Light-cone ordered perturbation theory in coordinate space

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1. Motivation

- **In QCD, the concepts of short and long distances are central to factorization, infrared safety.**

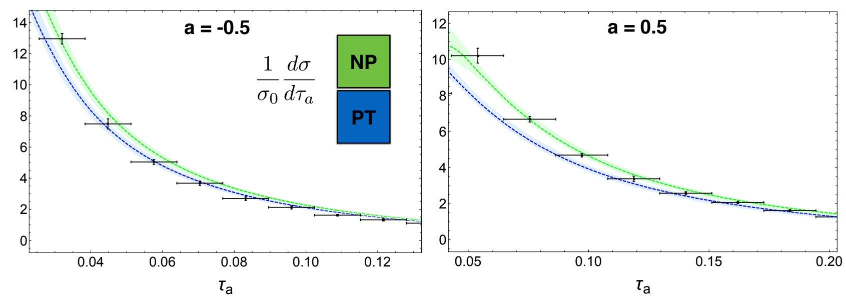
- **Resummed cross sections tie perturbation theory to power corrections (1990s: Webber, Korchemsky & GS, Akhoury & Zakharov, . . . Berger, Kucs & GS) for event shapes like "angularities", with (Berger & GS 2003)**

$$\tau_a(N) = \frac{1}{Q} \sum_{\text{all } i \in N} p_{i\perp}\, e^{-|\eta_i|(1-a)}$$

$$\begin{aligned}
\ln\left[\frac{1}{\sigma_{\text{tot}}}\tilde{\sigma}\left(\nu, Q, a\right)\right] &= 2\left[\int_{\kappa^2}^{Q^2}\frac{dp_\perp^2}{p_\perp^2} + \int_0^{\kappa^2}\frac{dp_\perp^2}{p_\perp^2}\right] A\left(\alpha_s(p_\perp)\right) \\
&\quad \times \int_{p_\perp^2/Q^2}^{p_\perp/Q} \frac{du}{u}\left(e^{-u^{1-a}\nu(p_\perp/Q)^a} - 1\right) + B\text{-term} \\
&\equiv \ln\left[\frac{1}{\sigma_{\text{tot}}}\tilde{\sigma}_{\text{PT}}\left(\nu, Q, \kappa, a\right)\right] \\
&\quad + \frac{2}{1-a}\sum_{n=1}^{\infty}\frac{1}{n\, n!}\left(-\frac{\nu}{Q}\right)^n \int_0^{\kappa^2}\frac{dp_\perp^2}{p_\perp^2} p_\perp^n A\left(\alpha_s(p_\perp)\right). \qquad (5)
\end{aligned}$$

- **where the power corrections "come from small transverse momenta".**

- **Analyses based on these considerations show some success in tying PT and NP together for LEP data (Bell, Hornig, Lee, Talbert, 2015)**



(Data: Achard et al, JHEP 1110 (2011) 143)

- **It might be nice if we can make these concepts tangible in another way. We could imagine posing questions like "Where in space-time do power corrections arise?"**

- **At the EIC, a transition will take place in cold nuclear matter.**

- **With this in mind, we'll develop a coordinate-space version of LCOPT**

2. Light-cone orderings for $\langle 0|T\prod_i \phi(x_i)|0\rangle$: paths

- **Recall: LCOPT in momentum space and LC ordering**

$$G(\{p_a\}) = (-ig)^N \int \prod_{\text{loops}\, i=1}^{\mathcal{L}} \frac{d^4 l_i}{(2\pi)^4} \prod_{\text{lines}\, j=1}^{L} \frac{i}{k_j^2(l_i,p_a) - m^2 + i\epsilon}$$
$$= \sum_{\mathcal{O}} G_{\mathcal{O}}(\{p_a\})$$

- **LC ordered products of states**

$$-i\, G_{\mathcal{O}}(\{p_a\}) = g^N \prod_{\text{loops}\ i} \int \frac{dq_i^+ d^2 q_{i\perp}}{(2\pi)^3} \prod_{\text{lines}\ j} \frac{\theta(k_j^+)}{2k_j^+}$$
$$\times \prod_{\text{states}\ \mathrm{s}\in\mathcal{O}=1}^{N-1} \frac{1}{P^-_{\text{ext}}(s) - \sum_{k_j\in s} \frac{k_{j,\perp}^2+m^2}{2k_j^+} + i\epsilon}$$

k_j^+ is defined in the direction of increasing LC time, and the denominators for each state are "LC energy" deficits.

- **Is there the corresponding form for $G(\{x_a\})$?**

- **Green functions with massless scalars**

$$G(\{x_a\}) = \frac{(-ig)^N}{(4\pi^2)^L} \prod_{\text{vertices } i=1}^{N} \int d^4 y_i \prod_{\text{lines } j=1}^{L} \frac{1}{-z_j^2(y_i, x_a) + i\epsilon}$$

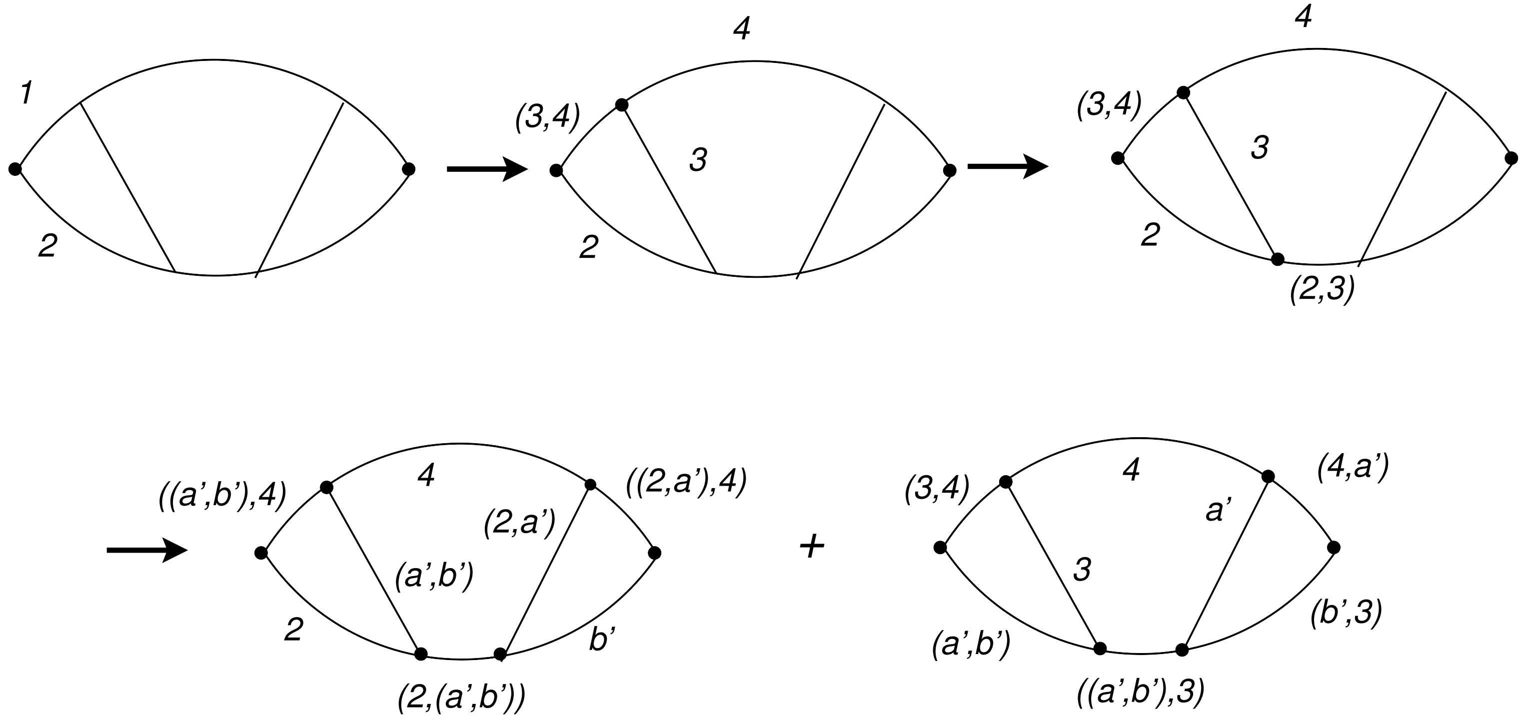
- **with causal propagators**

$$\frac{1}{4\pi^2} \frac{1}{-z^2 + i\epsilon} = \int \frac{d^4 p}{(2\pi)^4} e^{-ip\cdot z} \frac{i}{p^2 + i\epsilon}$$

- **As in momentum space, diagrams are LC ordered ($\mathcal{O}_A$). Each $G_{\mathcal{O}_A}$ is covered by paths from external "in" vertices to external "out" vertices:**

$$G\left(\{x_a\}\right) = \sum_{\mathcal{O}_A} \sum_{\mathcal{C} \in \mathcal{C}[\mathcal{O}_A]} G_{\mathcal{O}_A}^{(\mathcal{C})}\left(\{x_b\}_{\text{out}}, \{x_a\}_{\text{in}}\right)$$

• **Example of assigning 6 paths ($\mathcal{C}$) to a diagram with 8 lines and 6 vertices. There are 6 paths ($1, 2, 3, 4, a'$ and b').**



- **The procedure: write each coordinate propagator as a LC energy Fourier transform,**

$$\frac{1}{-z_j^2(y_i, x_a) + i\epsilon} = -i \int_{-\infty}^{\infty} dE_j^+ \, \frac{\theta\left(E_j^+ z_j^+\right)}{2|z_j^+|} e^{-iE_j^+\left(\eta_{ji} y_i^- + \eta'_{ja} x_a^- - \frac{z_{j\perp}^2 + i\epsilon}{2 z_j^+}\right)}$$

- **Then integrate over LC times (minus) of internal vertices, which give delta functions.**

- **Each internal vertex must receive positive light-cone energy from at least one earlier vertex, and must provide positive light-cone energy to at least one later vertex. Because of the theta functions linking z_j^+ with E_j^+ this eliminates, as in the case of momentum space LCOPT, vertices at which particles emerge from the vacuum or are absorbed by the vacuum.**

- **Finally, integrate over the remaining LC energies, which follow paths from incoming to outgoing vertices. (In momentum LCOPT, state denominators result from integrating over the times of vertices within a given order.)**

- **The result: coordinate LCOPT:**

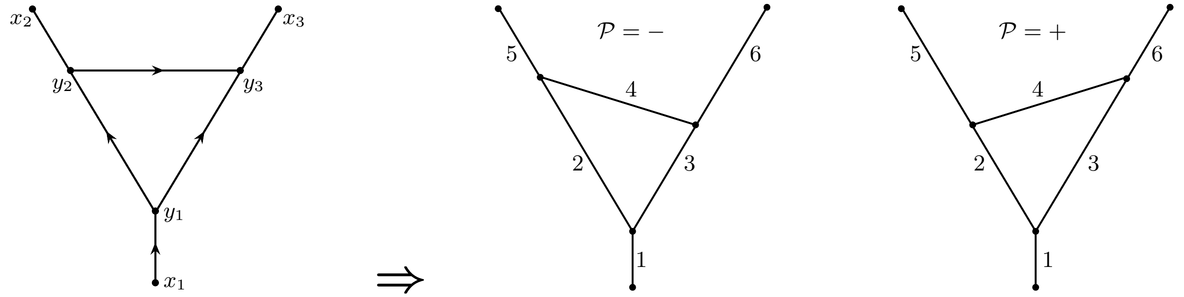
$$G^{(\mathcal{C})}_{\mathcal{O}_A}\left(\{x_b\}_{\rm out},\{x_a\}_{\rm in}\right) = (2\pi)^{N-2L}\,(-g)^N \prod_{i\in N}\int d^2y_{i\perp}\int_{\mathcal{O}_A} dy_i^+ \prod_{j\in L}\frac{\theta(z_j^+)}{2z_j^+}$$
$$\times \prod_{\alpha\in L-N}\frac{-1}{x^-_{b_\alpha} - x^-_{a_\alpha} - D^{(\mathcal{C})}_\alpha - i\epsilon}\,.$$

$$D^{(\mathcal{C})}_\alpha = \sum_{j\in\pi_\alpha}\frac{z^2_{j\perp}}{2z_j^+} \quad \text{where} \quad z_j^\mu \equiv \eta_{ji}y_i^\mu + \eta'_{ja}x_a^\mu$$

where $j \in \pi_\alpha$ labels the lines in path π_α.

- **As in momentum space, lines move forward in LC time, but the "LC energy deficits" are replaced by "LC time deficits" on paths, the difference between the LC time at the finish and start of the path and the sum of "on-shell" LC times between vertices along the path.**

- **An example**



ordering (a)

$$G^{(\mathcal{C}_a)}_{\mathcal{O}_a}\left(\{x_2,x_3\}_{\text{out}},\{x_1\}_{\text{in}}\right) = -\frac{(-g)^3}{(2\pi)^9}\prod_{i=1}^{3}\int d^2y_{i\perp}\int dy_i^+ \prod_{j=1}^{6}\frac{\theta(z_j^+)}{2z_j^+}$$
$$\times\frac{1}{(x_3^- - x_1^- - D_{631} - i\epsilon)}\frac{1}{(x_2^- - x_1^- - D_{521} - i\epsilon)}\frac{1}{(x_2^- - x_1^- - D_{5431} - i\epsilon)}$$

with

$$D_{i_n,\ldots,i_1} = \sum_{j\in\{i_n,\ldots i_1\}}\frac{z_{j\perp}^2 + i\epsilon}{2z_j^+}$$

ordering (b)

$$G^{(\mathcal{C}_b)}_{\mathcal{O}_b}\left(\{x_2,x_3\}_{\text{out}},\{x_1\}_{\text{in}}\right) = -\frac{(-g)^3}{(2\pi)^9}\prod_{i=1}^{3}\int d^2y_{i\perp}\int dy_i^+ \prod_{j=1}^{6}\frac{\theta(z_j^+)}{2z_j^+}$$
$$\times\frac{1}{(x_2^- - x_1^- - D_{521} - i\epsilon)}\frac{1}{(x_3^- - x_1^- - D_{631} - i\epsilon)}\frac{1}{(x_3^- - x_1^- - D_{6421} - i\epsilon)}$$

3. Inclusive & Weighted Cross sections

- Generic "total" cross section

$$\sigma_F(Q) \;=\; \int \prod_{i=1}^{n} \frac{d^4p_i}{(2\pi)^4}\, 2\pi\delta_+(p_i^2)|A(p_1,\dots p_n)|^2(2\pi)^4\delta^4\left(q-\sum_{j=1}^{n}p_j\right) \tag{1}$$

- Case of current induced

$$A(p_1,\dots p_n) = \langle p_1,\dots p_n|J(0)|0\rangle$$

- We have LCOPT for Fourier transformed amplitude

$$\tilde{A}(y_1,\dots y_n,0) \;\equiv\; \prod_{j=1}^{n}\int \frac{d^4p_i}{(2\pi)^4}\, e^{-ip_j\cdot y_j} A(p_1,\dots p_n)\,, \tag{2}$$

and its complex conjugate, starting from

$$\Delta^*(z) \;=\; \frac{1}{4\pi^2}\,\frac{1}{-z^2-i\epsilon}\,. \tag{3}$$

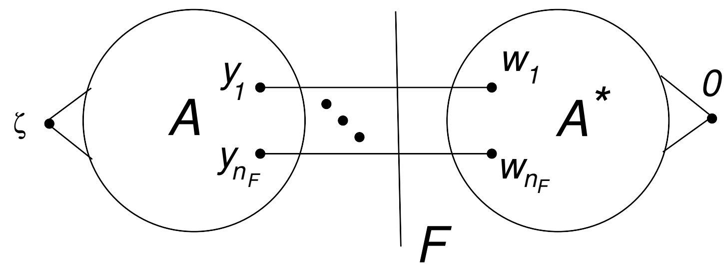
- for the cross section, just need Fourier transform of the final state ("cut") on-shell propagator,

$$\begin{aligned}\Delta_c(z) \;&=\; \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot z}(2\pi)\,\delta_+(p^2)\\ &=\; \frac{1}{4\pi^2}\,\frac{1}{-z^2+i\epsilon z^0}\\ &=\; \frac{1}{4\pi^2}\,\frac{1}{-2(z^+-i\epsilon)(z^--i\epsilon)+z_\perp^2}\,.\end{aligned}$$

- **This defines a coordinate picture for total cross sections**

$$\sigma_F(Q) \;=\; \int d^4\zeta\, e^{-iq\cdot\zeta} \prod_{i=1}^{n} \int d^4w_i\, d^4y_i\, \tilde{A}^*(w_1,\dots w_n,0) \times \prod_{j=1}^{n} \Delta_c\,(w_j - y_j)\; \tilde{A}(y_1,\dots y_n,\zeta)$$

- **represented by**



- **The same picture in terms of ordered paths: but now paths extend from ζ out to largest y_i^+ (always increasing) – then to the largest w_j and (always decreasing) back to 0.**

- **The “Schwinger-Keldysh” like paths follow from reverse order of complex conjugate propagators and lack of order for cut propagators**

$$\frac{1}{-z_j^2+i\epsilon} = \frac{-i}{2|z_j^+|}\int_{-\infty}^{\infty} dE_j^+\, e^{-iE_j^+\left(z_j^- - \frac{z_{j\perp}^2+i\epsilon}{2z_j^+}\right)}\, \theta\left(E_j^+ z_j^+\right)$$

$$\frac{1}{-2(z_j^+-i\epsilon)(z_j^--i\epsilon)+z_{j\perp}^2} = \frac{-i}{2(z_j^+-i\epsilon)}\int_0^{\infty} dE_j^+ e^{-iE_j^+\left(z_j^- - \frac{z_{j\perp}^2}{2(z_j^+-i\epsilon)} - i\epsilon\right)}$$

$$\frac{1}{-z^2-i\epsilon} = \frac{i}{2|z_j^+|}\int_{-\infty}^{\infty} dE_j^+\, e^{-iE_j^+\left(z_j^- - \frac{z_{j\perp}^2-i\epsilon}{2z_j^+}\right)}\, \theta\left(-E_j^+ z_j^+\right)$$

- **Otherwise, energy integration can be carried out as for amplitudes, leading to parth form for cross section.**

- **Current-induced cross section in terms of paths $\mathcal{O}_A, {\mathcal{O}_A}^*$ for final-state F at order $N_A + N_A^*$ is a sum over paths and integral over $+, \perp$ vertex positions:**

$$\sigma_F^{(N_A,N_{A^*})}(Q) = \frac{(-g)^{N_A}(g)^{N_{A^*}}}{(2\pi)^{2L-N}} \sum_{\mathcal{O}_A \mathcal{O}_{A^*}} \prod_{i=1}^{N_A} \int d^2 y_{i\perp} \int_{\mathcal{O}_A} dy_i^+$$
$$\times \prod_{i'=1}^{N_{A^*}} \int d^2 w_{i'\perp} \int_{\mathcal{O}_{A^*}} dw_{i'}^+ \int d^4\zeta \, e^{-iq\cdot\zeta}$$
$$\times \sum_{\mathcal{C}\in\mathcal{C}[\mathcal{O}_A \mathcal{O}_{A^*}]} G^{(\mathcal{C})}_{\mathcal{O}_A \mathcal{O}_{A^*}}(y_i^+, y_{i\perp}, w_{i'}^+, w_{i'\perp}, \zeta) \tag{4}$$

- **With integrand:**

$$G^{(\mathcal{C})}_{\mathcal{O}_A \mathcal{O}_{A^*}}(y_i^+, y_{i\perp}, w_{i'}^+, w_{i'\perp}, \zeta) = \prod_{j=1}^{L_A} \frac{\theta(z_j^+)}{2z_j^+} \prod_{k=1}^{L_F} \frac{1}{2(z_k^+ - i\epsilon)} \prod_{j'=1}^{L_{A^*}} \frac{\theta(-z_{j'}^+)}{2z_{j'}^+}$$
$$\times \prod_{\pi_\alpha \in \mathcal{C}} \frac{-1}{\left[0^- + \sum_{j'\in\pi_\alpha^{(A^*)}} \frac{z_{j'\perp}^2}{2|z_{j'}^+|} - \frac{z_{\pi_F\perp}^2}{2(z_{\pi_F}^+ - i\epsilon)} - \sum_{j\in\pi_\alpha^{(A)}} \frac{z_{j\perp}^2}{2z_j^+} - \zeta^- - i\epsilon\right]} .$$

- **LC time deficits follow the path out to infinity and back. If z_j^μ are collinear, can generate infrared divergences as times on the way back cancel those on the way out.**

4. Short distances and beyond

- Individual final states generally show infrared divergences, which should cancel with appropriate sums over state (KLN)

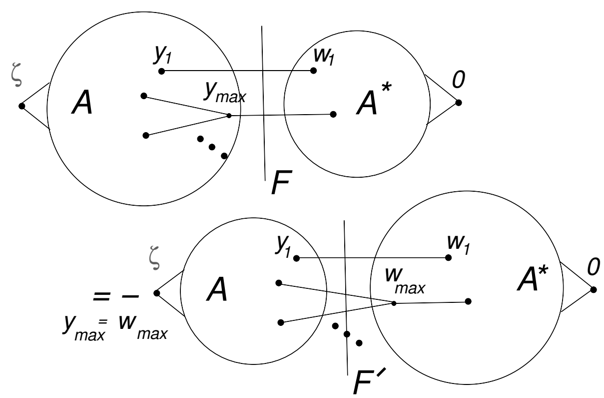
- For example, collinear singularities, when $\beta^2 \equiv z_{j\perp}^2/(2(z_j^+)^2)$ same for all lines on a path, then can have:

$$\sum_{j' \in \pi_\alpha^{(A^*)}} \frac{z_{j'\perp}^2}{2|z_{j'}^+|} - \frac{z_{\pi_F\perp}^2}{2(z_{\pi_F}^+ - i\epsilon)} - \sum_{j \in \pi_\alpha^{(A)}} \frac{z_{j\perp}^2}{2z_j^+} = -\beta^2 \left(\sum_{j' \in \pi_\alpha^{(A^*)}} z_{j'}^+ + z_{\pi_F}^+ + \sum_{j \in \pi_\alpha^{(A)}} z_j^+ \right)$$

- The $z_{j'}^+$ are all < 0, so the integrals over positions of vertices are infinite, giving same collinear singularities as in momentum space.
- This takes some power counting for scalar and gauge theories; let's assume that.
- So, what happens?

- **When the vertex with the largest LC time "crosses" the final state, the integrand is almost unchanged, only $(-1)^{N_A}$ changes sign in:**

$$(-1)^{N_A} \prod_{j=1}^{L_A} \frac{\theta(z_j^+)}{2z_j^+} \prod_{k=1}^{L_F} \frac{1}{2(z_k^+ - i\epsilon)} \prod_{j'=1}^{L_{A^*}} \frac{\theta(-z_{j'}^+)}{2z_{j'}^+}$$



- **This is the "largest-time" equation (Veltman). In this total cross section, only vertices between 0 and ζ^+ survive! This is an expression of unitarity.**

- **Where this is going . . .**

- **Infared safe cross wighted sections:**

$$\sigma_F[Q,h] = \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4}\, 2\pi\delta_+(p_i^2)\, h_n(p_1,\dots,p_n)\, |A(p_1,\dots p_n)|^2 (2\pi)^4 \delta^4\left(q - \sum_{j=1}^{n} p_j\right)$$

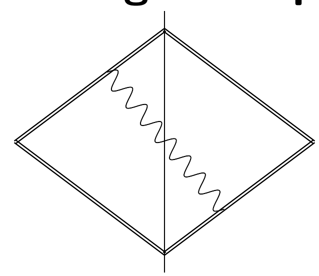
- **IR safety:** $h_n(\dots p_{n-1}, p_n) \;=\; h_n(\dots p_{n-1}, \alpha p_n, (1-\alpha)p_n)$, $0 \le \alpha \le 1$ **(**h_n **symmetric).**

In coordinate space, a convolution,

$$\begin{aligned} \sigma_F[Q,h] \;&=\; \int d^4\zeta\, e^{-iq\cdot\zeta} \prod_{i=1}^{n} \int d^4 w_i\, d^4 y_i\, \tilde{A}^*(w_1,\dots w_n,0) \\ &\times \prod_{j=1}^{n} \int d^4 r_j\, \Delta_c\,(w_j - y_j - r_j)\; \tilde{h}(r_1,\dots,r_n)\, \tilde{A}(y_1,\dots y_n,\zeta)\,. \end{aligned} \tag{5}$$

- **In principle, we can see how cancellation limits regions in coordinate space, depending on the weight function involved.**

- **Illustration, Laplace weghts of thrust at lowest order in the "soft function" (color-singlet product of light-like Wilson lines meeting at cusp vertex).**



- **Fourier transform for Laplace moments of thrust ($a = 0$ angularity):**

  Thrust weight for the soft function (Wilson lines in LC $\pm$ directions):

$$f_\tau(k) \;=\; \delta\left(\min\left(\mathrm{k}^+/\mathrm{Q}\,,\,\mathrm{k}^-/\mathrm{Q}\right) - \tau\right)$$

  Laplace moments (for each final-state particle)

$$f_N(k) \;=\; \int d\tau\, e^{-N\tau}\, f_\tau(k)$$

  Fourier transform ($\nu \equiv \frac{N}{Q}$) of the thrust:

$$\begin{aligned}\tilde{\tau}_\nu(r) \;&=\; \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot r}\, [e^{-\nu k^-}\,\theta(k^+ - k^-)\,\theta(k^-) + e^{-\nu k^+}\,\theta(k^- - k^+)\,\theta(k^+)] \\ &=\; \frac{\delta^2(r_\perp)}{(2\pi i)^2}\,\frac{1}{r^+ + r^- - i\nu}\left(\frac{1}{r^+ - i\epsilon} + \frac{1}{r^- - i\epsilon}\right)\end{aligned}$$

- **Real-gluon final state as a convolution**

$$\boldsymbol{\sigma}_{\rm LO}^{\rm real} = \frac{\alpha_g}{2\pi}\int_0^\infty d\lambda d\sigma \int_{-\infty}^{\infty} \frac{dr^+dr^-}{(2\pi i)^2}\frac{1}{r^+ + r^- - i\nu}\left(\frac{1}{r^+ - i\epsilon} + \frac{1}{r^- - i\epsilon}\right) \times \frac{1}{2(\lambda + r^+ + i\epsilon)(\sigma - r^- - i\epsilon)}$$

$$= \frac{\alpha_g}{4\pi}\int_0^\infty d\lambda d\sigma \; \frac{1}{\sigma - \lambda - i\nu}\left(\frac{1}{\lambda} - \frac{1}{\sigma}\right)$$

- **virtual (μ renormalzation scale for cusp)**

$$\sigma_{\rm LO}^{\rm vir} = -\frac{\alpha_g}{4\pi}\int_{1/\mu}^{\infty} d\lambda d\sigma \; \frac{1}{\lambda\sigma}$$

- **The real and virtual cancel beyond length scale given by $\nu = N/Q$, as expected from "largest time" . . .**

- **All together, with complex-conjugate diagrams**

$$\sigma(\nu, Q) \ = \ \frac{\alpha_g}{2\pi} \int_{1/Q}^{\infty} d\lambda d\sigma \ \frac{1}{\lambda\sigma} \left(\frac{\nu^2}{(\sigma - \lambda)^2 + \nu^2} \right)$$

- **Worth noting: $\lambda = \sigma$ enhanncement in denominator results in power correction like ν (linear) as for momentum space.**

- **Integral is suppressed when $\lambda\sigma \gg \nu^2$.**

- **For $\lambda\sigma \ll \nu^2$, it goes like $\ln^2(\mu\nu)$, and in summary,**

$$\sigma(\nu, Q) \ = \ \frac{\alpha_g}{2\pi} \int_{1/Q}^{\nu} d\lambda d\sigma \ \frac{1}{\lambda\sigma} \left(\frac{\nu^2}{(\sigma - \lambda)^2 + \nu^2} \right)$$

- **Natural projection to all orders (using an exact result for the virtual cusp form factor):**

$$\sigma(\nu, Q) \ = \ \int_{1/Q}^{\nu} d\lambda d\sigma \ \frac{A(\alpha_s(1/\lambda\sigma))}{\lambda\sigma} \left(\frac{\nu^2}{(\sigma - \lambda)^2 + \nu^2} \right)$$

with $A(\alpha)$ the cusp anomalous dimension of the theory in question.
Transition at $\lambda\sigma = 1/\Lambda^2_{\rm QCD}$

5. Summary

- **LCOPT theory in coordinate space gives a role to paths.**

- **IR cancellation can be realized in coordinate space.**

- **Suggests the possibility of a geometric picture of the perturbative-nonperturbative transition.**