int\frac{\mathrm{d}p'^+\,\mathrm{d}^2\boldsymbol{p}'_\perp}{2p'^+(2\pi)^3}\langle\psi|p',s'\rangle\underbrace{\langle p',s'|\hat{J}^\mu(x)|p,s\rangle}_{\text{form factors appear here}}\langle p,s|\psi\rangle$$

- Convolution formula uses matrices over spin indices:

$$\langle J^\mu_{2\mathrm{D}}(\boldsymbol{x}_\perp,x^+)\rangle = \sum_{s,s'}\int\frac{\mathrm{d}P^+\,\mathrm{d}^2R_\perp}{4\pi P^+}\underbrace{\Theta^\mu{}_\nu(\psi,P^+;s',s)}_{\text{smearing function}}\overbrace{\mathfrak{J}^\nu_{s's}(\underbrace{\boldsymbol{x}_\perp-\boldsymbol{R}_\perp}_{\text{invariant under boosts}},P^+)}^{\text{internal density}}$$

  - Move wave packet dependence into **smearing function**.
  - Call what remains the “internal” density.
  - Only possible on light front!

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AF & Miller, Phys Rev D 107 (2023) 074036 (proof it’s only possible on light front in appendix)

- Smearing matrices defined exactly as before—but with spin-dependent wave functions:

**Boost scheme**

$$\Theta^{\mu}{}_{\nu} = \psi^*_{s'}(P^+, \boldsymbol{R}_\perp, x^+) \begin{bmatrix} 1 & & & \\ V_x & 1 & & \\ V_y & 0 & 1 & \\ \frac{1}{2}\boldsymbol{V}_\perp^2 & V_x & V_y & 1 \end{bmatrix} \psi_s(P^+, \boldsymbol{R}_\perp, x^+)$$

**Butter scheme**

$$\Theta^{\mu}{}_{\nu} = \psi^*_{s'}(P^+, \boldsymbol{R}_\perp, x^+) \begin{bmatrix} 1 & & & \\ V_x & 1 & & \\ V_y & 0 & 1 & \\ \frac{1}{4}(\boldsymbol{v}_\perp^2 + \boldsymbol{v}_\perp'^2) & V_x & V_y & 1 \end{bmatrix} \psi_s(P^+, \boldsymbol{R}_\perp, x^+)$$

$$\boldsymbol{v}_\perp \equiv \frac{\boldsymbol{p}_\perp}{P^+}$$

$$\boldsymbol{v}'_\perp \equiv \frac{\boldsymbol{p}'_\perp}{P^+}$$

$$\boldsymbol{V}_\perp \equiv \frac{\boldsymbol{P}_\perp}{P^+}$$

$$\boldsymbol{P}_\perp \to -\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}$$

# Elementary fermions

- Galilean components the same in both schemes:

$$\mathfrak{J}^{+}(\boldsymbol{b}_\perp, P^+) = \delta^{(2)}(\boldsymbol{b}_\perp) \qquad \mathfrak{J}^{\perp}(\boldsymbol{b}_\perp, P^+) = \underbrace{\frac{(\boldsymbol{\sigma}\times\boldsymbol{\nabla}_\perp)_\perp}{2P^+}\delta^{(2)}(\boldsymbol{b}_\perp)}_{\text{Expected rotational current}}$$

- Different results for longitudinal current

$$\mathfrak{J}^{-}_{\text{boost}}(\boldsymbol{b}_\perp, P^+) = \left(\frac{m^2}{2(P^+)^2} - \underbrace{\frac{m(\boldsymbol{\sigma}\times\boldsymbol{\nabla}_\perp)_z}{2(P^+)^2}}_{\text{Expected rotational current}} + \underbrace{\frac{\boldsymbol{\nabla}_\perp^2}{8(P^+)^2}}_{\text{Extra term?}}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$$

$$\mathfrak{J}^{-}_{\text{butter}}(\boldsymbol{b}_\perp, P^+) = \left(\frac{m^2}{2(P^+)^2} - \overbrace{\frac{m(\boldsymbol{\sigma}\times\boldsymbol{\nabla}_\perp)_z}{2(P^+)^2}} + \overbrace{\frac{\boldsymbol{\nabla}_\perp^2}{4(P^+)^2}}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$$

  - Apparent extra term in both cases
  - There is a sensible physical explanation—which prefers the **boost scheme**

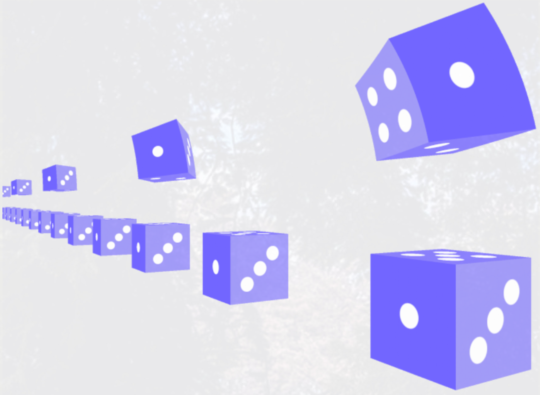
# Induced longitudinal motion

- Light front transverse boosts induce longitudinal motion!

$$\Lambda = \begin{bmatrix} 1 & & & \\ v_x & 1 & & \\ v_y & 0 & 1 & \\ \frac{1}{2}\boldsymbol{v}_\perp^2 & v_x & v_y & 1 \end{bmatrix}$$

$$B_x^{(\mathrm{LF})} = \frac{1}{\sqrt{2}}(K_x + J_y)$$

$$B_y^{(\mathrm{LF})} = \frac{1}{\sqrt{2}}(K_y - J_x)$$



Dice images by Ute Kraus,
https://www.spacetimetravel.org/

- Really a mix of transverse boost + Terrell counterrotation
- Mixture needed to leave transverse images invariant

# Induced longitudinal current

- Effective velocity operator:

$$\mathfrak{J}^{\perp}(\boldsymbol{b}_\perp, P^+) = \underbrace{\frac{(\boldsymbol{\sigma} \times \boldsymbol{\nabla}_\perp)_\perp}{2P^+}}_{\equiv \boldsymbol{v}_\perp} \delta^{(2)}(\boldsymbol{b}_\perp)$$

- Half velocity-squared operator:

$$\frac{1}{2}\boldsymbol{v}_\perp^2 \delta^{(2)}(\boldsymbol{b}_\perp) = \frac{\boldsymbol{\nabla}_\perp^2}{8(P^+)^2}\delta^{(2)}(\boldsymbol{b}_\perp)$$

- Longitudinal currents:

$$\mathfrak{J}^{-}_{\text{boost}}(\boldsymbol{b}_\perp, P^+) = \left(\frac{m^2}{2(P^+)^2} - \frac{m(\boldsymbol{\sigma}\times\boldsymbol{\nabla}_\perp)_z}{2(P^+)^2} + \overbrace{\frac{\boldsymbol{\nabla}_\perp^2}{8(P^+)^2}}^{=\frac{1}{2}\boldsymbol{v}_\perp^2}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$$

$$\mathfrak{J}^{-}_{\text{butter}}(\boldsymbol{b}_\perp, P^+) = \left(\frac{m^2}{2(P^+)^2} - \frac{m(\boldsymbol{\sigma}\times\boldsymbol{\nabla}_\perp)_z}{2(P^+)^2} + \underbrace{\frac{\boldsymbol{\nabla}_\perp^2}{4(P^+)^2}}_{=\boldsymbol{v}_\perp^2}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$$

- **Boost scheme** gives the correct answer in this case!

# Boost or butter?

## Boost scheme

- Internal density boosted by *average velocity of kets*
- ☑ Matrix element in Drell-Yan frame
- ☒ Unphysical spin-zero current
- ☑ Sensible spin-half current
- ☑ Preserves metric

## Butter scheme

- Internal density boosted by *average of boosts to ket velocities*
- ☒ No associated frame
- ☑ Sensible spin-zero current
- ☒ Unphysical spin-half current
- ☒ Does not preserve metric

- Complete agreement for **Galilean densities**:

$$\mathfrak{J}^A(\boldsymbol{b}_\perp, P^+) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \frac{\langle p', s'|\hat{J}^A(0)|p, s\rangle}{2P^+} \mathrm{e}^{-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{b}_\perp}\bigg|_{\boldsymbol{P}_\perp=0} \quad : \quad A \in \{+, 1, 2\}$$

- So far, boost scheme looks better
- But *energy density* prefers butter scheme


Energy-momentum tensor

# The energy-momentum tensor

- The energy-momentum tensor describes **density** and **flow** of energy & momentum.
- Also known as the stress-energy tensor.

Energy density

Momentum densities

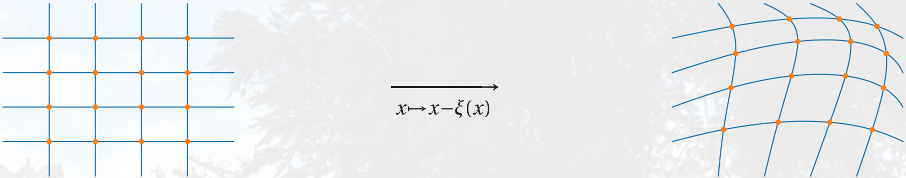
$$T^{\mu\nu}(x) = \begin{bmatrix} T^{++}(x) & T^{+1}(x) & T^{+2}(x) & T^{+-}(x) \\ T^{1+}(x) & T^{11}(x) & T^{12}(x) & T^{1-}(x) \\ T^{2+}(x) & T^{21}(x) & T^{22}(x) & T^{2-}(x) \\ T^{-+}(x) & T^{-1}(x) & T^{-2}(x) & T^{--}(x) \end{bmatrix}$$

Energy fluxes

Stress tensor

# Noether's theorems and spacetime distortions

- Conserved current from *local* spacetime translations (**Noether's second theorem**):




  - **Noether's theorems**: symmetries imply conservation laws
  - *Local* translation: move spacetime differently everywhere

- The **energy-momentum tensor (EMT)** is the associated conserved quantity

$$\delta_\xi S_{\mathrm{QCD}} = -\int \mathrm{d}^4x\, \underbrace{T^{\mu\nu}_{\mathrm{QCD}}(x)}_{\sum_q \left\{\frac{i}{2}\bar{q}(x)\gamma^\mu \overleftrightarrow{D}^\nu q(x)\right\} - 2\mathrm{Tr}[G^{\mu\lambda}G^\nu{}_\lambda] + \frac{1}{2}g^{\mu\nu}\mathrm{Tr}[G^{\lambda\sigma}G_{\lambda\sigma}]} \partial_\mu \xi_\nu(x) = 0$$

- This method gives an **asymmetric EMT**, but other methods give a symmetric EMT.

AF, Phys Rev D 113 (2026) 016011

# Energy-momentum-stress densities

- *Physical* four-current given by quantum expectation value:

$$\langle\psi|\hat{T}^{\mu\nu}(x)|\psi\rangle = \sum_{s,s'}\int\frac{\mathrm{d}p^+\,\mathrm{d}^2\boldsymbol{p}_\perp}{2p^+(2\pi)^3}\int\frac{\mathrm{d}p'^+\,\mathrm{d}^2\boldsymbol{p}'_\perp}{2p'^+(2\pi)^3}\langle\psi|p',s'\rangle\underbrace{\langle p',s'|\hat{T}^{\mu\nu}(x)|p,s\rangle}_{\text{form factors appear here}}\langle p,s|\psi\rangle$$

- Convolution formula:

$$\langle T^{\mu\nu}_{\mathrm{2D}}(\boldsymbol{x}_\perp,x^+)\rangle = \sum_{s,s'}\int\frac{\mathrm{d}P^+\,\mathrm{d}^2R_\perp}{4\pi P^+}\underbrace{\Phi^{\mu\nu}{}_{\alpha\beta}(\psi,P^+;s',s)}_{\text{smearing function}}\overbrace{\mathfrak{t}^{\alpha\beta}_{s's}(\underbrace{\boldsymbol{x}_\perp-\boldsymbol{R}_\perp}_{\text{invariant under boosts}},P^+)}^{\text{internal density}}$$

  - Move wave packet dependence into **smearing function**.
  - Call what remains the "internal" density.
  - Only possible on light front!

AF & Miller, Phys Rev D 107 (2023) 074036 (proof it's only possible on light front in appendix)

Li, Wang & Vary, 2405.06892

# Smearing function for rank-two tensor

## Boost scheme

- Boost to average velocity applied per index:

$$\Phi^{\mu\nu}{}_{\alpha\beta} = \psi^* \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \psi$$

$$\Lambda^\mu{}_\alpha = \begin{bmatrix} 1 & & & \\ V_x & 1 & & \\ V_y & 0 & 1 & \\ \frac{1}{2}\boldsymbol{V}_\perp^2 & V_x & V_y & 1 \end{bmatrix}$$

## Butter scheme

- Averaged boost applied per index:

$$\Phi^{\mu\nu}{}_{\alpha\beta} = \psi^* \bar{\Lambda}^\mu{}_\alpha \bar{\Lambda}^\nu{}_\beta \psi$$

$$\bar{\Lambda}^\mu{}_\alpha = \begin{bmatrix} 1 & & & \\ V_x & 1 & & \\ V_y & 0 & 1 & \\ \frac{1}{4}\left(\boldsymbol{v}_\perp^2 + \boldsymbol{v}_\perp'^2\right) & V_x & V_y & 1 \end{bmatrix}$$

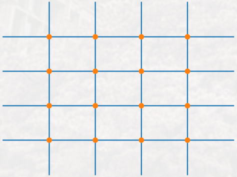
- ...with $\boldsymbol{P}_\perp \to -\frac{i}{2}\overleftrightarrow{\nabla}_\perp$ implicitly understood

- Result for elementary fermion differs between **symmetric EMT** and **asymmetric EMT**
- Matrix element for elementary fermion:

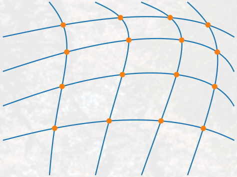
$$\langle p', s' | \hat{T}_S^{\mu\nu}(0) | p, s \rangle = \frac{1}{2} \bar{u}(p', s') \Big( \gamma^\mu P^\nu + P^\mu \gamma^\nu \Big) u(p, s)$$

$$\langle p', s' | \hat{T}_A^{\mu\nu}(0) | p, s \rangle = \bar{u}(p', s') \gamma^\mu P^\nu u(p, s)$$

- Either EMT can be found by a Belinfante procedure
- **Asymmetric EMT** follows from local translation invariance & Noether's second theorem



$$x \longmapsto x + \xi(x)$$



AF, Phys Rev D 113 (2026) 016011

# Energy density of an elementary fermion

| | Symmetric EMT | Asymmetric EMT |
|---|---|---|
| **Boost scheme** | $\left(\frac{m^2}{2P^+} + \frac{m(\boldsymbol{\sigma}\times\boldsymbol{\nabla}_\perp)_z}{4P^+}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$ | $\left(\frac{m^2}{2P^+} - \frac{\boldsymbol{\nabla}_\perp^2}{8P^+}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$ |
| **Butter scheme** | $\left(\frac{m^2}{2P^+} + \frac{m(\boldsymbol{\sigma}\times\boldsymbol{\nabla}_\perp)_z}{4P^+} + \frac{\boldsymbol{\nabla}_\perp^2}{8P^+}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$ | $\frac{m^2}{2P^+}\delta^{(2)}(\boldsymbol{b}_\perp)$ |

- Energy density given by $T^{+-}$ — a minus component that isn't a longitudinal flux
  - More important to get sensible energy density than any longitudinal flux
- ☒ Symmetric EMT gives energy density with angular modulations
- ☑ **Asymmetric EMT with butter scheme gives most sensible result**

# Why the asymmetric EMT?

- Asymmetric EMT has direct physical interpretation:

$$\langle p', s' | \hat{T}^{\mu\nu}_A(0) | p, s\rangle = \underbrace{\bar{u}(p', s')\gamma^\mu u(p,s)}_{\substack{\text{Four-current} \\ \text{(velocity times probability density)}}} \overbrace{P^\nu}^{\text{Four-momentum being transported}}$$

- EMT conserved with respect to first index:

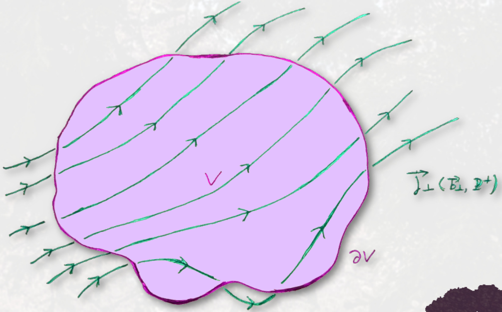
$$\partial_\mu T^{\mu\nu}(x) = 0$$

  - Describes density and flux of $P^\nu$—*second index*

- Integral form:

$$\frac{\mathrm{d}}{\mathrm{d}x^+}\int_V \mathrm{d}^2 x_\perp\, T^{+\nu}(\boldsymbol{x}_\perp, x^+) = -\oint_{\partial V} \mathrm{d}\hat{S}_i\, T^{i\nu}(\boldsymbol{x}_\perp, x^+)$$

  - Flux of $P^\nu$ in region—*second index*




- Symmetric EMT mixes up roles played by $v^\mu$ and $p^\nu$.

# Why the butter scheme?

- Boost scheme uses Drell-Yan frame, but individual kets are **not at rest**

$$\int \frac{\mathrm{d}^2 \Delta_\perp}{(2\pi)^2} \frac{\langle P^+, \frac{1}{2}\boldsymbol{\Delta}_\perp, s' | \hat{T}^{+-}(0) | P^+, -\frac{1}{2}\boldsymbol{\Delta}_\perp, s \rangle}{2P^+} \mathrm{e}^{-i\boldsymbol{\Delta}_\perp \cdot \boldsymbol{b}_\perp} = \int \frac{\mathrm{d}^2 \Delta_\perp}{(2\pi)^2} \left( \frac{m^2}{2(P^+)^2} + \frac{\boldsymbol{\Delta}_\perp^2}{8(P^+)^2} \right) \mathrm{e}^{-i\boldsymbol{\Delta}_\perp \cdot \boldsymbol{b}_\perp}$$

- Each ket has a transverse velocity

$$\boldsymbol{v}_\perp = -\frac{\boldsymbol{\Delta}_\perp}{2P^+} \qquad \boldsymbol{v}'_\perp = +\frac{\boldsymbol{\Delta}_\perp}{2P^+}$$

- Each ket has the same **kinetic energy**:

$$\frac{\boldsymbol{p}_\perp^2}{2P^+} = \frac{\boldsymbol{p}_\perp'^2}{2P^+} = \frac{\boldsymbol{\Delta}_\perp^2}{8P^+}$$

  - Boost scheme counts this as “internal structure”
  - Butter scheme moves this into smearing function

# Cross-check: elementary spin zero energy density

- Elementary spin-zero matrix element:

$$\langle p'|\hat{T}^{\mu\nu}(0)|p\rangle = 2P^\mu P^\nu + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{2P^+} D_0$$

  - $D_0 = -1$ for free Klein-Gordon equation
  - $D_0 = -\frac{1}{3}$ in conformal EMT
  - $D_0 = -\frac{1}{3}$ for $\phi^4$ theory — even as coupling goes to zero
  - Non-zero $D_0 \implies$ internal pressure = internal structure — use $D_0 = 0$ for “true” point

| | **Boost scheme** | **Butter scheme** |
|---|---|---|
| $\mathfrak{t}^{+-}(\boldsymbol{b}_\perp, P^+)$ | $\left(\frac{m^2}{2P^+} - (1+2D_0)\frac{\boldsymbol{\nabla}_\perp^2}{8P^+}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$ | $\left(\frac{m^2}{2P^+} - D_0\frac{\boldsymbol{\nabla}_\perp^2}{4P^+}\right)\delta^{(2)}(\boldsymbol{b}_\perp)$ |

- Butter scheme looks truly pointlike when $D_0 = 0$
- Non-pointlike for $D_0 \neq 0$ makes sense—internal pressure

J. Hudson & P. Schweitzer, Phys Rev D 96 (2017) 114013
B. Maynard, Phys Rev D 111 (2025) 076001


Summary

# Boost or butter?

## Boost scheme

- Internal density boosted by *average velocity of kets*
- ☑ Matrix element in Drell-Yan frame
- ☒ Unphysical spin-zero current (minus only)
- ☑ Sensible spin-half current
- ☒ **Unphysical energy density**
- ☑ Preserves metric

## Butter scheme

- Internal density boosted by *average of boosts to ket velocities*
- ☒ No associated frame
- ☑ Sensible spin-zero current
- ☒ Unphysical spin-half current (minus only)
- ☑ **Sensible energy density**
- ☒ Does not preserve metric ($--$ only)

- Complete agreement for **Galilean densities**:

$$\mathfrak{J}^A(\boldsymbol{b}_\perp, P^+) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} \frac{\langle p', s'|\hat{J}^A(0)|p, s\rangle}{2P^+} e^{-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{b}_\perp}\bigg|_{\boldsymbol{P}_\perp=0} \quad : \quad A \in \{+, 1, 2\}$$

- Minus-direction fluxes not so important—so I've grayed them out
- **Correct energy density more important**

# Energy density: composite system

- **Butter scheme:**

$$\mathrm{t}^{+-}_{\mathrm{butter}}(\boldsymbol{b}_\perp, P^+) = \frac{m^2}{2P^+}\int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\left(A(\boldsymbol{\Delta}_\perp^2) + \frac{\boldsymbol{\Delta}_\perp^2}{2m^2}D(\boldsymbol{\Delta}_\perp^2)\right)\mathrm{e}^{-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{b}_\perp}$$

- **Boost scheme:**

$$\mathrm{t}^{+-}_{\mathrm{boost}}(\boldsymbol{b}_\perp, P^+) = \frac{m^2}{2P^+}\int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\left(A(\boldsymbol{\Delta}_\perp^2) + \frac{\boldsymbol{\Delta}_\perp^2}{2m^2}D(\boldsymbol{\Delta}_\perp^2) + \frac{\boldsymbol{\Delta}_\perp^2}{4m^2}A(\boldsymbol{\Delta}_\perp^2)\right)\mathrm{e}^{-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{b}_\perp}$$

  - Contains artifact of barycentric motion
- Spin-zero expressions for simplicity
- Change to butter scheme *just means dropping red term from energy density*

# The End

Thank you for your time!