Rotating the CGC:
Paving the way for precision computations

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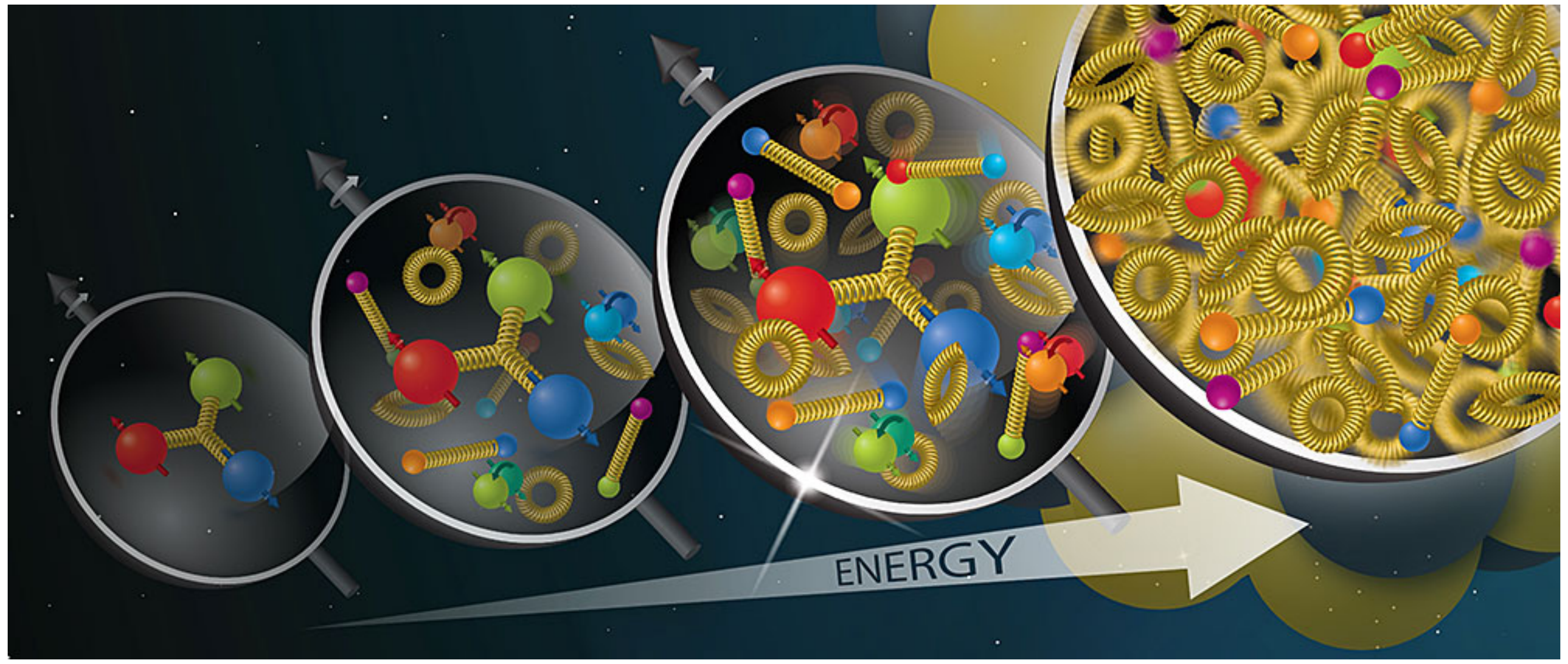
High Energy Limit of QCD

- At high energies (small x), the gluon density rises rapidly and is expected to saturate leading to a new state of QCD matter
- This regime is currently being explored at **RHIC** and the **LHC**, and will be a central focus of the **EIC** experimental program
- To reach predictive power, we need precision theoretical tools — yet progress has been hindered by several conceptual and technical roadblocks



Partons → Classical gluon fields

$$x_{\mathrm{Bj}} = \frac{Q^2}{s} \to 0$$

Work in collaboration with

R. Boussarie, P. Caucal and P. Korcyl

2509.20236 [hep-ph]

2511.07509 [hep-ph] (To appear in PRL)

Outline

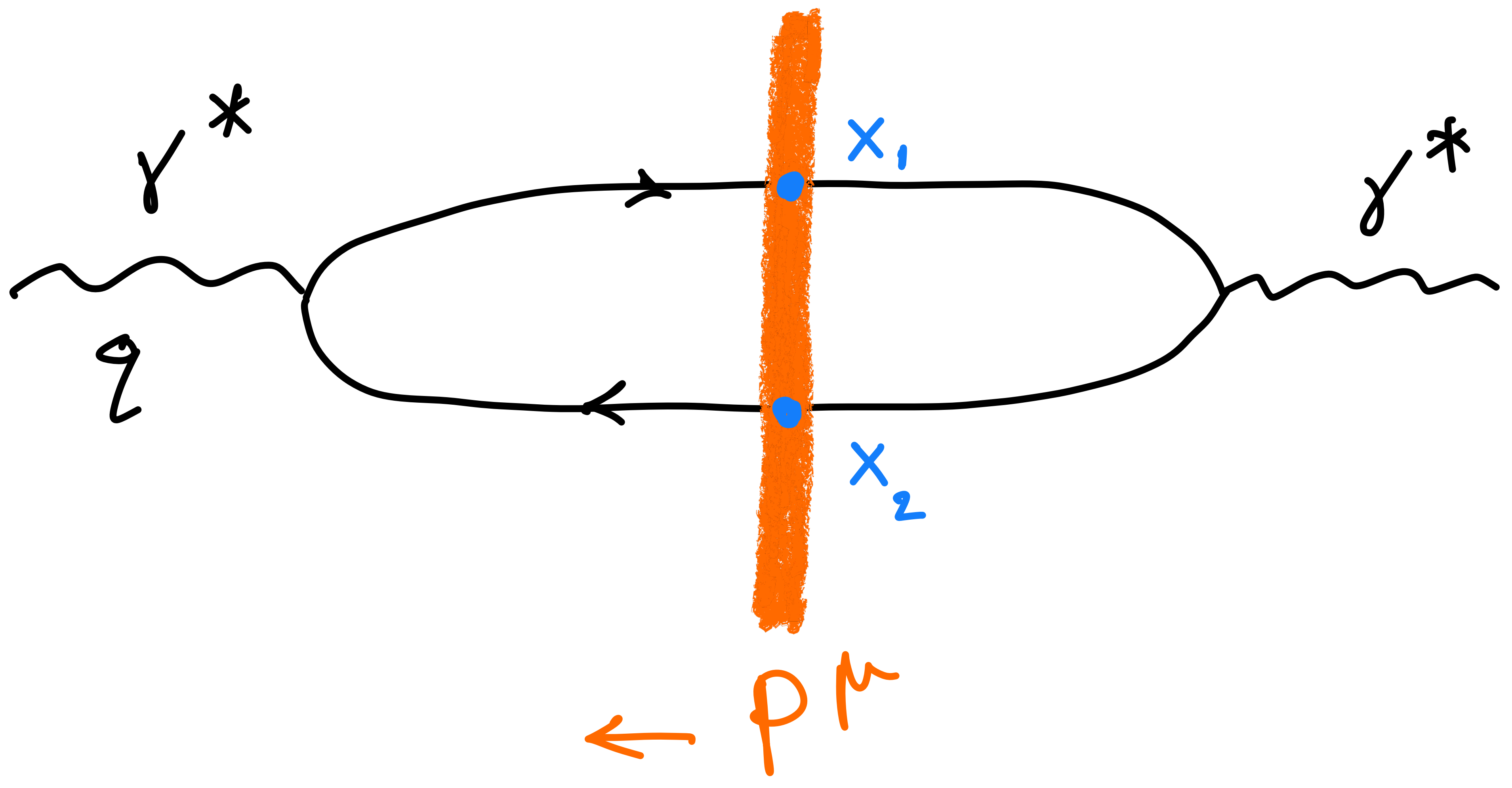
- The origin of instabilities in small-x evolution
- Factorization and scheme transformation
- The rotated dipole and the NLO BK equation
- Numerical simulations
- Outlook

The origin of instabilities in small-x evolution

High Energy Factorization in DIS

- In the dipole frame: DIS cross-section factorizes into a hard impact factor H and a hadronic matrix element S_{12}
- $S_{12} \sim \langle \mathrm{tr}\, U_2^\dagger U_1 \rangle$ encodes the instantaneous interaction of a $q\bar{q}$ dipole with the target (proton or nucleus)

"Dipole scattering in the target field"



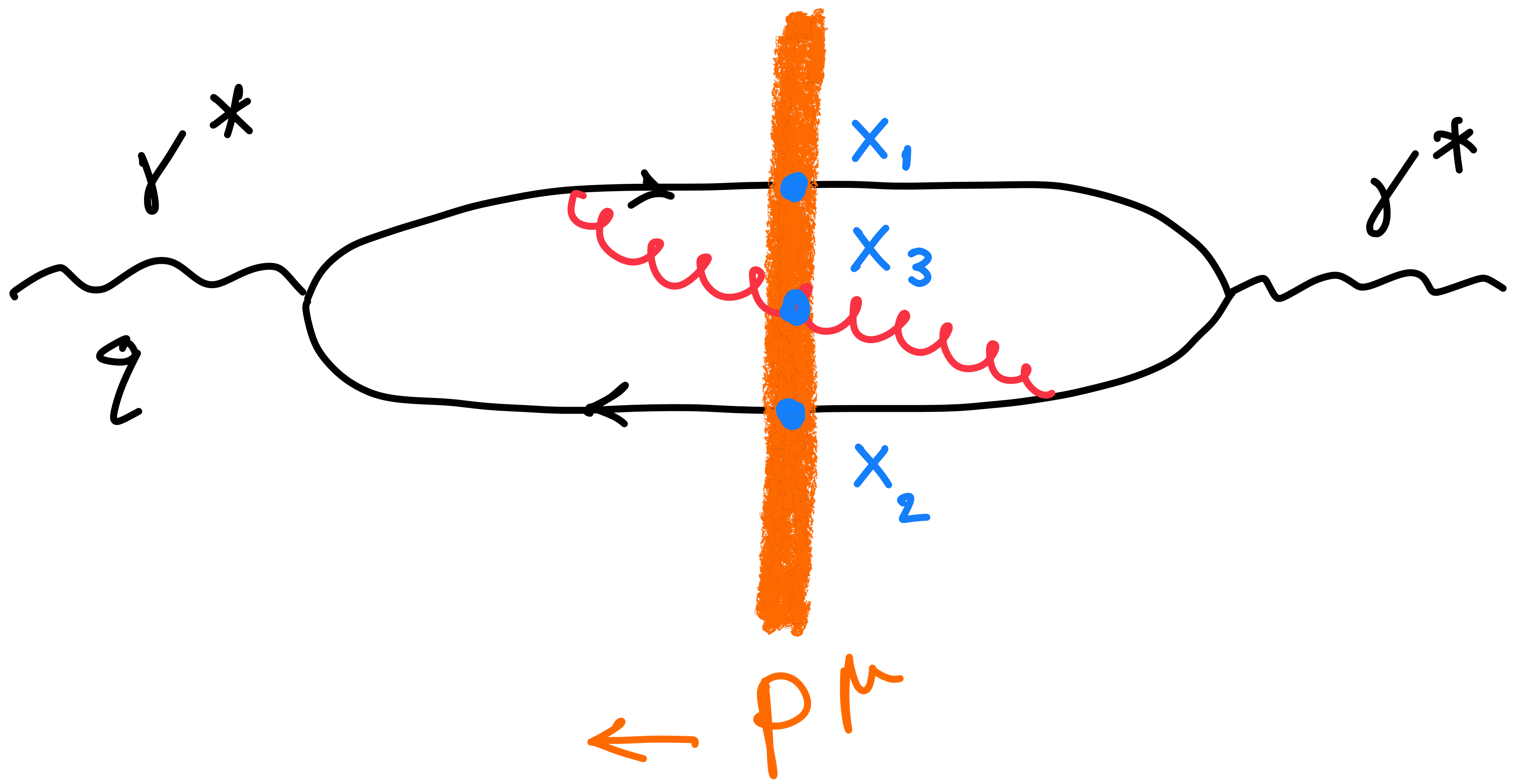
$$\boxed{\sigma_{\mathrm{LO}}(x, Q^2) = \int_{\boldsymbol{x}_1, \boldsymbol{x}_2} H(Q^2, \boldsymbol{x}_1, \boldsymbol{x}_2)\, S_{12}}$$

LO Balitsky-Kovchegov evolution (1997)

- Rapidity divergence at one loop

$$\int^{+\infty} \frac{\mathrm{d}k^+}{k^+} \to \int^{\rho^+} \frac{\mathrm{d}k^+}{k^+} \sim \ln \rho^+$$

"Dipole scattering in the target field"



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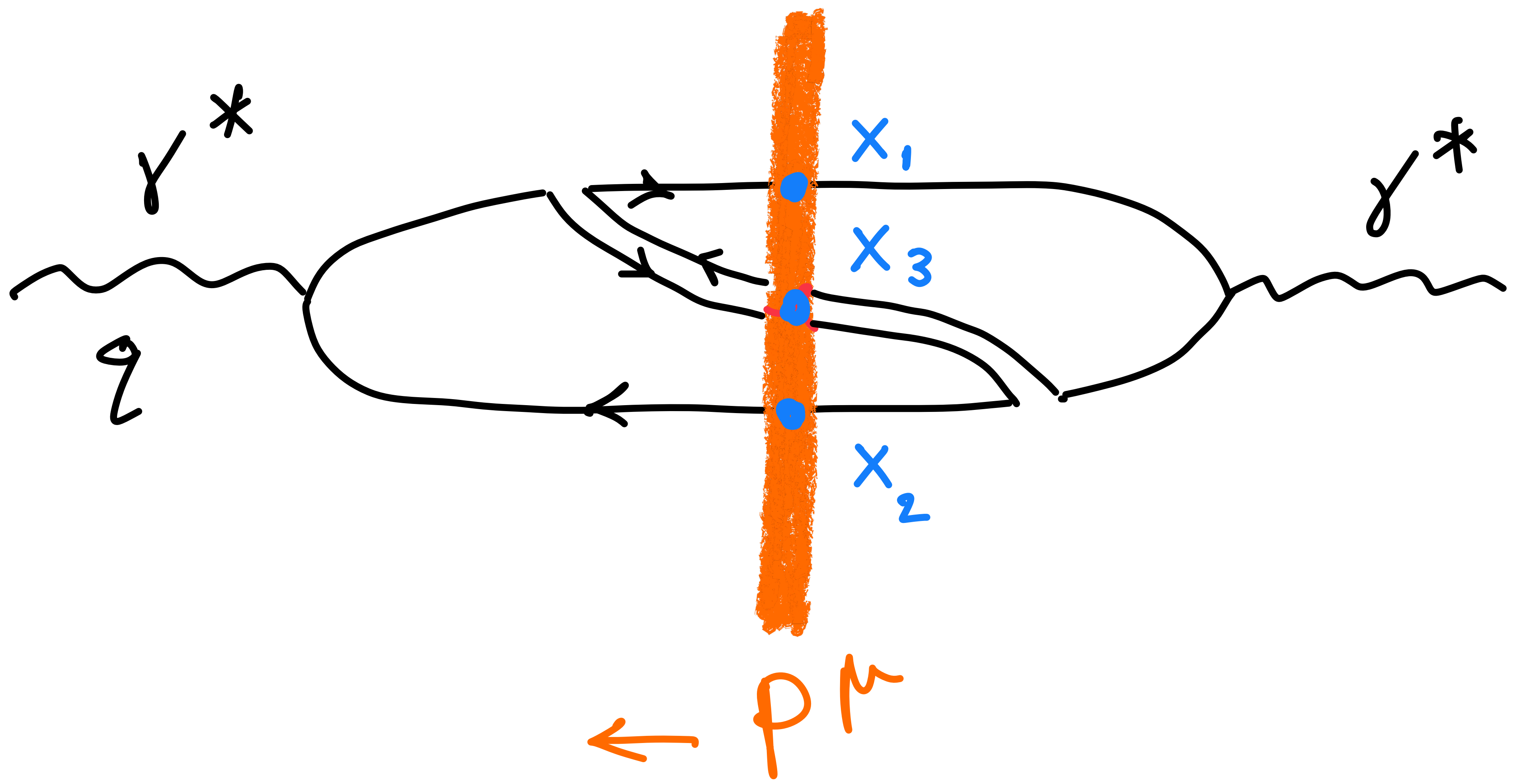
Large N_c limit

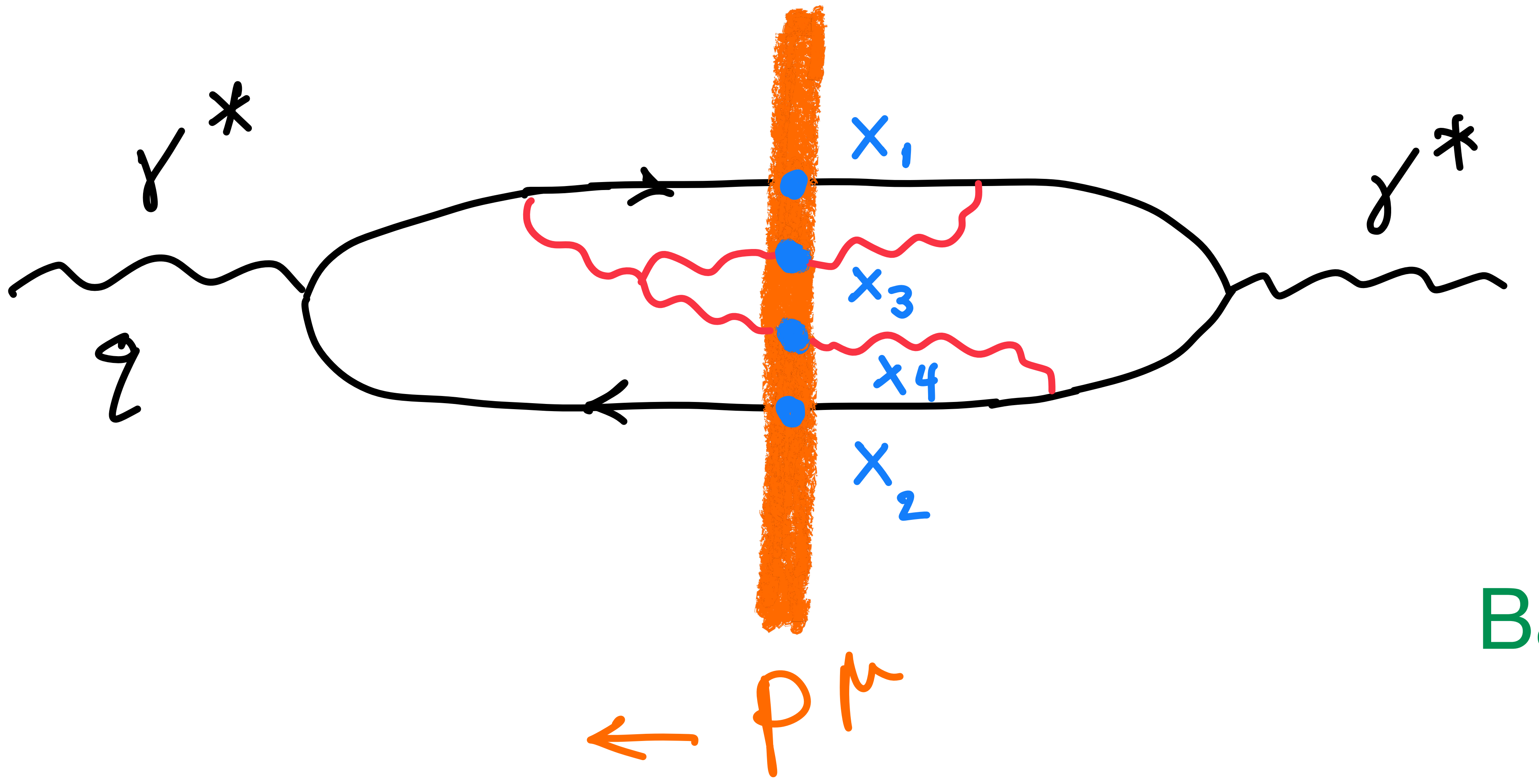


- Resummation via Balitsky-Kovchegov equation (1997)

$$\frac{\partial}{\partial \ln \rho^+} S_{12} = \bar{\alpha} \int_3 K_{123} \left[S_{13} S_{32} - S_{12} \right]$$

$$K_{123} = \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2}$$

NLO Balitsky-Kovchegov evolution



Balitsky, Chirilli (2007)

$$\frac{\partial}{\partial \ln \rho^+} S_{12} = \bar{\alpha} \int_3 K_{123}^{\mathrm{NLO}} \left[S_{13} S_{32} - S_{12}\right] + \bar{\alpha}^2 \int_3 K_{1234} \left[S_{13} S_{34} S_{42} - S_{13} S_{32}\right]$$

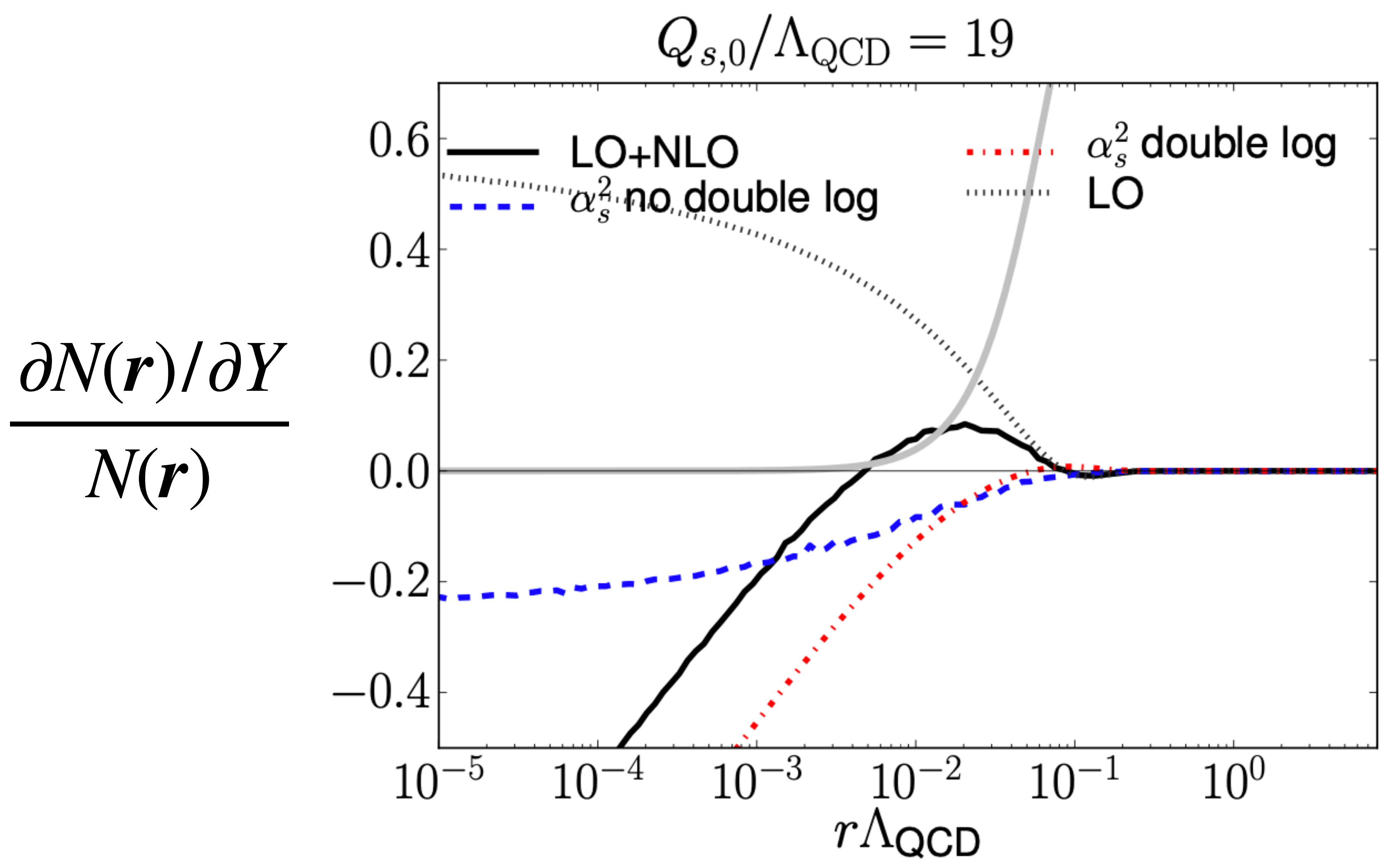
Similar color structure to LO

3 dipole term

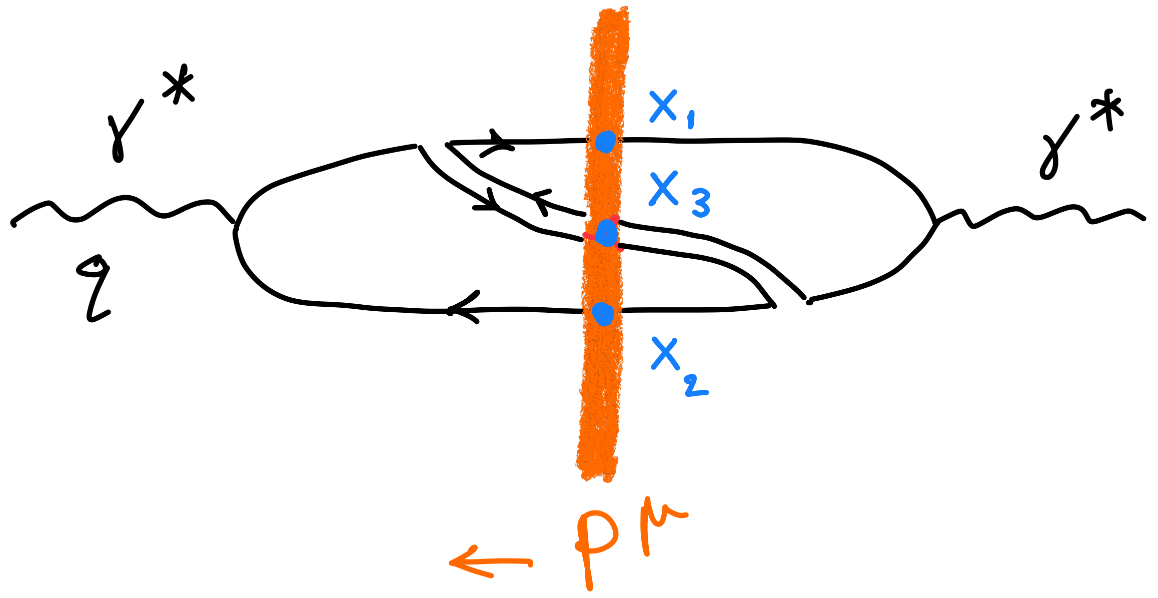
Unstable evolution

- NLO BK equations shown by Lappi and Mäntysaari (2015) to be unstable
- Instability driven by collinear double logarithms
- NB: $N = 1 - S$ and $\boldsymbol{r} \equiv \boldsymbol{x}_{12}$

"Evolution speed"



$$\frac{\partial}{\partial Y} S_{12} = \bar{\alpha} \int_3 K_{123} \left(1 - 2\bar{\alpha} \ln \frac{\boldsymbol{x}_{13}^2}{\boldsymbol{x}_{12}^2} \ln \frac{\boldsymbol{x}_{32}^2}{\boldsymbol{x}_{12}^2} \right) [S_{13} S_{32} - S_{12}] + O(\bar{\alpha}^2)$$



Large negative term when $\boldsymbol{x}_{12} \to 0$

Diagnosing small-x physics

- Similar instabilities were already observed in **NLO BFKL** [Salam, Ciafaloni, Colferai (1998)]
- Related issues:
  - **Negative cross sections** in forward hadron production in pA collisions [Shi, Wang, Wei, Xiao (2021); Liu, Xie, Kang, Liu, (2022)]
  - Wrong sign of Sudakov logarithms [Caucal, Salazar, Schenke, Venugopalan, (2022); Taels, Altinoluk, Beuf, Marquet (2022)]
- Proposed solutions:
  - i) *M*odification of the LO BK kernel to resum double logarithms [Beuf (2014)]
  - ii) Change of rapidity variable [Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos (2015-2019)]

- So far, proposed fixes remain formally unsatisfactory and offer no insight into the all-order structure of the high-energy OPE — its factorization, operator content, and universality

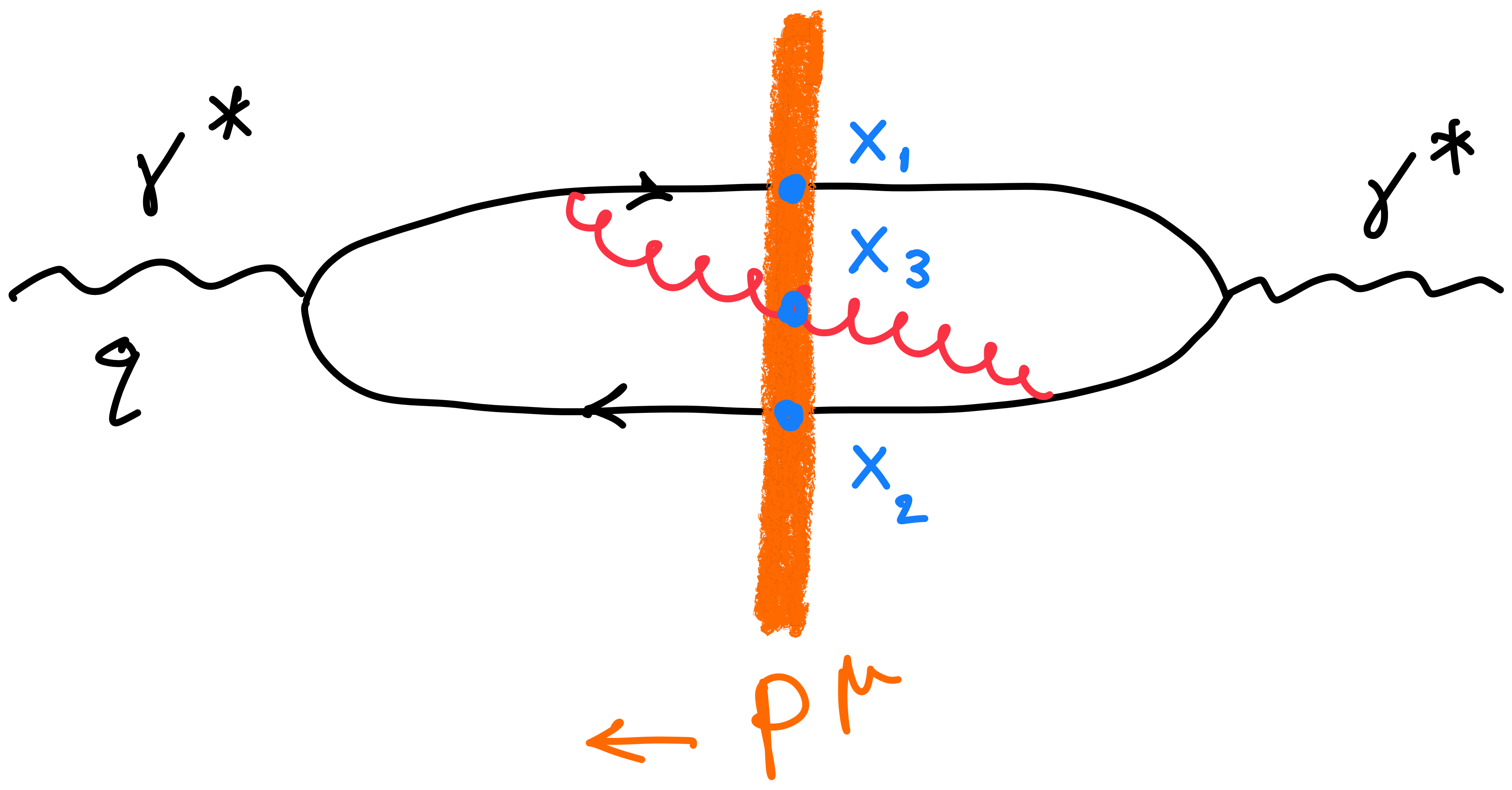
- In this work, we construct a systematic, OPE-consistent framework that aims at stabilizing small-x evolution

Caucal, Boussarie, Korcyl, MT, 2511.07509 [hep-ph] (To appear in PRL)

Emergence of collinear logs

- Standard high energy OPE ignores the constraint associated with the size of the target:

$$t_g > t_{\text{target}} \implies k^- < P^- \sim \sqrt{s}$$



- The double logarithmic phase space in high energy OPE ($k^- \sim k_\perp^2/k^+$)

$$\bar{\alpha}\int_{Q_0^2}^{Q^2}\frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2}\int_{q_0^+}^{\rho^+}\frac{\mathrm{d}k^+}{k^+} = \bar{\alpha}\ln\frac{\rho^+}{q_0^+}\ln\frac{Q^2}{Q_0^2}$$

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- The double logarithmic phase space in high energy OPE ($k^- \sim k_\perp^2/k^+$)

$$\bar{\alpha} \int_{Q_0^2}^{Q^2} \frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2} \int_{\boldsymbol{k}^2/P^-}^{\rho^+} \frac{\mathrm{d}k^+}{k^+} = \bar{\alpha} \ln \frac{\rho^+ P^-}{Q_0^2} \ln \frac{Q^2}{Q_0^2} - \frac{\bar{\alpha}}{2} \ln^2 \frac{Q^2}{Q_0^2}$$

Kinematic constraint generates double logs

Emergence of collinear logs

- The basic idea behind our approach is to construct a scheme transformation that eliminates the undesired collinear logarithms
- Introduce a collinear factorization scale μ to partition the phase-space:

$$\int_{\boldsymbol{k}_\perp^2/P^-}^{\rho^+} \frac{\mathrm{d}k^+}{k^+} = \int_{\mu^2/P^-}^{\rho^+} \frac{\mathrm{d}k^+}{k^+} + \int_{\boldsymbol{k}_\perp^2/P^-}^{\mu^2/P^-} \frac{\mathrm{d}k^+}{k^+} = \ln\frac{\rho^+ P^-}{\mu^2} - \ln\frac{\boldsymbol{k}_\perp^2}{\mu^2}$$

Rapidity logarithm

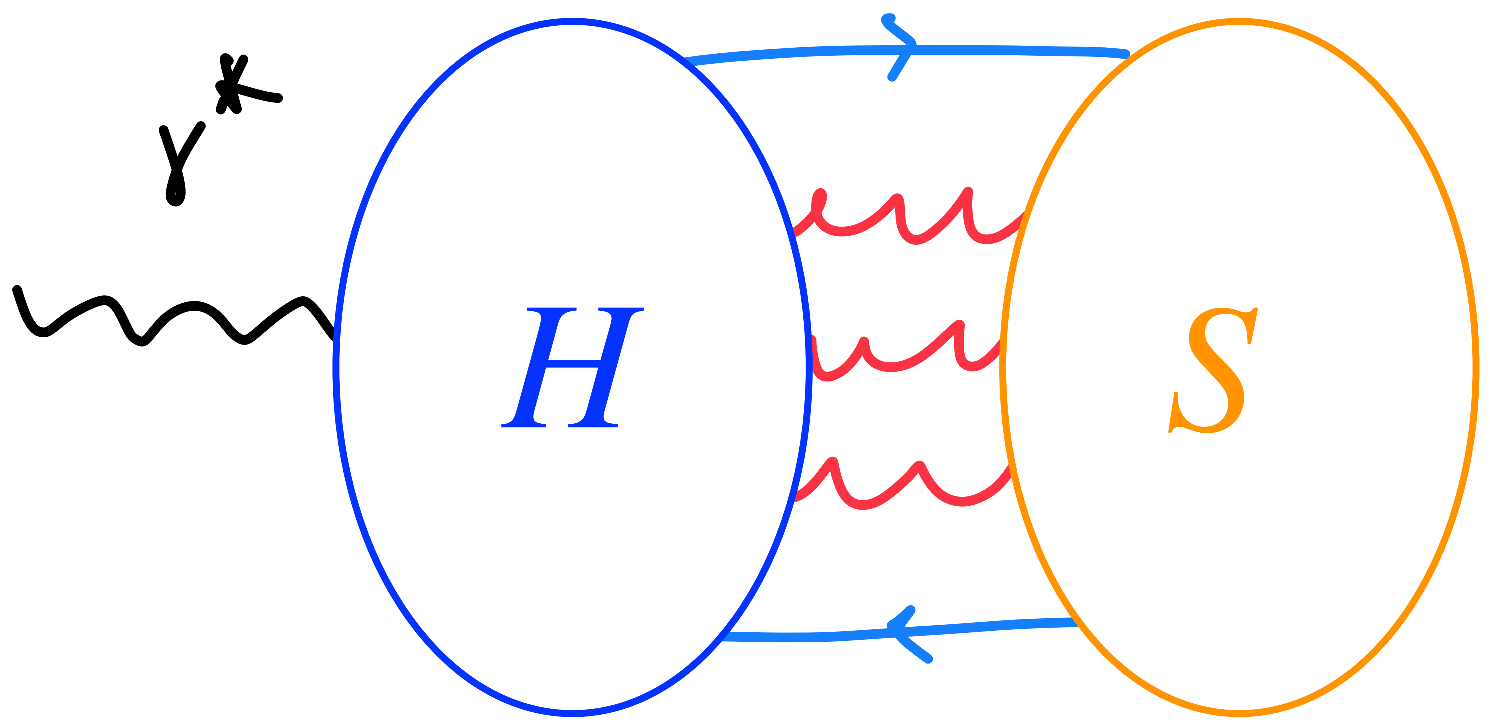
Collinear logarithm

- Subtraction is not straightforward due to non-linear dynamics

Factorization
and Scheme Transformation

Linear algebra for non-linear dynamics

- The all-order structure of the factorization reads:

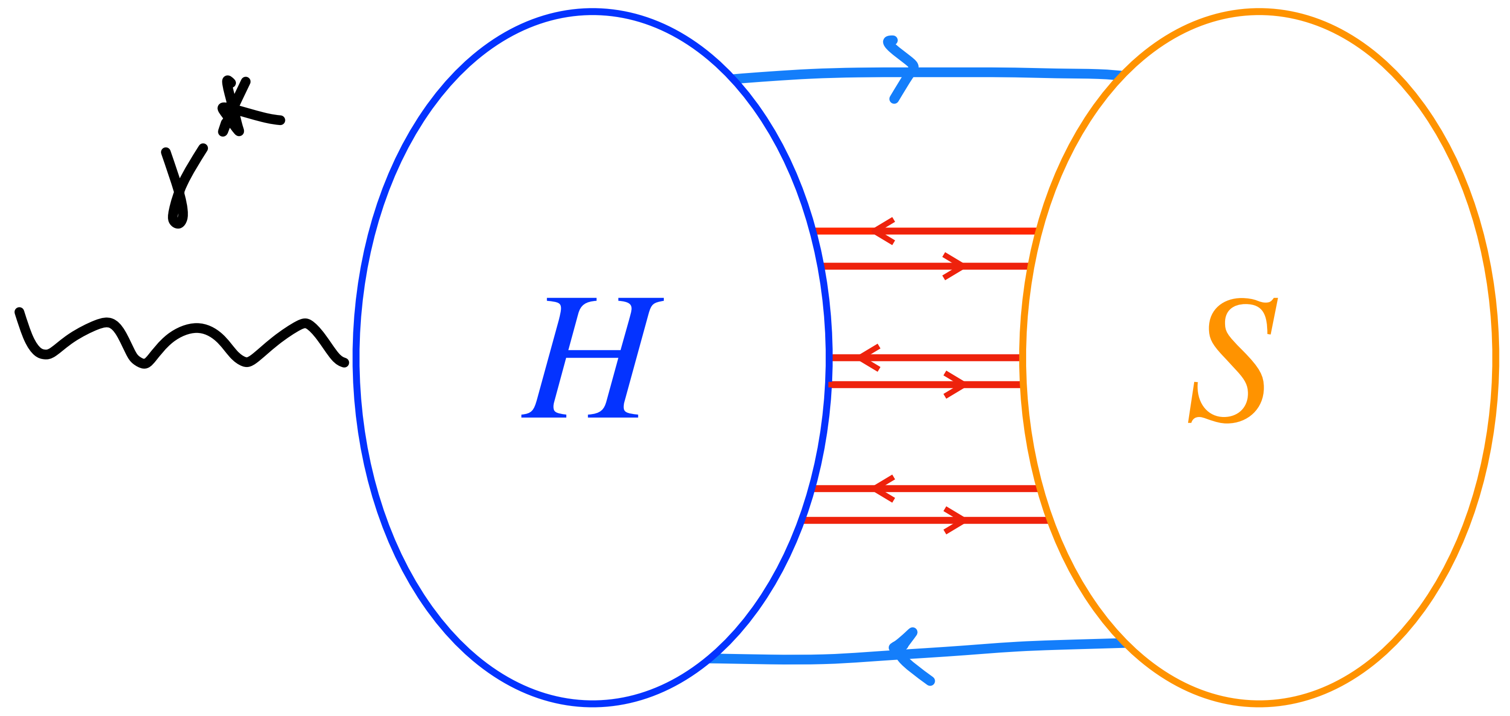


$$\boxed{\sigma(x,Q^2) = \sum_{n=2}^{\infty} \int_{\boldsymbol{x}_i} H^{(n)}(Q^2, \{\boldsymbol{x}_i\}_{i=1}^n, \zeta/s) \times \left(S^{(n)}(\{\boldsymbol{x}_i\}_{i=1}^n, \zeta) - 1\right)}$$

- $S^{(n)} \sim \langle U_1 U_2 \ldots U_n \rangle$ is the correlator of n Wilson lines and $\zeta = \rho^+ P^-$ the factorization scale

Linear algebra for non-linear dynamics

- In the large N_c limit the d.o.f.'s are dipoles

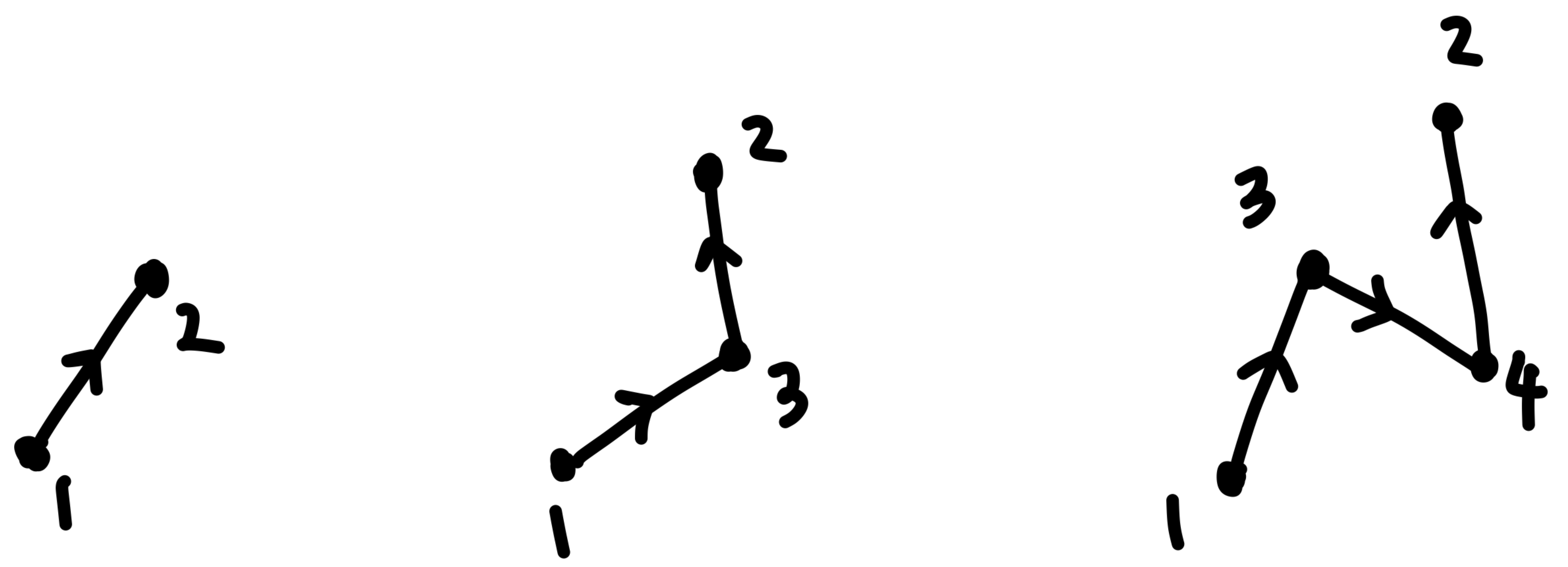


$$\boxed{\sigma(x, Q^2) = \sum_{n=2}^{\infty} \int_{\boldsymbol{x}_i} H^{(n)}(Q^2, \{\boldsymbol{x}_i\}_{i=1}^{n}, \zeta/s) \times (S^{(n)}(\{\boldsymbol{x}_i\}_{i=1}^{n}, \zeta) - 1)}$$

- $S^{(n)} \sim \langle U_1 U_2 \ldots U_n \rangle$ is the correlator of n Wilson lines and $\zeta = \rho^+ P^-$ the factorization scale

Linear algebra for non-linear dynamics

- We introduce the “dipole chain” basis



$|0)$ $\quad |\boldsymbol{x}_1\boldsymbol{x}_2)$ $\quad |\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{x}_2)$ $\quad |\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{x}_4\boldsymbol{x}_2)$ $\quad \ldots$

- We adopt the Dirac notation (Doi-Peliti formalism (1976-1986) for probability transitions in stochastic processes)

$$S^{(n)} \equiv (\boldsymbol{x}_1\boldsymbol{x}_3\ldots\boldsymbol{x}_n\boldsymbol{x}_2|S) = S_{13}S_{34}\ldots S_{n2}$$

Linear algebra for non-linear dynamics

- In these notations the solution to the BK equation exponentiates (as in the JIMWLK approach)

$$|S(\zeta)) = \mathrm{e}^{-\bar{\alpha}K(\bar{\alpha})\ln\frac{\zeta}{\zeta_0}}\,|S(\zeta_0))$$

- In the dipole chain formalism the LO BK equation reads

$$\frac{\partial}{\partial\ln\zeta}(2|S) = \bar{\alpha}\left[(2|K|3)(3|S) + (2|K|2)(2|S)\right]$$

- With the transition rates given by

$$(2|K|3) \sim \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2\boldsymbol{x}_{32}^2} \qquad (2|K|2) \sim -\int_3 \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2\boldsymbol{x}_{32}^2}$$

where

$$|2) \equiv |\boldsymbol{x}_1\boldsymbol{x}_2)$$

$$|3) \equiv |\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{x}_2)$$

The rotated dipole and the NLO BK equation

The rotated dipole

- A scheme transformation is achieve via a nonunitary operator

$$\boxed{|\bar{S}(\zeta,\mu^2)) = \mathrm{e}^{-\bar{\alpha}L(\bar{\alpha},\mu^2)}\,|S(\zeta))}$$

- The L operator is constructed to all order by imposing that the composite dipole obeys the property (to match collinear logs with rapidity logs)

$$\frac{\partial\bar{S}}{\partial \ln\zeta} = -\frac{\partial\bar{S}}{\partial \ln\mu^2} \qquad \Longrightarrow \qquad \bar{S}(\zeta,\mu^2) \equiv \bar{S}\left(\frac{\zeta}{\mu^2}\right)$$

- Note that choosing $\zeta = s$ and $\mu^2 = Q^2 \Longrightarrow$ $\boxed{Y = \ln(\zeta/\mu^2) = \ln(1/x_{Bj})}$

The rotated dipole

- Expanding in $\bar{\alpha}$: $K = K_1 + \bar{\alpha}\, K_2 + \ldots$ and $L = L_1 + \bar{\alpha}\, L_2 + \ldots$

- The scaling constraint yields at NLO

$$\frac{\partial}{\partial \ln \mu^2} L_1(\mu^2) = K_1$$

- Residual freedom in fixing the constant in L

$$(2|L_1(\mu^2)|3) \sim \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2} \ln\left(\mu^2 |\boldsymbol{x}_{13}||\boldsymbol{x}_{32}|\right)$$

The rotated dipole at NLO

- Expending the exponential to NLO the composite dipole reads

$$\boxed{\bar{S}_{12}(\zeta/\mu^2) = S_{12}(\zeta) + \bar{\alpha}\int_3 \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2\boldsymbol{x}_{32}^2}\ln\left(\frac{1}{|\boldsymbol{x}_{13}||\boldsymbol{x}_{32}|\mu^2}\right)[S_{13}S_{32} - S_{12}] + \mathcal{O}(\bar{\alpha}^2)}$$

- This approach encompasses Balitsky's conformal dipole (2008)

- Similar transformation for the Impact factor (new NLO term)

The rotated NLO evolution

- Transforming the NLO BK kernel using the Campbell identity:

$$\begin{aligned}\bar{K} &= \mathrm{e}^{-\bar{\alpha}L} K\, \mathrm{e}^{+\bar{\alpha}L} \\ &= K + \bar{\alpha}[K,L] + \frac{\bar{\alpha}^2}{2!}[[K,L],L] + \mathcal{O}(\bar{\alpha}^3)\end{aligned}$$

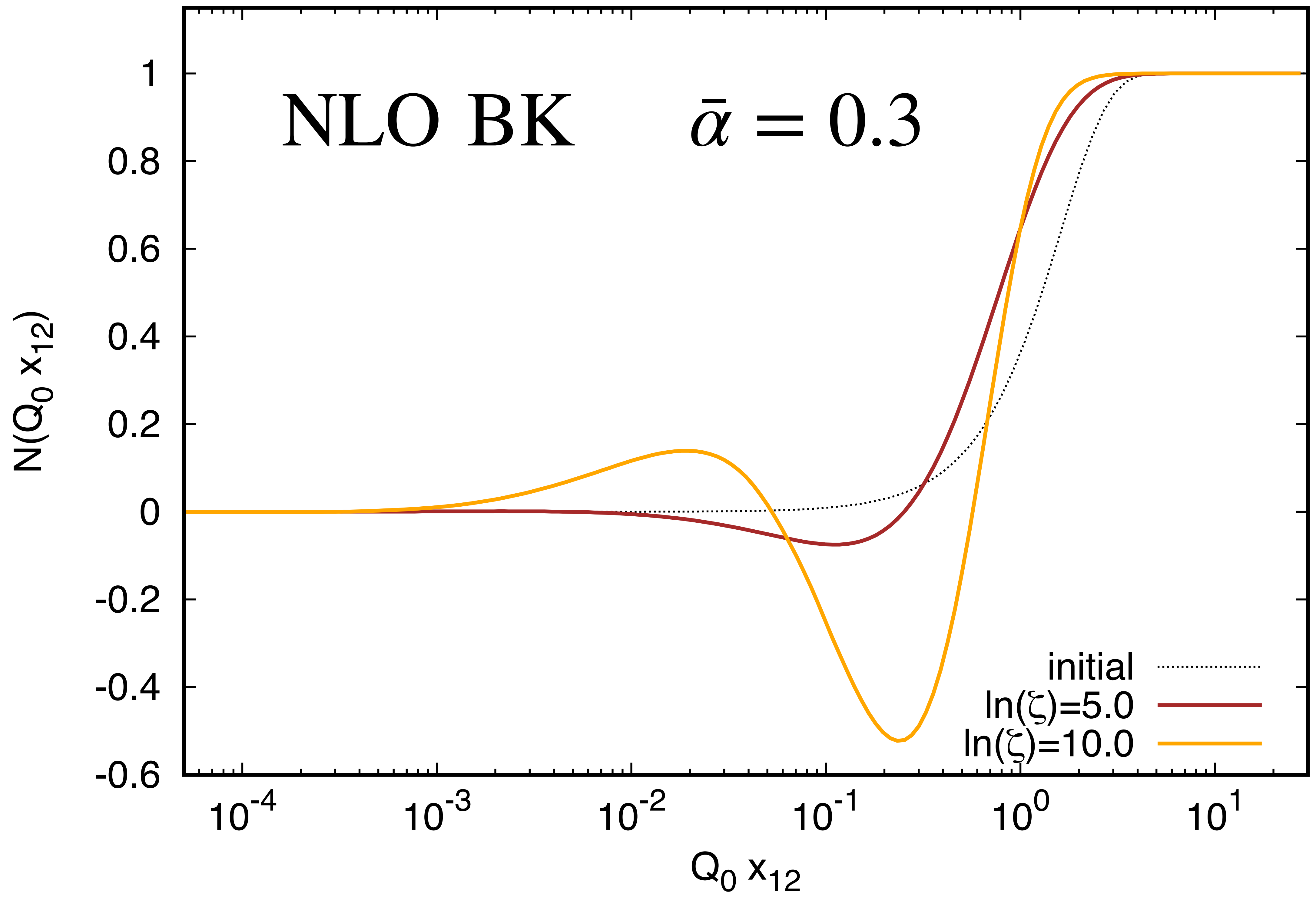
- We obtain the NLO evolution equation for the rotated dipole

$$\begin{aligned}\frac{\partial \bar{S}_{12}}{\partial Y} = \bar{\alpha} K^{\mathrm{NLO}} \otimes \bar{S} &+ \frac{\bar{\alpha}^2}{2}\int_3 \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{32}^2} \ln\frac{\boldsymbol{x}_{13}^2}{\boldsymbol{x}_{12}^2} \ln\frac{\boldsymbol{x}_{32}^2}{\boldsymbol{x}_{12}^2} (\bar{S}_{13}\bar{S}_{32} - \bar{S}_{12}) \\ &+ \frac{\bar{\alpha}^2}{2}\int_{3,4} \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{34}^2 \boldsymbol{x}_{42}^2} \ln\left(\frac{\boldsymbol{x}_{34}^2}{\boldsymbol{x}_{14}^2}\right) \bar{S}_{13}(\bar{S}_{34}\bar{S}_{42} - \bar{S}_{32}) + \frac{\bar{\alpha}^2}{2}\int_{3,4} \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{14}^2 \boldsymbol{x}_{43}^2 \boldsymbol{x}_{32}^2} \ln\left(\frac{\boldsymbol{x}_{43}^2}{\boldsymbol{x}_{42}^2}\right) (\bar{S}_{14}\bar{S}_{43} - \bar{S}_{13})\bar{S}_{32}\end{aligned}$$

- The double logs cancel precisely the pathological term in K^{NLO}

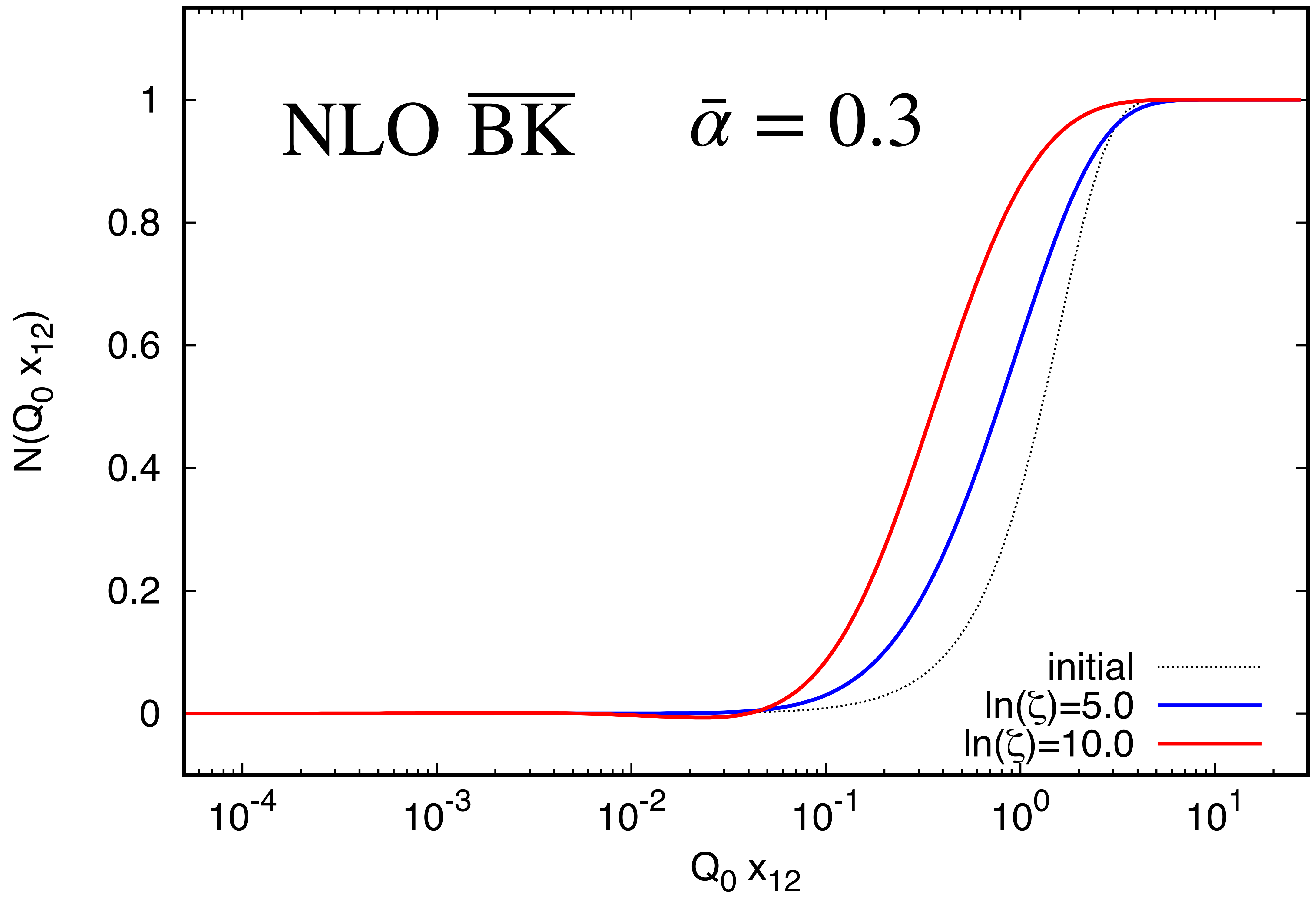
Numerical Simulations

NLO BK vs. NLO $\overline{\text{BK}}$



- The dipole amplitude $N = 1 - S$ becomes negative after a few units of rapidity

NLO BK vs. NLO $\overline{\text{BK}}$



- Stable evolution in the physical range up to $Y = \ln(\zeta) = 10$
- Anti-collinear logs/Single logs instabilities emerging at higher rapidities remain. To be addressed (mitigated by NNLO corrections when $\bar{\alpha}^3 Y \sim 1$)

Summary and Outlook

- We have developed a **systematic framework, consistent with the high-energy Operator Product Expansion (OPE)**, to subtract the pathological double logarithms that have long plagued small-x phenomenology
- In addition to the rapidity factorization scale, we introduce a **collinear scale** that enables a consistent mapping between different rapidity variables.
- **Outlook:**
  - Incorporate **running-coupling effects + collinear single logarithms**
  - Compute the **full cross section**, including the **rotated impact factor**
  - **Investigate NNLO** using recent result by S. Caron-Huot (2025)
  - Extend the framework to other processes: **inclusive dijet, forward hadron production, SIDIS**, etc.

Thank you!