

Small-x physics in DLS: An overview and recent developments



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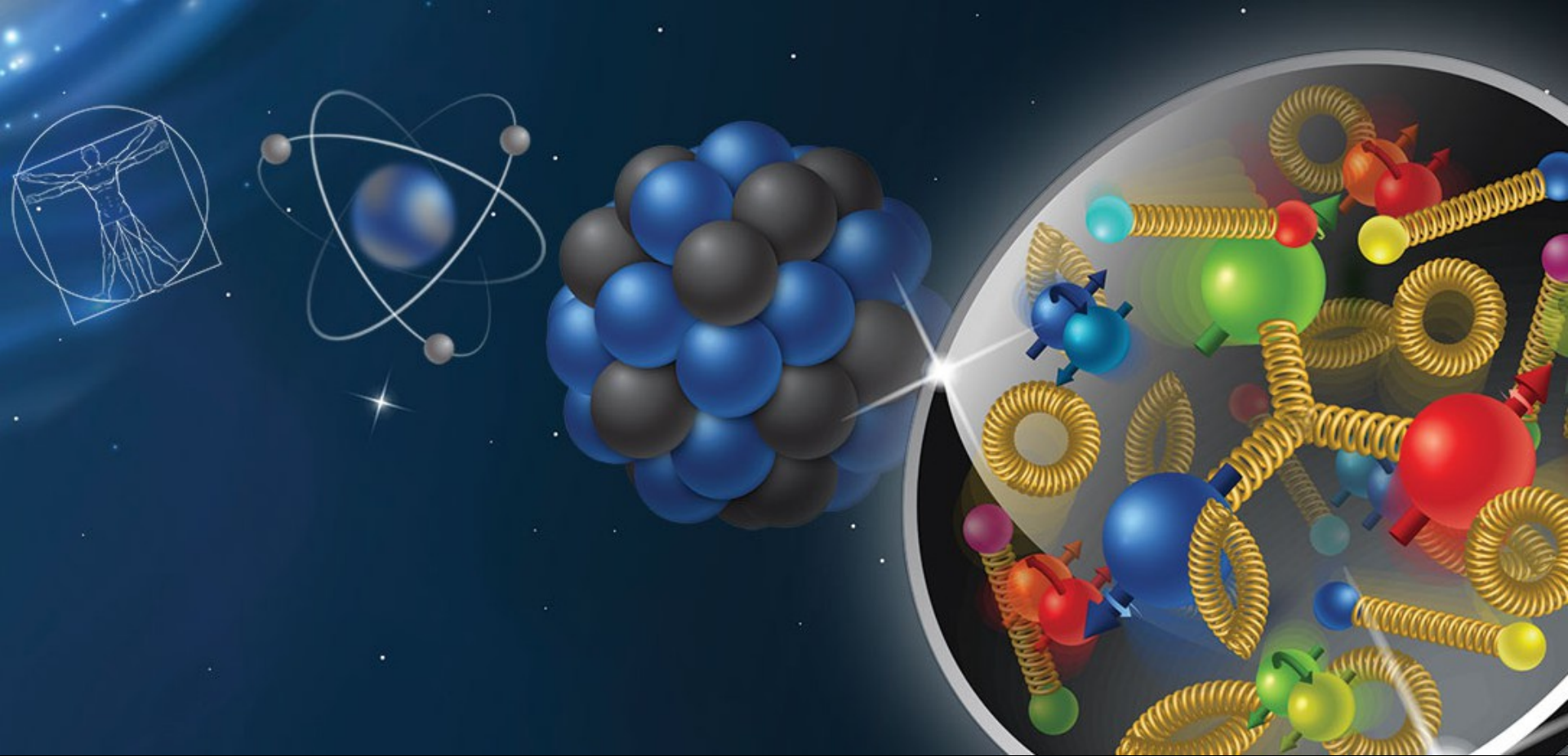
Outline

- **General concepts of small-x physics and saturation:**
 - Quarks and gluons in a proton; DIS in the dipole picture; BFKL Pomeron; Unitarity; GGM/MV formula
- **Non-linear small-x evolution:**
 - Non-linear BK and KL evolution equations; Solution of BK/KL equations; geometric scaling symmetry
 - Running coupling corrections; LT solution
- **Particle production at small-x**
 - AGK cross sections formula; The linear but with complicated kernel evolution of σ_n and its solutions
 - Applications at EIC era: Multiplicity distribution and entanglement entropy for produced gluons

Phys. Rev. D 110, 054045 (2024),

Phys. Rev. D 111, 096025 (2025),

Phys. Rev. D 113, 114021 (2026)



*General concepts of
small-x physics and
saturation*

Quarks and Gluons in the Proton ¹

atom 10^{-8} cm

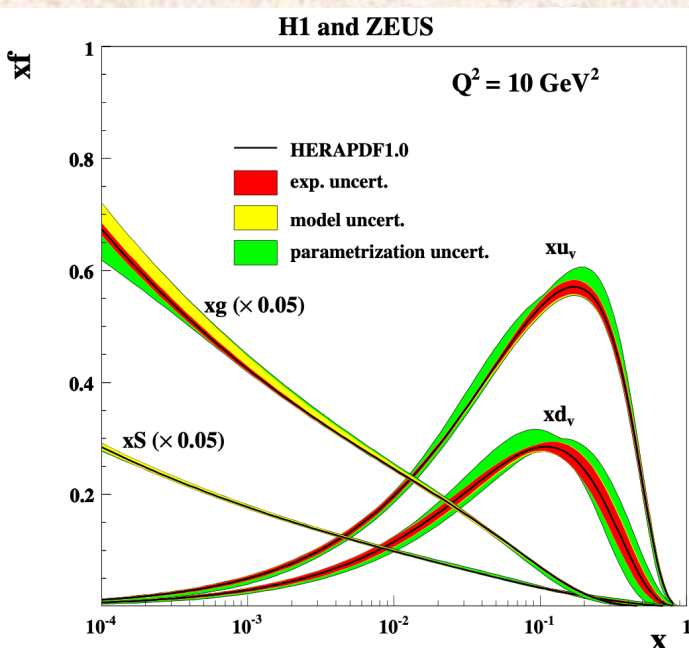
electron $< 10^{-16}$ cm

gluon $< 10^{-16}$ cm

nucleus 10^{-12} cm

quark $< 10^{-16}$ cm

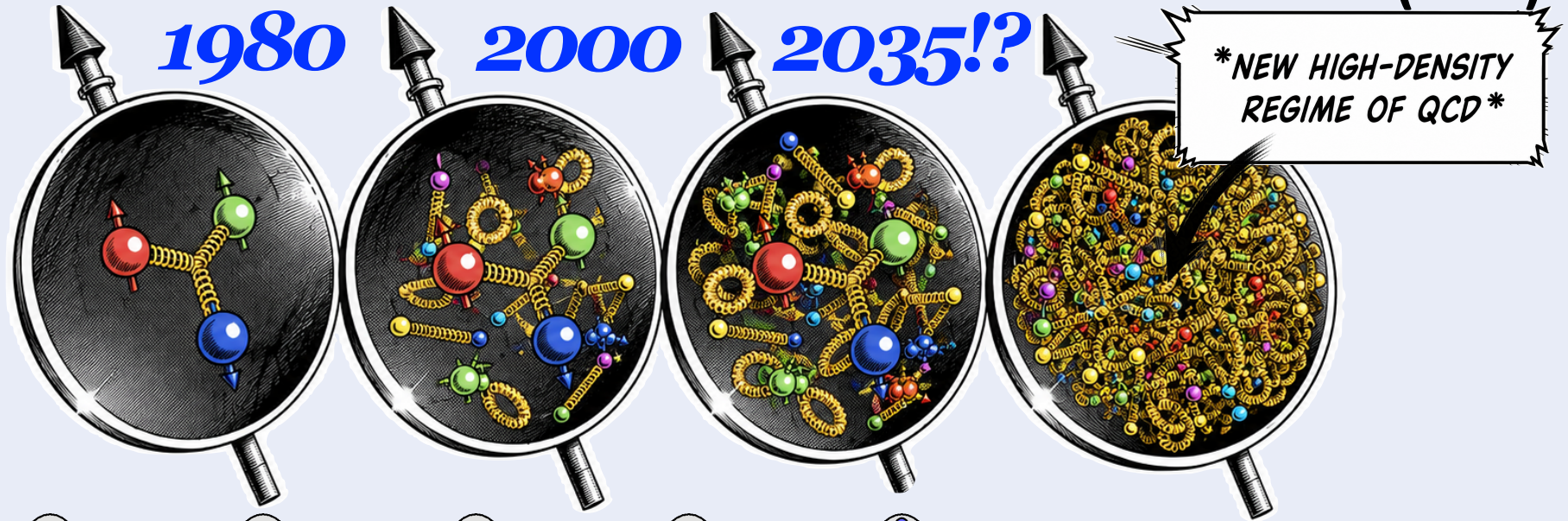
nucleon (proton/
neutron) 10^{-13} cm



- There is a large number of quarks, anti-quarks and gluons at small-x in a proton. We still don't have a clear picture of how they are distributed.

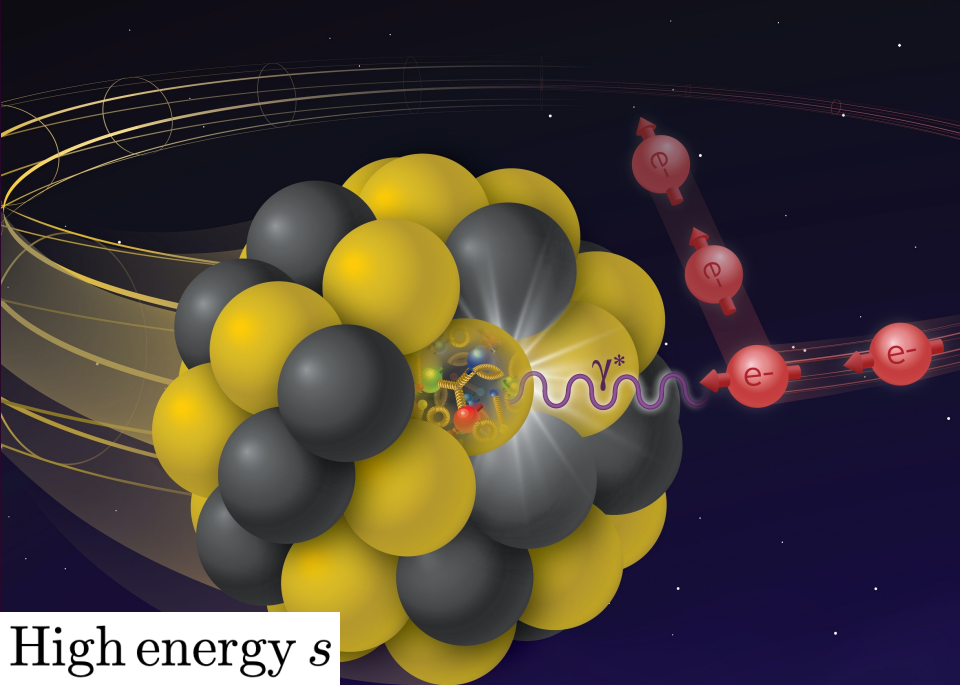
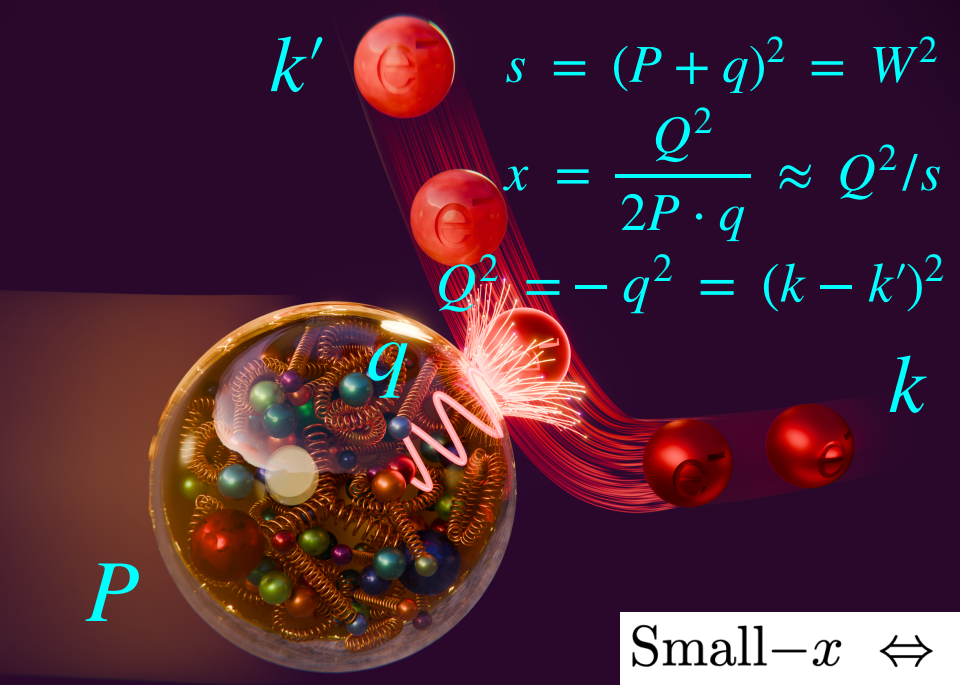
Quarks and Gluons in the Proton ²

- High number of gluons populates the transverse extent of the proton or nucleus, leading to a very dense saturated wave function known as the Color Glass Condensate (CGC)

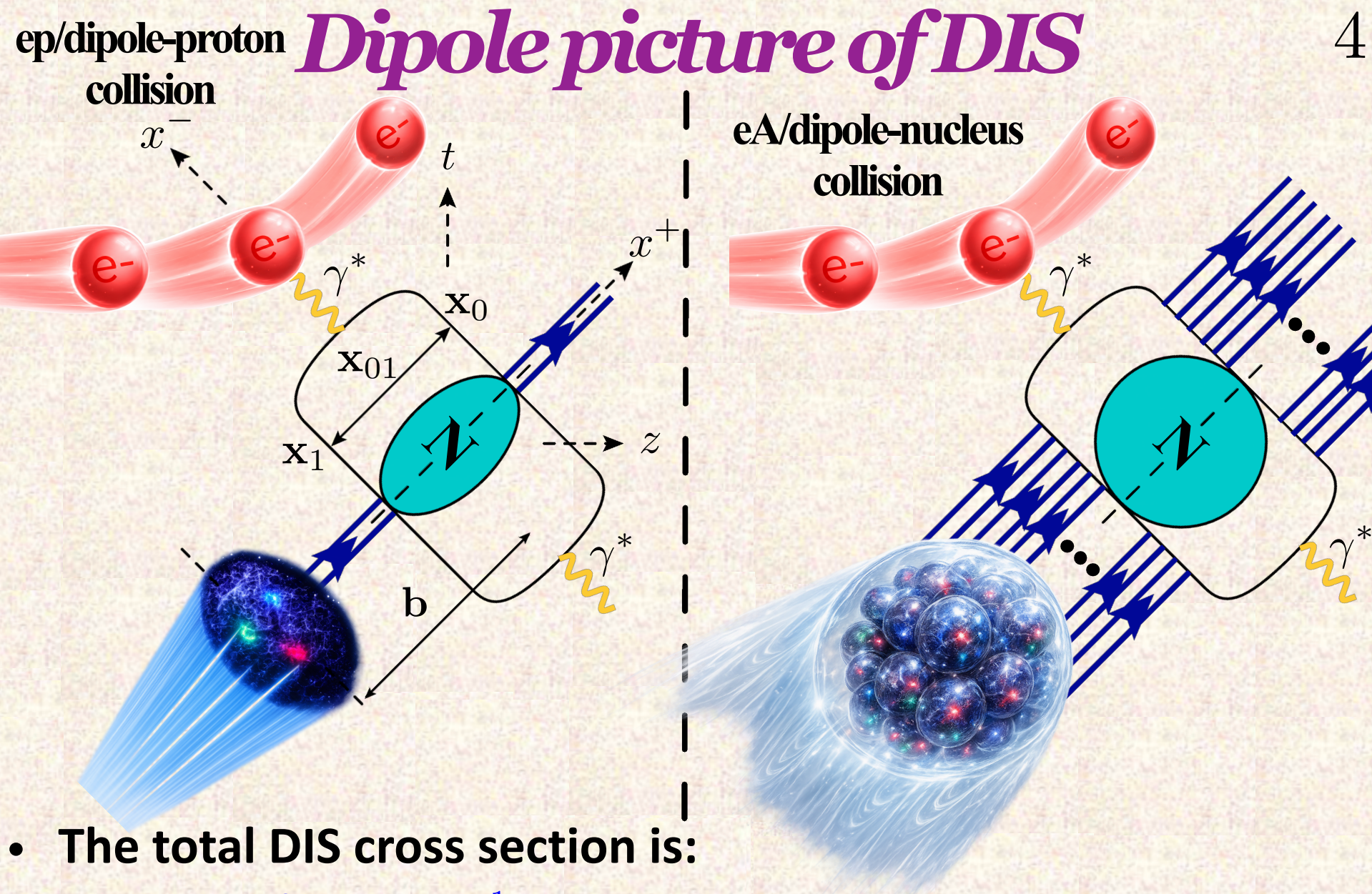


- As energy increases, gluons not only split into more gluons, but also recombine.

Deep inelastic scattering (DIS)



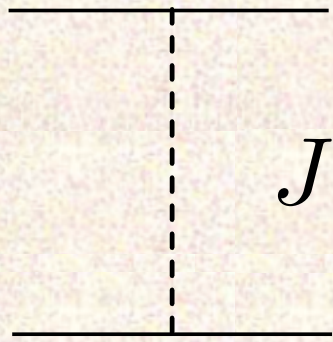
Small- $x \Leftrightarrow$ High energy s



$$\sigma_{tot}^{\gamma^* p/A} = \int \frac{d^2 x_{01}}{2\pi} d^2 b \int_0^1 \frac{dz}{z(1-z)} |\Psi^{\gamma^* \rightarrow q\bar{q}}(\mathbf{x}_{01}, z, Q^2)| N(Y; \mathbf{x}_{01}, \mathbf{b})$$

Dipole picture of DIS

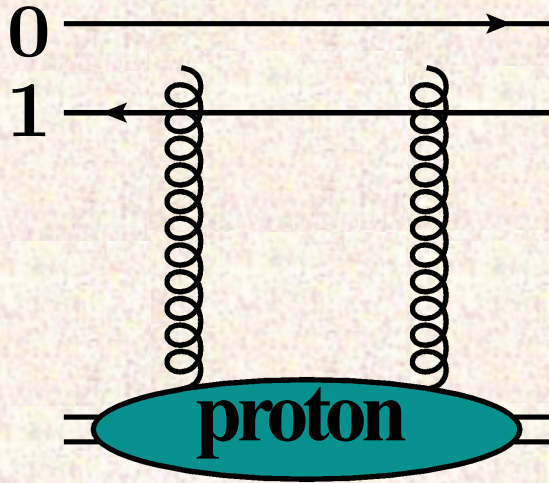
- How does the dipole interact with the proton?



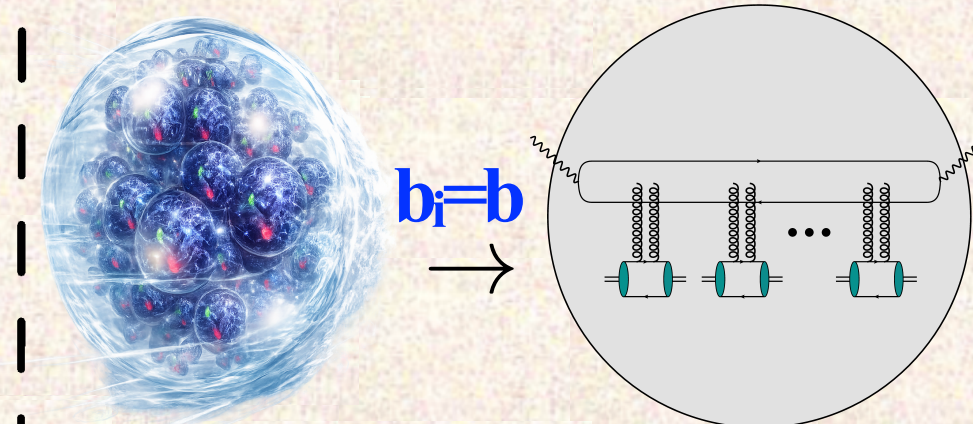
$$\sigma \sim \dots s^{J-1}$$

J = spin of exchanged particle

- Answer: the **(two) gluon exchange dominates ($J=1$)**. Spin-2 particles are weakly-coupled at nowadays energies.



$$N^{2\text{-gluons}} = \frac{1}{4} x_{01}^2 Q_{s0}^2$$

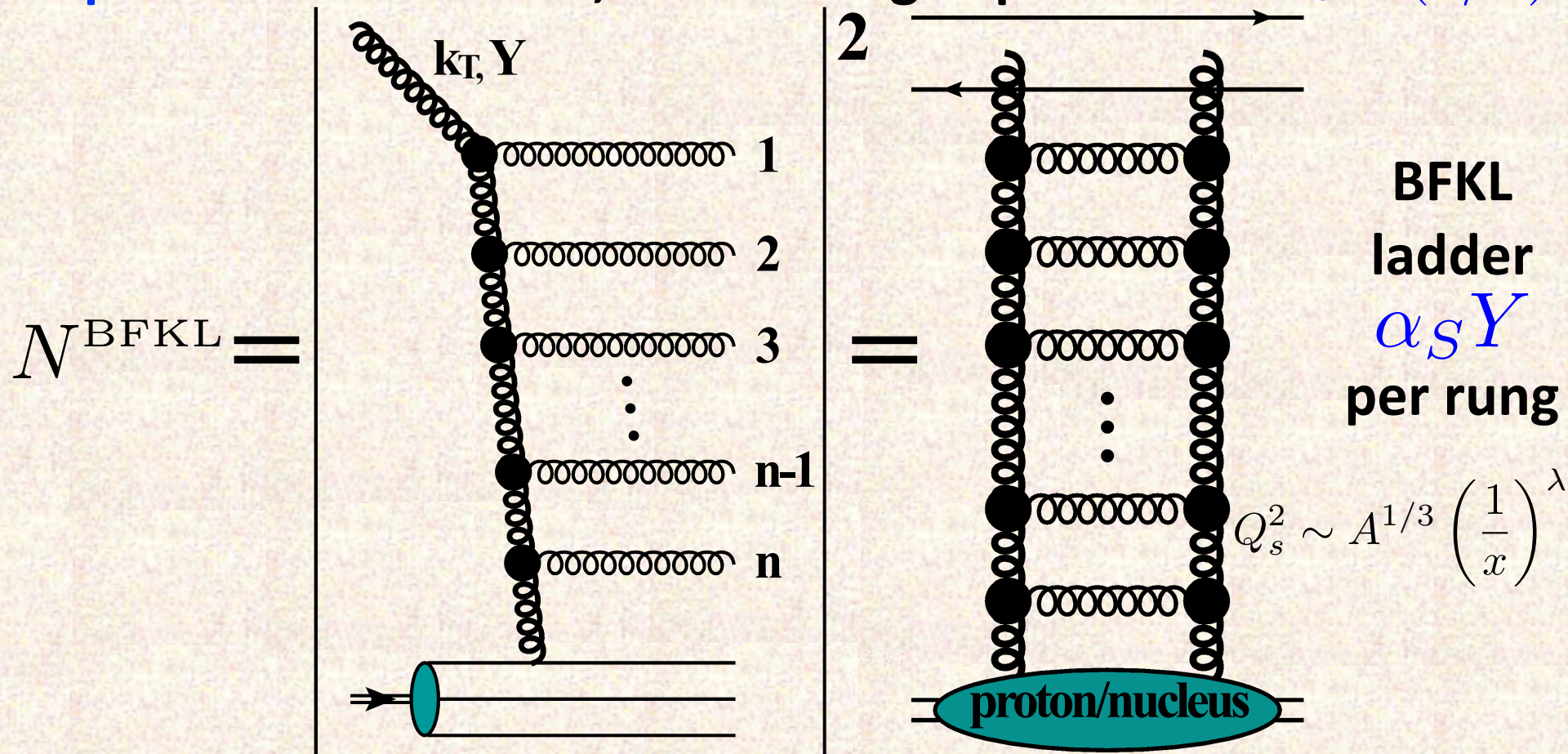


$$N^{(0)} = 1 - e^{-\frac{1}{4} x_{01}^2 Q_{s0}^2}$$

$$\equiv \gamma(\mathbf{x}_{01}, \mathbf{b})$$

The BFKL Pomeron

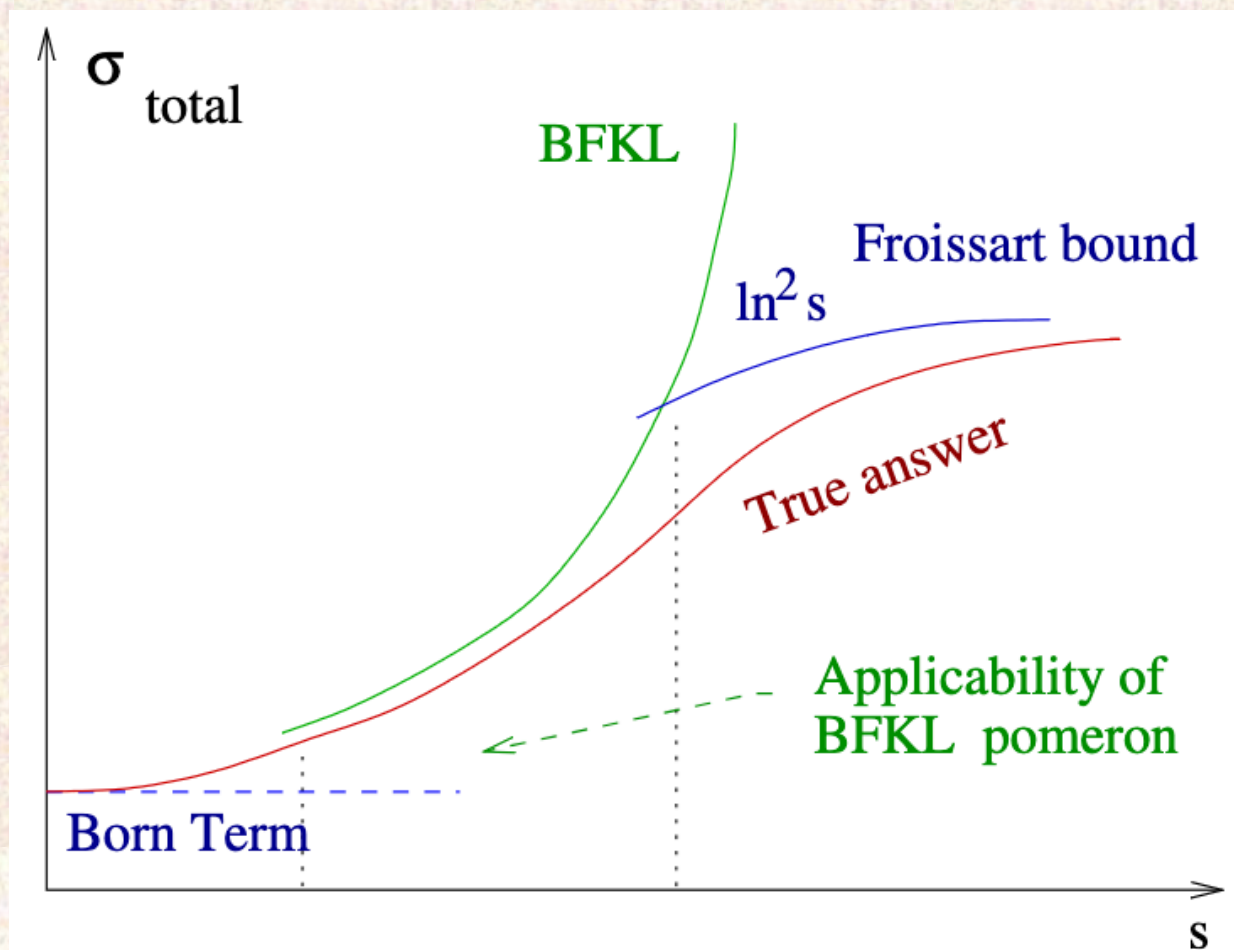
- The previous result does not include any energy dependence. Energy dependence comes in through **quantum corrections**, which bring in powers of $\alpha_S \ln(1/x)$



$$N^{\text{BFKL}}(Y, \mathbf{x}_{01}, \mathbf{b}) \approx N_0 (x_{01}^2 Q_s^2(Y, \mathbf{b}))^{1 - \gamma_{cr}} = N_0 e^{\bar{\gamma} z} \equiv G_P(z)$$

The BFKL Pomeron

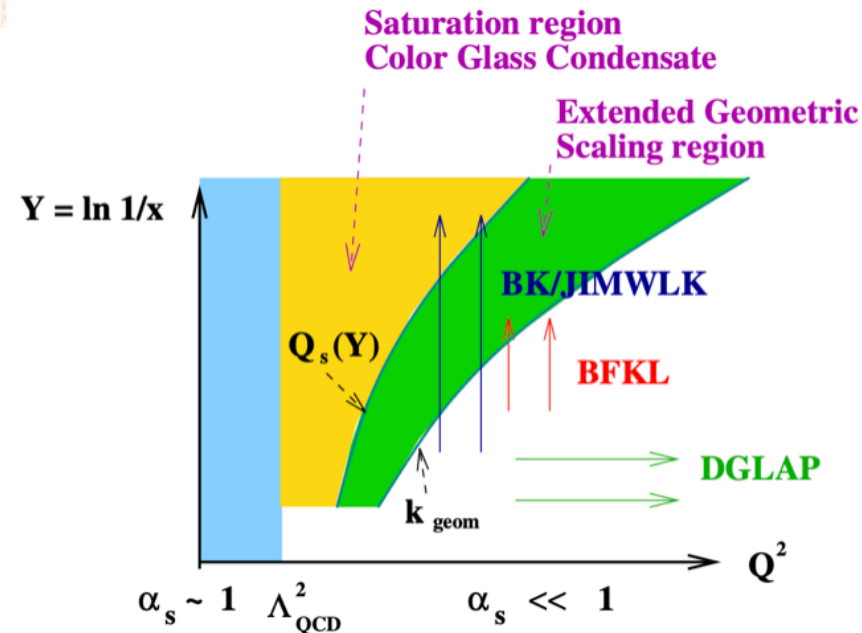
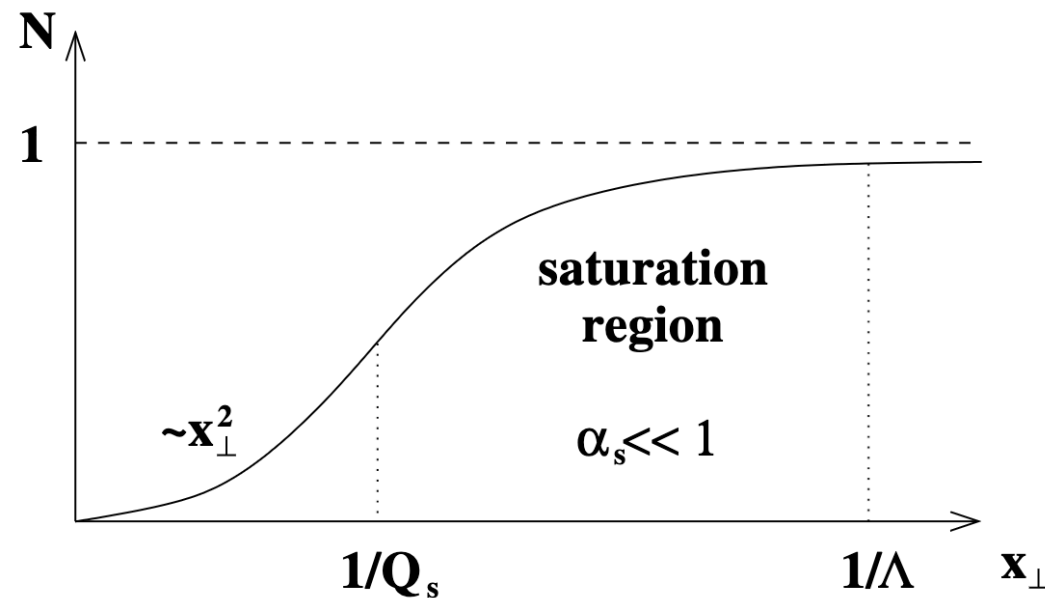
- As we increase energy, the **single BFKL Pomeron violates unitarity** of the total cross section



- We call the Froissart boundary the following limit of the energy growth of the total cross section: $\sigma_{\text{tot}} \leq C \ln^2 s$

Fixing the problems

- Let us define the **saturation scale** $Q_s(Y)$ like a transition point between the region where $N(x_\perp, Y)$ grows with $x_\perp$ and the region where this growth is tamed, that is, by requiring that $N(x_\perp = 1/Q_s(Y), Y) = \text{const}$, with some order-one constant.

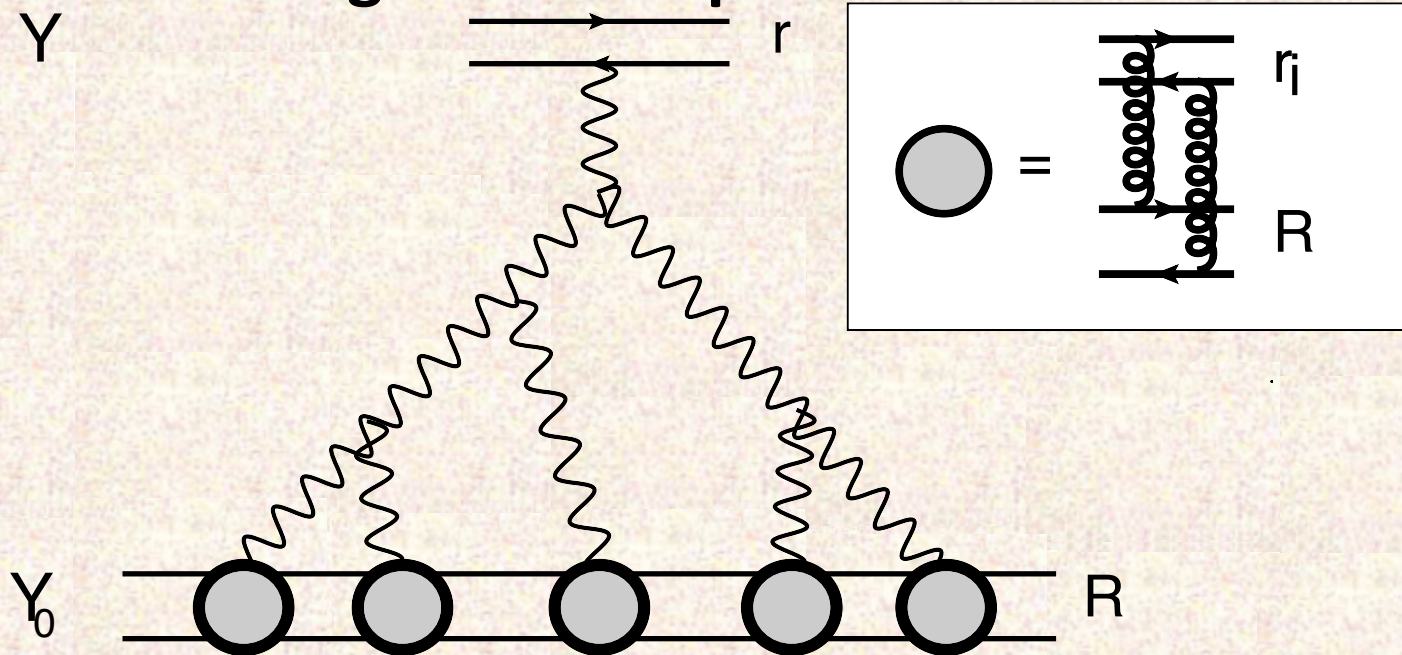


$$Q_s^2(Y) = Q_{s0}^2 \exp \left\{ \bar{\alpha}_S \kappa Y - \frac{3}{2(1 - \gamma_{cr})} \ln \bar{\alpha}_S Y + \mathcal{O} \left(\frac{1}{\sqrt{Y}} \right) \right\}$$

where $\kappa \equiv \chi(\gamma_{cr})/(1 - \gamma_{cr})$, $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)$, $\gamma_{cr} = 0.37$

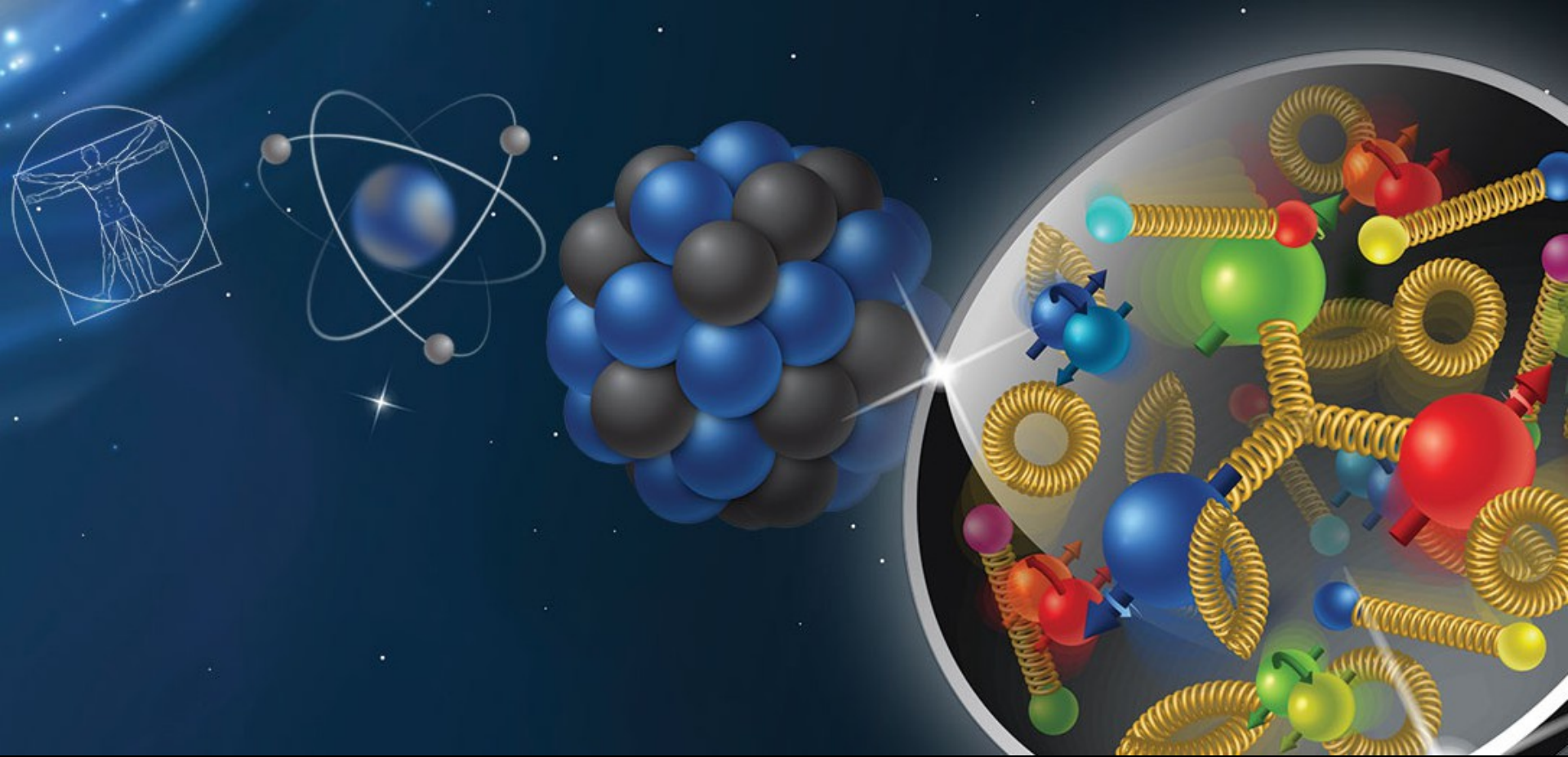
Fixing the problems

- To reconcile with unitarity, we say that the dipole system in the saturation region develops the **BFKL Pomeron cascade**:



$$S_{p/A}^d(Y, \mathbf{x}_{01}, \mathbf{b}) = \sum_{n=0}^{\infty} (-1)^n C_n G_P^n(Y, \mathbf{x}_{01}, \mathbf{b}) = \oint_C \frac{dn}{2\pi i} \frac{\pi}{\sin(\pi n)} C_n G_P^n$$

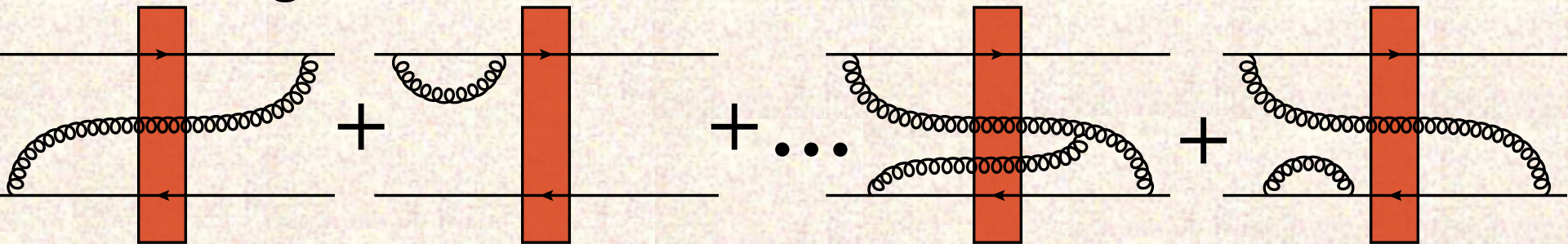
- S=1-N.** The wavy lines denote the BFKL Pomeron exchanges. The circles denote the Born amplitude.
- This is a non-Borel summable series.



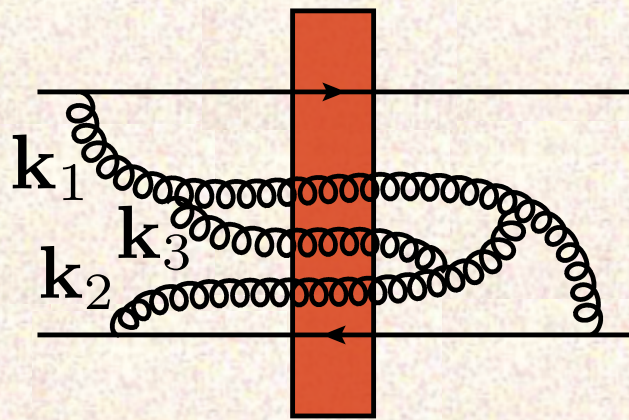
*Non-linear
small- x evolution*

Small- x Evolution

- Let us denote by the red rectangle the GGM exchanges. The quantum corrections are the virtual and long-lived s-channel gluons:



- In general we don't know how to sum this gluon cascade, only at leading $\log-1/x$ and next to leading log. Also due to color factors, still hard to resume. To resume the cascade we take the large- N_c limit ('t Hooft '77):

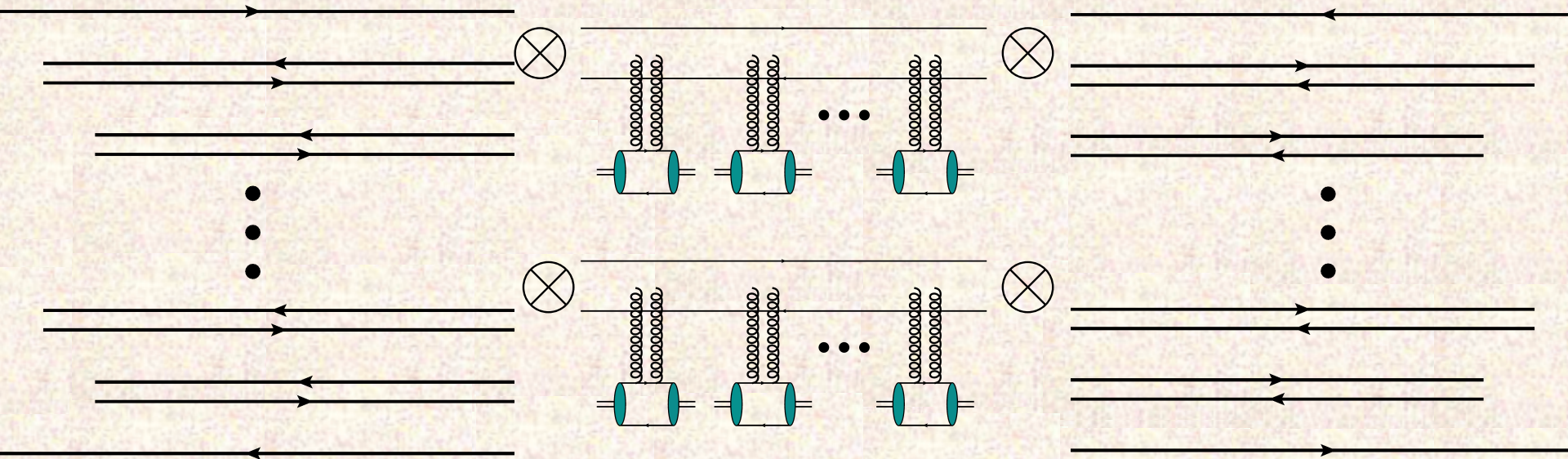


$$N_c \gg 1 \quad \& \quad \text{wavy line} \rightarrow \text{double line}$$

$$N_c \rightarrow \text{large}, \alpha_S N_c = \text{const}$$

Small- x Evolution

- The original dipole has developed a **dipole cascade**, each of which interacts with the nucleus via GGM scattering



$$S(Y; \mathbf{x}_{01}, \mathbf{b}) = \sum_{n=0}^{\infty} (-1)^n \int \prod_{i=1}^n \gamma^{BA}(\mathbf{x}_{01,i}, \mathbf{x}_{0'1'}, \mathbf{b}_i) \\ \times d^2 x_{01,i} d^2 b_i \rho_n(Y; \mathbf{x}_{01}, \mathbf{b}, \{\mathbf{x}_{01,i}, \mathbf{b}_i\})$$

- Where $S=1-N$ and ρ_n denotes the dipole densities (probability to find n dipoles in the amplitude)

Small- x Evolution

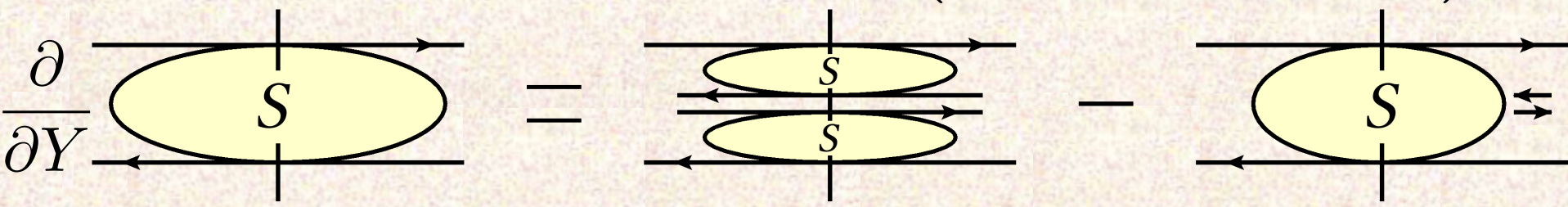
- In arriving to that expression, the dipole–nucleus S-matrix is found to obey the following **non-linear evolution equation** (Balitsky '96; Kovchegov '99)

$$\frac{\partial S_{dA}(Y, \mathbf{x}_{01}, \mathbf{b})}{\partial Y} = \frac{\alpha_S N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \times [S_{dA}(Y; \mathbf{x}_{12}, \mathbf{b}) S_{dA}(Y; \mathbf{x}_{02}, \mathbf{b}) - S_{dA}(Y; \mathbf{x}_{01}, \mathbf{b})]$$

where $Y = \ln 1/x$ and $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, $x_{ij} = |\mathbf{x}_{ij}|$

- The initial condition with energy is given by

$$S_{dA}(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}) = \exp \left(-\frac{1}{4} x_{01}^2 Q_s^2(Y_A, \mathbf{b}) \right)$$



Solution of BK equation

- For the leading twist BFKL kernel (E. Levin and K. Tuchin '00)

$$\frac{\bar{\alpha}_S}{2\pi} \int d^2x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \rightarrow \frac{\bar{\alpha}_S}{2} \int_{Q_s^{-2}(Y,b)}^{x_{01}^2} \frac{dx_{02}^2}{x_{02}^2} + \frac{\bar{\alpha}_S}{2} \int_{Q_s^{-2}(Y,b)}^{x_{01}^2} \frac{dx_{12}^2}{x_{12}^2} \equiv \frac{\bar{\alpha}_S}{2} \int_{-\xi_s}^{\xi} d\xi_{02} + \frac{\bar{\alpha}_S}{2} \int_{-\xi_s}^{\xi} d\xi_{12}$$

where $\xi_{ij} = \ln(x_{ij}^2 Q_s^2(Y = Y_A, \mathbf{b}))$ and $\xi_s = \bar{\alpha}_S \kappa (Y - Y_A)$

- The BK evolution equation can be rewritten in terms of ξ_s and ξ as

$$\frac{\partial S_{dA}(\xi_s, \xi)}{\partial \xi_s} = \frac{1}{\kappa} S_{dA}(\xi_s, \xi) \int_{-\xi_s}^{\xi} d\xi' (1 - S_{dA}(\xi_s, \xi'))$$

- Introducing $S_{dA}(\xi_s, \xi) = \exp(-\Omega(\xi_s, \xi))$

$$\frac{\partial^2 \Omega(\xi_s, \xi)}{\partial \xi_s \partial \xi} = \frac{1}{\kappa} \left(1 - e^{-\Omega(\xi_s, \xi)} \right)$$

Solution of BK equation

- For the geometric scaling variable $z=\xi_s+\xi$ it reduces to

$$\frac{d^2 \Omega(z)}{dz^2} = \frac{1}{\kappa} \left(1 - e^{-\Omega(z)} \right)$$

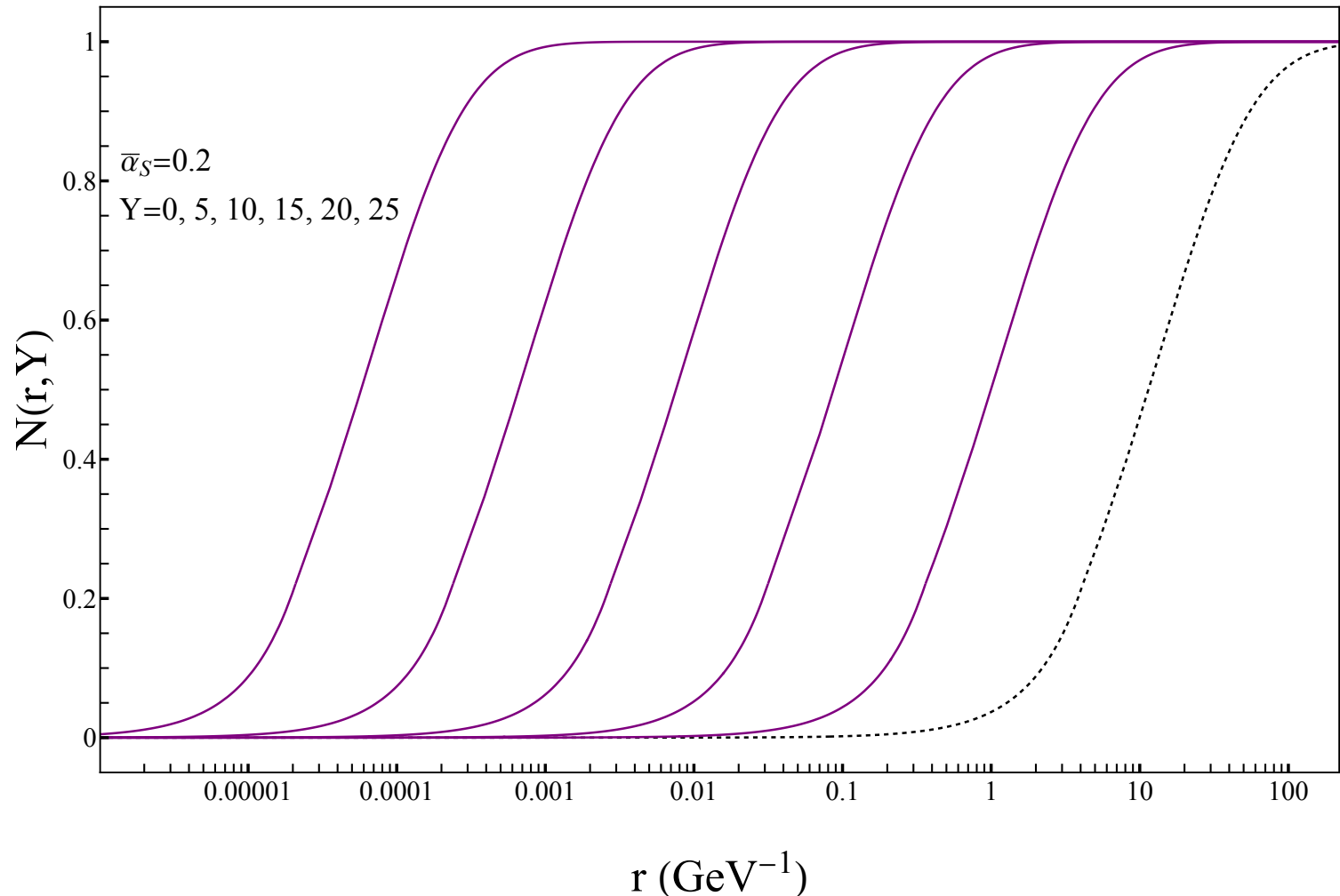
$$\Rightarrow \mathcal{U}(\Omega) \equiv \int_{\Omega_0(\xi)}^{\Omega(z)} \frac{d\Omega'}{\sqrt{\Omega' + e^{-\Omega'} - 1 + \frac{\kappa}{2} C_1}} = \sqrt{\frac{2}{\kappa}} (z + C_2)$$

- Constants C_1 and C_2 can be found from matching with perturbative QCD region:

$$\Omega(z=0) = \Omega_0 \quad \left. \frac{d \ln(\Omega(z))}{dz} \right|_{z=0} = \bar{\gamma}$$

Solution of BK equation

$$N(Y, \mathbf{x}_{01}, \mathbf{b}) = N_0 (x_{01}^2 Q_s^2(Y, \mathbf{b}))^{\bar{\gamma}} \Theta(1 - x_{01} Q_s(Y, \mathbf{b})) \\ + (1 - e^{-\Omega(Y, \mathbf{x}_{01}, \mathbf{b})}) \Theta(x_{01} Q_s(Y, \mathbf{b}) - 1)$$



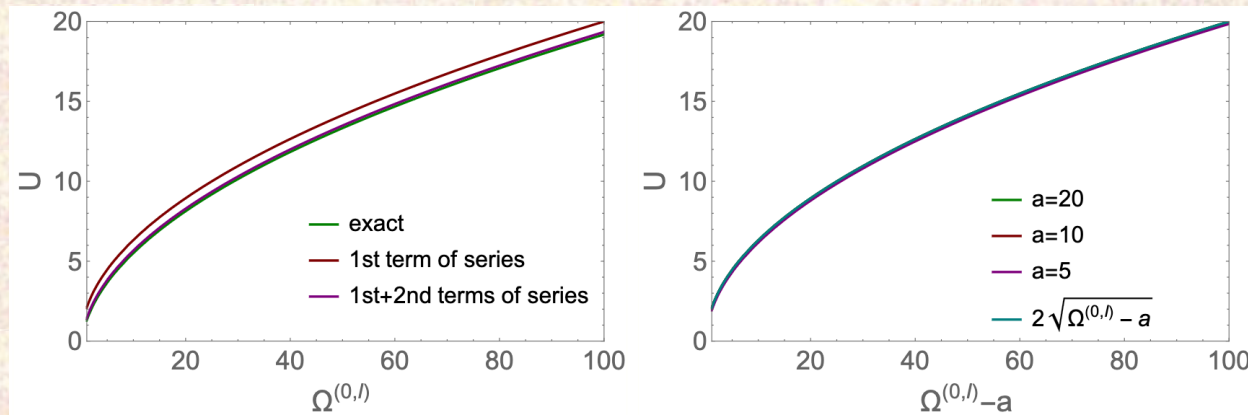
Solution of BK equation

- Actually, for large Ω' we have

$$\frac{1}{\sqrt{\Omega' + \exp(-\Omega)} - a} = \frac{1}{\sqrt{\Omega' - a}} \left(1 + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{k!} \left(-\frac{e^{-\Omega'}}{2(\Omega' - a)} \right)^k \right)$$

- Plugging this in the equation we reduce it to the form

$$\mathcal{U}(\Omega) = 2\sqrt{\Omega - a} + \sum_{k=1}^{\infty} \frac{(-1)^k k^{k-\frac{1}{2}} (2k-1)!!}{2^k k!} e^{-a k} \left(\frac{(-1)^k 2^k \sqrt{\pi}}{(2k-1)!!} - \Gamma\left(\frac{1}{2} - k, k(\Omega - a)\right) \right)$$



- Resolving this equation at large z we have

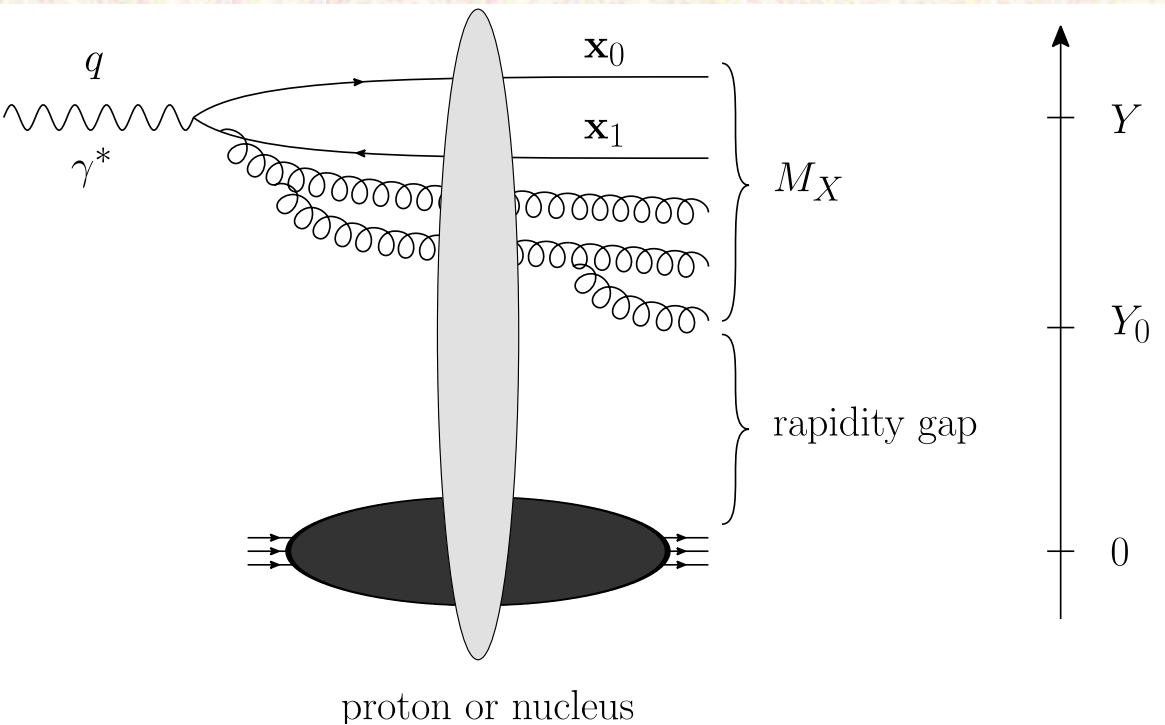
$$S(z) = \Delta_{LT}(z) \exp \left(-a + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2k-1)!!}{2^k k k!} \left(\frac{2\kappa}{z^2} \right)^{k+\frac{1}{2}} \Delta_{LT}^k(z) \right)$$

where $\Delta_{LT}(z) = \exp \left(-\frac{z^2}{2\kappa} \right) = \exp \left(-\frac{\ln^2(x_{01}^2 Q_s^2(Y, \mathbf{b}))}{2\kappa} \right)$

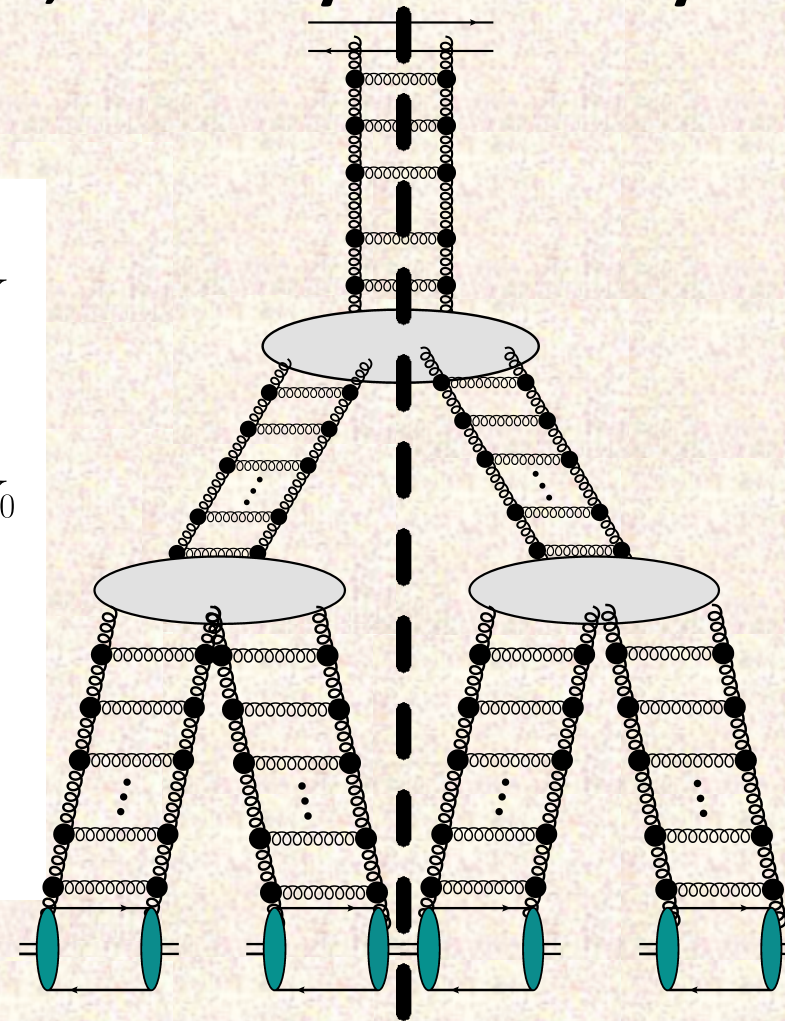
Single diffractive dissociation

17

- At high M_X , in addition to the $q\bar{q}$, one may have many more gluons produced ($y > Y_0$):



The process



Diffraction evolution as
fan diagrams

Single diffractive dissociation 18

- The single diffractive cross section is

$$M_X^2 \frac{d\sigma_{diff}^{\gamma^* A}}{dM_X^2} = \int d^2 x_0 d^2 x_1 \int_0^1 dz |\Psi^{\gamma^* \rightarrow q\bar{q}}(\mathbf{x}_{01}, z, Q^2)| \frac{\partial S^D(Y, Y_0; \mathbf{x}_{01}, \mathbf{b})}{\partial Y_0}$$

- S^D is the diffractive S-matrix with the rapidity gap greater than or equal to Y_0 . It is descomposed as $S^D=1-2N+\sigma_{SD}$.
- S^D is a complicated object:

$$\begin{aligned} S_{dA}^D(Y, Y_0; \mathbf{x}_{01}, \mathbf{b}) &= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{\infty} (-1)^k \\ &\times \int \prod_{i=1}^n d^2 x_{01,i} d^2 b_i \prod_{j=1}^k d^2 x_{01,j} d^2 b_j \\ &\times \gamma^{BA}(\mathbf{x}_{01,i}, \mathbf{x}_{0'1'}, \mathbf{b}_i) \gamma^{BA}(\mathbf{x}_{01,j}, \mathbf{x}_{0'1'}, \mathbf{b}_j) \\ &\times \rho_{n,k}^D(Y_m, Y_0; \{\mathbf{x}_{01,i}, \mathbf{b}_i\}) \rho_n(Y_0; \{\mathbf{x}_{01,i}, \mathbf{b}_i\}) \end{aligned}$$

Diffractive S-matrix

- Fortunately, doing one step in evolution and using the following cancellations of final state interactions for gluons with $y > Y_0$ (Z. Chen, A. Mueller '95)

The image displays two sets of Feynman diagrams representing cancellations of final state interactions for gluons. The top set consists of three diagrams, each with a horizontal line representing a quark line and a vertical dashed line representing a gluon line. The first diagram has a gluon loop in the final state, with labels 0, ∞ , and 0 below the horizontal line. The second and third diagrams have similar configurations but with different vertical line placements. They are summed to zero. The bottom set consists of two diagrams with a gluon loop in the final state, summed to zero.

- The equation for the single diffraction production in QCD satisfies again the BK equation, but with different I.C.:

$$\frac{\partial S_{dA}^D(Y, Y_0; \mathbf{x}_{01}, \mathbf{b})}{\partial Y} = \frac{\alpha_S N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \times [S_{dA}^D(Y, Y_0; \mathbf{x}_{12}, \mathbf{b}) S_{dA}^D(Y, Y_0; \mathbf{x}_{02}, \mathbf{b}) - S_{dA}^D(Y, Y_0; \mathbf{x}_{01}, \mathbf{b})]$$

I.C. $S_{dA}^D(Y = Y_0, Y_0; \mathbf{x}_{01}, \mathbf{b}) = S_{dA}^2(Y = Y_0; \mathbf{x}_{01}, \mathbf{b})$

Solution of KL equation

- Introducing $S_{dA}^D(\xi, \xi_s) = \exp(-\Omega^D(\xi, \xi_s))$

$$\frac{\partial^2 \Omega^D(\xi_s, \xi)}{\partial \xi_s \partial \xi} = \frac{1}{\kappa} \left(1 - e^{-\Omega^D(\xi_s, \xi)} \right)$$

- The solution to this equation has the form:

$$S_{dA}^D(z) = S_{IP}^D(z_m) S_{dA}^2(z_0)$$

- This notation reflects that the production of gluons with mass M_X occurs in the dipole-Pomeron scattering
- We can use our previous formula found in the elastic case but for S^2 . That means we use $a=2\Omega$
- Then we can calculate quantities such as $\sigma_{\text{diff}} = \frac{\partial S^D(Y, Y_0)}{\partial Y_0}$

Running Coupling BK

- we calculate the $\alpha_S N_f$ corrections to the evolution kernel to all orders and then complete N_f to the QCD beta function

$$+ \dots N_f \rightarrow -6 \pi \beta_2 + \dots$$

- Here's the BK equation with the running coupling corrections (H. Weigert, Yu. K. '06; I. Balitsky, '06):

$$\frac{\partial S(Y; \mathbf{x}_{01}, \mathbf{b})}{\partial Y} = \frac{N_c}{2 \pi^2} \int d^2 x_2 K_{\text{rcBK}}^{\text{Bal/KW}}(\mathbf{x}_0, \mathbf{x}_1; \mathbf{x}_2) \times [S(Y; \mathbf{x}_{12}, \mathbf{b}) S(Y; \mathbf{x}_{02}, \mathbf{b}) - S(Y; \mathbf{x}_{01}, \mathbf{b})]$$

where

$$K_{\text{rcBK}}^{\text{Bal}}(\mathbf{x}_0, \mathbf{x}_1; \mathbf{x}_2) = \alpha_S(1/x_{01}^2) \left[\frac{x_{01}^2}{x_{02}^2 x_{12}^2} + \frac{1}{x_{02}^2} \left(\frac{\alpha_S(1/x_{02}^2)}{\alpha_S(1/x_{12}^2)} - 1 \right) + \frac{1}{x_{12}^2} \left(\frac{\alpha_S(1/x_{12}^2)}{\alpha_S(1/x_{02}^2)} - 1 \right) \right]$$

- There is also the KW prescription, they differ in what we call the subtraction term.

Running Coupling BK

- In the vicinity of the saturation scale where

$$x_{01}^2 \approx x_{12}^2 \approx x_{02}^2 \approx 1/Q_s^2$$

we can consider $\alpha_S(1/x_{ij}^2) \rightarrow \alpha_S(1/x_{01}^2)$ so that

$$K_{\text{rcBK}}^{\text{Bal}}(\mathbf{x}_0, \mathbf{x}_1; \mathbf{x}_2) \approx \alpha_S(1/x_{01}^2) \frac{x_{01}^2}{x_{02}^2 x_{12}^2}$$

- The LT equations becomes more complex

$$\frac{\partial S(Y, \mathbf{x}_{01}, \mathbf{b})}{\partial Y} = S(Y, \mathbf{x}_{01}, \mathbf{b}) \int_{Q_s^{-2}(Y, \mathbf{b})}^{x_{01}^2} \frac{dx_{01}^2}{x_{01}^2} \bar{\alpha}_S(1/x_{01}^2) (1 - S(Y, \mathbf{x}_{01}, \mathbf{b}))$$

(Compare it with fixed coupling case)

where $Q_s^2(Y) = \Lambda_{QCD}^2 e^{\sqrt{\frac{8N_c}{b_0} \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}} Y}$

- We can try to find a solution with geometric scaling z , but it will give us (with $\tilde{z} = \sqrt{16N_c/b_0} z$)

$$\sqrt{\frac{16N_c}{b_0}} \frac{\tilde{z}}{\sqrt{2Y}} \frac{d^2 \Omega(\tilde{z})}{d\tilde{z}^2} + \frac{d^2 \Omega(\tilde{z})}{d\tilde{z}^2} = 1 - e^{-\Omega(\tilde{z})}$$

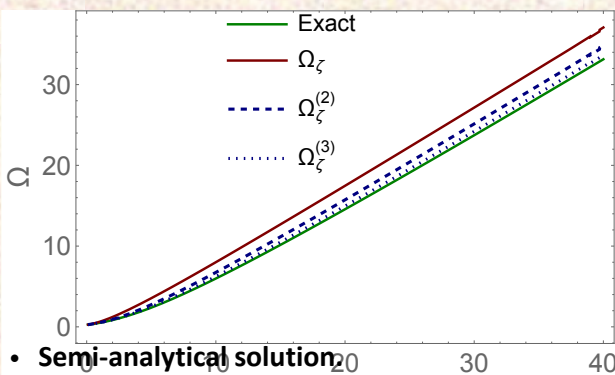
Running Coupling BK

- We are not able to find a solution with geometric scaling (GS) for the model LT kernel. We defined instead a new variable “zeta”, and wrote the equation in terms of this variable:

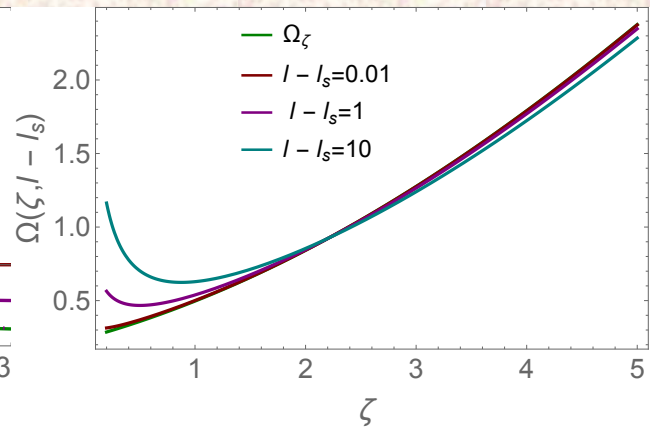
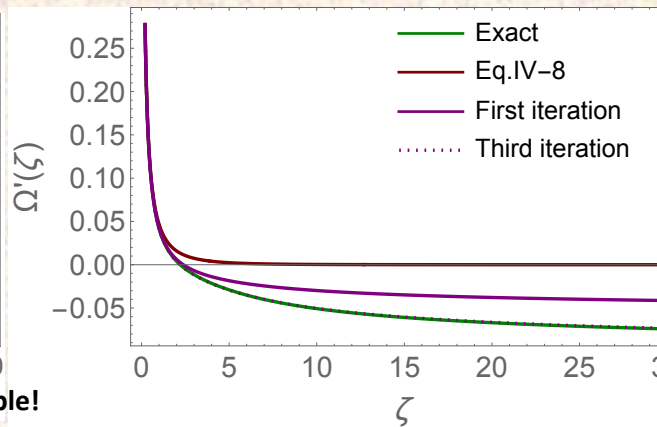
$$\zeta = Y(l - l_s) = -\frac{4N_c}{b_0} Y \ln(\bar{\alpha}_S(Q_s^2(Y))/\bar{\alpha}_S(1/x_{01}^2))$$

$$\Omega(Y, \mathbf{x}_{01}, \mathbf{b}) = \Omega_\zeta(\zeta) + \Omega'(Y, \zeta)$$

$$\zeta \frac{d^2 \Omega_\zeta(\zeta)}{d\zeta^2} + \frac{d\Omega_\zeta(\zeta)}{d\zeta} = 1 - e^{-\Omega_\zeta(\zeta)} ; \frac{\partial^2 \Omega'(\zeta, l - l_s)}{\partial Y \partial (l - l_s)} = e^{-\Omega_\zeta(\zeta)} \Omega'(\zeta, l - l_s)$$



- Semi-analytical solution
- No geometric scaling even with this new variable!



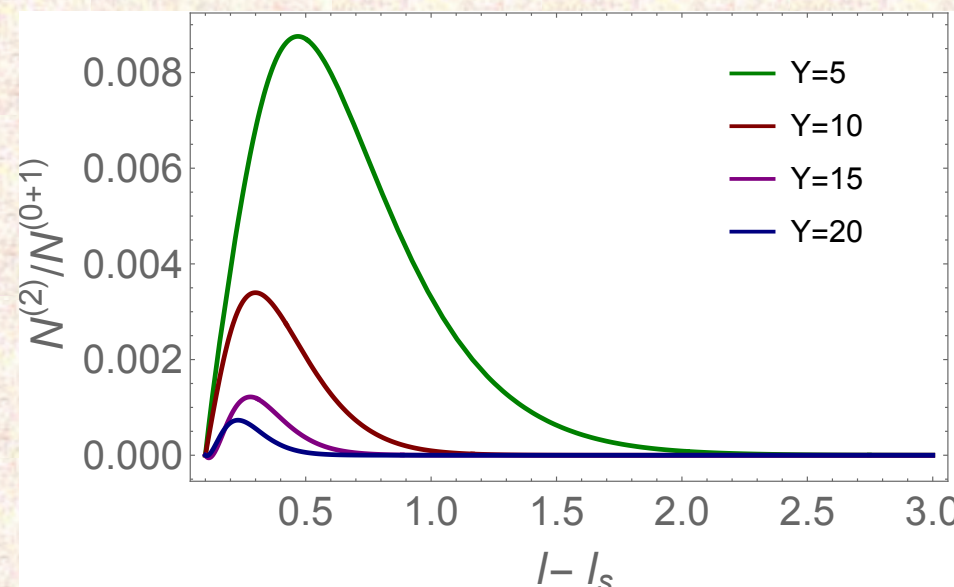
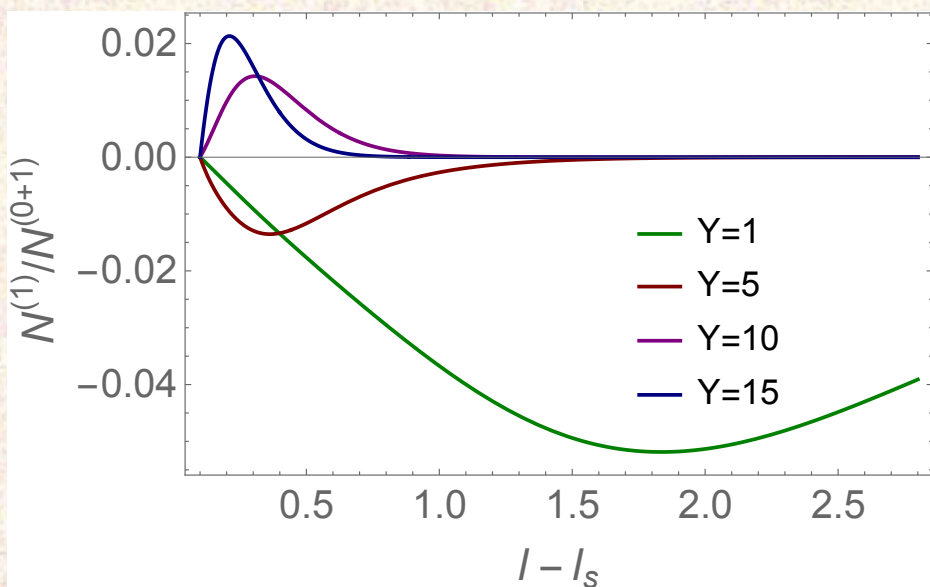
- No geometric scaling even with this new variable!

Running Coupling BK

- In general, you need to compute the effects of the “general corrections” of the approximation. We did this using the homotopy approach

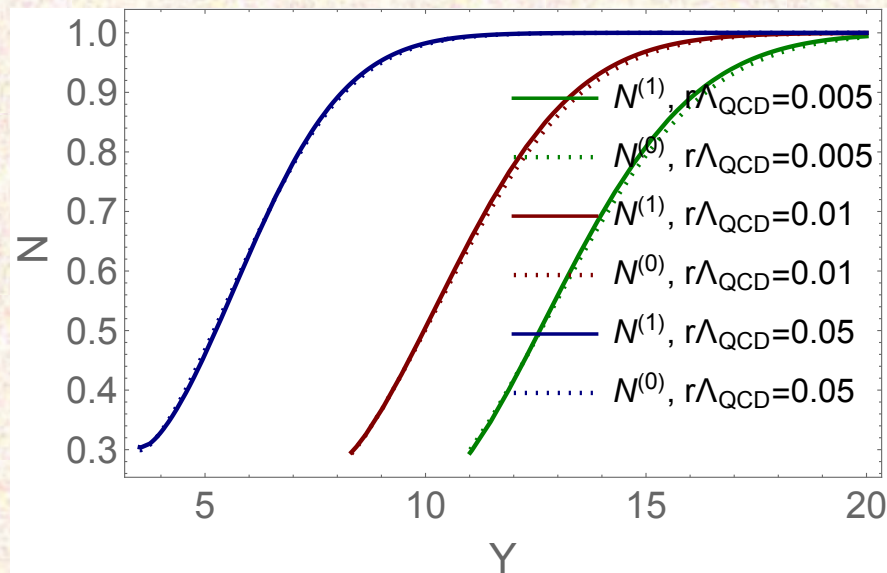
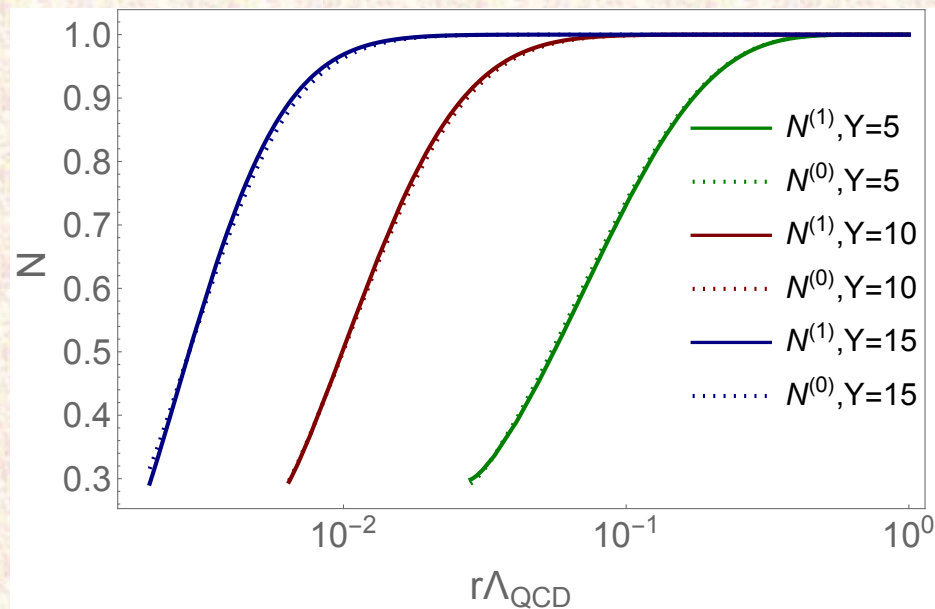
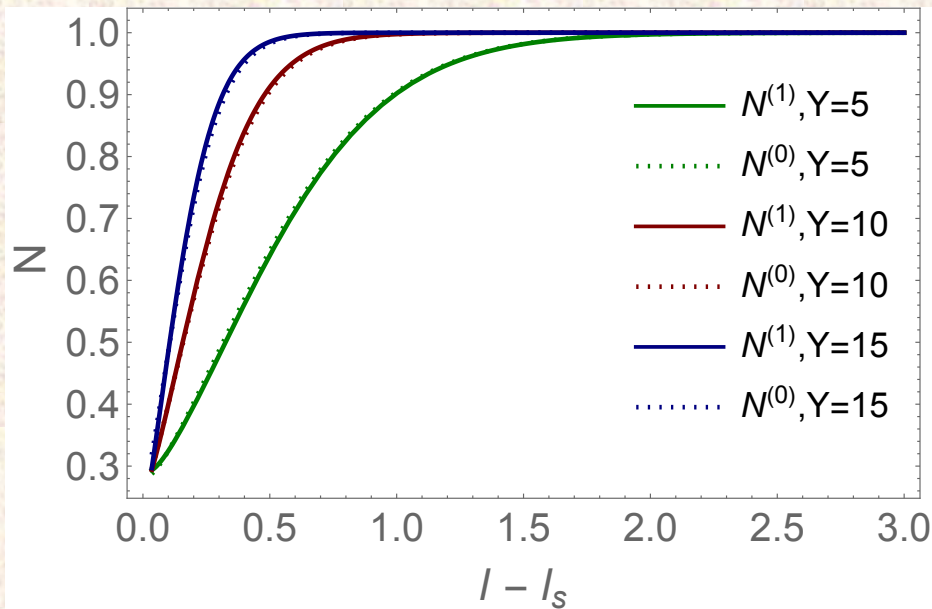
$$\mathcal{L}[\Omega^{(p)}] = \frac{\partial^2 \Omega^{(p)}(Y, r, b)}{\partial Y \partial l} - 1 + e^{-\Omega^{(p)}(Y, r, b)}$$

$$\mathcal{N}_{\mathcal{L}}[\Omega^{(p)}] = \int_{r_1 > r}^{1/\Lambda_{QCD}} \frac{d^2 r_1}{2\pi} K(r; r_1, r_2) \exp \left(-\Omega^{(p)} \left(r_1, Y; \mathbf{b} - \frac{1}{2} \mathbf{r}_2 \right) - \Omega^{(p)} \left(r_2, Y; \mathbf{b} - \frac{1}{2} \mathbf{r}_1 \right) \right)$$



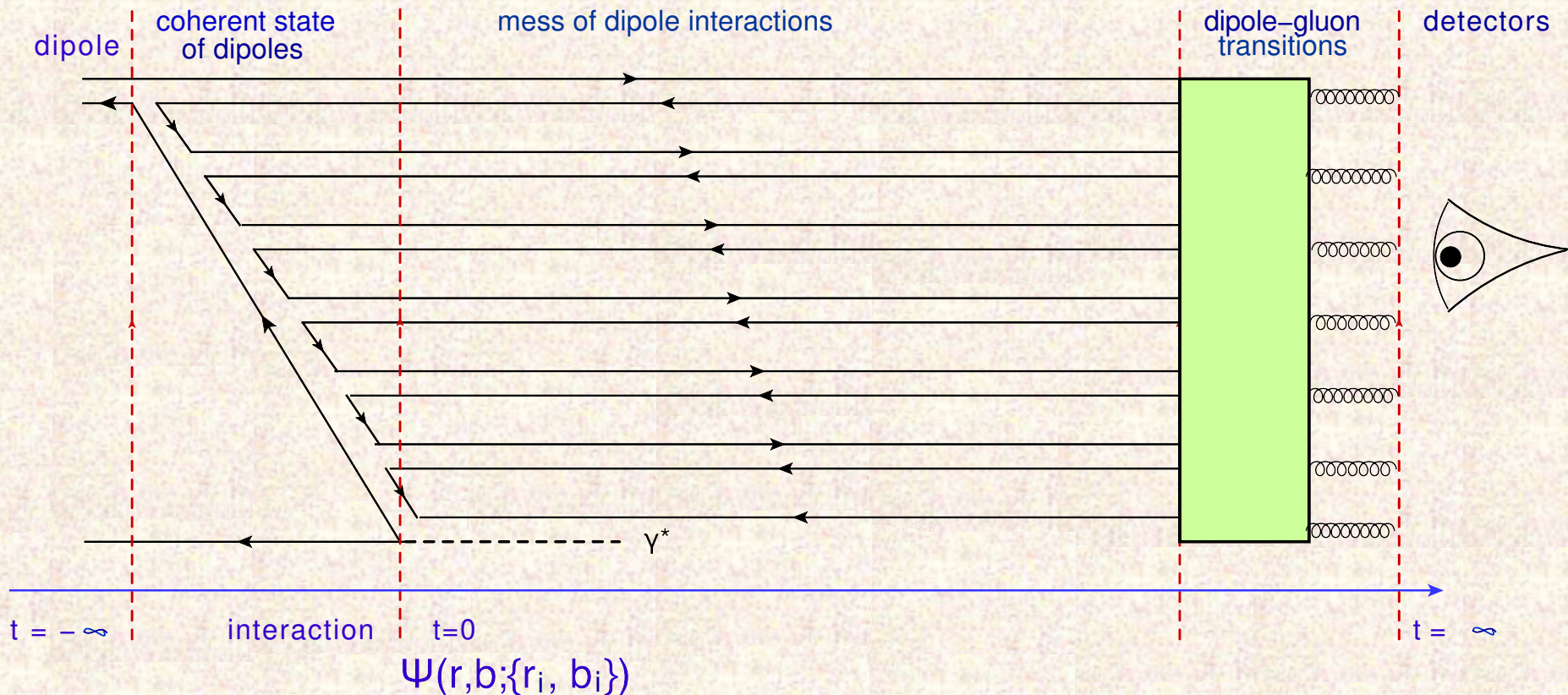
Running Coupling BK

- No traveling wave solution ->





*Particle production
at small- x*



- The interaction of fast hadron (dipole) with the virtual photon (γ^*). The coherence of the partonic wave function of the fast hadron is destroyed at $t = 0$, while the gluons can be measured at $t = \infty$.

The Cut Pomeron

- Goal: To find the multiplicity distributions of produced gluons in the final states. For that, we calculate first cross section of productions of n gluons in the final state.

$$2 N^{\text{BFKL}} = \sigma_{in}^{\text{BFKL}}$$

The diagram illustrates the relationship between the Pomeron Green's function and the cross section of produced gluons. On the left, a Pomeron Green's function is shown as a vertical chain of gluons (wavy lines) connected by a central dashed line, with a large number 2 to its left. On the right, a cross section of produced gluons is shown as a single vertical chain of gluons, with a large number 2 to its right. An equals sign is placed between the two diagrams.

- This relation gives us connection between the Pomeron Green's function and the cross section of produced gluons and determines the so-called cut Pomeron.

The AGK cutting rules

- The **Abramovsky-Gribov-Kancheli (AGK) cutting rules** allows us to calculate the cross sections for production of n-cut Pomerons if we know F_k : the contribution of the exchange of k-Pomerons to the cross section

$$\sigma_n^{\text{AGK}}(Y; \mathbf{r}, \mathbf{b}) = \sum_{k=n}^{\infty} \sigma_n^k(Y; \mathbf{r}, \mathbf{b})$$

where

$$\sigma_n^k(Y; \mathbf{r}, \mathbf{b}) = \begin{cases} (-1)^k (2^k - 2) F_k(Y; \mathbf{r}, \mathbf{b}) & \text{for } n = 0 \\ (-1)^{k-n} \frac{k!}{(k-n)! n!} 2^k F_k(Y; \mathbf{r}, \mathbf{b}) & \text{for } n \geq 1 \end{cases}$$

and $F_k(z) = C_k G_P^k(z)$. The multiplicity distribution of particles in the final states is the convolution of σ_k with the distribution of particles inside k-cut Pomerons

$$\sigma_n^{f.s.}(Y, \mathbf{x}_{01}, \mathbf{b}) = \sum_{k=1}^{\infty} \sigma_k^{\text{AGK}}(Y, \mathbf{x}_{01}, \mathbf{b}) P_n^{\text{Poisson}}(k \langle n_P(Y) \rangle)$$

- The I.C. from the GGM formula

$$\begin{aligned}
 N^{(0)}(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}) &= 1 - \exp \left(-\frac{1}{4} x_{01}^2 Q_s^2(Y_A, \mathbf{b}) \right) \\
 &= \sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{\frac{1}{k!} \left(\frac{x_{01}^2 Q_s^2(Y_A, \mathbf{b})}{4} \right)^k}_{F_k}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{n,01}^{\text{AGK}}(Y = Y_A) &= \sum_{k=n}^{\infty} (-1)^{k-n} \frac{k!}{(k-n)! n!} 2^k \frac{1}{k!} \left(\frac{x_{01}^2 Q_{s0}^2}{4} \right)^k \\
 &= \frac{\left(\frac{1}{2} x_{01}^2 Q_{s0}^2 \right)^n}{n!} \exp \left\{ -\frac{1}{2} x_{01}^2 Q_{s0}^2 \right\}
 \end{aligned}$$

- For $\xi = \ln(x_{01}^2 Q_{s0}^2)$

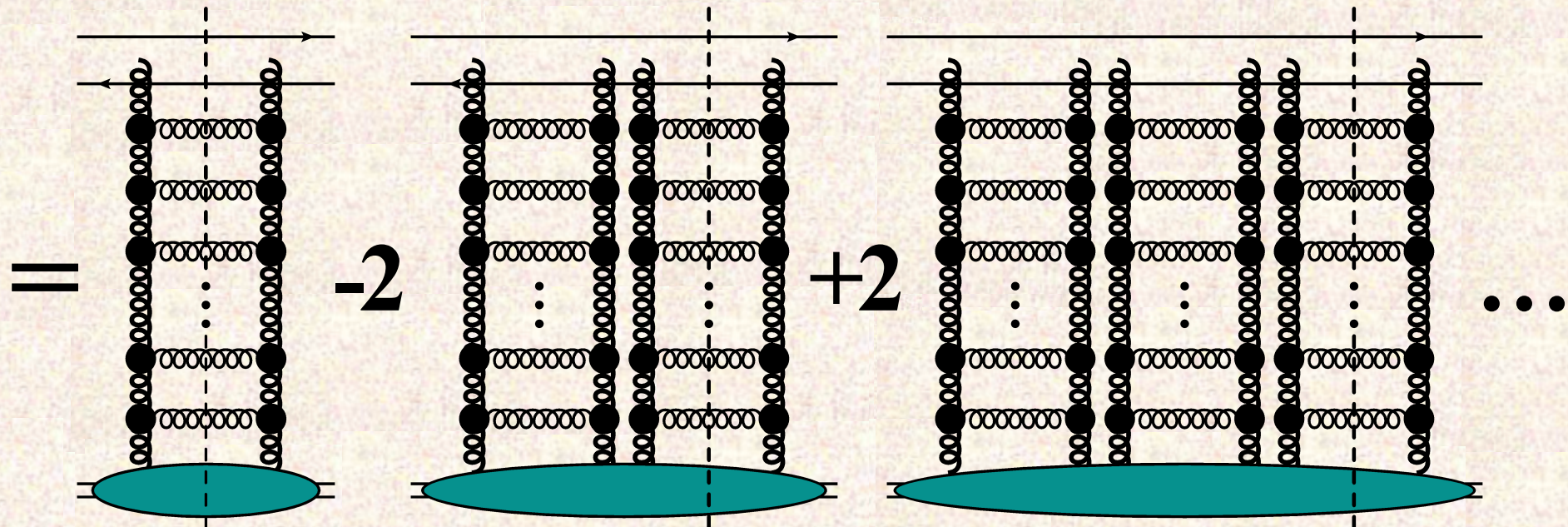
$$\sigma_n^{\text{AGK}}(Y = Y_A; \mathbf{x}_{01}, \mathbf{b}) = \frac{\left(\frac{1}{2} e^{\xi} \right)^n}{n!} \exp \left\{ -\frac{1}{2} e^{\xi} \right\}$$

The AGK cutting rules

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- General case $n=1$

$$\begin{aligned}\sigma_1^{\text{AGK}} &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k!}{(k-1)! 1!} 2^k C_k (\text{Im } G_{\text{BFKL}})^k \\ &= C_1 \sigma_{in}^{\text{BFKL}} + C_2 2 (2 \text{Im } G_{\text{BFKL}}) \sigma_{in}^{\text{BFKL}} \\ &\quad + C_3 3 (2 \text{Im } G_{\text{BFKL}})^2 \sigma_{in}^{\text{BFKL}} + \dots\end{aligned}$$



The AGK evolution equations

- The AGK cutting rules can be replaced by the evolution equations for σ_n . From the s-channel unitarity:

$$2 \operatorname{Im} A(Y; \mathbf{x}_{01}, \mathbf{b}) = |A(Y; \mathbf{x}_{01}, \mathbf{b})|^2 + G_{in}(Y; \mathbf{x}_{01}, \mathbf{b})$$

- For $N = \operatorname{Im} A$

$$2 N(Y; \mathbf{x}_{01}, \mathbf{b}) = \sigma_{SD}(Y; \mathbf{x}_{01}, \mathbf{b}) + \sigma_{in}(Y; \mathbf{x}_{01}, \mathbf{b})$$

- We define a new quantity – the particle production S-matrix “ S_{in} ”: it includes σ_n along with the no-interaction term (=1) and all the interaction terms (N and σ_{sd}) on either side (or both sides) of the final state cut:

$$S_{in} = 1 - 2 N + \sigma_{SD} + \sum_{n=1}^{\infty} \sigma_n$$

- From the final state cancellations, the evolution equation for S_{in} is of BK type too

$$\underbrace{\frac{\partial}{\partial Y} S_{in}(Y; \mathbf{x}_{01}, \mathbf{b})}_{\text{select fixed multiplicity terms}} = \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \underbrace{[S_{in}(Y; \mathbf{x}_{12}, \mathbf{b}) S_{in}(Y; \mathbf{x}_{02}, \mathbf{b}) - S_{in}(Y; \mathbf{x}_{01}, \mathbf{b})]}_{\text{select fixed multiplicity terms}}$$

select fixed multiplicity terms

select fixed multiplicity terms

The AGK evolution equations

- Replacing $S_{in}=1-2N+\sigma_{SD}+\Sigma_n\sigma_n$ in the equation for the S_{in} -matrix, we obtain the evolution equation for σ_n :

$$\begin{aligned}
 \frac{\partial}{\partial Y} \sigma_n (Y; \mathbf{x}_{01}, \mathbf{b}) &= \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \\
 &\times [\sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) + \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) - \sigma_n (Y; \mathbf{x}_{01}, \mathbf{b}) \\
 &+ \sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) \sigma_{SD} (Y; \mathbf{x}_{02}, \mathbf{b}) + \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) \sigma_{SD} (Y; \mathbf{x}_{12}, \mathbf{b}) \\
 &+ \sum_{k=1}^{n-1} \sigma_{n-k} (Y; \mathbf{x}_{02}, \mathbf{b}) \sigma_k (Y; \mathbf{x}_{12}, \mathbf{b}) \\
 &- 2 \sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) N (Y; \mathbf{x}_{02}, \mathbf{b}) - 2 \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) N (Y; \mathbf{x}_{12}, \mathbf{b})]
 \end{aligned}$$

Evolution of σ_n at large N_c

$$\begin{aligned}
 \frac{\delta}{\delta Y} \sim & \left[\text{Diagram 1} \right] = \left[\text{Diagram 2} \right] + \left[\text{Diagram 3} \right] \\
 - & \left[\text{Diagram 4} \right] + \left[\text{Diagram 5} \right] + \left[\text{Diagram 6} \right] \\
 + \sum_{k=1}^{n-1} & \left[\text{Diagram 7} \right] - 2 \left[\text{Diagram 8} \right] - 2 \left[\text{Diagram 9} \right]
 \end{aligned}$$

The diagrams are represented as follows:

- Diagram 1:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.
- Diagram 2:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.
- Diagram 3:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.
- Diagram 4:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.
- Diagram 5:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_{SD} . Below the line is a blue circle labeled σ_n . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.
- Diagram 6:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n . Below the line is a blue circle labeled σ_{SD} . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.
- Diagram 7:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_{n-k} . Below the line is a red circle labeled σ_k . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.
- Diagram 8:** A rectangle with a horizontal line. Above the line is a blue circle labeled N . Below the line is a blue circle labeled σ_n . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.
- Diagram 9:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n . Below the line is a blue circle labeled N . The top-left corner is labeled 0 and the bottom-right corner is labeled 1.

- We can see the equation is linear but with complicated kernel ($K_{\text{BFKL}} + 2 - 4 N$) and non-homogeneous

Evolution of σ_n at large N_c

- **Noticing that**

$$\begin{aligned}
 &= \sigma_n^{02}(Y) (1 + \sigma_{SD}^{12}(Y) - 2N_{12}(Y)) = \sigma_n^{02}(Y) S_{12}^D(Y, Y_0 = 2) \\
 &= \sigma_n^{12}(Y) (1 + \sigma_{SD}^{02}(Y) - 2N_{02}(Y)) = \sigma_n^{12}(Y) S_{02}^D(Y, Y_0 = 2)
 \end{aligned}$$

- **we reduce to**

$$\begin{aligned}
 \frac{\partial}{\partial Y} \sigma_n(Y, \mathbf{x}_{01}, \mathbf{b}) &= \frac{\bar{\alpha}_S}{2\pi} \int d^2x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \\
 &\times \left[-\sigma_n(Y, \mathbf{x}_{01}, \mathbf{b}) + \sigma_n(Y, \mathbf{x}_{12}, \mathbf{b}) S^D(Y, Y_0 = 2, \mathbf{x}_{02}, \mathbf{b}) \right] \\
 &+ S^D(Y, Y_0 = 2, \mathbf{x}_{12}, \mathbf{b}) \sigma_n(Y, \mathbf{x}_{02}, \mathbf{b}) + \sum_{k=1}^{n-1} \sigma_{n-k}(Y, \mathbf{x}_{02}, \mathbf{b}) \sigma_k(Y, \mathbf{x}_{12}, \mathbf{b})
 \end{aligned}$$

- **For model LT kernel and GS scaling we have**

$$\begin{aligned}
 \kappa \frac{d\sigma_n(z)}{dz} &= - \underbrace{\left(z - \int dz' S^D(z', z_0 = 2) \right)}_{=T(z)} \sigma_n(z) \\
 &+ S^D(z, z_0 = 2) \int_{\xi_0^A}^z dz' \sigma_n(z') + \sum_{k=1}^{n-1} \int_{\xi_0^A}^z dz' \sigma_{n-k}(z') \sigma_k(z)
 \end{aligned}$$

Evolution of σ_n at large N_c

- Defining $\Sigma_n(z) = \int_z^{\xi_0^A} dz' \sigma_n(z')$, we can rewrite it as

$$\kappa \frac{d^2 \Sigma_n(z)}{dz^2} = - \frac{d}{dz} (T(z) \Sigma_n(z)) + \Sigma_n(z) + \sum_{k=1}^{n-1} \Sigma_{n-k}(z) \sigma_k(z)$$

Rewriting the sum as $\sum_{k=1}^{n-1} \Sigma_{n-k}(z) \sigma_k(z) = \frac{1}{2} \frac{d}{dz} \sum_{k=1}^{n-1} \Sigma_{n-k}(z) \Sigma_k(z)$

and introducing $\mathcal{S}_n(z) = \int_z^{\xi_0^A} dz' \Sigma_n(z')$, we have

$$\kappa \frac{d^2 \Sigma_n(z)}{dz^2} = - \frac{d}{dz} (T(z) \Sigma_n(z)) + \frac{d \mathcal{S}_n(z)}{dz} + \frac{1}{2} \frac{d}{dz} \left[\sum_{k=1}^{n-1} \Sigma_{n-k}(z) \Sigma_k(z) \right]$$

The large $n \geq \langle n(z) \rangle$ Solution

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- We suggest that the solution has the following form

$$\Sigma_n(z) = \phi(z) \exp(-n \Phi(z))$$

- Functions $\Phi(z)$ and $\phi(z)$ we will find from the equation.
Replacing and taking the large z and n limit we find

$$\kappa \frac{d\Sigma_n}{dz} = -(z - \Sigma_\Delta(z))\Sigma_n(z) + \mathcal{S}_n(z) + \frac{n-1}{2}\phi(z)\Sigma_n(z)$$

- We can now solve the equation and find for DIS:

$$\sigma_n(z) = \left(\phi'(z) - n\phi(z)\Phi'(z) \right) \exp(-n\Phi(z))$$

KNO structure

$$\xrightarrow{n \geq \langle n(z) \rangle} \frac{2}{\kappa} \frac{z^2}{\langle n(z) \rangle} \Psi \left(\xi = \frac{n}{\langle n(z) \rangle} \right)$$

but without KNO scaling

where $\Psi(\xi) = \xi e^{-\xi}$ and $\langle n(z) \rangle = 2n_0 z e^{\frac{(z-z_\Delta)^2}{2\kappa}}$

Entropy of produced gluons

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- The probability to have n-cut BFKL Pomerons in the final state is equal to

$$P_n^{\text{AGK}}(z) \equiv \frac{\sigma_n^{\text{AGK}}(z)}{\sum_{n=1}^{\infty} \sigma_n^{\text{AGK}}(z)} = \frac{2}{\kappa} \frac{z^2}{\langle n(z) \rangle} \Psi \left(\frac{n}{\langle n(z) \rangle} \right)$$

- The probability to have n-gluons in the final state is

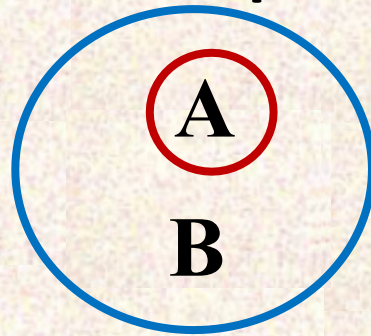
$$P_n^{\text{gluons}}(z) = \sum_{k=1}^{\infty} P_k^{\text{AGK}}(z) P_n^{\text{Poisson}}(k \langle n_P(z) \rangle)$$

$$\stackrel{z \gg 1}{=} \frac{2}{\kappa} \frac{z^2}{\langle n(z) \rangle} \Psi \left(\frac{n}{\langle n(z) \rangle \langle n_P(z) \rangle} \right)$$

Entropy of produced gluons

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- DIS probes only a part of the proton's wave function (region A).



$$r^2 \sim 1/Q^2$$

- The new observable of DIS — the entanglement entropy — is given by the von Neumann's formula

$$\begin{aligned} S_E(z) &= - \sum_n P_n(z) \ln(P_n(z)) \\ &= \ln \langle n(z) \rangle \xrightarrow{z \gg 1} \frac{z^2}{2\kappa} \end{aligned}$$

- In agreement with Kharzeev and Levin (2017) predictions.
- Using “local parton-hadron duality (LPHD)”, we have

$$S_{\text{hadron}} \approx S_{\text{gluons}}$$

Homotopy solution

- For σ_1 in region I (GS) (here $S^D = \Delta$)

$$\begin{aligned}
 \kappa \frac{d\sigma_1(z)}{dz} &= -z \sigma_1(z) + \sigma_1(z) \int_{\xi_0^A}^z dz' \Delta(z') + \Delta(z) \int_{\xi_0^A}^z dz' \sigma_1(z') \\
 &= - \left(z - \underbrace{\int_{\xi_0^A}^z dz' \Delta(z')}_{T(z)} \right) \sigma_1(z) + \Delta(z) \underbrace{\int_{\xi_0^A}^z dz' \sigma_1(z')} \\
 &\quad \int_{\xi_0^A}^z dz' \sigma_1(z') = \underbrace{\int_{\xi_0^A}^{\infty} dz' \sigma_1(z')}_{=\sigma_{0,1}} - \underbrace{\int_z^{\infty} dz' \sigma_1(z')}_{=\tilde{\Sigma}_1(z)}
 \end{aligned}$$

$$= -T(z) \sigma_1(z) + \sigma_{0,1} \Delta(z) - \Delta(z) \tilde{\Sigma}_1(z)$$

$$\Rightarrow \underbrace{\kappa \frac{d\sigma_1(z)}{dz} + T(z) \sigma_1(z) - \sigma_{0,1} \Delta(z)}_{\mathcal{L}[\sigma_1]} + \underbrace{\Delta(z) \tilde{\Sigma}_1(z)}_{\mathcal{N}_{\mathcal{L}}[\sigma_1]} = 0$$

Homotopy solution

- We have divided the equation in two parts

$$\mathcal{L}[\sigma_1] = \kappa \frac{d\sigma_1(z)}{dz} + T(z) \sigma_1(z) - \sigma_{0,1} \Delta(z)$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_1] = \Delta(z) \tilde{\Sigma}_1(z)$$

- As a solution, we introduce

$$\sigma_1 = \sigma_1^{(0)} + p \sigma_1^{(1)} + p^2 \sigma_1^{(2)} + \dots$$

such that $\mathcal{L}[\sigma_1^{(p)}] + p \mathcal{N}_{\mathcal{L}}[\sigma_1^{(p)}] = 0$

- The 0th iteration is

$$\mathcal{L}[\sigma_1^{(0)}]; \quad \kappa \frac{d\sigma_1^{(0)}(z)}{dz} = -T(z) \sigma_1^{(0)}(z) + \sigma_{0,1} \Delta(z)$$

Homotopy solution

- The solution is a sum of the solution to the homogeneous part of the equation and the particular solution of the non-homogeneous one. We obtain

$$\sigma_1^{(0,I)}(z) = \underbrace{\exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right)}_{\tilde{\sigma}_1^{(0)}} \left\{ \underbrace{\frac{\sigma_{0,1}}{\kappa} \int_{\xi_0^A}^z dz' \Delta(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right)}_{\tilde{\sigma}_1^{(0,I)}} + C_\phi^{(1)} \right\}$$

where $C_\phi^{(1)} = \frac{1}{2} e^{\xi_0^A} \exp\left(-\frac{1}{2} e^{\xi_0^A}\right)$

- For the next homotopy iteration we need to account for the linear terms in p : $\mathcal{L}[\sigma_1^{(0)} + p \sigma_1^{(1)}] + p \mathcal{N}_{\mathcal{L}}[\sigma_1^{(0)}] = 0$
- The equation takes the form

$$\kappa \frac{d\sigma_1^{(1,I)}(z)}{dz} = -T(z) \sigma_1^{(1,I)}(z) - \Delta(z) \tilde{\Sigma}_1^{(0)}(z)$$

where $\tilde{\Sigma}_1^{(0)}(z) = \int_z^\infty dz' \sigma_1^{(0)}(z')$. **The general solution is**

$$\sigma_1^{(1,I)}(z) = -\frac{1}{\kappa} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \int_{\xi_0^A}^z dz' \Delta(z') \tilde{\Sigma}_1^{(0)}(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right)$$

- In general, the equation for the p-iteration ($p \geq 1$) in region I takes the form**

$$\kappa \frac{d\sigma_1^{(p,I)}(z)}{dz} = -T(z) \sigma_1^{(p,I)}(z) - \Delta(z) \tilde{\Sigma}_1^{(p-1)}(z)$$

with solution

$$\sigma_1^{(p,I)}(z) = -\frac{1}{\kappa} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \int_{\xi_0^A}^z dz' \Delta(z') \tilde{\Sigma}_1^{(p-1)}(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right)$$

Homotopy solution for σ_n

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- Now we need to add to our definition of $\mathcal{L}[\sigma_1]$ the non-homogeneous term

$$\begin{aligned}\mathcal{L}[\sigma_n] &= \kappa \frac{d\sigma_n(z)}{dz} + T(z) \sigma_n(z) - \underbrace{\sigma_{0,n} \Delta(z) - \sum_{k=1}^{n-1} \Sigma_{n-k}(z) \sigma_k(z)}_{-U_n(z)} \\ \mathcal{N}_{\mathcal{L}}[\sigma_n] &= \Delta(z) \tilde{\Sigma}_n(z)\end{aligned}$$

- We solve the equation in a similar way to σ_1 , yielding

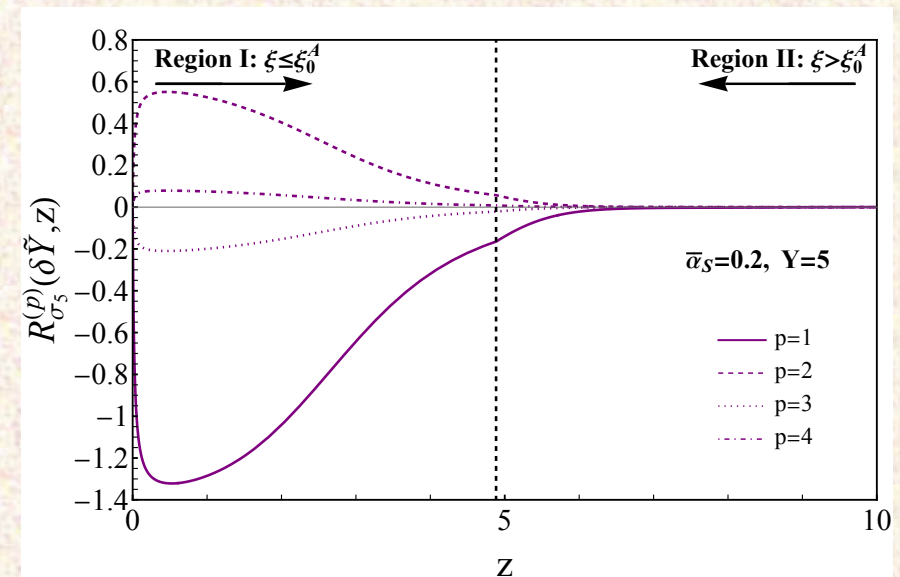
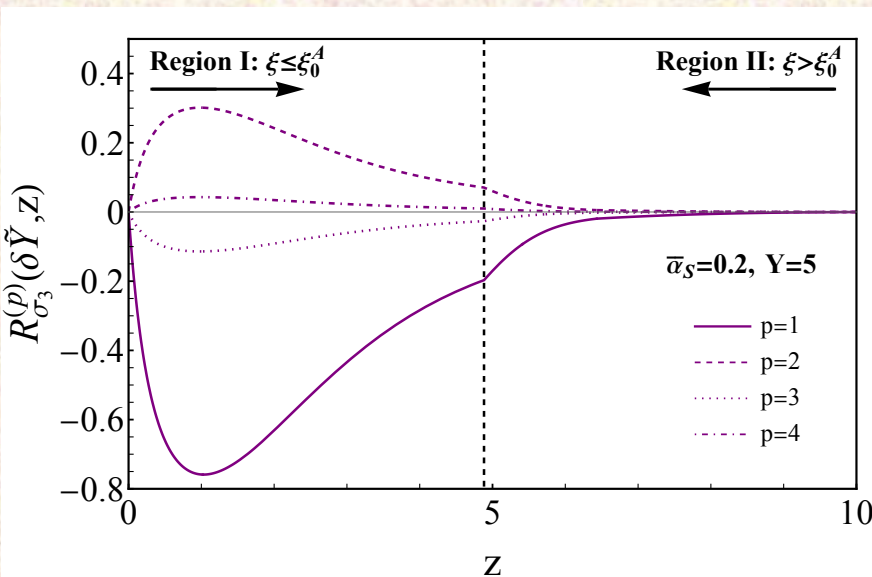
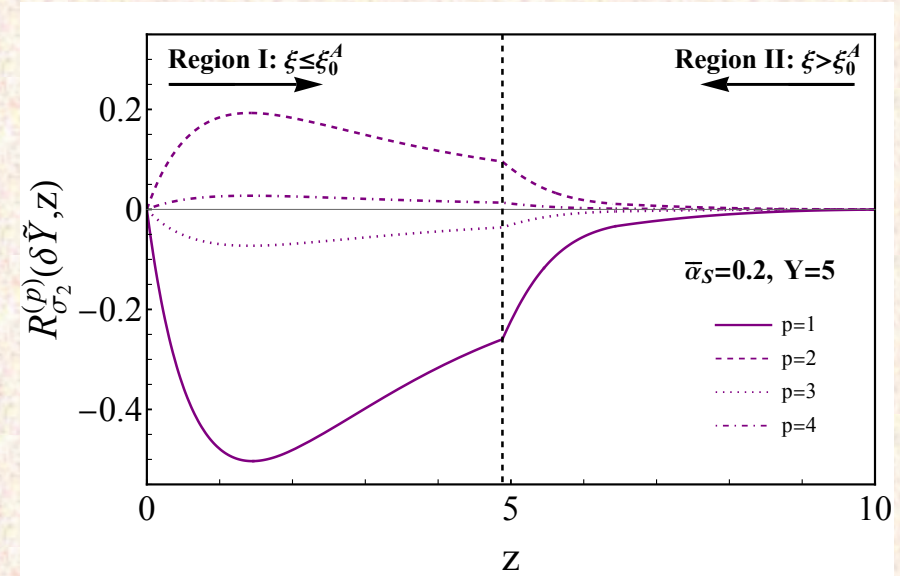
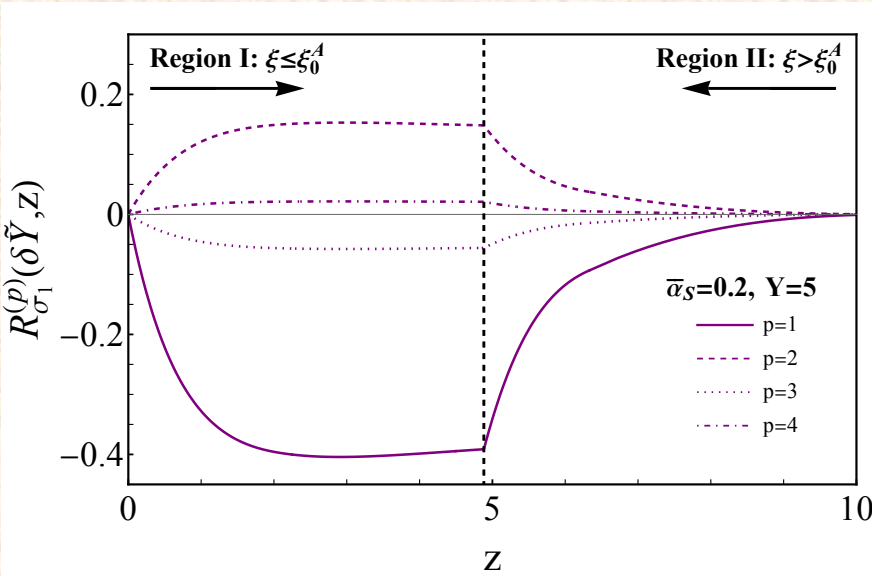
$$\sigma_n^{(0,I)}(z) = \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \left\{ \frac{1}{\kappa} \int_{\xi_0^A}^z dz' U_n(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right) + C_\phi^{(n)} \right\}$$

where $C_\phi^{(n)} = \frac{\left(\frac{1}{2}e^{\xi_0^A}\right)^n}{n!} e^{-\frac{1}{2}e^{\xi_0^A}}$. The formula for the p-iteration is

$$\sigma_n^{(p,I)}(z) = \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \left\{ \frac{1}{\kappa} \int_{\xi_0^A}^z dz' U_n(z') \tilde{\Sigma}_n^{(p-1)}(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right) + C_\phi^{(n)} \right\}$$

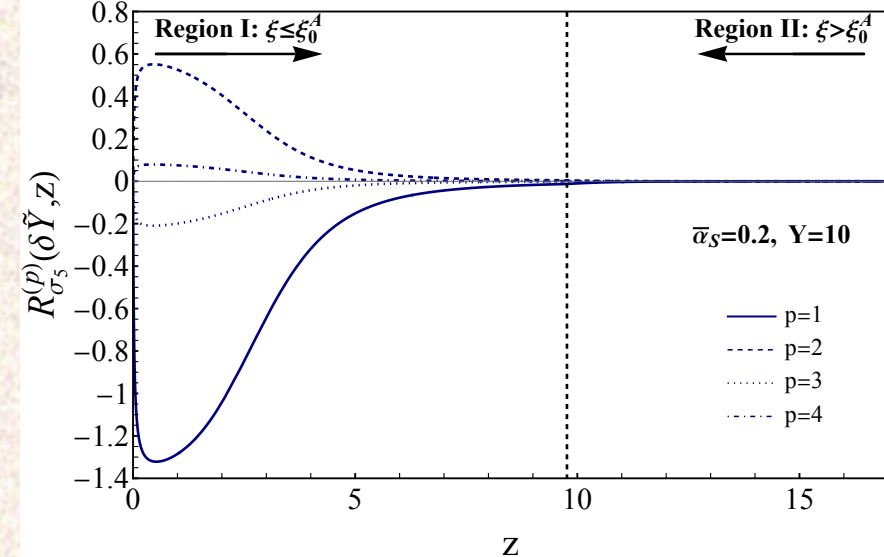
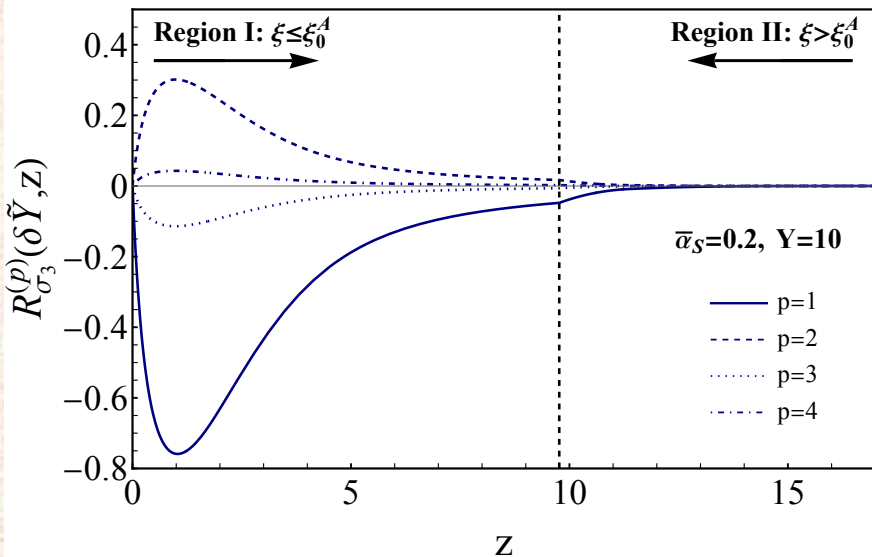
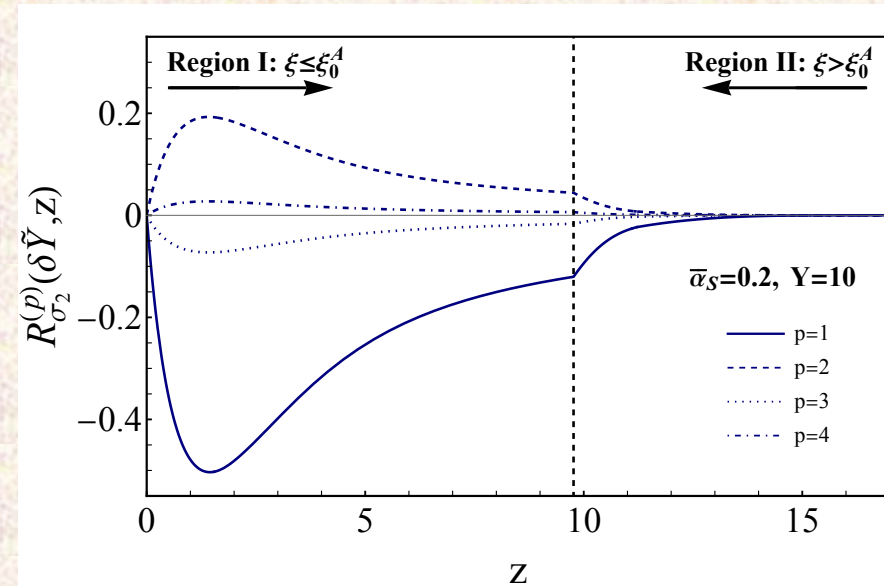
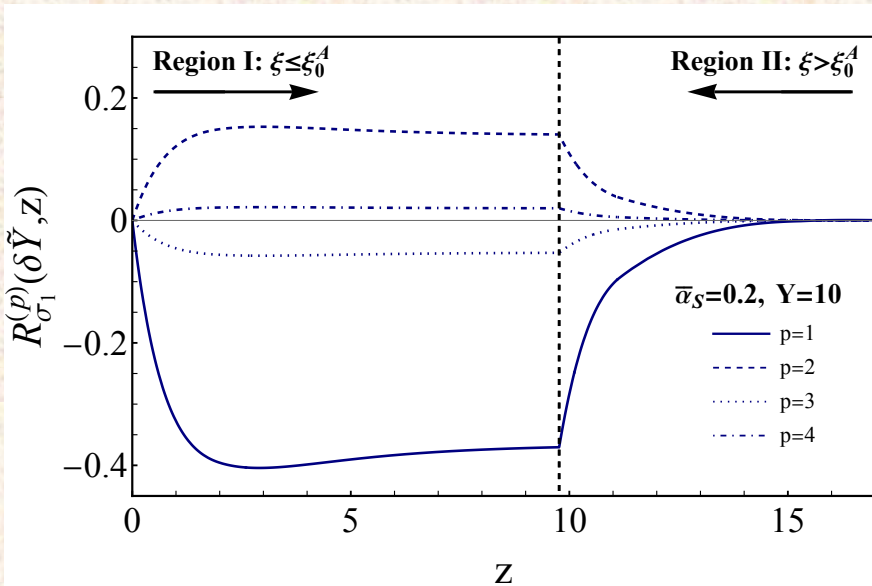
Numerical estimates

- Ratios of $\sigma_1, \sigma_2, \sigma_3$ and σ_5 for $\xi_0^A=0, \alpha_S=0.2, z_0=2$.



Numerical estimates

- Ratios of $\sigma_1, \sigma_2, \sigma_3$ and σ_5 for $\xi_0^A=0, \alpha_S=0.2, z_0=2$.



Numerical estimates

- One can see that the first iteration turns out to be rather large in the small- z region which corresponds to region I.
- Despite the fact that after an appropriate number of iterations the corrections become small, the approach may break down for a sufficiently large n . For this reason, we use this method only for small values of n .
- We solve the equation in the large- n limit, and study the possible matching with this solution.

Applications at EIC era

- Having these cross sections, we can calculate the multiplicity moments using Mueller's formula

$$\langle n^k(z) \rangle = \sigma_k^{\text{AGK}}(z) \langle n_P(z) \rangle^k$$

where $\langle n_P(z) \rangle = \bar{\gamma} z$

- And therefore, all cumulants can be calculated:

$$C_1(z) = \langle n(z) \rangle \quad \text{Mean}$$

$$C_2(z) = \langle n^2(z) \rangle - \langle n(z) \rangle^2 \quad \text{Variance}$$

$$C_3(z) = \langle n^3(z) \rangle - 3 \langle n^2(z) \rangle \langle n(z) \rangle \quad \text{Third central moment of our distributions}$$

- Finally we can obtain the multiplicity distribution of the produced gluons in the saturation region as follows

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{C_n}{n!} (\lambda - 1)^n ; \quad P_n(z) = \oint \frac{d\lambda}{2\pi i} \frac{e^{f(\lambda)}}{\lambda^{n+1}}$$



*Solution to the equation of sigma n
in momentum space?*

The momentum space approach ⁴⁸

- We could also solve the evolution equations by going to momentum space, using the following transformations:

$$N(Y, \mathbf{x}_{01}, \mathbf{b}) = x_{01}^2 \int_0^\infty dk_T k_T J_0(k_T x_{01}) \tilde{N}(Y, \mathbf{k}_T, \mathbf{b})$$

$$\tilde{N}(Y, \mathbf{k}_T, \mathbf{b}) = \int_0^\infty \frac{dx_{01}}{x_{01}} J_0(k_T x_{01}) N(Y, \mathbf{x}_{01}, \mathbf{b})$$

- For the initial condition, we have

$$\begin{aligned} \tilde{N}(Y = Y_A, \mathbf{k}_T, \mathbf{b}) &= \int_0^\infty \frac{dx_{01}}{x_{01}} J_0(k_T x_{01}) \left(1 - \exp \left(-\frac{1}{4} x_{01}^2 Q_{s0}^2 \right) \right) \\ &= \frac{1}{2} \Gamma \left(0, \frac{k_T^2}{Q_{s0}^2} \right) \end{aligned}$$

This is a positive initial condition.

*The momentum space approach*⁴⁹

- **For σ_n**

$$\tilde{\sigma}_n^{\text{AGK}}(Y, \mathbf{k}_T, \mathbf{b}) = \int_0^\infty \frac{dx_{01}}{x_{01}} J_0(k_T x_{01}) \sigma_n^{\text{AGK}}(Y, \mathbf{x}_{01}, \mathbf{b})$$

- **For the initial condition, however**

$$\begin{aligned} \tilde{\sigma}_n^{\text{AGK}}(Y, \mathbf{k}_T, \mathbf{b}) &= \int_0^\infty \frac{dx_{01}}{x_{01}} J_0(k_T x_{01}) \left[\frac{\left(\frac{1}{2} x_{01}^2 Q_{s0}^2\right)^n}{n!} \exp\left(-\frac{1}{2} x_{01}^2 Q_{s0}^2\right) \right] \\ &= \frac{1}{2n} \exp\left(-\frac{1}{2} \frac{k_T^2}{Q_{s0}^2}\right) L_{n-1}\left(\frac{1}{2} \frac{k_T^2}{Q_{s0}^2}\right) \end{aligned}$$

- **Unfortunately, this gives a negative initial cross section for large n . What can we do?**

Conclusions and outlook

- We calculated the entropy and the multiplicity distributions directly from QCD evolution equations and the s-channel unitarity constraints. These results are in agreement with the predictions of Kharzeev and Levin.
- σ_1 , σ_2 and σ_3 are cross sections for 1, 2 and 3 cut Pomeron production which are closely related to the moments and cumulants of the multiplicity distributions for DIS experiment. Calculations of the moments $\langle n^k \rangle$ for small k will be subject of next papers.
- As our understanding of high-energy QCD deepens, new challenges and questions arise. Stay tuned.

Conclusions and outlook

2025

2026

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1. Region I

In the region I in Fig. 2 Eq. (III.8) has the solution which shows the geometric scaling behaviour. For such a solution both z_Δ and σ_0 are constant and solution has a simple form:

$$\sigma_1^{(0,I)}(Y, r_{01}, b) = \sigma_1^{(0,I)}(z_0) \exp \left(\underbrace{-\frac{(z - z_\Delta^I)^2}{2\kappa}}_{\tilde{\sigma}_1^{(0)}} \right) \left\{ \underbrace{\frac{\sigma_0^I}{\kappa} \int_{z_0}^z dz' \frac{\Delta(z')}{\tilde{\sigma}_1^{(0)}(z')} + C_1 \right\} \quad (\text{III.9})$$

where $z_\Delta^I = \int_0^\infty dz' \Delta^I(z')$ and $\sigma_0^I = \int_0^\infty dz' \sigma(z')$ are constant (independent of $\delta\tilde{Y}$). From initial condition at $z = z_0$ (see Eq. (I.11)) we find: $C_1 \sigma_1^{(0,I)}(z_0) \exp \left(\frac{z_\Delta^I}{2\kappa} \right) = \frac{1}{2} e^{\xi^A} \exp \left(-\frac{1}{2} e^{\xi^A} \right)$.

Using Eq. (III.9) we can find the second iteration of Eq. (III.6) for $\mathcal{N}_L(\sigma_1) = \mathcal{N}_L'$ (see Eq. (III.4)). The equation takes the form:

$$\kappa \frac{\partial \sigma_1^{(1)}(z)}{\partial z} = - (z - z_\Delta) \sigma_1^{(1)}(z) - \sigma_1^{(0)}(z) \int_z^\infty dz' \Delta(Y, z') - \Delta(z) \int_z^\infty dz' \sigma_1^{(0)}(z') \quad (\text{III.10})$$

Searching for the solution in the form: $\sigma_1^{(1)}(Y, r_{01}, b) = \tilde{\sigma}_1^{(0)}(z) \tilde{\sigma}_1^{(1)}(z)$ we obtain

$$\tilde{\sigma}_1^{(1)}(z) = - \frac{1}{\kappa} \int_0^z dz' \frac{\sigma_1^{(0)}(z') \int_z^\infty dz'' \Delta(z'') + \Delta(z) \int_z^\infty dz' \sigma_1^{(0)}(z')}{\tilde{\sigma}_1^{(0)}(z')} \quad (\text{III.11})$$

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$$\times \int d^2 \delta b_i \gamma^{BA}(r_1, r'_1, b_i - b'_i \equiv \delta b_i) \int d^2 \delta b_j \gamma^{BA}(r_1, r'_j, b_j - b'_j \equiv \delta b_j) \rho_{n,k}^D(r'_1, b'_1, \dots, r'_n, b'_n; \dots, r'_k, b'_k)$$

FIG. 3: Summing large Pomeron loops for diffraction in the dipole-dipole scattering. The wavy lines denote Pomeron exchanges. The orange circles denote the amplitude γ . The black circle denote the triple Pomeron vertices show the single diffraction cross section. For simplicity we put only the BFKL Pomeron exchange for the production mass $\ln M_{m1} = Y - Y_{01}$. Fig. 3-b describe the double diffraction. Only the BFKL Pomeron exchanges are shown for $\ln M_{01} = Y - Y_{01}$ and $\ln M_{m2} = Y_{02} - 0$. S and S^* stand for S-matrix and its conjugated.

III. DOUBLE DIFFRACTION IN ONE DIMENSIONAL MODEL

2027?



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