ITMD beyond small-x: dijets in DIS

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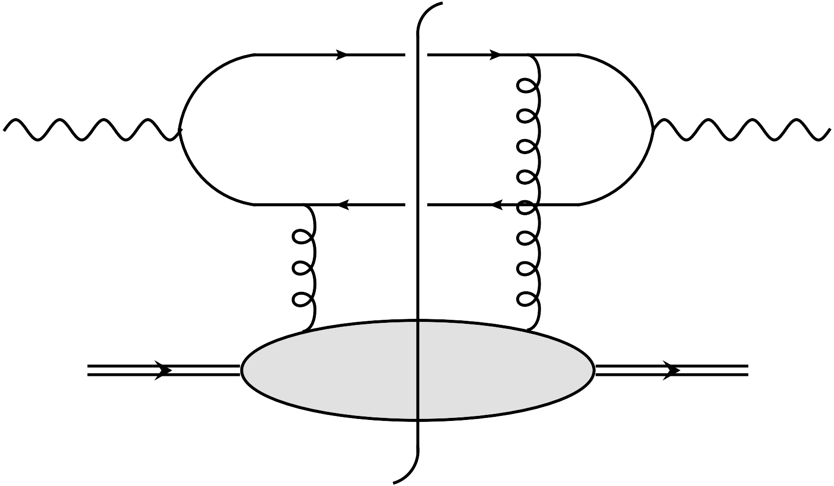
Thanks to: Swagato Mukherjee, Vladimir Skokov, Andrey Tarasov, Fei Yao.

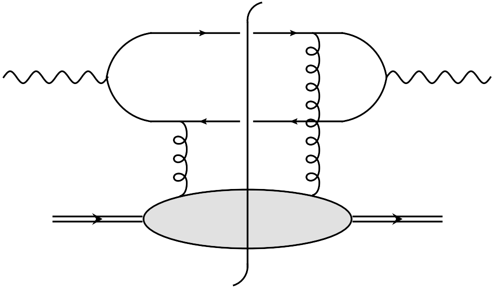
Based on: 2602.15137, 2607.12268 and to appear ..

Motivation



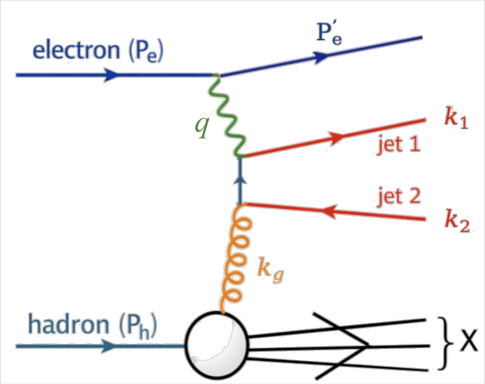
Dijet production

- Studied at **small-x** and **large-x**
- **At small-x**: CGC in (sub-)eikonal limit
- **At large-x**: TMD in back-to-back limit



Very interesting process to study the interplay between small and large-x formalisms

Dijet production

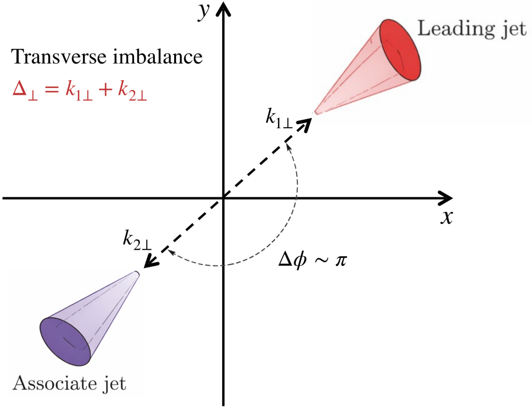


$$\boxed{\text{Hard scale: } Q \sim P_\perp}$$

Back to back kinematics

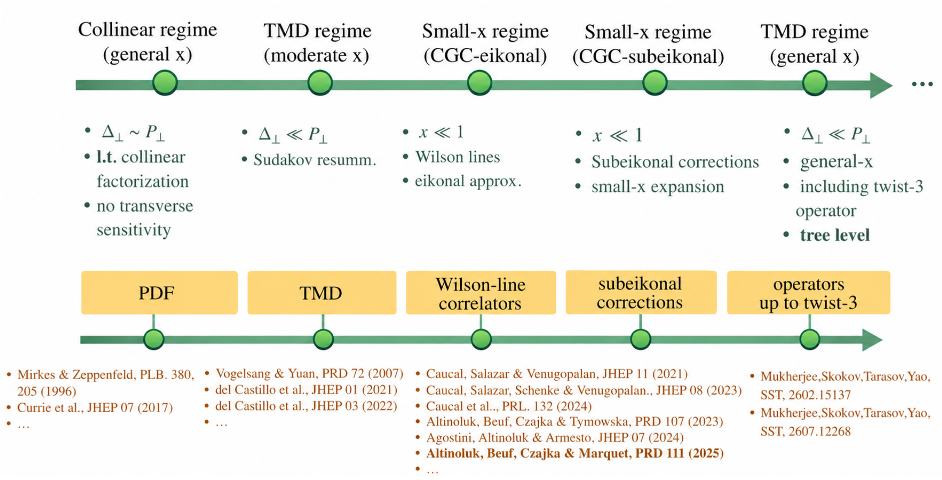
$$\boxed{\text{Imbalance: } \Delta_\perp = k_{1\perp} + k_{2\perp}}$$

$$\boxed{\text{Back to back limit: } \Delta_\perp \ll P_\perp}$$



TMD factorization has been shown for dijet production in back to back kinematics

Current status of calculations



Back-to-back dijet production in DIS at arbitrary Bjorken-x: TMD gluon distributions to twist-3 accuracy

Swagato Mukherjee,[1] Vladimir V. Skokov,[2] Andrey Tarasov,[2,3] Shaswat Tiwari,[1,2,*] and Fei Yao[1,†]

- Background field method
- Leading order in α_s
- Both quark and gluon operators upto twist-3
- Reproduces the (sub)eikonal results

Mukherjee, Skokov, Tarasov, Yao, SST (2026, 2026)

What operators are considered?

Kinematic twist

expansion in $\Delta_\perp / P_\perp$

dynamic twist

$\tau = d - s$

	twist-2 (leading)	twist-3 (linear)
twist-2	$C_1 \otimes F_{-i}F_{-j}$ ✔	$C_2 \otimes F_{-i}F_{-j}$ $C_2 \propto \mathcal{O}\left(\Delta_\perp / P_\perp\right)$ ✔
twist-3	$C_3 \otimes F_{ij}F_{-j}$ $C_4 \otimes F_{+-}F_{-j}$ $C_5 \otimes F_{-i}F_{-k}F_{-l}$ ✔	$C_6 \otimes F_{ij}F_{-j}\,, \, \ldots$ ✘

The procedure summarized

Dijet amplitude

Arbitrary number of insertions $\big\downarrow$ Identify the background propagators

Perform a twist expansion in terms of field strength tensors

Dynamical background $\big\downarrow$ Includes twist-three operators

Substitute back into the amplitude

Construct TMD with full phase $\big\downarrow$ Retain terms upto $\mathcal{O}(\frac{\Delta}{P})$

Dijet cross-section upto twist-three accuracy

Mukjerjee, Skokov, Tarasov, Yao, SST (2026, 2026)

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Does the methodology work at NLO?

TMD factorization at arbitrary bjorken x

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$\boxed{\text{The answer is no!}}$

ITMD beyond small-x

$\boxed{\text{Methodology developed under kinematic twist expansion not applicable}}$

- NLO in CGC: No problem since all twist
- ITMD formalism: resum kinematic corrections at small-x

$$\mathrm{d}\sigma_{\mathrm{CGC}} = \underbrace{\mathrm{d}\sigma_{\mathrm{TMD}} + \overbrace{\mathcal{O}\left(\frac{\Delta_\perp}{Q_\perp}\right) + \mathcal{O}\left(\frac{Q_s}{\Delta_\perp}\right)}^{\text{kinematic}}}_{\mathrm{d}\sigma_{\mathrm{ITMD}}} + \overbrace{\mathcal{O}\left(\frac{Q_s}{Q_\perp}\right)}^{\text{genuine}}$$

kotko, kutak, Marquet, Petreska, Sapeta, van Hameren (2015)
Altinoluk, Boussarie (2019)
Boussarie, Mehtar-Tani (2020)
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$\boxed{\text{We propose a new background field method that generalizes } \textbf{ITMD beyond small-x}}$

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$\boxed{\text{Let us understand our back-to-back calculation in more detail}}$

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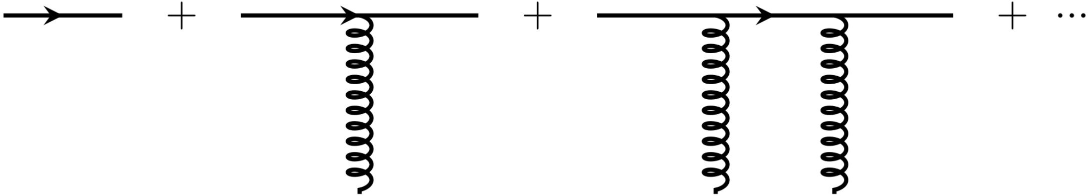
Quark propagator

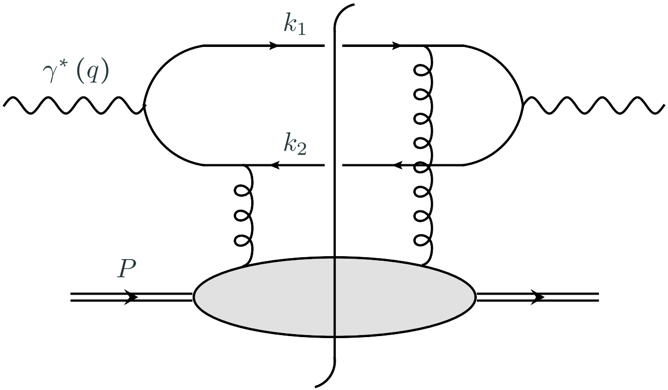


$$(x|\frac{i}{\not{P}}|y) = (x|\frac{i}{\not{p}}|y) + (x|\frac{i}{\not{p}}\not{A}\frac{1}{\not{p}}|y) + (x|\frac{i}{\not{p}}\not{A}\frac{1}{\not{p}}\not{A}\frac{1}{\not{p}}|y) + \dots$$

$$\boxed{P^\mu = p^\mu + A^\mu}$$

The dijet amplitude

- $P = (P^+, 0, 0_\perp)$
- $A^+ = 0$



$$i\mathcal{M} = -ie\,\epsilon_\rho(q) \lim_{k_1^2, k_2^2 \to 0} \int d^4y\, e^{-iq\cdot y}\, \bar{u}(k_1) \underbrace{\boxed{(k_1|\not{p}\frac{i}{\not{P}}|y)}}_{\text{quark}} \gamma^\rho \underbrace{\boxed{(y|\frac{i}{\not{P}}\not{p}|-k_2)}}_{\text{anti-quark}} v(k_2)$$

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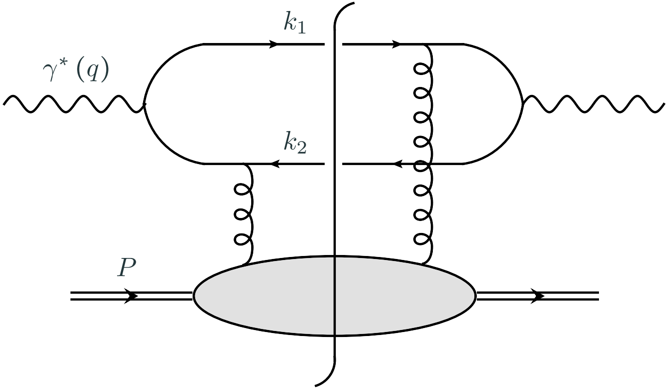
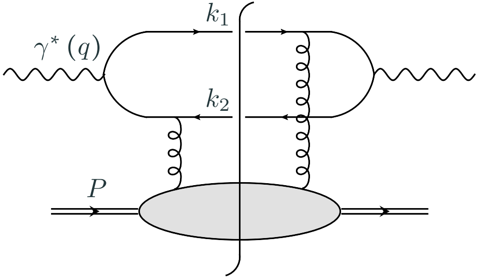


$$\bar{u}(k_1)\,(k_1|\not{p}\,\frac{i}{\not{\mathrm{P}}}|y) = \bar{u}(k_1)\,(k_1|\not{p}\not{\mathrm{P}}\,\frac{i}{\not{\mathrm{P}}\not{\mathrm{P}}}|y) = \bar{u}(k_1)k_1^2\,(k_1|\frac{i}{\mathrm{P}^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|y)\,,$$

The dijet amplitude

- $P = (P^+, 0, 0_\perp)$

- $A^+ = 0$



$$i\mathcal{M} = ie\,\epsilon_\rho(q)\int d^4y\,e^{-iq\cdot y}\lim_{k_1^2,k_2^2\to 0}\bar{u}(k_1)\underbrace{\boxed{k_1^2\,(k_1|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|y)}}_{\text{quark}}\gamma^\rho\underbrace{\boxed{(y|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|-k_2)\,k_2^2}}_{\text{anti-quark}}v(k_2)$$

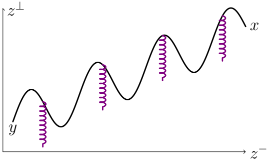
$\boxed{\text{We shall work with these propagators now}}$

The propagator and its expansion

$$(x|\frac{1}{\mathrm{P}^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} + i\epsilon}|y) = (x|\frac{1}{2p^- \boxed{\mathrm{P}^+} - \boxed{\mathrm{P}_\perp^2} - i\boxed{\partial^- A_-} + \boxed{\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}} + i\epsilon}|y)$$

$$\boxed{\text{Propagator} \underset{\text{gradient expansion}}{=} \text{Wilson line} + \text{F insertions} + \text{target dynamics} + \text{fermion}}$$

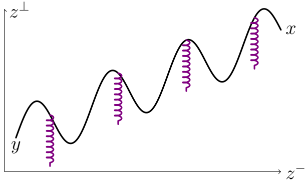
Expanding the scalar propagator



$$(x|\frac{1}{\mathrm{P}^{+}-\frac{\mathrm{P}_{\perp}^{2}}{2k^{-}}+i\epsilon}|y) = (x|\Big[\frac{1}{\mathrm{P}^{+}+i\epsilon}+\frac{1}{\mathrm{P}^{+}+i\epsilon}\frac{\mathrm{P}_{\perp}^{2}}{2k^{-}}\frac{1}{\mathrm{P}^{+}+i\epsilon}+\cdots\Big]|y)$$

$\boxed{\text{Let us simplify the eikonal } P^{+} \text{ propagator now}}$

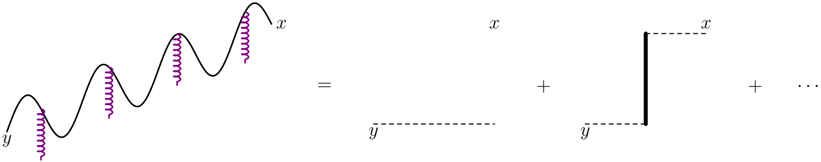
Expanding the scalar propagator: Wilson lines



$$(x^-|\frac{1}{\mathrm{P}^+ + i\epsilon}|y^-) = -i\theta(x^- - y^-)[x^-, y^-]$$

$$\boxed{\text{Plugging it back into the propagator}}$$

Expanding the scalar propagator

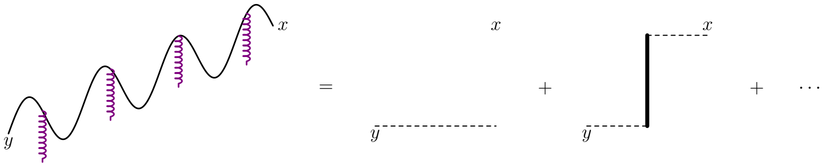


$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y) = \delta(x^+ - y^+)\,\theta(x^- - y^-)\,(x_\perp|\Big[-i\,[x^-, y^-]$$

$$- \int_{y^-}^{x^-} dz^-\,[x^-, z^-]\,\frac{\mathrm{P}_\perp^2(z^-)}{2k^-}\,[z^-, y^-] + \cdots\Big]|y_\perp)$$

$$\boxed{\text{Let us now generate F insertions}}$$

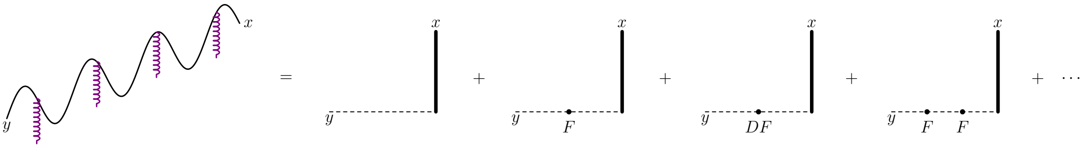
Generating F insertions



$$[x^-, y^-]\,\mathrm{P}_i(y^-) = \mathrm{P}_i(x^-)\,[x^-, y^-] - \int_{y^-}^{x^-} dz^-\,[x^-, z^-]\,F_{-i}(z^-)\,[z^-, y^-]$$

$\boxed{\text{Let us now generate F insertions}}$

Intermediate form of the scalar propagator

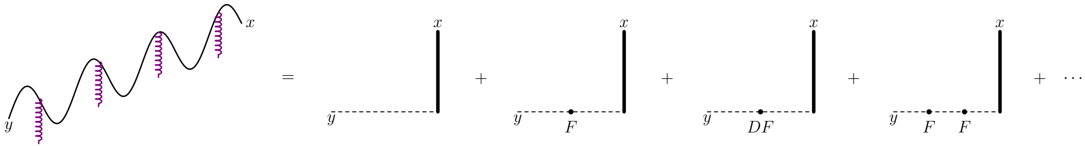


$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y) = \delta(x^+ - y^+)\,\theta(x^- - y^-)\,(x_\perp|\Big[-i\,[x^-,y^-] - (x^- - y^-)\frac{\mathrm{P}_\perp^2(x^-)}{2k^-}[x^-,y^-]$$

$$+\int_{y^-}^{x^-} dz_1^-\,(z_1^- - y^-)\frac{2\mathrm{P}_i(x^-)}{2k^-}[x^-,z_1^-]F_{-i}(z_1^-)[z_1^-,y^-] + \cdots\Big]|\,y_\perp)$$

We can resum the exponential at the boundary: phase ignored in small-x/CGC

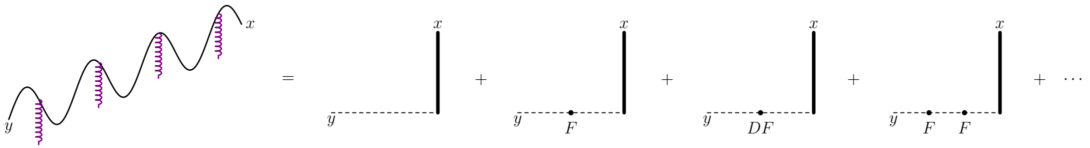
Final form of the scalar propagator



$$
\begin{aligned}
(x|\frac{1}{\mathrm{P}^{+}-\frac{\mathrm{P}_{\perp}^{2}}{2k^{-}}+i\epsilon}|y) &= \delta(x^{+}-y^{+})\theta(x^{-}-y^{-})(x_{\perp}|e^{-i(x^{-}-y^{-})\frac{\mathrm{P}_{\perp}^{2}(x^{-})}{2k^{-}}}\Big[-i\,[x^{-},y^{-}] \\
&+\frac{1}{2k^{-}}\int_{y^{-}}^{x^{-}}dz^{-}(z^{-}-y^{-})\Big(2\mathrm{P}_{k}(x^{-})\,[x^{-},z^{-}]\,F_{-k}(z^{-})\Big)\,[z^{-},y^{-}]\Big]|y_{\perp})\,,
\end{aligned}
$$

We can resum the exponential at the boundary: phase ignored in small-x/CGC

Going back to quark propagator

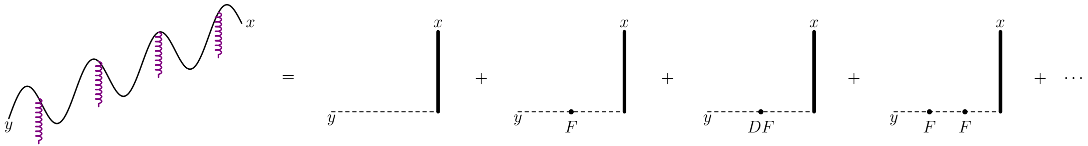


$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + \frac{\sigma F}{4k^-} + i\epsilon}|y) = (x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y) + \delta(x^+ - y^+)\,\theta(x^- - y^-)$$

$$(x_\perp|e^{-i(x^- - y^-)\frac{\mathrm{P}_\perp^2(x^-)}{2k^-}}\left[\frac{1}{4k^-}\int_{y^-}^{x^-} dz^-[x^-, z^-]\,\sigma F(z^-)\,[z^-, y^-] + \ldots\right]|y_\perp)$$

$\boxed{\text{Carrying out amputation (LSZ)}}$

Quark propagator amputated



$$
\lim_{k^2\to 0} k^2 (k|\frac{1}{P^2 + \frac{1}{2}\sigma F + i\epsilon}|y)
$$

$$
= i e^{iky}\Big\{-i[\infty, y^-]_{y_\perp} + \int_{y^-}^{\infty} dz^- \Big[\frac{k_k}{k^-}(z^- - y^-)\bar{F}_{-k}(z^-, y_\perp) + \frac{1}{4k^-}\sigma\bar{F}(z^-, y_\perp)\Big][\infty, y^-]_{y_\perp}\Big\}
$$

where $\bar{F}(z^-) = [\infty, z^-]F(z^-)[z^-, \infty]$

$\boxed{\text{Similar procedure for target dynamics}}$

Summary of different terms

Four layers of gradient expansion

(i) Longitudinal propagation — Wilson line

- Resumption of multiple A^- interactions
- Defines the gauge-invariant backbone

(ii) dynamics in target — F_{+-}

- Encodes longitudinal field gradients
- Generate twist-2 F_{+-} insertions
- Absent in strict eikonal limit

(iii) Field-strength insertions — F_{-i}

- merges from transverse momentum expansion
- Generates twist-1 F_{-i}, two-gluon body $F_{-k}F_{-l}$, and kinematic twist-2 from D_iF_{-j}

(iv) Spin-field coupling — $\sigma^{\mu\nu}F_{\mu\nu}$

- Pauli spin–field coupling
- Produces additional transverse twist-2 structures F_{ij}
- Absent for longitudinal photon

Final propagators

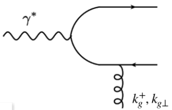
Quark propagator: $$\lim_{k^2\to 0} k^2 (k|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma F+i\epsilon}|y) = ie^{iky}\Big\{-i[\infty,y^-]_{y_\perp} + c_1(k)\,F + c_2(k)\,\sigma F + c_3(k)\,DF + c_4(k)\,D(\sigma F) + c_5(k)\,FF + c_6(k)\,(\sigma F)F + \ldots\Big\}$$

Antiquark propagator: $$\lim_{k^2\to 0} k^2 (y|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma F+i\epsilon}|-k) = -ie^{iky}\Big\{i[y^-,\infty]_{y_\perp} + c_1^*(-k)\,F + c_2^*(-k)\,\sigma F + c_3^*(-k)\,DF + c_4^*(-k)\,D(\sigma F) + c_5^*(-k)\,FF + c_6^*(-k)\,(\sigma F)F + \ldots\Big\}$$

Dijet amplitude

quark **antiquark**

$$i\mathcal{M} = ie\,\epsilon_\rho(q)\int d^4y\,e^{-iq\cdot y}\lim_{k_1^2,k_2^2\to 0}\bar{u}(k_1)\,k_1^2\,(k_1|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|y)\gamma^\rho(y|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|-k_2)\,k_2^2\,v(k_2)\,.$$



$$\begin{aligned} i\mathcal{M} = &-e\,\epsilon_\rho(q)\int\frac{dx^- d^2x_\perp}{2q^-}e^{i(k_g^+x^- - \Delta_\perp x_\perp)}\bar{u}(k_1)\Big\{\mathbb{C}_1^i\gamma^\rho\overline{F}_{-i}(x^-,x_\perp) + i\left(\mathbb{S}_1\sigma^{\mu\nu}\gamma^\rho + \mathbb{S}_2\gamma^\rho\sigma^{\mu\nu}\right)\overline{F}_{\mu\nu}(x^-,x_\perp) \\ &+\mathbb{C}_2^+\,\gamma^\rho\,\overline{F}_{+-}(x^-,x_\perp) + \Big[\left(\mathbb{T}_1^{kl}\,\gamma^\rho + i\,\mathbb{C}_3^l\,\sigma^{-k}\,\gamma^\rho + i\,\mathbb{C}_4^k\,\sigma^{-l}\gamma^\rho + i\,\mathbb{C}_5^k\,\gamma^\rho\,\sigma^{-l}\right)\mathbb{D}_{kl}(l^+,x^-,x_\perp) \\ &+\left(\mathbb{T}_2^{kl}\,\gamma^\rho + i\,\mathbb{C}_6^l\,\sigma^{-k}\,\gamma^\rho + i\,\mathbb{C}_7^l\,\gamma^\rho\,\sigma^{-k} + i\,\mathbb{C}_8^k\,\gamma^\rho\,\sigma^{-l}\right)\mathbb{D}_{kl}^\dagger(-l^+,x^-,x_\perp)\Big]_{l^+=0}\Big\}v(k_2) \end{aligned}$$

Two-body (twist-2) operator:

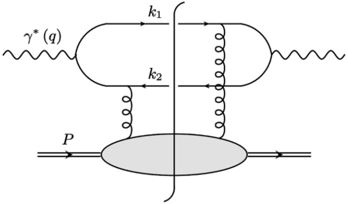
$$\mathbb{D}_{kl}(l^+,x^-,x_\perp) = i\int\frac{dk^+}{(2\pi)}\frac{1}{k^+ - l^+ + i\epsilon}\int dx'^-\,e^{ik^+(x^- - x'^-)}[\infty,x'^-]_{x_\perp}\,F_{-k}(x'^-,x_\perp)\,[x'^-,x^-]_{x_\perp}\,F_{-l}(x^-,x_\perp)\,[x^-,\infty]_{x_\perp}$$

Cross-section

Differential cross-section

$$k_1^- k_2^- \frac{d\sigma}{\text{đ} k_1^- \, \text{đ}^2 k_{1\perp} \, \text{đ} k_2^- \, \text{đ}^2 k_{2\perp}} = \pi \, q^- \, \delta(k_1^- + k_2^- - q^-) \, |i\mathcal{M}|^2$$

Longtudinal polarized	Transverse polarized
$\epsilon_-(q)\,\epsilon_-^*(q) = \frac{Q^2}{q^{-2}}$	$\frac{1}{2}\sum_{\lambda=\pm 1} \epsilon_k^{\lambda}(q)\,\epsilon_l^{\lambda *}(q) = -\frac{1}{2}\, g_{kl}$



Leading-twist contribution to the cross section.

Longitudinal photon cross-section

complete phase

$$|i\mathcal{M}|^2_L \approx e^2 \int dx^- d^2x_\perp dy^- d^2y_\perp \, e^{ik_g^+(x^- - y^-) - i\Delta_\perp(x_\perp - y_\perp)} \frac{16\, z^2 \bar{z}^2 \epsilon_f^2}{(P_\perp^2 + \epsilon_f^2)^4} \mathrm{Tr}\left[O_1 + O_2 + O_3 + O_4\right] \qquad \epsilon_f^2 = z\bar{z}Q^2$$

Kinematic twist-3

$$O_1 = \left[2P^iP^j - (\bar{z} - z)(P^i\Delta^j + P^j\Delta^i) + 8(\bar{z} - z)\frac{(\Delta\cdot P)P^iP^j}{P_\perp^2 + \epsilon_f^2}\right]\overline{F}_{-i}(x^-, x_\perp)\,\overline{F}_{-j}(y^-, y_\perp),$$ dynamic leading-twist

$$O_2 = \frac{(z - \bar{z})}{z\bar{z}\, q^-} P^iP_\perp^2\left[\overline{F}_{-i}(x^-, x_\perp)\,\overline{F}_{+-}(y^-, y_\perp) + \overline{F}_{+-}(x^-, x_\perp)\,\overline{F}_{-i}(y^-, y_\perp)\right],$$ dynamic twist-3

$$O_3 = P^i\left[\mathcal{H}_{kl}(z)\,\overline{F}_{-i}(x^-, x_\perp)\,\mathbb{D}_{kl}{}^\dagger(l^+, y^-, y_\perp) + \mathcal{H}_{kl}(z)\,\mathbb{D}_{kl}(l^+, x^-, x_\perp)\,\overline{F}_{-i}(y^-, y_\perp)\right]_{l^+=0},$$

dynamic twist-3

$$O_4 = P^i\left[\mathcal{H}_{kl}(\bar{z})\,\overline{F}_{-i}(x^-, x_\perp)\,\mathbb{D}_{kl}(-l^+, y^-, y_\perp) + \mathcal{H}_{kl}(\bar{z})\,\mathbb{D}_{kl}^\dagger(-l^+, x^-, x_\perp)\,\overline{F}_{-i}(y^-, y_\perp)\right]_{l^+=0},$$

where we defined

$$\mathcal{H}_{kl}(z) \equiv \delta_{kl} - \frac{4P^kP^l}{P_\perp^2 + \epsilon_f^2} + \frac{2P^kP^l}{zq^-}\frac{\partial}{\partial l^+}.$$

Similarly for transverse photon polarization

TMD factorization at small-x

- Full longitudinal phase (**general-x** TMD framework):

$$e^{\,i\,x\,k_g^+\,(x^- - y^-)}$$

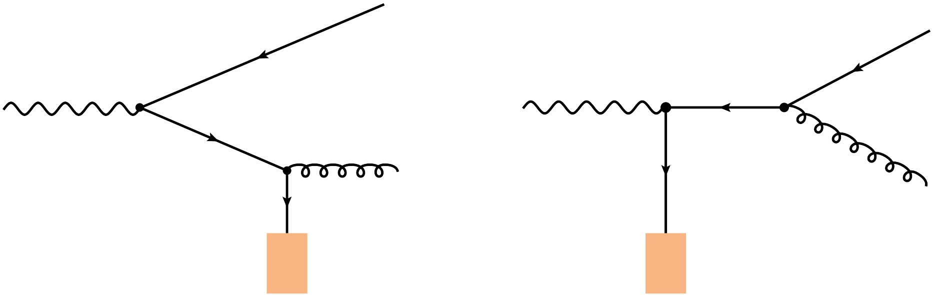
- Small-x limit ($x \to 0$), taking phase expansion:

$$e^{ixP^+(x^- - y^-)} \approx 1 + i\,x\,k_g^+\,(x^- - y^-)$$

- Operator reduction:
  - — $F_{-i}F_{-j} \to$ eikonal, subeikonal
  - — $F_{-i}F_{-j}F_{-k} \to$ eikonal, subeikonal
  - — $F_{+-}F_{-j}, F_{ij}\,F_{-j} \to$ subeikonal
- Reproduces eikonal and sub-eikonal CGC expressions in Ref. [Altinoluk, Beuf, Czajka & Marquet, PRD 111 (2025)]

What about quarks?

At large-x we also need to consider dijet in quark background



Amplitude

$$i\mathcal{M}^1 = -e \lim_{k^2,\, k_1^2\to 0} \int d^4z\, d^4y\, e^{-iq.y}\, \epsilon_i^\lambda(k)\, \epsilon_\mu(q)\, \underbrace{\boxed{k^2\, (k|\frac{1}{P^2 g_{\lambda\nu} + iF_{\lambda\nu} + i\epsilon}|z)^{ij}}}_{\text{gluon}}\, \bar{\psi}^c(z)\, t^j_{cb}\, \gamma^\nu$$

$$\underbrace{\boxed{(z|\frac{i}{P^2 + \frac{1}{2}\sigma F + i\epsilon}\not{P}|y)^{ba}}}_{\text{quark}}\, \gamma^\mu\, \underbrace{\boxed{(y|\frac{i}{P^2 + \frac{1}{2}\sigma F + i\epsilon}|k_1)^{ad}\, k_1^2}}_{\text{LSZ'd quark}}\, v^d(k_1)$$

$$i\mathcal{M}^2 = -ie \lim_{k^2,\, k_1^2\to 0} \int d^4y\, d^4z\, e^{-iq.y}\, \bar{u}^d(k_1)\, k_1^2 (k_1|\frac{i}{P^2 + \frac{\sigma F}{2} + i\epsilon}|z)^{de} \gamma^\nu t^i_{ef}\, (z|\frac{i}{P^2 + \frac{\sigma F}{2} + i\epsilon}\not{P}|y)^{fg}$$

$$\gamma^\mu\, \psi^g(y)\, \epsilon_j^\lambda(k)\, \epsilon_\mu(q)\, (z|\frac{-i}{P^2 g_{\nu\lambda} + iF_{\nu\lambda} + i\epsilon}|k)^{ij}$$

Longitudinal cross-section at leading twist

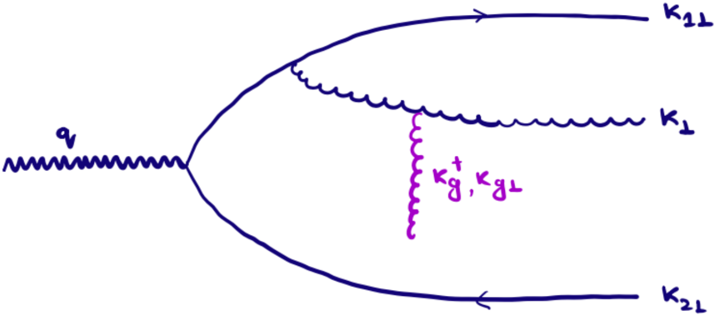
$$d\sigma^L = -8e^2 \left(\frac{\bar{z}z^2 q^- \epsilon_f^2}{(P_\perp^2 + \epsilon_f^2)^2} \right) \int dy^- dz^- d^2 y_\perp d^2 z_\perp \, e^{-i(y^- - z^-)k_g^+ + i(y_\perp - z_\perp)k_{g\perp}}$$
$$\bar{\psi}(y) \, [y^-, \infty]_{z_\perp} \, \gamma^+ \, [\infty, z^-]_{z_\perp} \psi^c(z)$$

- Matches with sub-eikonal results at small-x
- Can be used to systematically go beyond leading twist ($\bar{\psi}\gamma^i\psi$, $\bar{\psi}\gamma^+ F\psi$)
- Similar procedure for transverse cross-section

Altinoluk, Armesto, Beuf (2023)
Mukherjee, Skokov, Tarasov, Yao, SST(2026)

Dijet NLO

What about dijet NLO?

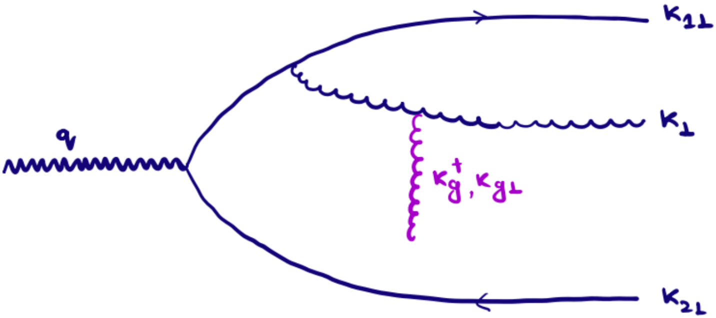


$$\lim_{k^2\to 0} k^2 (k| \frac{1}{P^2 g_{\nu\lambda} + iF_{\nu\lambda} + i\epsilon} |z)^{ij}$$

$$(z| \frac{i}{P^2 + \frac{1}{2}\sigma F + i\epsilon} \not{P} |y)^{ba}$$

$$(k_1| \frac{1}{\mathrm{P}^2 + \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu}} |y)$$

However the gradient expansion is not valid anymore!



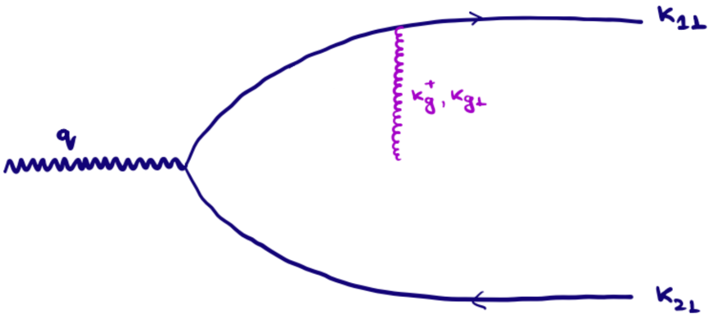
$$k_{g\perp} \propto k_\perp$$

$$DF \propto k_{g\perp} F$$

Need to keep all $D^n F$ contributions

Q: Is it possible to sum these contributions?

Framing it at leading order



$$k_{g\perp} \propto \Delta_\perp$$

$$DF \propto \Delta_\perp F$$

Need to keep all $D^n F$ contributions

$\boxed{\text{Q: Is it possible to sum these kinematic twist contributions?}}$

$\boxed{\text{The answer is } \textbf{yes} \text{ at small-x: } \textbf{ITMD formalism}}$

Improved TMD factorization for forward dijet production in dilute-dense hadronic collisions

P. Kotko,[a] K. Kutak,[b] C. Marquet,[c] E. Petreska,[c,d] S. Sapeta[e] and A. van Hameren[b]

$$d\sigma_{\rm CGC} = \underbrace{d\sigma_{\rm TMD} + \overbrace{\mathcal{O}\left(\frac{\Delta_\perp}{Q_\perp}\right) + \mathcal{O}\left(\frac{Q_s}{\Delta_\perp}\right)}^{\text{kinematic}}}_{d\sigma_{\rm ITMD}} + \overbrace{\mathcal{O}\left(\frac{Q_s}{Q_\perp}\right)}^{\text{genuine}}$$

Altinoluk, Boussarie(2019)
Boussarie, Mehtar-Tani(2020)
Boussarie,Mantysaari,Salazar,Schenke(2021)

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$$\boxed{\text{Can we develop a ITMD formalism beyond small-x?}}$$

An improved expansion

$$\boxed{\text{Let us exponentiate}}$$

$$(x|\frac{1}{\mathrm{P}^{+}-\frac{\mathrm{P}_{\perp}^{2}}{2k^{-}}+i\epsilon}|y) = \delta(x^{+}-y^{+})\,\theta(x^{-}-y^{-})\,(x_{\perp}|\Big[-i\,[x^{-},y^{-}]$$

$$-\int_{y^{-}}^{x^{-}} dz^{-}\,[x^{-},z^{-}]\,\frac{\mathrm{P}_{\perp}^{2}(z^{-})}{2k^{-}}\,[z^{-},y^{-}]+\cdots\Big]|\,y_{\perp})\,,$$

An improved expansion

$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y)$$

$$= -i\delta(x^+ - y^+)\,\theta(x^- - y^-)\,(x_\perp|\mathcal{P}\exp\Big(-i\int_{y^-}^{x^-} dz^-\,H(z^-;x^-)\Big)[x^-,y^-]|\,y_\perp)$$

$$H(z^-;x^-) = [x^-,z^-]\frac{P_\perp^2(z^-)}{2k^-}[z^-,x^-]$$

$$\boxed{\text{Let us simplify the hamiltonian}}$$

$\boxed{\text{Commuting the covariant derivative}}$

$$H(z^-;x^-) = H_0(x^-) + V(z^-;x^-)$$

with

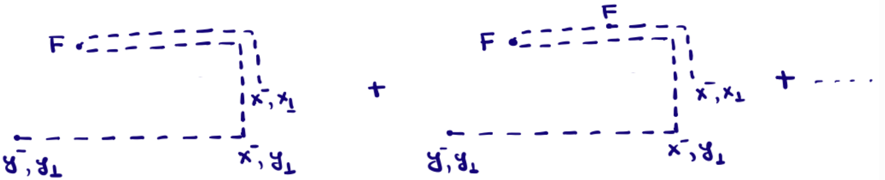
$$\begin{aligned} H_0(x^-) &= \frac{P_\perp^2(x^-)}{2k^-}, \\ V(z^-;x^-) &= -\frac{1}{2k^-}\left\{P_i(x^-), \int_{z^-}^{x^-} dz'^-\,[x^-,z'^-]F_{-i}(z'^-)[z'^-,x^-]\right\} \\ &\quad + \frac{1}{2k^-}\int_{z^-}^{x^-} dz'^-\,dz''^-\,[x^-,z'^-]F_{-i}(z'^-)[z'^-,z''^-]F_{-i}(z''^-)[z''^-,x^-]\,. \end{aligned}$$

An improved expansion

Carry out a duhamel expansion of the propagator

$$(x|\frac{1}{\mathrm{P}^{+}-\frac{\mathrm{P}_{\perp}^{2}}{2k^{-}}+i\epsilon}|y) = -i\delta(x^{+}-y^{+})\,\theta(x^{-}-y^{-})$$

$$\times\,(x_{\perp}\,|e^{-i(x^{-}-y^{-})H_0}\left(1-i\int_{y^{-}}^{x^{-}}dz^{-}\,\underbrace{e^{-i(y^{-}-z^{-})H_0}}_{\textbf{resummed}}\,V(z^{-};x^{-})\,\underbrace{e^{-i(z^{-}-y^{-})H_0}}_{\textbf{resummed}}+\mathcal{O}(V^{2})\right)[x^{-},y^{-}]|\,y_{\perp})$$

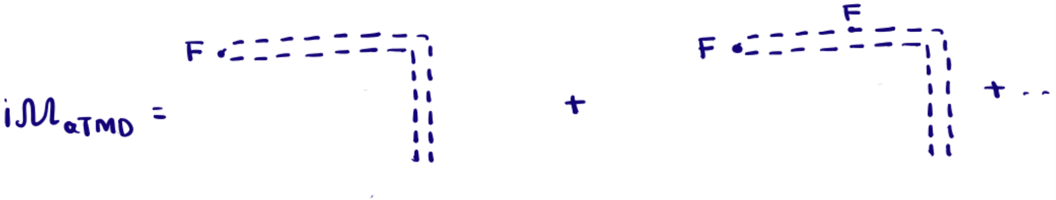


The amplitude

$$
\begin{aligned}
i\mathcal{M}_{\mathrm{aTMD}} = e\,\epsilon_\rho(q)\bar{u}(k_1)\int d^4y\, e^{ik_g y}\Bigg(&\mathbb{C}_i^\rho\, \bar{F}_{-i}(y^-, y_\perp) \\
&+ \mathbb{D}^\rho \int_{y^-}^{\infty} dz^- \left\{\bar{F}_{-i}(y^-, y_\perp), \bar{F}_{-i}(z^-, y_\perp)\right\} + i\mathbb{C}^\rho_{\mu\nu}\frac{\bar{F}^{\mu\nu}(y^-, y_\perp)}{2}\Bigg) v(k_2)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{C}_i^\rho &= \frac{2q^- z\bar{z}(k_1 - k_2)_i}{(k_{1\perp}^2 + \epsilon_f^2)(k_{2\perp}^2 + \epsilon_f^2)}\gamma^\rho, \\
\mathbb{D}^\rho &= -\frac{2q^- z\bar{z}}{(k_{1\perp}^2 + \epsilon_f^2)(k_{2\perp}^2 + \epsilon_f^2)}\gamma^\rho, \\
\mathbb{C}^\rho_{\mu\nu} &= \frac{\bar{z}}{k_2^2 + \epsilon_f^2}\sigma_{\mu\nu}\gamma^\rho + \frac{z}{k_1^2 + \epsilon_f^2}\gamma^\rho\sigma_{\mu\nu}
\end{aligned}
$$

$\boxed{\text{Does it give ITMD at small x? The answer is \textbf{no}}}$



The operators have **small dipole size**

At small-x

$\boxed{\text{Does not resum the dense corrections at small-x}}$

$$\mathrm{d}\sigma_{\mathrm{CGC}} = \underbrace{\mathrm{d}\sigma_{\mathrm{TMD}} + \overbrace{\mathcal{O}\left(\frac{\Delta_\perp}{Q_\perp}\right)}^{\text{kinematic}}}_{\mathrm{d}\sigma_{\mathrm{sTMD}}} + \overbrace{\mathcal{O}\left(\frac{Q_s}{\Delta_\perp}\right)}^{\text{dilute}} + \overbrace{\mathcal{O}\left(\frac{Q_s}{Q_\perp}\right)}^{\text{genuine}}$$

$\boxed{\text{Can we construct a better operator basis?}}$

A different expansion

$\boxed{\text{Let us reintroduce the dipole size into the game}}$

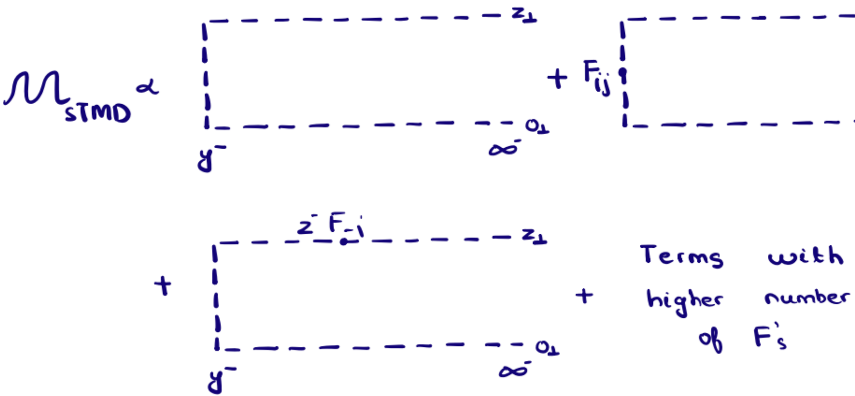
$$
\begin{aligned}
&(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y) \\
&= -i\delta(x^+ - y^+)\,\theta(x^- - y^-)\,(x_\perp|\mathcal{P}\exp\Big(-i\int_{y^-}^{x^-} dz^-\,H(z^-;x^-)\Big)[x^-,y^-]|\,y_\perp) \\
&= -i\delta(x^+ - y^+)\,\theta(x^- - y^-)\,(x_\perp|R(x^-)\Bigg[\mathcal{P}\exp\Big(-i\int_{y^-}^{x^-} dz^-\,H_s(z^-;x^-)\Big)[x^-,y^-]\Bigg]R^{-1}(y^-)|\,y_\perp)
\end{aligned}
$$

$$
R(x^-) = e^{-i\frac{P_\perp^2(x^-)}{2k^-}x^-}
$$

The factor $R^{-1}(y^-)$ generates transverse link at photon splitting $\Longrightarrow$ finite dipole size

The amplitude

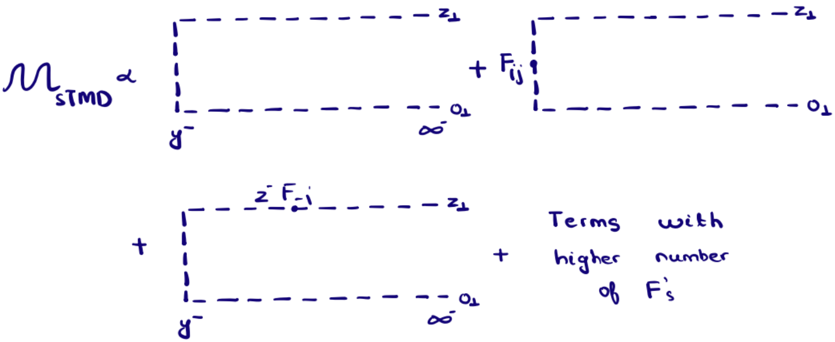
$$H_s(z^-;x^-) = -\frac{1}{2k^-}\int_0^1 d\alpha\,[x^-,z^-]R^{-1}(\alpha\,z^-)\,\{P_i(z^-),z^-F_{-i}(z^-)\}\,R(\alpha\,z^-)[z^-,x^-]$$



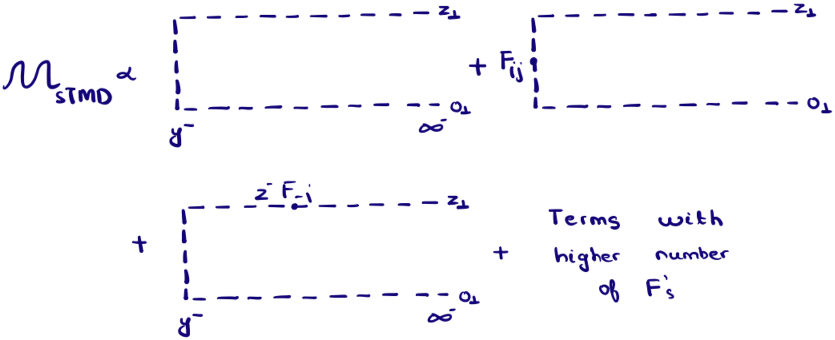
Organizing the expansion

$$\boxed{\text{Terms should be retained based on eikonality and twist}}$$

$$i\mathcal{M}_s = i\mathcal{M}_0 + i\mathcal{M}_1 + i\mathcal{M}_2 \ldots.$$



Organizing the expansion



We shall now compute the ITMD expansion starting with this amplitude

Going to ITMD

$$\boxed{\text{We can use the small-x method to generate F while keeping saturation corrections}}$$

- Construct F from wilson lines without gradient expansion

$$f(x) = f(y) + (f(x) - f(y)) = f(y) + \int d^2b \left[\delta^{(2)}(x-b) - \delta^{(2)}(y-b)\right] f(b)$$

- We use the following identity

$$f(x) = f(y) - i \int d^2b \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik\cdot(x-b)} - e^{ik\cdot(y-b)}}{r\cdot k} r^\alpha \partial_\alpha f(b)$$

- $f(x) = [y^-, \infty]$ gives us F

ITMD at all bjorken x

$\boxed{\text{Amplitude for single field strength insertion}}$

$$i\mathcal{M}_{\text{sTMD}} = i\mathcal{M}_0^{F_{-i}} + i\mathcal{M}_0^{F_{ij}} + i\mathcal{M}_1^{F_{-i}+F_{ij}}$$

$$i\mathcal{M}_0^{F_{-i}} = e\,\epsilon_\rho(q)\bar{\delta}(k_g^-)\;\bar{u}(k_1)\,\gamma^\rho\,v(k_2)\int_0^1 dt\,\frac{q^-}{k_1^-k_2^-(k_{p_\perp}^+)^2}\int d^2b_\perp$$
$$\int dz^-\,e^{ik_{p_\perp}^+ z^-}e^{-ik_g\cdot b_\perp}\,p_\alpha\bar{F}_{-\alpha}(z^-,b_\perp)\left(ik_{p_\perp}^+ z^- - 1\right).$$

$$k_{p_\perp}^+ = \frac{p^2+\epsilon_f^2}{2q^-z\bar{z}},$$
$$p = t\,k_{1\perp} - \bar{t}\,k_{2\perp}\,.$$

ITMD at all bjorken x

$$\boxed{\text{In general we have all eikonal orders}}$$

$$i\mathcal{M}_1^{F_{-i}+F_{ij}} = -\frac{ie\,\epsilon_\rho(q)}{4k_1^-\left(q^+ - k_{2\perp}^2\left(\frac{1}{2k_1^-}+\frac{1}{2k_2^-}\right)\right)}\int dy^+\,dz^-\,d^2z_{4\perp}\,e^{i(k_1^-+k_2^--q^-)y^+}\,e^{-iz_{4\perp}(k_{1\perp}+k_{2\perp})}\bar{u}(k_1)$$

$$\left(2k_1^-(k_{1i}-k_{2i})\frac{\left(e^{ik_g^+z^-}-e^{-i\left(q^+-k_{2\perp}^2\left(\frac{1}{2k_1^-}+\frac{1}{2k_2^-}\right)\right)z^-}\right)}{i(k_{1\perp}^2-k_{2\perp}^2)}\bar{F}_{-i}(z^-,z_{4\perp})+e^{ik_g^+z^-}\sigma\bar{F}(z^-,z_{4\perp})\right)\gamma^\rho v(k_2)$$

$$i\mathcal{M}^{F_{ij}} = -2ie\,\epsilon_\rho(q)\int dy^+dy^-\,e^{-iq\cdot y}\,e^{i(k_1^-+k_2^-)y^+}\,\bar{u}(k_1)\int d^2\zeta_\perp\int_0^s d\sigma\int_0^1 d\alpha\,\alpha^2\,e^{-i(k_{1\perp}+k_{2\perp})\zeta_\perp}$$
$$e^{i(s-\sigma)k_{1\perp}^2}\,\sigma\left(k_{1\perp}-\alpha(k_{1\perp}+k_{2\perp})\right)e^{i\sigma(k_{1\perp}-\alpha(k_{1\perp}+k_{2\perp}))^2}[\infty,y^-]_{\zeta_\perp}D_iF_{ij}(y^-,\zeta_\perp)$$
$$[y^-,\infty]_{\zeta_\perp}\gamma^\rho\,v(k_2)$$

ITMD at all bjorken x

$\boxed{\text{Reduces to ITMD at small-x}}$

$$i\mathcal{M}_0^{F_{-i}} = -e\,\epsilon_\rho(q)\bar{\delta}(k_g^-)\,\bar{u}(k_1)\,\gamma^\rho\,v(k_2)\int_0^1 dt\,\frac{q^-}{k_1^-k_2^-(k_{p_\perp}^+)^2}\int d^2b_\perp \int dz^-\,e^{-ik_g\cdot b_\perp}\,p_\alpha\bar{F}_{-\alpha}(z^-,b_\perp)$$

$$\mathrm{d}\sigma_{\mathrm{CGC}} = \underbrace{\mathrm{d}\sigma_{\mathrm{TMD}} + \overbrace{\mathcal{O}\left(\frac{\Delta_\perp}{Q_\perp}\right) + \mathcal{O}\left(\frac{Q_s}{\Delta_\perp}\right)}^{\text{kinematic}}}_{\mathrm{d}\sigma_{\mathrm{sITMD}}} + \overbrace{\mathcal{O}\left(\frac{Q_s}{Q_\perp}\right)}^{\text{genuine}}$$

Conclusions

- TMD factorization of dijet cross-section at LO upto twist 3 accuracy
- Propagators not enough for NLO processes
- Develop propagators which resum kinematic twists at LO
- Construct an extension of ITMD beyond eikonal accuracy
- Recover the standard ITMD result at small-x

Mukherjee, Skokov, Tarasov, Yao, SST (2026, 2026)

Outlook

- Compute NLO dijet with arbitrary background
- Understanding the sTMD expansion: Bridging small-x and large-x?
- Phenomenological applications?

Mukherjee, Skokov, Tarasov, Yao, SST (2026, 2026)
Boussarie, Mehtar-Tani (2021, 2023)

Thank you for your attention!

Propagators

$$\lim_{k^2\to 0} k^2 (k|\frac{1}{P^2 g_{\nu\lambda} + iF_{\nu\lambda} + i\epsilon}|z)^{ij} = ie^{ik^+z^- + ik^-z^+}(k_\perp| -i\, g_{\nu\lambda}[\infty, z^-]_A + \frac{1}{2k^-}\int_{z^-}^{\infty} dz'^- \\ (z'^- - z^-)\,[\infty, z'^-]_A\Big[g_{\nu\lambda}\, 2p_k F_{-k}(z'^-) + \frac{F_{\nu\lambda}(z'^-)}{(z'^- - z^-)}\Big][z'^-, z^-]_A|z_\perp)^{ij}$$

$$(z|\frac{i}{\not{P}}|y)^{ba}_{\beta\gamma} = \int_0^\infty đp^-\, e^{-ip^-(z^+ - y^+)}\,(z_\perp|\gamma^\mu \frac{P_\mu(z^-)}{2p^-} e^{-i(z^- - y^-)\frac{\mathrm{P}_\perp^2(z^-)}{2p^-}}\Big[-i[z^-, y^-] + \frac{1}{2p^-}\int_{y^-}^{z^-} dz'^- \\ (z'^- - y^-)[z^-, z'^-]\Big(\{\mathrm{P}_k, F_{-k}\}(z'^-) + \frac{\sigma F(z'^-)}{2(z' - y^-)} - i\frac{\delta^\mu_-\, p^-\, \sigma F(z^-)}{(z'^- - y^-)(z^- - y^-)}\Big)[z'^-, y^-]\Big]|y_\perp)^{ba}_{\beta\gamma}$$

Transverse photon cross-section

complete phase

$$|i\mathcal{M}|^2_T = 4\,e^2 \int \frac{dx^- d^2x_\perp dy^- d^2y_\perp}{(2q^-)^2}\, e^{ik_g^+(x^- - y^-) - i\Delta_\perp(x_\perp - y_\perp)}\, \mathrm{Tr}\Big[A^{ij}\,\overline{F}_{-i}(x^-,x_\perp)\,\overline{F}_{-j}(y^-,y_\perp)$$
$$+ B^i\left(\overline{F}_{-i}(x^-,x_\perp)\,\overline{F}_{+-}(y^-,y_\perp) + \overline{F}_{+-}(x^-,x_\perp)\,\overline{F}_{-i}(y^-,y_\perp)\right)$$
$$+ C^i\left(\overline{F}_{ij}(x^-,x_\perp)\overline{F}_{-j}(y^-,y_\perp) + \overline{F}_{-j}(x^-,x_\perp)\overline{F}_{ij}(y^-,y_\perp)\right)$$

dynamic twist-3

$$+ \left(E^{ikl}\,\overline{F}_{-i}(x^-,x_\perp)\,\mathbb{D}^\dagger_{kl}(l^+,y^-,y_\perp) + E^{ikl}\,\mathbb{D}_{kl}(l^+,y^-,y_\perp)\,\overline{F}_{-i}(y^-,y_\perp)\right)_{l^+=0}$$
$$+ \left(G^{ikl}\,\overline{F}_{-i}(x^-,x_\perp)\,\mathbb{D}_{kl}(-l^+,y^-,y_\perp) + G^{ikl}\,\mathbb{D}^\dagger_{kl}(-l^+,x^-,x_\perp)\,\overline{F}_{-i}(y^-,y_\perp)\right)_{l^+=0}\Big]$$

Bianchi identity $\rightarrow$ operator $F_{ij}F_{-j}$ reduction: kinematic twist-3 + three-body operator