High-energy evolution to three loops in (planar) QCD

Mathieu Giroux

Columbia University & IAS Princeton

Based on [2508.03794] + work in progress

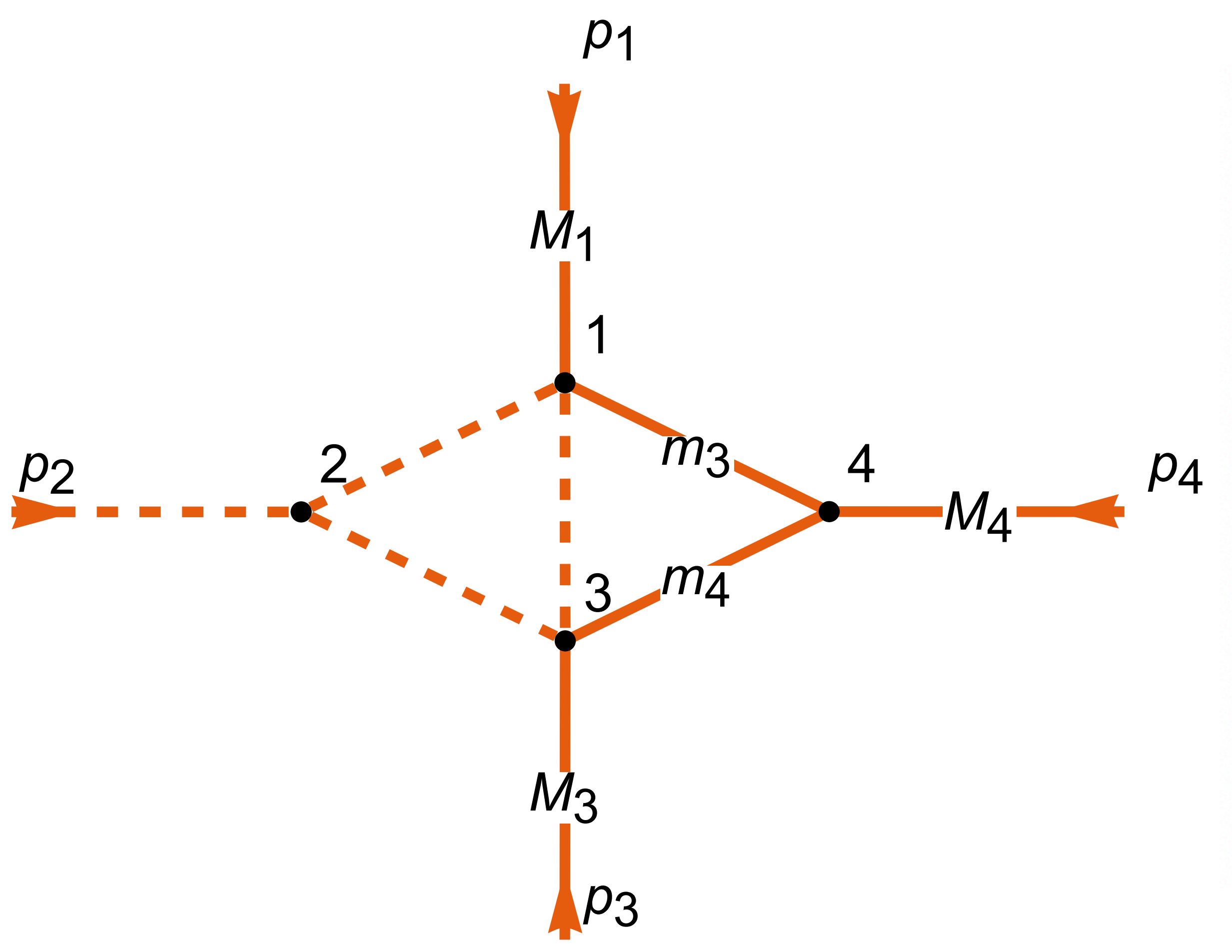
with G. Brunello, S. Caron-Huot, G. Crisanti, S. Smith

My background is in *formal* QFT with a focus on developing new *tools* to compute scattering observables

(observables you can build from S-matrices)

An example of *tool* is `SubTropica`, e.g., draw your Feynman diagram, then `Shift + Enter`

w/ Mizera (Columbia) & Salvatori (IAS) [2604.20954] — subtropi.ca

`STIntegrate[`  `]`

Try it yourself!

"A simple 2 loops 7 scales example"

» Overall timing: 30.1s

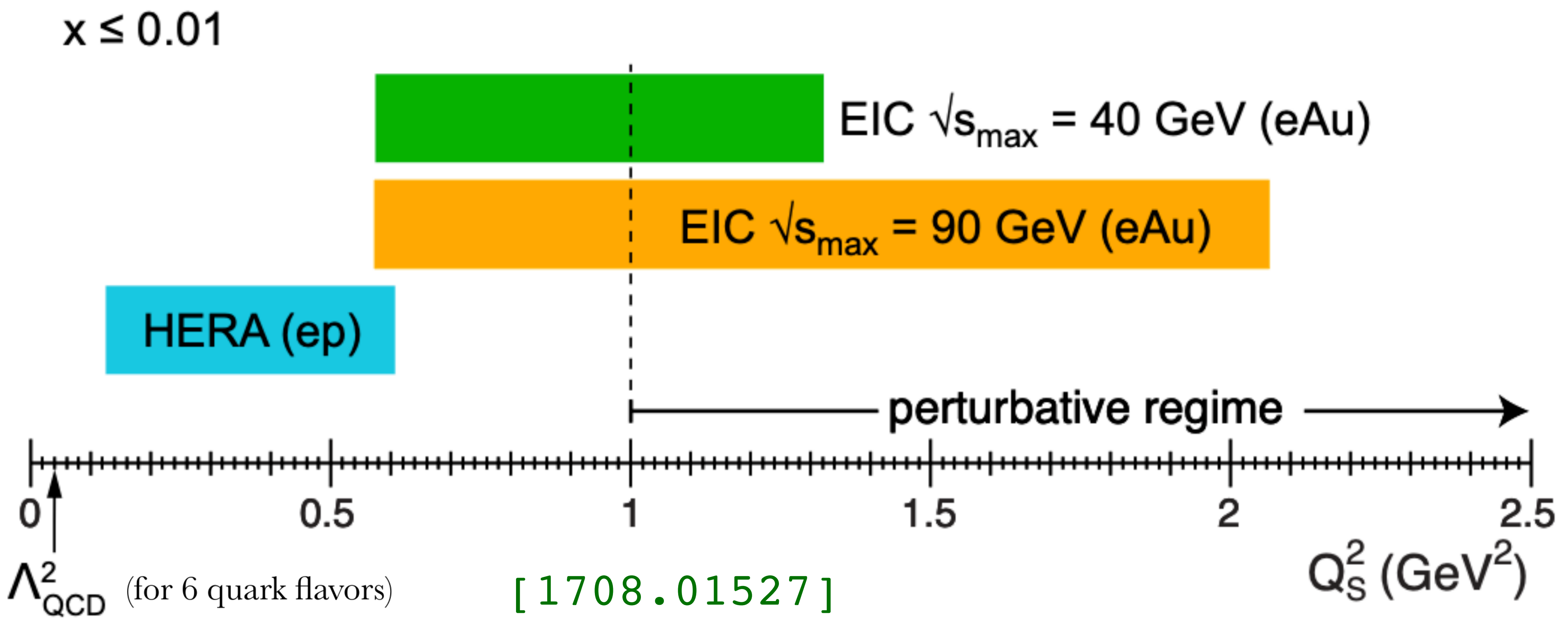
$$
\begin{aligned}
&\frac{1}{12\,(\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23})}\Bigg(12\,\mathrm{Hlog}\Big[1,\Big\{0,1-\frac{\mathrm{mm4}}{\mathrm{MM3}}\Big\}\Big]\times\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wm}[1]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+12\,\mathrm{Hlog}\Big[1,\Big\{1-\frac{\mathrm{mm4}}{\mathrm{MM3}},\frac{\mathrm{MM3}-\mathrm{s23}}{\mathrm{MM3}+\mathrm{mm4}-\mathrm{s23}}\Big\}\Big]\times\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wm}[1]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-\\
&2\pi^2\,\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wm}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{0,1-\frac{\mathrm{mm4}}{\mathrm{MM3}}\Big\}\Big]\times\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wm}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{1-\frac{\mathrm{mm4}}{\mathrm{MM3}},1-\frac{\mathrm{s23}}{\mathrm{MM3}}\Big\}\Big]\times\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wm}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+\\
&2\pi^2\,\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wm}[3]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+12\,\mathrm{Hlog}\Big[1,\Big\{0,1-\frac{\mathrm{mm4}}{\mathrm{MM3}}\Big\}\Big]\times\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wp}[1]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+12\,\mathrm{Hlog}\Big[1,\Big\{1-\frac{\mathrm{mm4}}{\mathrm{MM3}},\frac{\mathrm{MM3}-\mathrm{s23}}{\mathrm{MM3}+\mathrm{mm4}-\mathrm{s23}}\Big\}\Big]\times\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wp}[1]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-\\
&2\pi^2\,\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wp}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{0,1-\frac{\mathrm{mm4}}{\mathrm{MM3}}\Big\}\Big]\times\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wp}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{1-\frac{\mathrm{mm4}}{\mathrm{MM3}},1-\frac{\mathrm{s23}}{\mathrm{MM3}}\Big\}\Big]\times\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wp}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+\\
&2\pi^2\,\mathrm{Hlog}\Big[1,\Big\{\frac{1}{1+\mathrm{Wp}[3]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{\mathrm{mm3}-\mathrm{s12}}{\mathrm{mm3}+\mathrm{MM3}-\mathrm{mm4}-\mathrm{s12}},\frac{1}{1+\mathrm{Wm}[1]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{\mathrm{mm3}-\mathrm{s12}}{\mathrm{mm3}+\mathrm{MM3}-\mathrm{mm4}-\mathrm{s12}},\frac{1}{1+\mathrm{Wm}[3]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+\\
&12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{\mathrm{mm3}-\mathrm{s12}}{\mathrm{mm3}+\mathrm{MM3}-\mathrm{mm4}-\mathrm{s12}},\frac{1}{1+\mathrm{Wp}[1]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{\mathrm{mm3}-\mathrm{s12}}{\mathrm{mm3}+\mathrm{MM3}-\mathrm{mm4}-\mathrm{s12}},\frac{1}{1+\mathrm{Wp}[3]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{\mathrm{MM1}-\mathrm{mm3}}{\mathrm{MM1}-\mathrm{mm3}+\mathrm{mm4}-\mathrm{s23}},\frac{1}{1+\mathrm{Wm}[1]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+\\
&12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{\mathrm{MM1}-\mathrm{mm3}}{\mathrm{MM1}-\mathrm{mm3}+\mathrm{mm4}-\mathrm{s23}},\frac{1}{1+\mathrm{Wm}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{\mathrm{MM1}-\mathrm{mm3}}{\mathrm{MM1}-\mathrm{mm3}+\mathrm{mm4}-\mathrm{s23}},\frac{1}{1+\mathrm{Wp}[1]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{\mathrm{MM1}-\mathrm{mm3}}{\mathrm{MM1}-\mathrm{mm3}+\mathrm{mm4}-\mathrm{s23}},\frac{1}{1+\mathrm{Wp}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-\\
&12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{1}{1+\mathrm{Wm}[1]},\frac{1}{1+\mathrm{Wm}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]+12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{1}{1+\mathrm{Wm}[1]},\frac{1}{1+\mathrm{Wm}[3]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]-12\,\mathrm{Hlog}\Big[1,\Big\{0,\frac{1}{1+\mathrm{Wm}[1]},\frac{1}{1+\mathrm{Wp}[2]},\frac{\mathrm{MM1}-\mathrm{s12}}{\mathrm{MM1}+\mathrm{MM3}-\mathrm{s12}-\mathrm{s23}}\Big\}\Big]
\end{aligned}
$$

$+ 248$ terms $+ \mathcal{O}(\epsilon)$

All the integrals we will face in this talk will be doable systematically in `SubTropica`

As new tools come along, timely experimental developments with the EIC

which promises a new high-energy QCD precision era to study, e.g., saturation



HERA sits only marginally above the non-perturbative scale:
the EIC (eA) reaches decisively beyond

	non-linear $\longrightarrow$	linear
SU(N) $\downarrow$	~~JIMWLK~~ not practical	~~BFKL~~ physically limited cases
SU(big)	BK	~~planar BFKL~~

full color = infinite hierarchy of Wilson-line correlators;
the planar limit closes it on the dipole: BK

So my talk will be organized around the point of view that a realistic precision direction for saturation is BK
My goal will be to use recently developed <u>tools</u> to push it to three loops

In formula:

the non-conformal part of the three-loop (NNLO) BK Hamiltonian in an arbitrary gauge theory

$$H^{(3,0)}_{\rm BK} \simeq \boxed{H^{\overline{\rm MS}\,(3)}_{\rm NGL}} + \int \frac{d^2 z_0}{\pi} \frac{z_{12}^2}{z_{10}^2 z_{02}^2} (\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}) \left[b_0 \tilde{H}^{(2,1)}_{[102]} + b_1 \tilde{H}^{(1,1)}_{[102]} - b_0^2 \tilde{H}^{(1,2)}_{[102]} \right] + b_0 \int \frac{d^2 z_0\, d^2 z_{0'}}{\pi^2} \frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2} \left[(\mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2} - \mathcal{U}_{12}) \tilde{H}^{(2,1)}_{[100'2]} \right] \propto \beta$$

[2508.03794]

all kernel functions $\tilde{H}$ known in closed form functions of transverse distances $z_{ij} = |z_i - z_j|$ and $n^s_{\rm adj}$ and $n^f_{\rm adj}$

Planar QCD	$\mathcal{N} = 4$ SYM	$\mathcal{N} = 1$ SYM
$n^s_{\rm adj} = 0$	$n^s_{\rm adj} = 6\,, n^f_{\rm adj} = 4$	$\Delta n^s_{\rm adj} = 2\Delta n^f_{\rm adj}$

In progress. matter part of the NGL piece $\rightarrow$ NNLO BFKL eigenvalues and Pomeron trajectory

[w/ Caron-Huot]

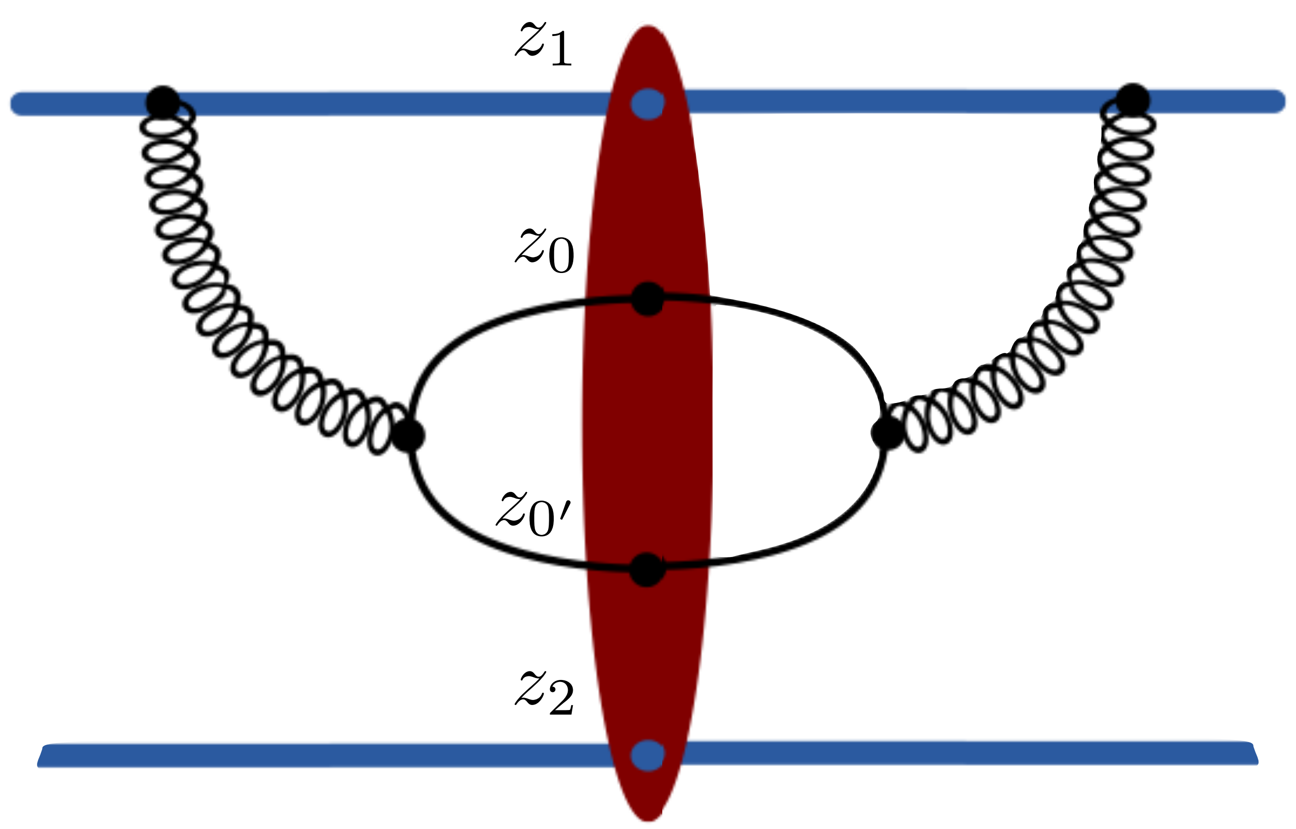
What do I mean by planar QCD ?

't Hooft–Veneziano large-N limit with $n_{adj}^{s,f} = n^{s,f}/N$ and $a = \dfrac{\alpha_s N}{4\pi}$ fixed

such that: $\quad 1/N^2 \ll a \ll 1 \sim n_{\text{adj}}^{s,f}$

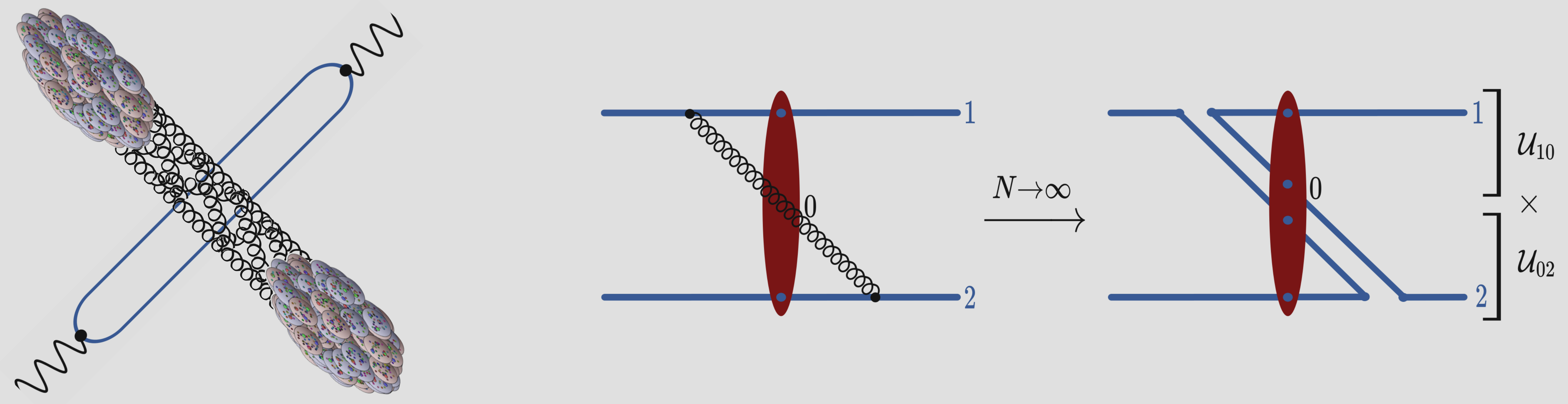
(actual QCD is more like: $\quad 1/N^2 \lesssim a \ll 1 \sim n_{\text{adj}}^{f}$)

In particular, planar QCD will keep quarks! That is, I won't drop those diagrams

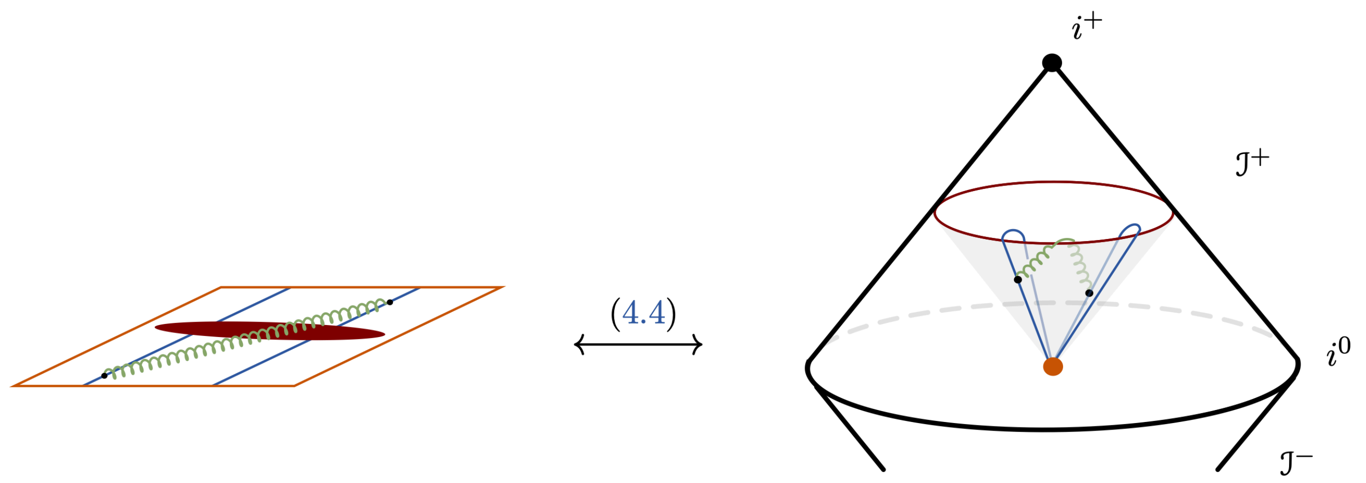


$\sim \mathcal{U}_{10}\mathcal{U}_{0'2} \quad$ (fundamental quarks)

Same as working from the get-go with adjoint matter $\sim \mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2}$ and setting $\mathcal{U}_{00'} \to 1$ in the end



1. Background and why planar?



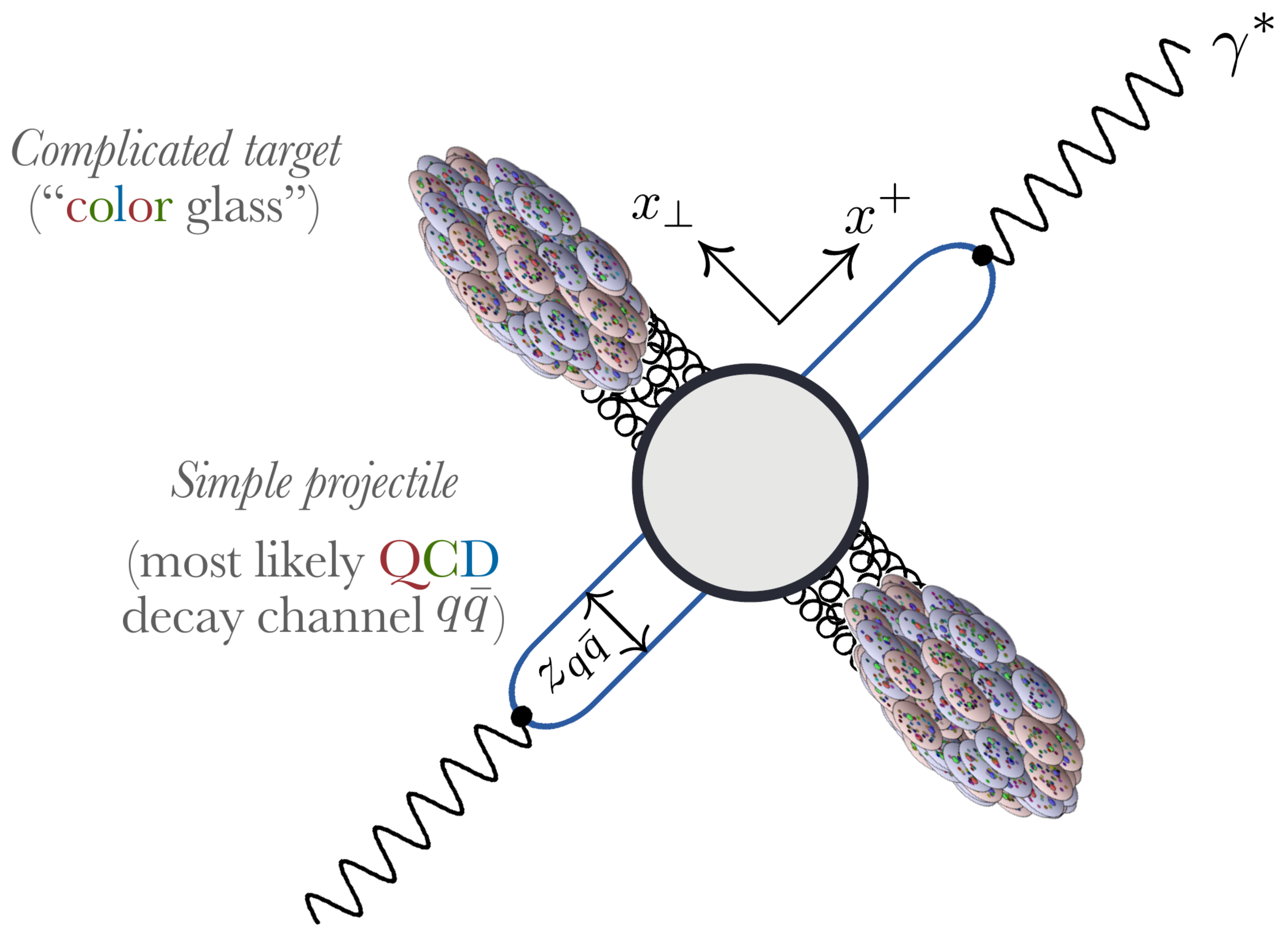
2. Spacelike-timelike correspondence

$$\begin{aligned} H_{\rm BK}^{(3,0)} \simeq H_{\rm NGL}^{\overline{\rm MS}\,(3)} &+ \int \frac{\mathrm{d}^2 z_0}{\pi}\frac{z_{12}^2}{z_{10}^2 z_{02}^2}(\mathcal{U}_{10}\mathcal{U}_{02}-\mathcal{U}_{12})\left[b_0\tilde{H}^{(2,1)}_{[102]} + b_1\tilde{H}^{(1,1)}_{[102]} - b_0^2\tilde{H}^{(1,2)}_{[102]}\right] \\ &+ b_0\int\frac{\mathrm{d}^2 z_0\,\mathrm{d}^2 z_{0'}}{\pi^2}\frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2}\begin{bmatrix} (\mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2}-\mathcal{U}_{12})\tilde{H}^{(2,1)}_{[100'2]} \\ -(\mathcal{U}_{10}\mathcal{U}_{02}-\mathcal{U}_{12})\tilde{H}^{(2,1)\rm coll}_{[100'2]} \end{bmatrix}. \end{aligned}$$

3. Results and what they enable

The setup in one minute

At eikonal accuracy, the large-p^-/static target is so slim that the large-p^+ DIS probe *effectively* fluctuates into dipole long before it interacts with it



shorthand for transverse space distances

$$z_{ij} = |x_i^\perp - x_j^\perp|$$

Think of this non-perturbative object as a high-energy OPE:

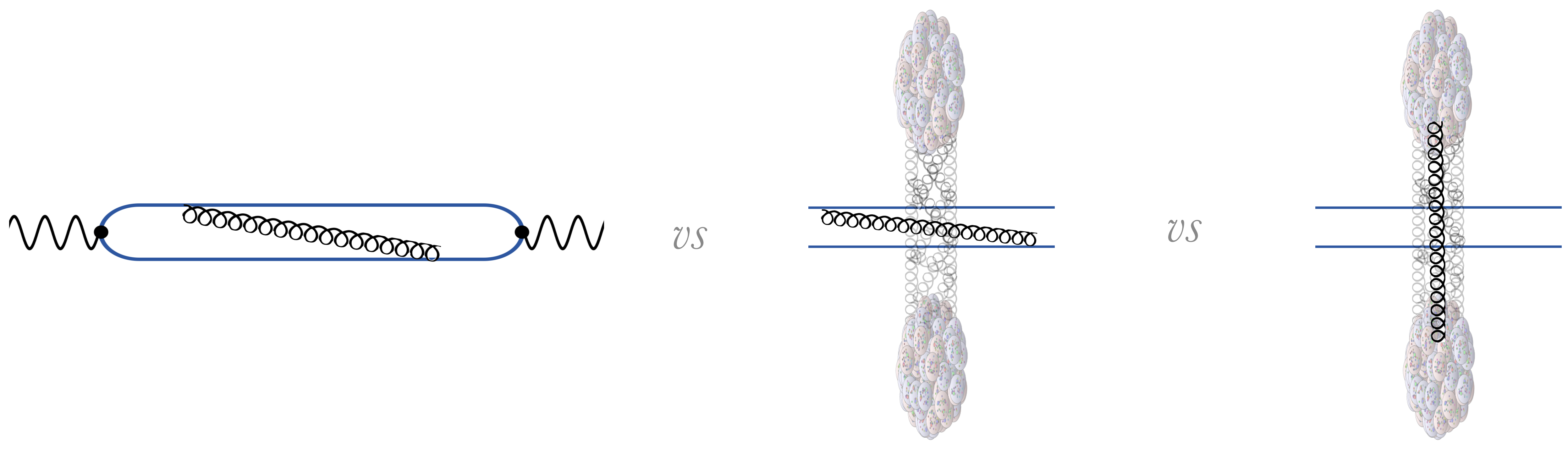
perturbative impact factors $\otimes$ dipole correlators of *infinite* Wilson lines

[9509348]

$$\langle T|\mathcal{T}(J^\dagger_\mu(x)J_\nu(y))\,|T\rangle \approx \int \mathrm{d}z_1\mathrm{d}z_2 I^{\mathrm{LO}}_{\mu\nu}(z_1,z_2|x,y)\,\langle T|\,\underbrace{\mathrm{Tr}[\hat{U}(z_1)U^\dagger(z_2)]}_{=\hat{\mathcal{U}}_{12}}\,|T\rangle$$

$$+\int \mathrm{d}z_1\mathrm{d}z_0\mathrm{d}z_2 I^{\mathrm{NLO}}_{\mu\nu}(z_1,z_0,z_2|x,y)\,\langle T|\,\mathrm{Tr}[\hat{U}(z_1)U^\dagger(z_0)]\mathrm{Tr}[\hat{U}(z_0)U^\dagger(z_2)]\,|T\rangle + \ldots \;+\mathcal{O}(x_B^0)$$

the factorization offers a way to separate radiative emissions in their rapidity:



partons with $\eta' > \eta$
(ψ corrections)

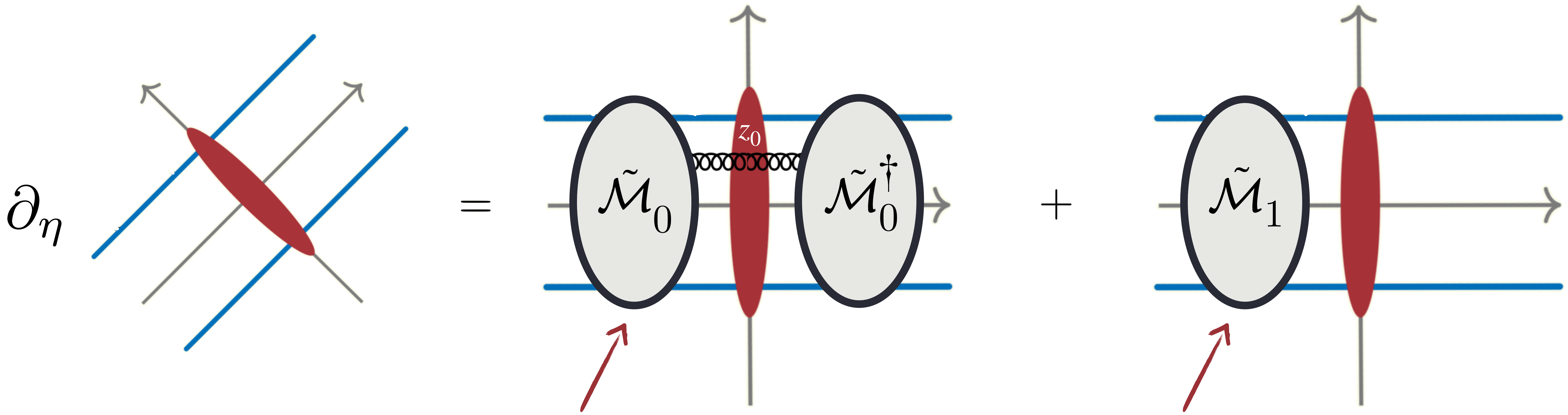
partons with $\eta_T < \eta' < \eta$
(rad. corr. to ∞ WL $\Rightarrow$ div!)

partons with $\eta' < \eta_T$
(non. pert. target corr.)

A fact

An important property of each of these target matrix elements is that, while they cannot be computed pertubatively, their evolution in rapidity *can*

For the dipole $\mathcal{U}_{12} = \langle T|\,\mathrm{Tr}[\hat{U}(z_1)U^\dagger(z_2)]\,|T\rangle$, for example

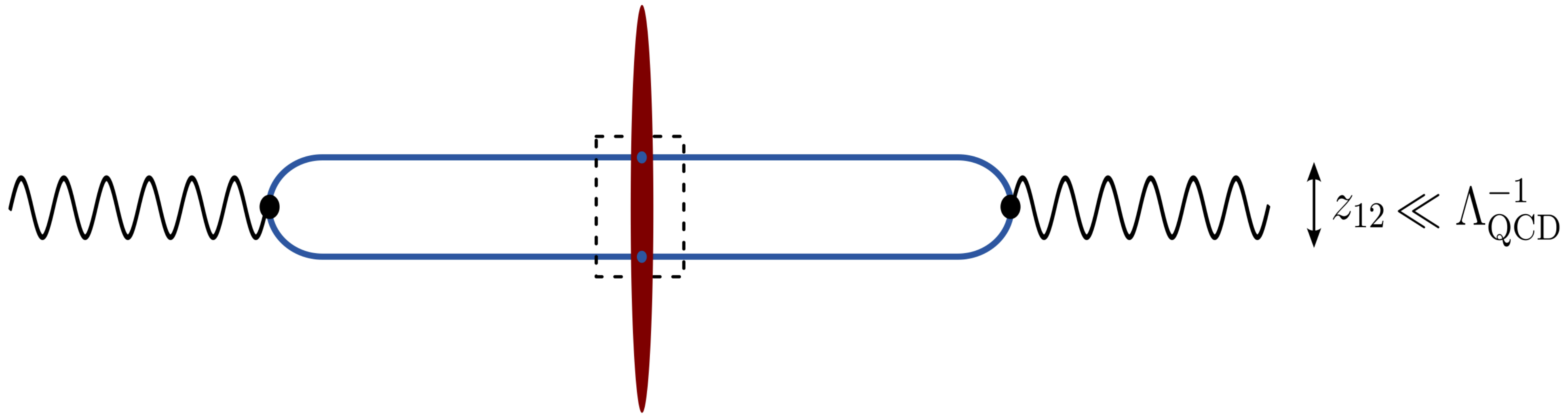


$+$ perm. $\quad + \mathcal{O}(\alpha_s^2)$

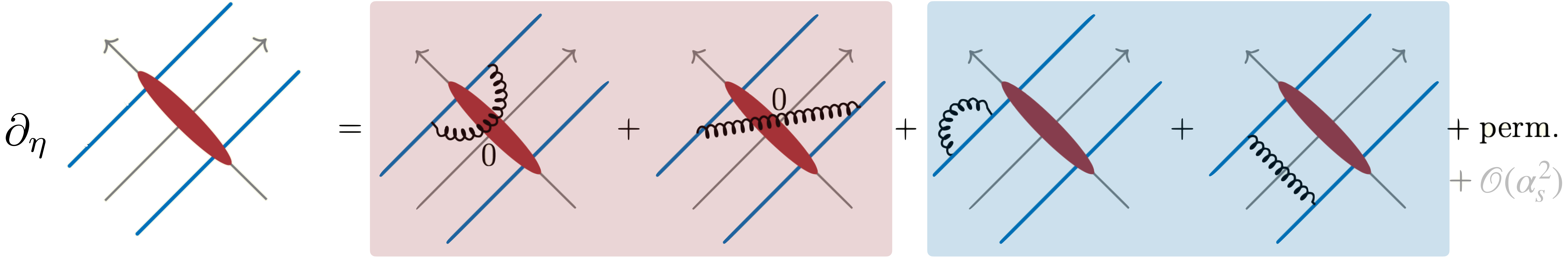
Tree-level impact parameter space amplitude

One loop impact parameter space amplitude

red blob = shorthand for boosted target



Expanding the amplitude blobs in terms of Feynman diagrams gives



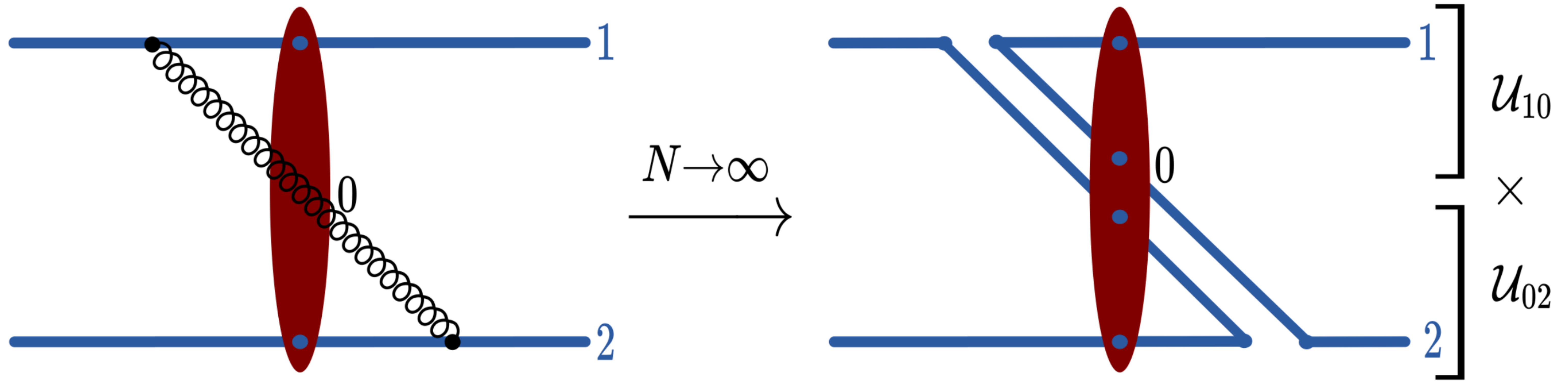
$$= \frac{\alpha_s}{2\pi}\int \frac{\mathrm{d}^2 z_0}{\pi}\,\frac{z_{12}^2}{z_{10}^2 z_{02}^2}\Big[\overbrace{\langle T|\,\mathrm{Tr}[\hat{U}(z_1)U^\dagger(z_0)]\mathrm{Tr}[\hat{U}(z_0)U^\dagger(z_2)]\,|T\rangle}^{\textit{non-linear term}} - \overbrace{\mathcal{U}_{12}}^{\textit{term} \,\propto\, \textit{bare dipole}}\Big] + \mathcal{O}(\alpha_s^2)$$

$$z_{ij} = |x_i^\perp - x_j^\perp| \qquad\qquad \mathrm{d}^2 z = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i}$$

At full color, have an *infinite* (☠️) tower of equations like this: for the 2-trace, 3-trace, … $\Longrightarrow$ Balitsky hierarchy / B-JIMWLK eq.

[9701284, 9709432, 1309.7644, 1310.0378, 1405.0418]

In the planar limit, the only evolution equation we need is that of the dipole (BK)



$\langle T|\,\mathrm{Tr}[\hat{U}(z_1)U^\dagger(z_0)]\mathrm{Tr}[\hat{U}(z_0)U^\dagger(z_2)]\,|T\rangle$ $\quad$ $\langle T|\,\mathrm{Tr}[\hat{U}(z_1)U^\dagger(z_0)]\,|T\rangle \times \langle T|\,\mathrm{Tr}[\hat{U}(z_0)U^\dagger(z_2)]\,|T\rangle$

BK:

$$\partial_\eta \mathcal{U}_{12} = \frac{\alpha_s N}{2\pi}\int \frac{\mathrm{d}^2 z_0}{\pi}\,\frac{z_{12}^2}{z_{10}^2 z_{02}^2}\left[\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}\right]$$

[9509348 (Balitsky), 9901281 (Kovchegov)]

✅ planar BK is simplest saturation sensitive option: infinite hierarchy closes on the dipole — NLO 2007, simulated 2015, used in fits

[0710.4330, 1502.02400, 1601.06598]

⛔ full color (JIMWLK): infinite hierarchy of multi-trace correlator DEs — NLO (2013) never solved (numerically ?)

[1405.0418]

Why three loops (NNLO)?

Non-linear small-x evolution (BK or B-JIMWLK) is a cornerstone of saturation physics: EIC!

The NLO correction was big & numerically unstable: blamed on large collinear logarithms

[1502.02400, 1601.06598]

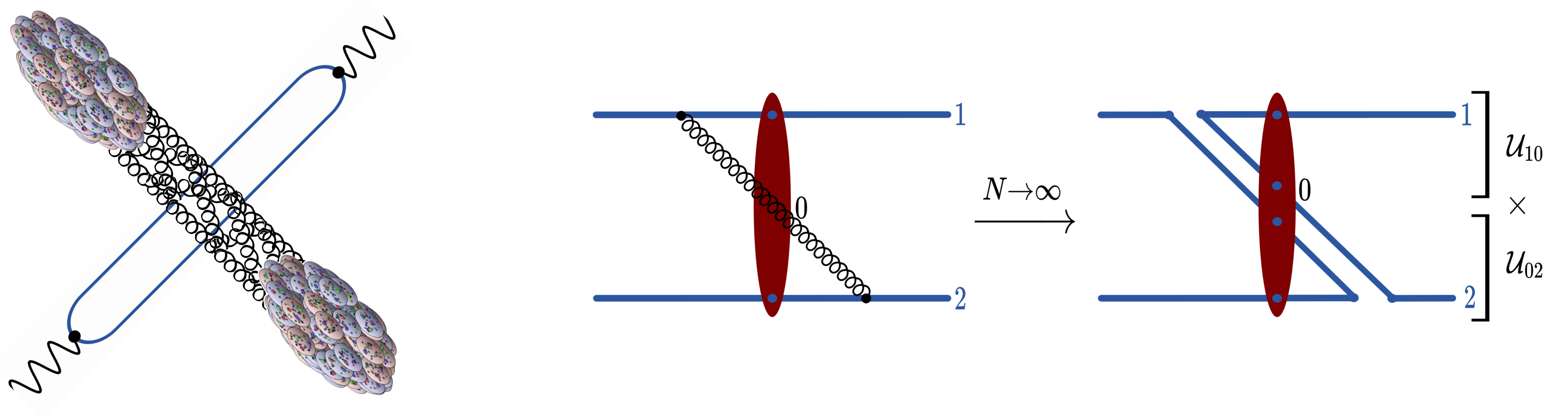
Resumming those logs fixes NLO, but NLO is the only data point it was built to reproduce

NNLO would really be the first chance for the resummation to fail (or convince)

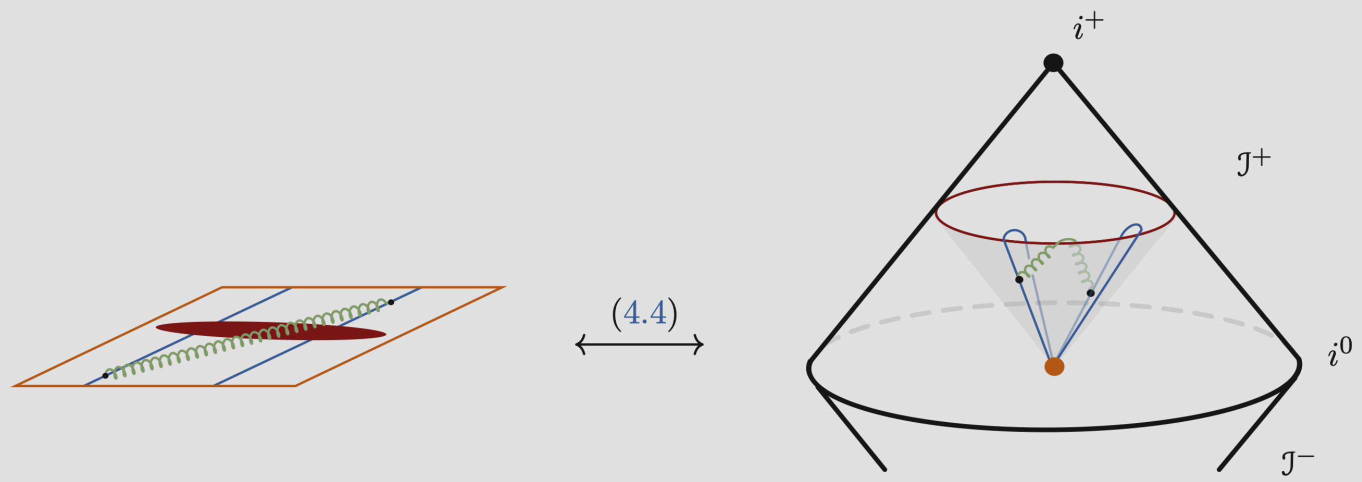
Byproduct: extract unknown (?) NNLO Pomeron trajectory = solutions of linearized BK
(give info on how the DIS cross section "scales" in some energy window of the forward limit)

Said differently:

Confirm overall consistency of the *formalism* — OPE is an *assumption* after all!



1. Background and why planar?



2. Spacelike-timelike correspondence

$$
\begin{aligned}
H_{\rm BK}^{(3,0)} \simeq H_{\rm NGL}^{\overline{\rm MS}\,(3)} &+ \int \frac{\mathrm{d}^2 z_0}{\pi}\,\frac{z_{12}^2}{z_{10}^2 z_{02}^2}(\mathcal{U}_{10}\mathcal{U}_{02}-\mathcal{U}_{12})\Big[b_0\tilde{H}^{(2,1)}_{[102]} + b_1\tilde{H}^{(1,1)}_{[102]} - b_0^2\tilde{H}^{(1,2)}_{[102]}\Big] \\
&+ b_0\int \frac{\mathrm{d}^2 z_0\,\mathrm{d}^2 z_{0'}}{\pi^2}\,\frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2}\left[\begin{array}{l}(\mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2}-\mathcal{U}_{12})\tilde{H}^{(2,1)}_{[100'2]} \\ -(\mathcal{U}_{10}\mathcal{U}_{02}-\mathcal{U}_{12})\tilde{H}^{(2,1)\rm coll}_{[100'2]}\end{array}\right].
\end{aligned}
$$

3. Results and what they enable

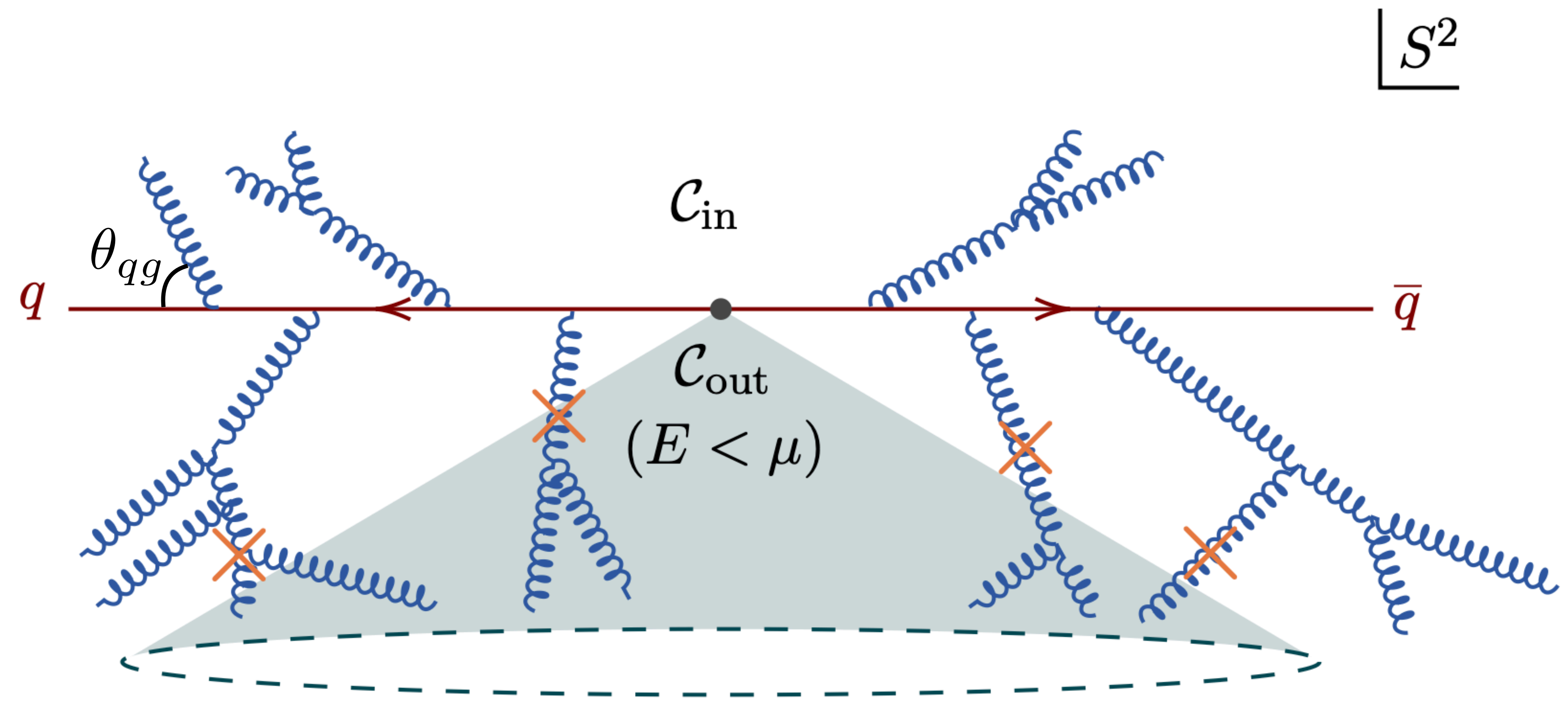
To compute three-loop BK, we borrow from an unexpected place: **jet physics**

there, one resums **non-global logarithms** (NGLs) — soft radiation vetoed in a region of the detector

Why? NGL resummation and BK rapidity evolution are the same problem*

**almost, as we will see*

For example, in back-to-back dijet production (e.g., $e^+e^- \to \gamma^* \to q\bar{q}$)



NGLs arise when we consider, e.g., the probability $\mathcal{V}_{12}$ that the parent $q\bar{q}$ dipole does *not* radiate quanta of energy greater than a veto energy scale μ in an exclusion region $\mathcal{C}_{\text{out}}$ of the detector sphere

At leading order in the strong coupling, large $\log(\mu \to 0)$ of the veto scale are resummed by the **BMS equation**

$$-\mu\partial_\mu \mathcal{V}_{12} = \frac{\alpha_s N}{2\pi} \int \frac{\mathrm{d}^2\Omega_0}{4\pi} \frac{\alpha_{12}}{\alpha_{10}\alpha_{02}} \Big[\mathcal{V}_{10}\mathcal{V}_{02} - \mathcal{V}_{12}\Big]$$

[0206076]
(in planar limit)

with $\alpha_{ij} = \dfrac{1-\cos\theta_{ij}}{2}$ measuring angular separation between partons i, j

There is a clear quantitative equivalence between LO (one loop) BK and NGL

BMS: $$-\mu\partial_\mu \mathcal{V}_{12} = \frac{\alpha_s N}{2\pi}\int \frac{\mathrm{d}^2\Omega_0}{4\pi}\,\frac{\alpha_{12}}{\alpha_{10}\alpha_{02}}\Big[\mathcal{V}_{10}\mathcal{V}_{02}-\mathcal{V}_{12}\Big]$$

(physics of *timelike*/real radiation at null infinity)

BK: $$\partial_\eta \mathcal{U}_{12} = \frac{\alpha_s N}{2\pi}\int \frac{\mathrm{d}^2 z_0}{\pi}\,\frac{z_{12}^2}{z_{10}^2 z_{02}^2}\Big[\mathcal{U}_{10}\mathcal{U}_{02}-\mathcal{U}_{12}\Big]$$

(physics of *spacelike*/virtual radiation disrupted by targets)

given that $\mu \propto \mathrm{e}^{-\eta},\ \mathcal{V} \leftrightarrow \mathcal{U}$ and by identifying angular with transverse separations

$$\alpha_{ij} = \frac{1-\cos\theta_{ij}}{2} \leftrightarrow z_{ij}^2 = (x_i^\perp - x_j^\perp)^2 \qquad \frac{\mathrm{d}^2\Omega}{4\pi} \leftrightarrow \frac{\mathrm{d}^2 z}{\pi}$$

Summary: a physical dictionary

	BMS (jets, *timelike*)	**BK** (small-x, *spacelike*)
OBJECT	$\mathcal{V}_{12}$ = probability of *no* radiation into the vetoed region	$\mathcal{U}_{12}$ = probability the dipole crosses the target color-intact
EVOLVES IN	veto scale: $-\mu\,\partial_\mu$	boost/rapidity: ∂_η
NONLINEARITY	soft gluon 0 splits the dipole, $(12) \to (10)(02)$; at large N the daughters veto *independently*: $\mathcal{V}_{10}\,\mathcal{V}_{02}$	the same splitting in the transverse plane; the daughters scatter *independently*: $\mathcal{U}_{10}\,\mathcal{U}_{02}$
FIXED POINT $= 1$	transparency: never radiates into the vetoed region (unstable: evolution drives it down)	color transparency: never exchanges color with the target (unstable)
FIXED POINT $= 0$	black body: radiates into $\mathcal{C}_{\text{out}}$ with certainty (stable attractor as $\mu \to 0$)	black disc = saturation: scatters with certainty (stable attractor as $x_B \to 0$)

both equations count the odds that nothing happens & the quadratic term is what keeps probabilities ≤ 1

Giving names to the integrals from earlier for later convenience

BMS:

$$-\mu\partial_\mu \mathcal{V}_{12} = \frac{\alpha_s N}{2\pi} \boxed{\int \frac{\mathrm{d}^2\Omega_0}{4\pi} \frac{\alpha_{12}}{\alpha_{10}\alpha_{02}} \Big[\mathcal{V}_{10}\mathcal{V}_{02} - \mathcal{V}_{12}\Big]}$$

$$= H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}(1)} \mathcal{V}_{12}$$

BK:

$$\partial_\eta \mathcal{U}_{12} = \frac{\alpha_s N}{2\pi} \boxed{\int \frac{\mathrm{d}^2 z_0}{\pi} \frac{z_{12}^2}{z_{10}^2 z_{02}^2} \Big[\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}\Big]}$$

$$= H_{\mathrm{BK}}^{(1)}\Big|_{\mathrm{D}=4} \mathcal{U}_{12}$$

the ‘spacelike-timelike correspondence’: the following equality after relabeling

$$H_{\mathrm{BK}}^{(1)}\Big|_{\mathrm{D}=4} = H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}(1)}$$

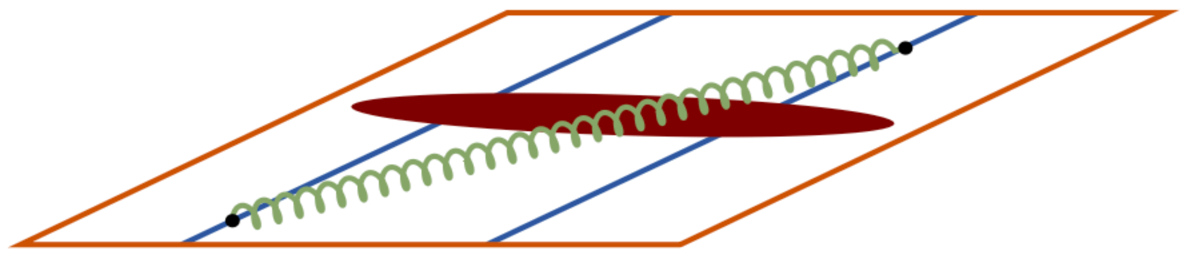
Claim: this relation holds and can be explained, order-by-order in α_s, by conformal invariance

[Previous works: Hoffman, Maldacena | Hatta | Caron-Huot, Herranen | Mueller | Vladimirov | Balitsky, Chirilli]

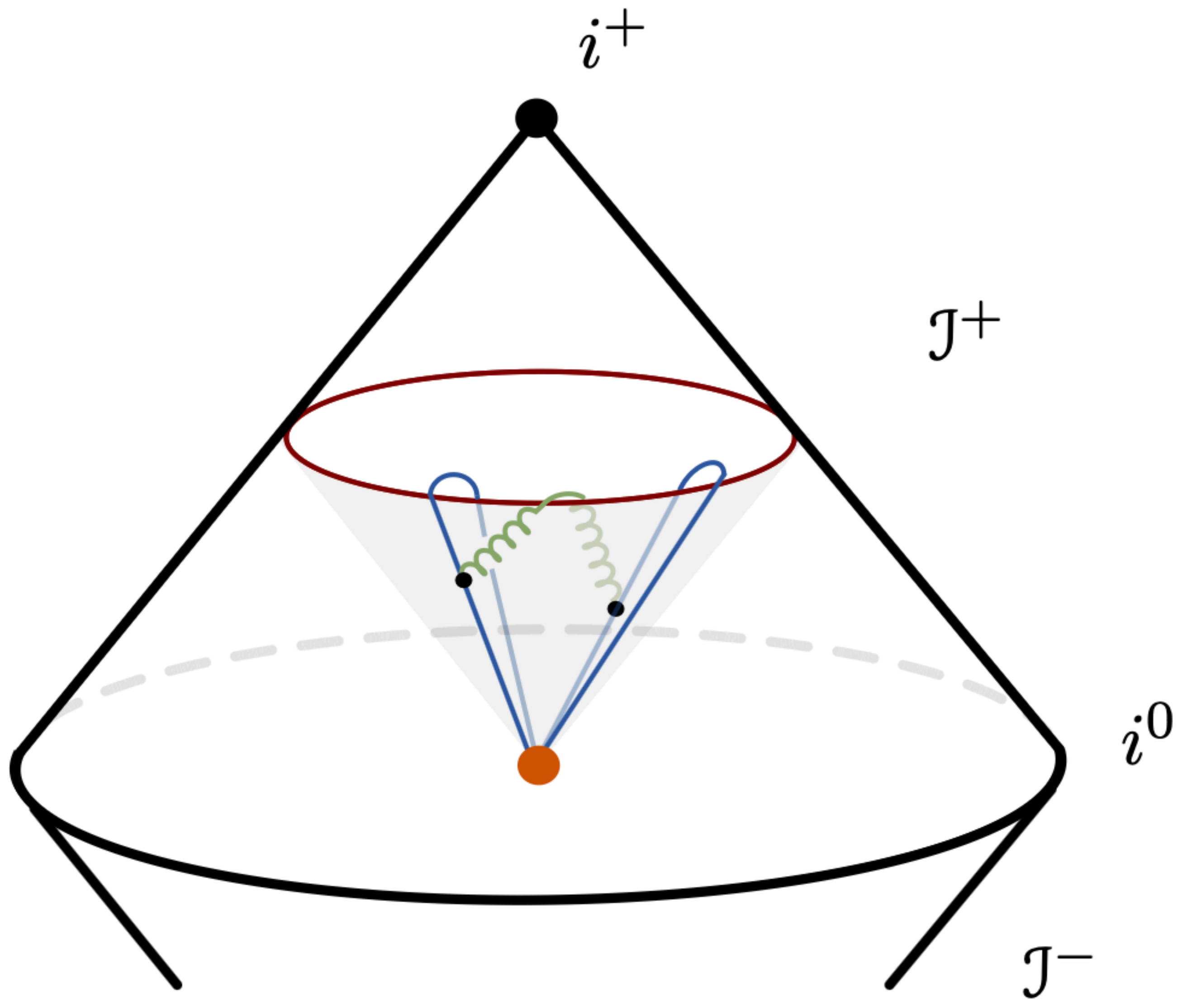
In particular, *assuming* we are allowed to do a conformal transformation on the BK and NGL kernels

Inversion: $$x^+ = -\frac{1}{y^+}\,, \quad x^- = y^- - \frac{y_\perp^2}{y^+}\,, \quad z^i = \frac{y^i}{y^+}$$ takes

Target on null sheet at small-x $\longleftrightarrow$ *Null infinity where soft radiation ends*



(x^+, x^-, z^i)



$(y^+, y^-, y_\perp^i)$ (related to angles by a change of variables)

But wait ! QCD is ***not*** conformal !?! 🤯

Why are we allowed to do conformal transformations ?

Yet, *perturbative* QCD can be in dimensional regularization $\mathrm{D} = 4 - 2\varepsilon$, where

$$\frac{1}{g_s}\beta(g_s) = -\varepsilon - \bar{\beta}(a) = -\varepsilon - (ab_0 + a^2 b_1 + \ldots)\,, \qquad a \equiv \frac{\alpha_s N}{4\pi} = \frac{g_s^2 N}{16\pi^2} \qquad b_0 = \frac{11}{3} - \frac{2}{3}n^f_{\mathrm{adj}} - \frac{1}{6}n^s_{\mathrm{adj}}$$

There is a Wilson-Fisher point for a particular value of ε:

$$\varepsilon_* \equiv -\bar{\beta}(a) \sim \mathcal{O}(\alpha_s) \qquad \text{(conformal dimension)}$$

Key observation behind the geometric picture: all-order **spacelike-timelike correspondence**

$$H_{\mathrm{BK}}(\varepsilon_* = -\bar{\beta}(a)) \simeq H^{\overline{\mathrm{MS}}}_{\mathrm{NGL}} \qquad \text{(up to scheme)}$$

Means equal up to a conformal transformation

Note: NGLs (just like soft or cusp anomalous dimensions) are *independent* of ε in $\overline{\mathrm{MS}}$ scheme, while rapidity divergences exist in all dimensions since *not* subtracted by the $\overline{\mathrm{MS}}$ scheme (i.e., remains a function of ε)

Writing both terms in $H_{\rm BK}(\varepsilon_* = -\bar\beta(a)) \simeq H^{\overline{\rm MS}}_{\rm NGL}$, respectively, as double and single expansions

$$\partial_\eta \mathcal{U}_{12} = H_{\rm BK} = \sum_{\ell=1}^{\infty}\sum_{k=0}^{\infty} a^\ell \varepsilon^k H^{(\ell,k)}_{\rm BK} \quad \text{and} \quad -\mu\partial_\mu \mathcal{V}_{12} = H^{\overline{\rm MS}}_{\rm NGL} = \sum_{\ell=1}^{\infty} a^\ell H^{\overline{\rm MS}\,(\ell)}_{\rm NGL}$$

we find, after substituting $\varepsilon_* = -(ab_0 + a^2 b_1 + \ldots)$, expanding and `//Collect[#,a^__]&`

that we can break BK into *simpler** building blocks

$$\boxed{H^{(1,0)}_{\rm BK}} - H^{\overline{\rm MS}(1)}_{\rm NGL} \simeq 0$$ ✓ [few slides ago]

(up to scheme)

$$\boxed{H^{(2,0)}_{\rm BK}} - b_0 H^{(1,1)}_{\rm BK} - H^{\overline{\rm MS}(2)}_{\rm NGL} \simeq 0$$ ✓ [Balitsky, Chirilli] [our recent paper]

Order in a and ε

$$\boxed{H^{(3,0)}_{\rm BK}} - b_0 H^{(2,1)}_{\rm BK} - b_1 H^{(1,1)}_{\rm BK} + b_0^2 H^{(1,2)}_{\rm BK} - H^{\overline{\rm MS}(3)}_{\rm NGL} \simeq 0$$

$$\vdots$$

*I will tell you why "simpler" later

Before moving to the more technical details: why is this perspective simpler?

1. Buys a loop order + we learned how to efficiently deal with dim. regulated Fourier transforms (FT)

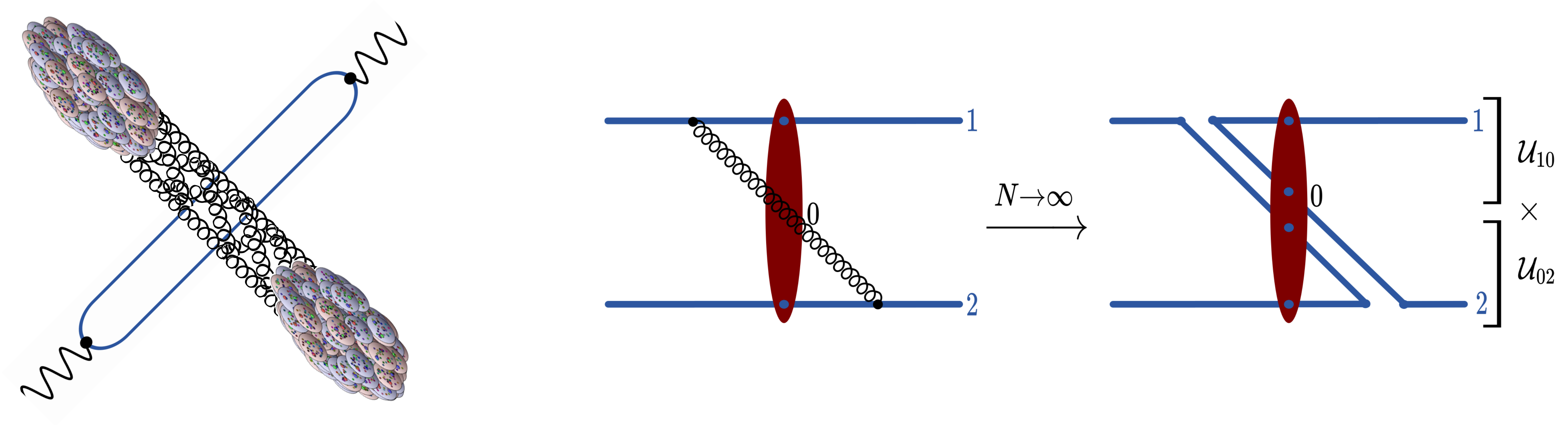
[w/ Brunello, Crisanti, Mastrolia, Smith] + automation in SubTropica [w/ Mizera, Salvatori]

2. NGLs are simpler to compute: no FT + leverage the large body of knowledge about soft currents

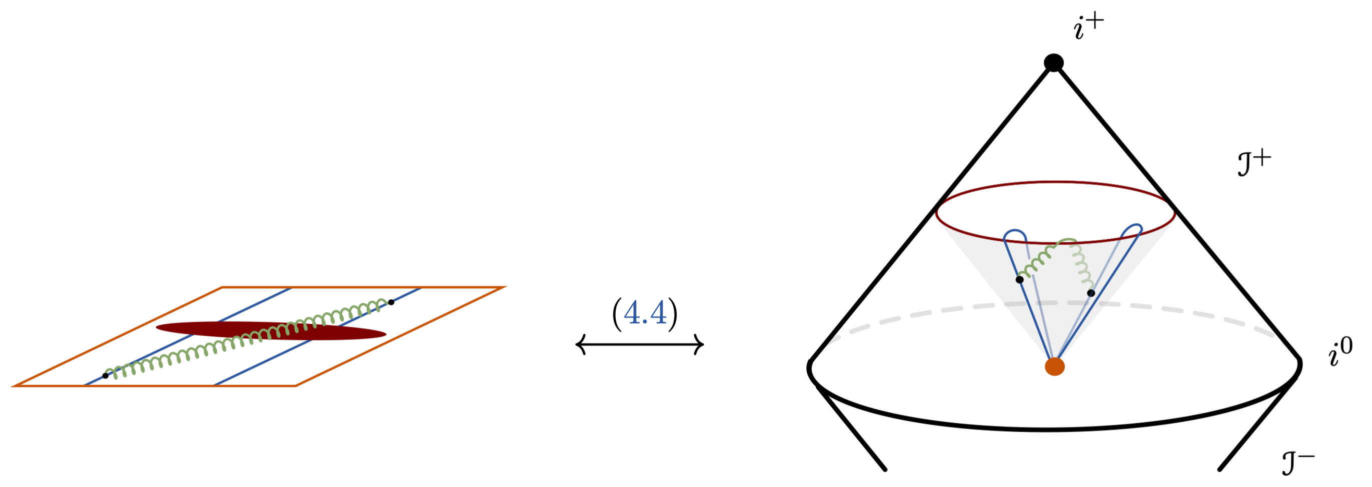
[slides from Caron-Huot]

Already known in N=4 at NNLO!

[QCD matter part: WIP w/ Caron-Huot]



1. Background and why planar?



2. Spacelike-timelike correspondence

$$H_{\mathrm{BK}}^{(3,0)} \simeq H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}\,(3)} + \int \frac{\mathrm{d}^2 z_0}{\pi} \frac{z_{12}^2}{z_{10}^2 z_{02}^2} (\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}) \Big[b_0 \tilde{H}_{[102]}^{(2,1)} + b_1 \tilde{H}_{[102]}^{(1,1)} - b_0^2 \tilde{H}_{[102]}^{(1,2)} \Big]$$
$$+ b_0 \int \frac{\mathrm{d}^2 z_0\, \mathrm{d}^2 z_{0'}}{\pi^2} \frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2} \left[\begin{array}{l} (\mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2} - \mathcal{U}_{12}) \tilde{H}_{[100'2]}^{(2,1)} \\ -(\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}) \tilde{H}_{[100'2]}^{(2,1)\mathrm{coll}} \end{array} \right] .$$

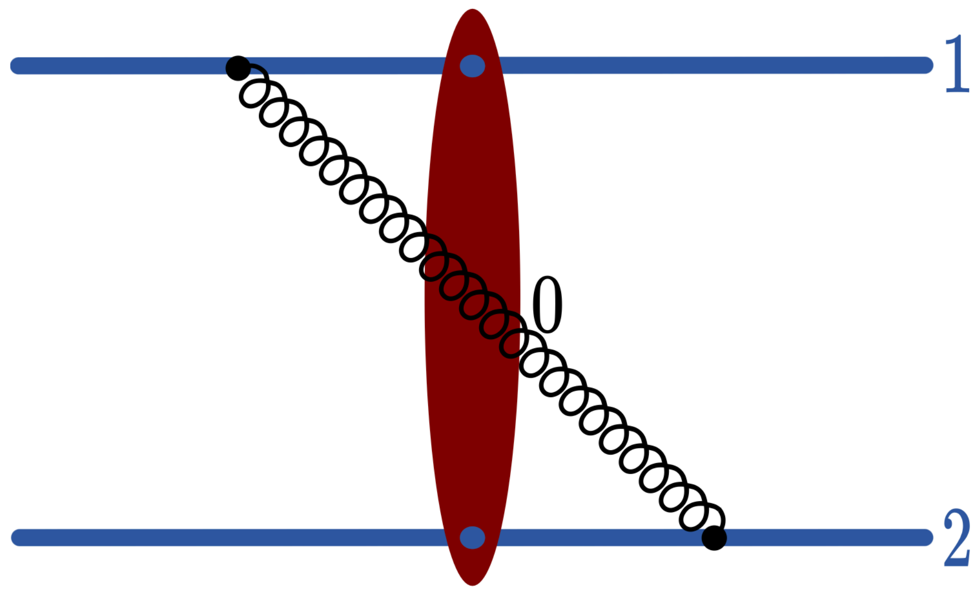
3. Results and what they enable

Our job: compute LO and NLO BK up to $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\varepsilon)$, respectively

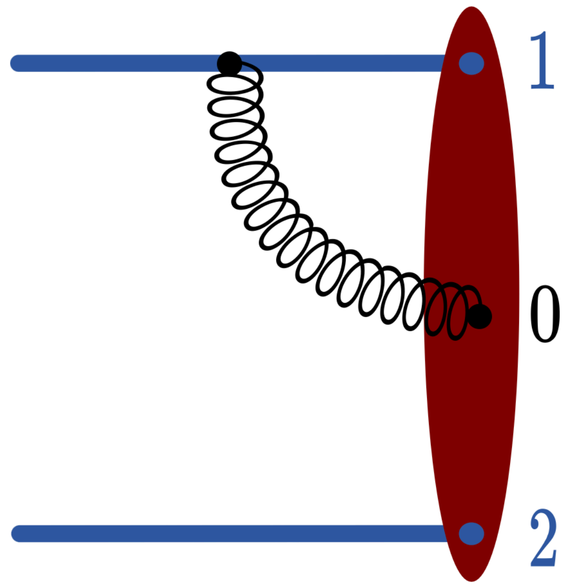
$$H_{\rm BK}^{(3,0)} \simeq \boxed{b_0 H_{\rm BK}^{(2,1)}} + \boxed{b_1 H_{\rm BK}^{(1,1)} - b_0^2 H_{\rm BK}^{(1,2)}} + H_{\rm NGL}^{\overline{\rm MS}(3)}$$

How? Let me illustrate the basic idea at LO where we can get a result exact in ε

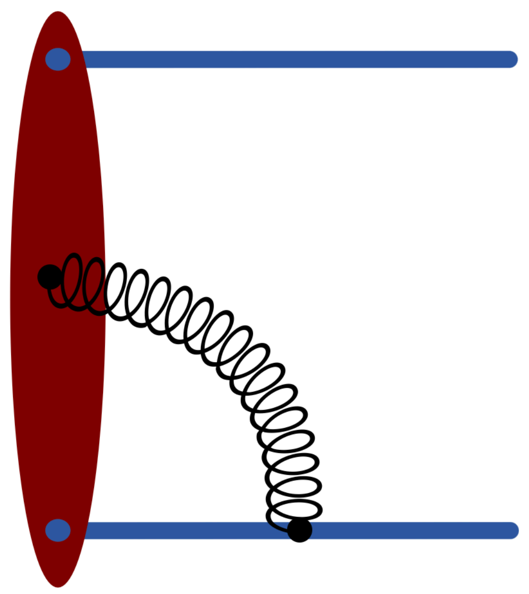
$H_{\rm BK}^{(1)} \supset$



$=$



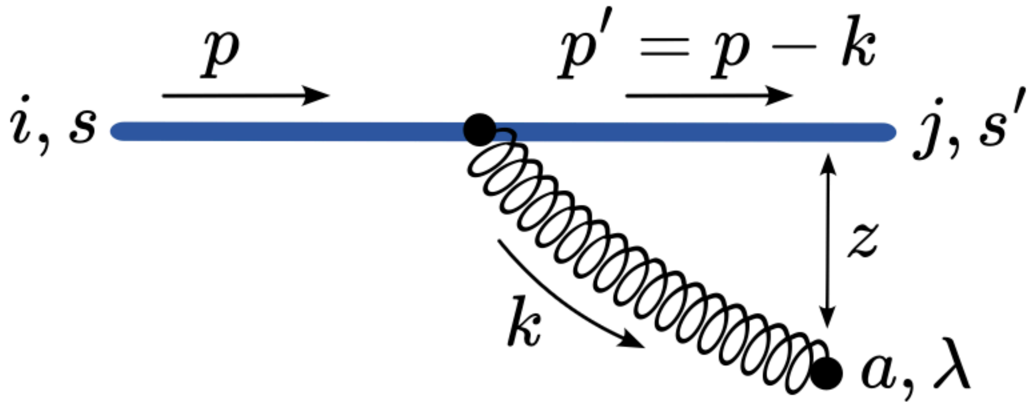
$\times$



$\times \mathcal{U}_{10}\, \mathcal{U}_{02}$

Construct integrand w/ *standard* Feynman rules and integrate loop momenta

$\mathcal{M}(z, k^+) =$

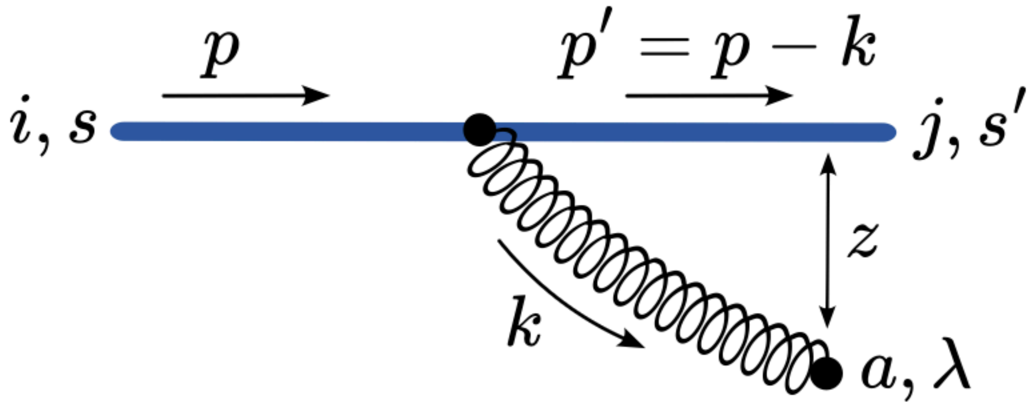


$$= \int \frac{\mathrm{d}k^-}{2\pi}\, \text{đ}^{\mathrm{D}_\perp} k \, \frac{\overline{u}_{s'}(p')\,(g f^{aij})\, \not{\epsilon}(k)\, u_s(p)}{[k^2 - i0][(p-k)^2 - i0]}\, \mathrm{e}^{i k_\perp \cdot z}$$

$\mathrm{D}_\perp = 2 - 2\varepsilon$

Construct integrand w/ *standard* Feynman rules and integrate loop momenta

$$\mathcal{M}(z, k^+) =$$



$$= \int \frac{\mathrm{d}k^-}{2\pi}\, \text{đ}^{\mathrm{D}_\perp} k \, \frac{\overline{u}_{s'}(p')\,(g f^{aij})\,\not\epsilon(k)\,u_s(p)}{[k^2 - i0][(p-k)^2 - i0]}\, \mathrm{e}^{ik_\perp \cdot z}$$

$\mathrm{D}_\perp = 2 - 2\varepsilon$

Some unusual features

For each daughter parton **crossing the target**, we need to Fourier transform to ⊥-position space since these are the variables on which the dipoles depend on

In LC gauge, *only* k^+ is conserved across the target: it is only integrated at the amplitude-squared level

Do the numerator algebra and perform the k^- integral by residues

$$\mathcal{M}(z,k^+) = g f^{aij} \int đ^{\mathrm{D}_\perp} k \; \frac{[2i(\delta_{\lambda,2s}+(1-\xi)\delta_{\lambda,-2s})(k_\perp-\xi p_\perp)\cdot\varepsilon^\lambda_\perp(k)]\theta(k^+)\theta(p^+-k^+)}{\xi\sqrt{1-\xi}\, k^+\,[p^+-k^+][\frac{k_\perp^2}{k^+}+\frac{(p_\perp-k_\perp)^2}{p^+-k^+}]} \; \mathrm{e}^{ik_\perp\cdot z}$$

$(p')^+ = (1-\xi)\,p^+$
$k^+ = \xi\,p^+$

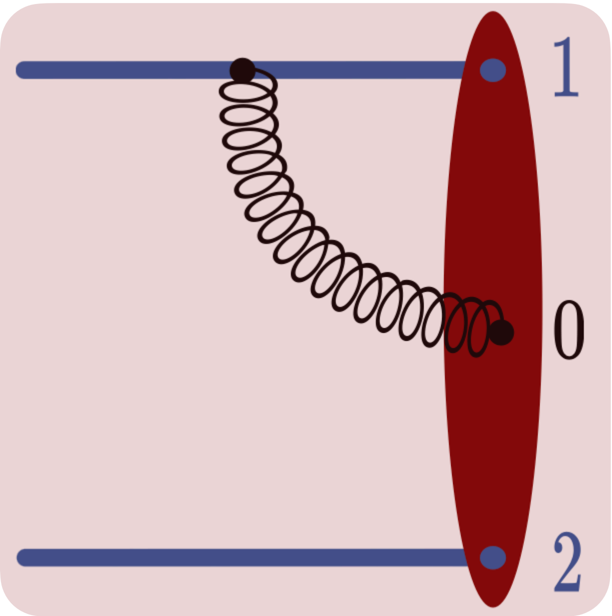
take the eikonal limit $k^+ \ll p^+$ (the parent quark is the fastest = follows a straight line)

$$\mathcal{M}(z,k^+) \approx g f^{aij}\frac{\theta(k^+)}{k^+}\int đ^{\mathrm{D}_\perp} k \; \tilde{\mathcal{A}}^i(k)\cdot\varepsilon_\perp(k)\;\mathrm{e}^{ik_\perp\cdot z} \quad \text{with} \quad \boxed{\tilde{\mathcal{A}}^i(k) \equiv \frac{2ik^i_\perp}{k^2_\perp}}$$

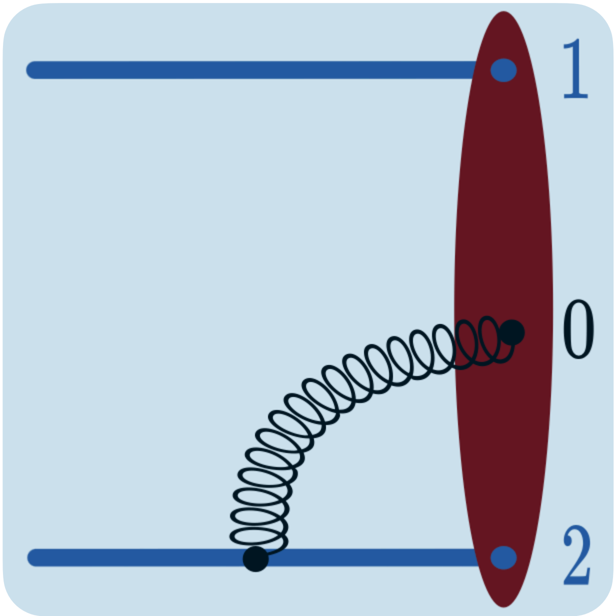
Once we integrate over k^+ at the amplitude *squared* level, this gives the rapidity log resumed by taking ∂_η (i.e., we never do this integral)

Using this result, the full single-real amplitude is given by the simple Fourier transform

$$\mathcal{A}^{(g,0)i}_{[102]}(z_0) =$$



$$+$$



$$= \int \frac{\mathrm{d}^{2-2\varepsilon} k_0}{(2\pi)^{2-2\varepsilon}} \frac{2ik_0^i}{k_0^2} \left(\mathrm{e}^{ik_0\cdot z_{01}} - \mathrm{e}^{ik_0\cdot z_{02}} \right)$$

Simple recipe to do them that will work for one-loop and (surprisingly) most of two-loop Fourier

Step 1. Use the Schwinger trick $\displaystyle \frac{1}{k_\perp^2} = \int_0^\infty \mathrm{d}a\, \mathrm{e}^{-a k_\perp^2}$

Step 2. Integrate over $k_\perp$ (Gaussian = trivial!)

(keep $k_\perp^i = -i\partial/\partial z_i$ numerator for the end)

Step 3. `Mathematica`'s `Integrate[]` over a

Result: $$\mathcal{A}^{(g,0)i}_{[102]}(z_0) = C_\varepsilon \left[\frac{z_{10}^i}{(z_{10}^2)^{1-\varepsilon}} - \frac{z_{20}^i}{(z_{20}^2)^{1-\varepsilon}} \right]$$

$$C_\varepsilon \equiv \frac{\Gamma(1-\varepsilon)}{\pi^{1-\varepsilon}}$$

This gives ingredient #1 to extract 3-loop BK in D = 4: one-loop BK at $\mathcal{O}(\varepsilon^{1,2})$

$$H_{\mathrm{BK}}^{(3,0)} \simeq b_0 H_{\mathrm{BK}}^{(2,1)} + b_1 H_{\mathrm{BK}}^{(1,1)} - b_0^2 H_{\mathrm{BK}}^{(1,2)} + H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}(3)}$$

Derived this way in any dimension [known already from Balitsky & Chirilli, 2024]

$$\partial_\eta \mathcal{U}_{12} = H_{\mathrm{BK}}^{(1,0)} + \varepsilon H_{\mathrm{BK}}^{(1,1)} + \varepsilon^2 H_{\mathrm{BK}}^{(1,2)} + \ldots$$

$$= C_\varepsilon \int_{z_0} (\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}) \left(\frac{z_{12}^2}{z_{10}^2 z_{02}^2}\right)^{1-\varepsilon} \tilde{H}_{[102]}$$

Just the square of previous amp.

$$\mathcal{A}_{[102]}^{(g,0)i}(z_0) = C_\varepsilon \left[\frac{z_{10}^i}{(z_{10}^2)^{1-\varepsilon}} - \frac{z_{20}^i}{(z_{20}^2)^{1-\varepsilon}}\right]$$

$$\tilde{H}_{[102]}^{(1)} = 2 + 2\varepsilon\left(L_\mu - \frac{z_{01}^2 - z_{02}^2}{z_{12}^2}\log\frac{z_{01}^2}{z_{02}^2}\right) + \varepsilon^2\left(\frac{\pi^2}{6} + L_\mu^2 - 2L_\mu \frac{z_{01}^2 - z_{02}^2}{z_{12}^2}\log\frac{z_{01}^2}{z_{02}^2} + \frac{z_{01}^2 + z_{02}^2}{z_{12}^2}\log^2\frac{z_{01}^2}{z_{02}^2}\right) + \mathcal{O}(\varepsilon^3)$$

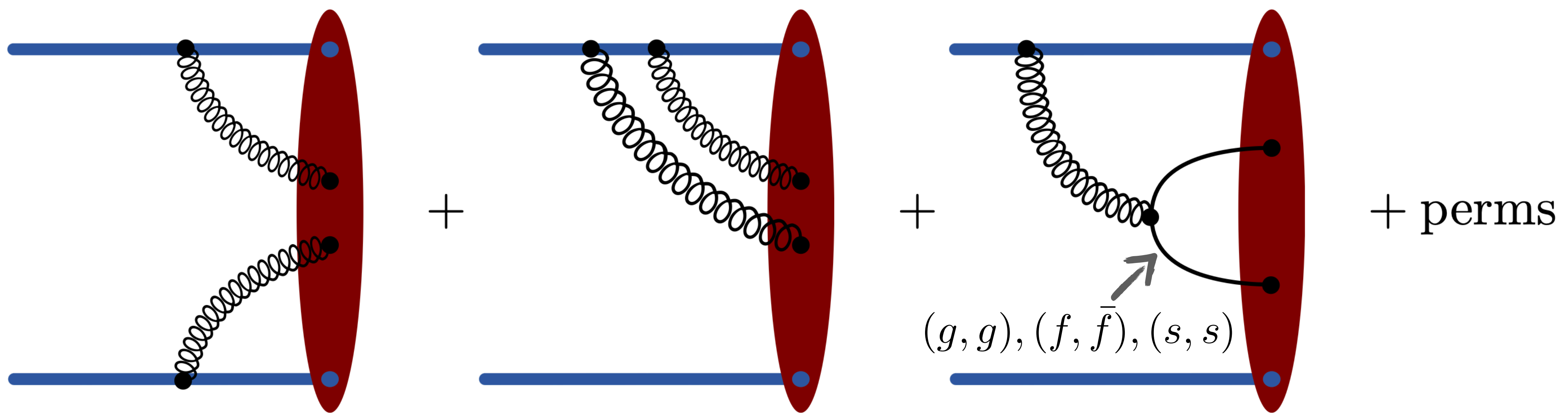
$$L_\mu = \log\frac{\overline{\mu}^2 z_{12}^2}{4\mathrm{e}^{-2\gamma_{\mathrm{E}}}}$$

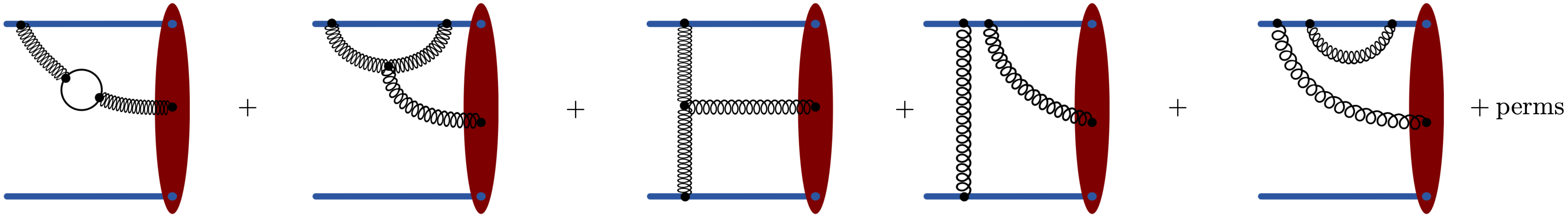
Note for the experts: thanks to the conformal measure, transcendental weight of $\tilde{H}$ increases with ε

Next, ingredient #2 to extract 3-loop BK in D = 4: two-loop BK at $\mathcal{O}(\varepsilon)$

$$H^{(3,0)}_{\mathrm{BK}} \simeq \boxed{b_0 H^{(2,1)}_{\mathrm{BK}}} + b_1 H^{(1,1)}_{\mathrm{BK}} - b_0^2 H^{(1,2)}_{\mathrm{BK}} + H^{\overline{\mathrm{MS}}(3)}_{\mathrm{NGL}}$$

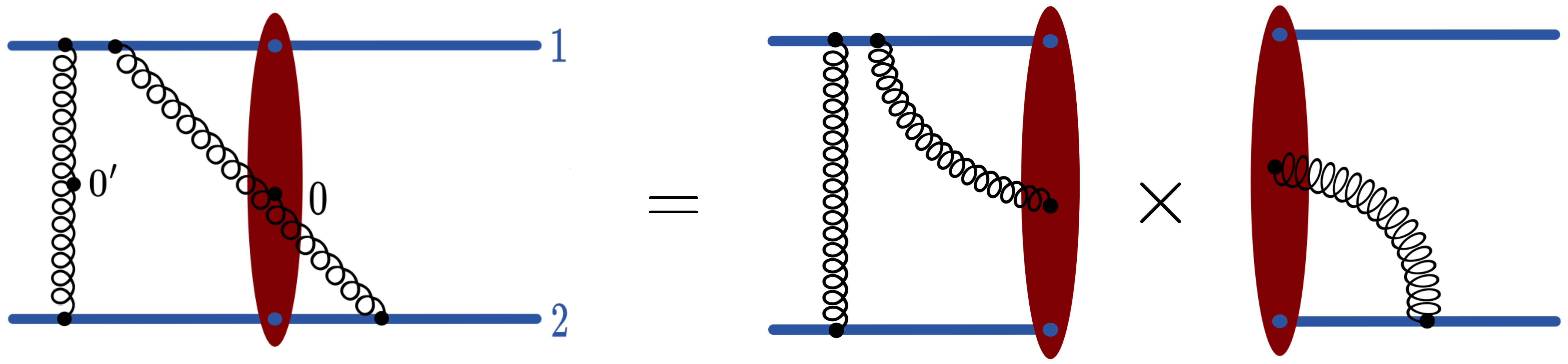
Here we have way more diagrams to consider: **double-real (top)** and **real-virtual (bottom)** diagrams





Example diagram: what are the expected integrals?

$H^{(2)}_{\rm BK} \supset$



$\times \mathcal{U}_{10}\mathcal{U}_{02}$

1. There are 3 virtuality integrals $\mathrm{d}k^-$: **trivial** by residues

2. There are two ‘energy’ integrals $\mathrm{d}\log(k_0^+)\,\mathrm{d}\log(k_{0'}^+)$: **easy one-fold integrals**

Variables extracting rapidity divergence: overall (geometric mean) $\nu_{[102]}$ and energy fraction τ

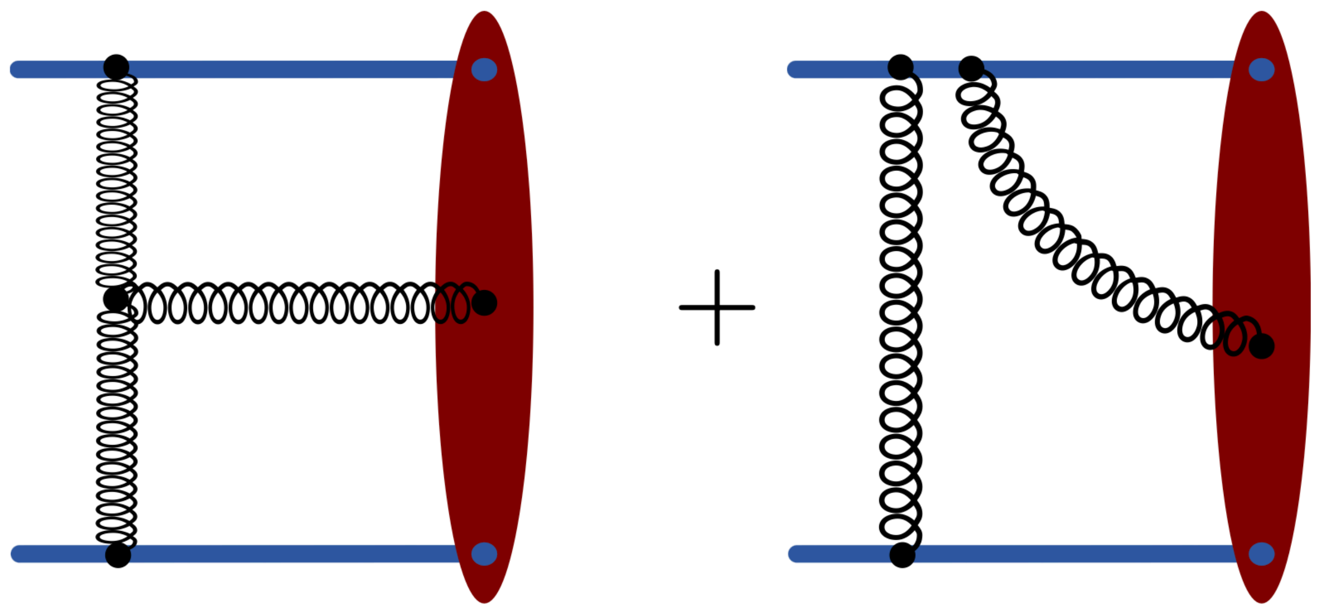
$$(k_0^+, k_{0'}^+) \xrightarrow{\text{c.v.}} \left(\nu_{[102]} = \sqrt{k_0^+ k_{0'}^+}, \tau = k_0^+/k_{0'}^+\right)$$

Integration over $\nu_{[102]}$ gives a rapidity log!
Only integrate the energy fraction @ amplitude-level

3. There are 2 Fourier transforms $\mathrm{d}^{2-2\varepsilon}k_0^{\perp}$: **challenging**

(one on each side of the target)

After performing the k^- residues the rapidity stripped amplitude is



$(\top)$

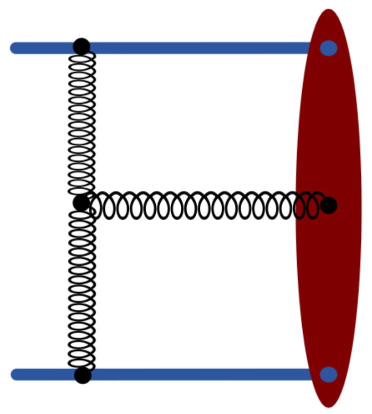
$$\mathcal{A}^{(g,1)i}_{[102]}(z_{01},z_{02};\tau)_\top = -4i\,\tau\int_{k_1,k_2} \mathrm{e}^{ik_1\cdot z_{01}+ik_2\cdot z_{02}}\left[\frac{\frac{k_2^i}{k_2^2}+\frac{(k_1^2+k_2^2)(k_1+k_2)^i}{k_2^2(k_1+k_2)^2}+\frac{2(k_1+k_2)^i}{\tau k_1^2}}{\tau k_1^2+(k_1+k_2)^2} + \frac{k_2^i}{(1+\tau)k_1^2k_2^2}\right]$$

Integrating* this in $0 \leq \tau \leq \infty$ is trivial and gives leaves us with a complicated Fourier transform

$$\frac{\mathcal{A}^{(g,1)i+}_{[102]}\Big|_\top}{-8\pi i} = \int_{k_1,k_2} \mathrm{e}^{i(k_1\cdot z_{01}+k_2\cdot z_{12})}\left[\frac{(k_1-k_2)^i}{k_2^2(k_1-k_2)^2}+\frac{k_1^i}{k_1^2(k_1-k_2)^2}-\frac{k_1^i}{k_1^2k_2^2}\right]\log\frac{k_1^2}{k_2^2} - (z_1\leftrightarrow z_2)$$

*This will depend on a subtraction scheme; ask me later for the details

Those Fourier transforms *used* to be the main challenge! Now just a few `Shift + Enter`'s in `SubTropica`



$$I = \frac{e^{-2\varepsilon\gamma_E}}{\pi^{D}} \int \frac{d^{D}\ell_1\, d^{D}\ell_2}{\ell_1^2(\ell_1-\ell_2)^2} \log\left(\frac{\ell_1^2}{\ell_2^2}\right) e^{i(\ell_1\cdot X_1+\ell_2\cdot X_2)}$$

Mathematica code snipet

```
(* Jacobian between d[Log[x]] and d[x] *)
jac = (Times @@ xvars)

(* Tropical analysis *)
analysis = STPreAnalysis[jac integrand /. q → q eps, xvars, coeffs]

(* Ray {1,1}: x1 → x1/λ, x2 → x2/λ *)
STFactor[jac integrand /. {x1 → x1/λ, x2 → x2/λ}]
  (* → λ^q: regulated by q only *)

(* Ray {0,1}: x1 fixed, x2 → x2/λ *)
STFactor[jac integrand /. {x1 → x1, x2 → x2/λ}]
  (* → λ^-ε: regulated by ε *)

(* Ray {1,0}: x1 → x1/λ, x2 fixed *)
STFactor[jac integrand /. {x1 → x1/λ, x2 → x2}]
  (* → λ^(-ε+q): regulated by ε *)

{{pref1, int1}, {pref2, int2}} =
    STTropicalContinuation[
      {{pref, jac integrand}}, xvars, {{1,1}}]

(* Extract O(q) coefficient of each term *)
int1 = SeriesCoefficient[pref1 int1, {q, 0, 1}];
int2 = SeriesCoefficient[pref2 int2, {q, 0, 1}];

(* Tropical subtraction in eps *)
serI1 = STExpandIntegral[int1, xvars, coeffs]
serI2 = STExpandIntegral[int2, xvars, coeffs]
```

$$I = \frac{1}{2\varepsilon^3} + \frac{\zeta_2}{2\varepsilon} + 4\,\mathrm{Re}[\mathrm{Li}_3(1-z)] + 4\,\mathrm{Re}[\log(1-z)]\,\mathrm{Re}[\mathrm{Li}_2(z)]$$
$$+ 4\,\mathrm{Re}[\log(1-z)]\big(\mathrm{Re}[\log(1-z)\log z] - \zeta_2\big) + \frac{\zeta_3}{3} + \mathcal{O}(\varepsilon)$$

Try it yourself!

w/ Mizera (Columbia) & Salvatori (IAS) [2604.20954] — subtropi.ca

Summary of new results after all the other amplitudes-squared are computed

$$H_{\mathrm{BK}}^{(3,0)} \simeq \boxed{b_0 H_{\mathrm{BK}}^{(2,1)}} + b_1 H_{\mathrm{BK}}^{(1,1)} - b_0^2 H_{\mathrm{BK}}^{(1,2)} + H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}(3)}$$

Double-real contributions

$$H_{\mathrm{BK}}^{(2,1)}\Big|_{\mathrm{DR}} = \int \frac{\mathrm{d}^2 z_0\, \mathrm{d}^2 z_{0'}}{\pi^2} \frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2} \left[(\mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2} - \mathcal{U}_{12}) \tilde{H}_{[100'2]}^{(2,1)} \right]$$

$$\tilde{H}_{[100'2]}^{(2)} \equiv \begin{bmatrix} (n_{\mathrm{adj}}^s - 2n_{\mathrm{adj}}^f + 2)\tilde{H}_{[100'2]}^{(2)\mathcal{N}=0} \\ + (n_{\mathrm{adj}}^f - 4)\tilde{H}_{[100'2]}^{(2)\mathcal{N}=1} \\ + \tilde{H}_{[100'2]}^{(2)\mathcal{N}=4} \end{bmatrix}$$

Organizes naturally as a

linear combination of SUSY multiplets

Real-virtual contributions

$$H_{\mathrm{BK}}^{(2,1)}\Big|_{\mathrm{RV}} = \int \frac{\mathrm{d}^2 z_0}{\pi} \frac{z_{12}^2}{z_{10}^2 z_{02}^2} (\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}) \tilde{H}_{[102]}^{(2,1)}$$

Explicit transverse function results for non-conformal contributions from two-loop ***double-real***

$$
\begin{aligned}
\frac{z_{12}^2 z_{00'}^2}{z_{10}^2 z_{20'}^2}\tilde{H}^{(2)\mathcal{N}=1}_{[100'2]} &= \frac{2u}{v-1}\left[\log v\,(1-\varepsilon\log uv+2\varepsilon L_\mu)-4\varepsilon\,\mathrm{Li}_2(1-v)\right]\\
&\quad+\varepsilon(1+\mathcal{P})\Bigg\{z_{00'}^2\mathcal{N}_1\left[\frac{F_1^{\mathrm{long}}}{(z_{10}z_{20'}^2-z_{10'}z_{20}^2)^2}+\frac{1}{z_{10'}^2 z_{20}^2 z_{20'}^2}B_2\!\left(\frac{z_{10}^2}{z_{10'}^2},\frac{z_{00'}^2}{z_{10'}^2}\right)\right]\\
&\quad+\frac{2z_{00'}^2(z_{20}^2-z_{12}^2-z_{10}^2)}{z_{10'}^2 z_{20}^2}B_2\left(\frac{z_{10}^2}{z_{10'}^2},\frac{z_{00'}^2}{z_{10'}^2}\right)\Bigg\}+\mathcal{O}(\varepsilon^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{z_{12}^2 z_{00'}^2}{z_{10}^2 z_{20'}^2}\tilde{H}^{(2)\mathcal{N}=0}_{[100'2]} &= \left(\frac{v+1}{v-1}\log v-2-8\varepsilon\right)(1-\varepsilon\log uv+2\varepsilon L_\mu)+\frac{8\varepsilon}{1-v}\mathrm{Li}_2(1-v)\\
&\quad+\varepsilon(1+\mathcal{P})\Bigg\{z_{20}^2\frac{z_{20'}^2(z_{10}^2+z_{10'}^2-z_{00'}^2)-2z_{10'}^2 z_{20}^2}{(z_{10}z_{20'}^2-z_{10'}z_{20}^2)^2}F_1^{\mathrm{long}}\\
&\quad+\frac{z_{20'}^2(z_{00'}^2(z_{10}^2+z_{10'}^2)-(z_{10}^2-z_{10'}^2)^2)-2z_{02}^2 z_{10'}^2 z_{00'}^2}{z_{10'}^2 z_{20'}^2 z_{00'}^2}B_2\left(\frac{z_{10}^2}{z_{10'}^2},\frac{z_{00'}^2}{z_{10'}^2}\right)\\
&\quad+\frac{z_{00'}^2+z_{10'}^2-z_{10}^2}{z_{00'}^2}\left[2\mathrm{Li}_2\left(1-\frac{z_{10}^2}{z_{10'}^2}\right)+\log\frac{z_{10}^2}{z_{10'}^2}\log\frac{z_{00'}^2}{z_{10'}^2}\right]\\
&\quad+\log\frac{z_{10}^2}{z_{10'}^2}\left[\log\frac{z_{10}^2 z_{00'}^4}{z_{10'}^2 z_{20}^4}-2\right]\Bigg\}+\mathcal{O}(\varepsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{H}^{(2)\mathcal{N}=4}_{[100'2]} &= 2\left(2\log u+\frac{u+v-1}{v-1}\log v\right)(1-\varepsilon\log uv+2\varepsilon L_\mu)\\
&\quad+\varepsilon\Bigg[\left(2(2-\mathcal{X})(1+v-u)-8\frac{u+v-1}{v-1}\right)\mathrm{Li}_2(1-v)+2\zeta_2(\mathcal{X}-2)(1+u-v)\\
&\qquad+(1+(v-u)(2-\mathcal{X}))\log^2 v+6\log u\log v+4\log u\log\frac{z_{00'}^4}{z_{10}^2 z_{20'}^2}\Bigg]\\
&\quad+\varepsilon(1+\mathcal{P})\Bigg\{\frac{z_{20}^2}{z_{20'}^2}(z_{20}^2-z_{00'}^2-3z_{20'}^2)\left[\frac{z_{10}^2 z_{20'}^2 F_1^{\mathrm{long}}}{(z_{10}z_{20'}^2-z_{10'}z_{20}^2)^2}+\frac{z_{10}^2}{z_{10'}^2 z_{20}^2}B_2\left(\frac{z_{10}^2}{z_{10'}^2},\frac{z_{00'}^2}{z_{10'}^2}\right)\right]\\
&\quad+\left(\frac{z_{20'}^2}{z_{20}^2}(z_{00'}^2-z_{20'}^2+3z_{20}^2)+z_{00'}^2\mathcal{X}\right)\left[\frac{z_{10}^2 z_{20'}^2 F_1^{\mathrm{long}}}{(z_{10}z_{20'}^2-z_{10'}z_{20}^2)^2}-\frac{z_{10}^2}{z_{10'}^2 z_{20}^2}B_2\left(\frac{z_{10}^2}{z_{10'}^2},\frac{z_{00'}^2}{z_{10'}^2}\right)\right]\\
&\quad+B_2\left(\frac{z_{10}^2}{z_{10'}^2},\frac{z_{00'}^2}{z_{10'}^2}\right)\left[\frac{z_{10}^2+z_{00'}^2-z_{10'}^2}{z_{10'}^2}\mathcal{X}+\frac{z_{10}^2+z_{10'}^2-z_{00'}^2}{z_{10'}^2}\left(\frac{z_{20'}^2-z_{00'}^2}{z_{20}^2}+\frac{z_{10}^2-z_{00'}^2}{z_{10'}^2}\right)\right]\\
&\quad+\frac{z_{10}^2-z_{00'}^2-z_{10'}^2}{z_{10'}^2}\left[2\log u\log\frac{z_{00'}^2}{z_{10}^2}+\left(\log\frac{z_{10}^2}{z_{10'}^2}\log\frac{z_{00'}^2 z_{10'}^2}{z_{20}^2 z_{20'}^2}+(1\leftrightarrow 2)\right)-4\zeta_2\right]\\
&\quad+\frac{z_{20}^2-z_{10}^2}{z_{12}^2}\left[\log\frac{z_{10}^2}{z_{20}^2}\log\frac{u^2 z_{20'}^2}{z_{20}^2}+\log\frac{z_{10}^2}{z_{00'}^2}\log v\right]+\log\frac{z_{10}^2}{z_{10'}^2}\log\frac{z_{20'}^2 z_{00'}^4}{z_{20}^6}\Bigg\}+\mathcal{O}(\varepsilon^2)\,.
\end{aligned}
$$

Baffling: at $\varepsilon \neq 0$, $\mathcal{N}=4$ is the most complicated piece!

$$
u=\frac{z_{12}^2 z_{00'}^2}{z_{10'}^2 z_{20}^2}\,,\qquad v=\frac{z_{10}^2 z_{20'}^2}{z_{10'}^2 z_{20}^2}
$$

$$
\mathcal{N}_1 = z_{12}^2(z_{20'}^2-z_{20}^2)+z_{10}^2 z_{20'}^2-z_{10'}^2 z_{20}^2
$$

$$
\mathcal{X} = (z_{10}^2-z_{10'}^2-z_{20}^2+z_{20'}^2)/z_{12}^2
$$

$$
\begin{aligned}
F_1^{\mathrm{long}} \equiv{}& 2\mathrm{Li}_2\left(1-\frac{z_{10}^2 z_{20'}^2}{z_{10'}^2 z_{20}^2}\right)-2\mathrm{Li}_2\left(1-\frac{z_{20}^2}{z_{20'}^2}\right)+\log\frac{z_{10}^2 z_{20'}^4}{z_{10'}^2 z_{20}^4}\log\frac{z_{00'}^2}{z_{10'}^2}\\
&+\left(\frac{z_{10}^2 z_{20'}^2}{z_{10'}^2 z_{20}^2}-\frac{z_{20}^2}{z_{20'}^2}\right)B_2\left(\frac{z_{10}^2}{z_{10'}^2},\frac{z_{00'}^2}{z_{10'}^2}\right)
\end{aligned}
$$

$$
B_2(u,v)\equiv\frac{2[\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar z)]+(\log(1-z)-\log(1-\bar z))\log z\bar z}{z-\bar z}\quad\text{with}\quad\begin{cases}u=z\bar z\,,\\ v=(1-z)(1-\bar z)\end{cases}
$$

Explicit transverse function results for non-conformal contributions from two-loop ***real-virtual***

$$\tilde{H}^{(2,1)}_{[102]} = 8\frac{94-n^f_{\text{adj}}-13n^s_{\text{adj}}}{27} - 24\zeta_3 + c^{(2,0)}\tilde{H}^{(1,1)}_{[102]} + b_0(2\tilde{H}^{(1,2)}_{[102]}+L_\mu^2 - \zeta_2) + (L_1-2b_0)L_2$$
$$+ \left(2\zeta_2+\frac{4}{9}(n^s_{\text{adj}}-2n^f_{\text{adj}}+2)\right)(\tilde{H}^{(1,1)}_{[102]}+2L_1-2L_\mu) + \left(\frac{1}{2}\tilde{H}^{(1,1)}_{[102]} - L_\mu\right)(b_0(2L_\mu-L_1) + L_2)$$
$$+ 2\frac{z_{12}^2+z_{20}^2-z_{10}^2}{z_{12}^2}\mathcal{L}_{1,0,1}(\zeta,\overline{\zeta}) + 2\frac{z_{12}^2+z_{10}^2-z_{20}^2}{z_{12}^2}\mathcal{L}_{1,0,1}(1-\zeta,1-\overline{\zeta})\,.$$

$$c^{(2,0)} \equiv 2\frac{59-2n^f_{\text{adj}}-8n^s_{\text{adj}}}{9} - \frac{4\pi^2}{3} \qquad L_\mu = \log\frac{\overline{\mu}^2 z_{12}^2}{4\mathrm{e}^{-2\gamma_\mathrm{E}}}\,, \quad L_1 = \log\frac{z_{12}^4}{z_{01}^2 z_{02}^2}\,, \quad L_2 = \log\frac{z_{12}^2}{z_{10}^2}\log\frac{z_{12}^2}{z_{20}^2}$$

$$\mathcal{L}_{1,0,1}(\zeta,\bar{\zeta}) = H_{1,0,1} + \overline{H}_{1,0,1} + H_{1,0}\overline{H}_1 + H_1\overline{H}_{1,0} \qquad z_{10}^2 = \zeta\overline{\zeta}z_{12}^2\,, \quad z_{20}^2 = (1-\zeta)(1-\overline{\zeta})z_{12}^2$$

$$H_{1,0,1} = 2\mathrm{Li}_3(1-\zeta) - \log(1-\zeta)\,(\mathrm{Li}_2(1-\zeta)+\zeta_2) - 2\zeta_3, \quad H_{1,0} = \mathrm{Li}_2(1-\zeta) - \zeta_2, \quad H_1 = -\log(1-\zeta),$$

The three-loop BK Hamiltonian predicted by the spacelike-timelike correspondence is thus

$$H_{\mathrm{BK}}^{(3,0)} \simeq H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}\,(3)} + \int \frac{\mathrm{d}^2 z_0}{\pi} \frac{z_{12}^2}{z_{10}^2 z_{02}^2} (\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}) \Big[b_0 \tilde{H}_{[102]}^{(2,1)} + b_1 \tilde{H}_{[102]}^{(1,1)} - b_0^2 \tilde{H}_{[102]}^{(1,2)} \Big] + b_0 \int \frac{\mathrm{d}^2 z_0\, \mathrm{d}^2 z_{0'}}{\pi^2} \frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2} \Big[(\mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2} - \mathcal{U}_{12}) \tilde{H}_{[100'2]}^{(2,1)} \Big] \quad \propto \beta$$

known [2508.03794]

where all the $\tilde{H}$ are known explicitly in terms of the transverse functions shown on last two slides

in progress: matter part of the NGL piece [w/ Caron-Huot]

$$H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}\,(3)}\Big|_{\text{planar QCD}} = H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}\,(3)}\Big|_{\text{planar }\mathcal{N}=4} + \left(H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}\,(3)}\Big|_{\text{planar QCD}} - H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}\,(3)}\Big|_{\text{planar }\mathcal{N}=4} \right)$$

known [1604.07417] — in progress

→ NNLO BFKL eigenvalues & Pomeron trajectory (in progress as well)

Enables tests and predictions: Pomeron trajectory from the NNLO kernel

1. linearize BK in the dilute regime ($\mathcal{U} = 1 - W \sim 1 - 1/N^2$) using *conformal* eigenfunctions W diagonalize BK kernel to BFKL

For L < 4, we can package the result as

$$\Rightarrow H^{(L)}_{\mathrm{BK}} W_{12} = \gamma^{(L)}_{\mathrm{cusp}} H^{(1)}_{\mathrm{BK}} W_{12} + \int_{\mathbb{C}} \frac{\mathrm{d}y \wedge \mathrm{d}\bar{y}}{2i} \underbrace{\left[\int_{\mathbb{C}} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2i} H^{(L)}_{\mathrm{BK},[1\; z\; (z-y)\; 0]} \right]}_{\equiv \mathcal{H}^{(L)}(y)} W_{12} \qquad (*)$$

$$z_1 = 0$$
$$z_0 = z$$
$$z_{0'} = z - y$$
$$z_{0''} = z_{0''}$$
$$z_{0'''} = z_{0'''}$$
$$z_2 = 1$$

For example, at two loops (because it fits on the slide)

$$\mathcal{H}^{(2)}(y)|_{\mathcal{N}=4} = \int_{\mathbb{C}} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2i\pi} \frac{2}{|1-z|^2|y-z|^2} \left[2\log\frac{|y|^2}{|z-1-y|^2|z|^2} + \left(1 + \frac{|y|^2}{|1-z|^2|y-z|^2 - |1+y-z|^2|z|^2}\right) \log\frac{|1-z|^2|y-z|^2}{|1+y-z|^2|z|^2} \right]$$

2. the Pomeron trajectory is (one minus) the Mellin transform of the impact-parameter Hamiltonian $\mathcal{H}(y)$

Note. Getting to (*) from the L-loop BK equation is non trivial because it requires integrating over at most $L-1$ transverse distances

Over hundreds of z-$\bar{z}$ integrals todo… automate in a public package ! (in preparation)

⏰

$\mathbb{C}$`Integrate` package: extracts the symbol of integrals of the form rational × (multi)polylogarithms

```
Q = y1 yb1 - yb1 z - y1 yb1 z + yb1 z^2 - y1 zb - y1 yb1 zb + y1 zb^2;
                                  (* = Abs[1-z]^2 Abs[y-z]^2 - Abs[1+y-z]^2 Abs[z]^2 *)

integrand = (1 + y1 yb1/Q) Log[((1-z)(z-y1)(1-zb)(zb-yb1))/((1+y1-z) z (1+yb1-zb) zb)]/(Pi (z-1)(z-y1)(zb-1)(zb-yb1));

built = FlatConnectionGraded[integrand, {z, zb}];
triangularSys = BuildTriangularSystem[built];
triangularSys["Flat"]             (* True: flatness check *)

bdry   = {0, 0, 0, 1};            (* fixed on the locus y1, yb1 -> -1 *)
symbol = BuildSymbol[triangularSys["Amat"], bdry, 4];
```

$$H^{(2)}(y) = 4\frac{\log^2(|y|^2)}{|1-y|^2} + 8\mathrm{Re}\left[\frac{\mathrm{Li}_2(y) - \mathrm{Li}_2(\bar{y}) + \mathrm{Li}_2(-y\bar{y}) - \log(y\bar{y})\log\frac{1-\bar{y}}{1+y\bar{y}} + \frac{1}{2}\zeta_2}{(1-\bar{y})(1+y)}\right]$$

`matches Caron-Huot & Herranen [1604.07417, eq. (B.9)]`

pipeline now running on the 3-loop QCD kernel; stay tuned for the NNLO Pomeron trajectories

Conclusion

We predicted the *non-conformal part* of the three-loop BK Hamiltonian in planar QCD

$$H_{\mathrm{BK}}^{(3,0)} \simeq H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}\,(3)} + \int \frac{\mathrm{d}^2 z_0}{\pi} \frac{z_{12}^2}{z_{10}^2 z_{02}^2} (\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12}) \left[b_0 \tilde{H}_{[102]}^{(2,1)} + b_1 \tilde{H}_{[102]}^{(1,1)} - b_0^2 \tilde{H}_{[102]}^{(1,2)} \right] + b_0 \int \frac{\mathrm{d}^2 z_0\, \mathrm{d}^2 z_{0'}}{\pi^2} \frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2} \left[(\mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2} - \mathcal{U}_{12}) \tilde{H}_{[100'2]}^{(2,1)} \right]$$

We explicitly check it in the large n_F where gluons are suppressed: $\mathcal{O}(\alpha_s^3 n_F^2)$ matches $H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}(3)}\Big|_{n_F\to\infty}$

Remains to compute the matter part of $H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}(3)}$ in planar QCD + linearized eigenvalues

[WIP w/ Caron-Huot]

Solve NNLO BK numerically ? Looks complicated, but doable ?

Expected check of the full three-loop BK:

Collinear limit of linearized eigenvalues (e.g., Pomeron trajectories) w/ DGLAP

Spin puzzle and polarized/sub-eikonal BK? $\mathcal{N} = 4$ BK from *subleading* Weinberg theorem?

Extra slides

EIC PROSPECT

On the experimental side, the EIC will extend this range and promises decisive tests for saturation!

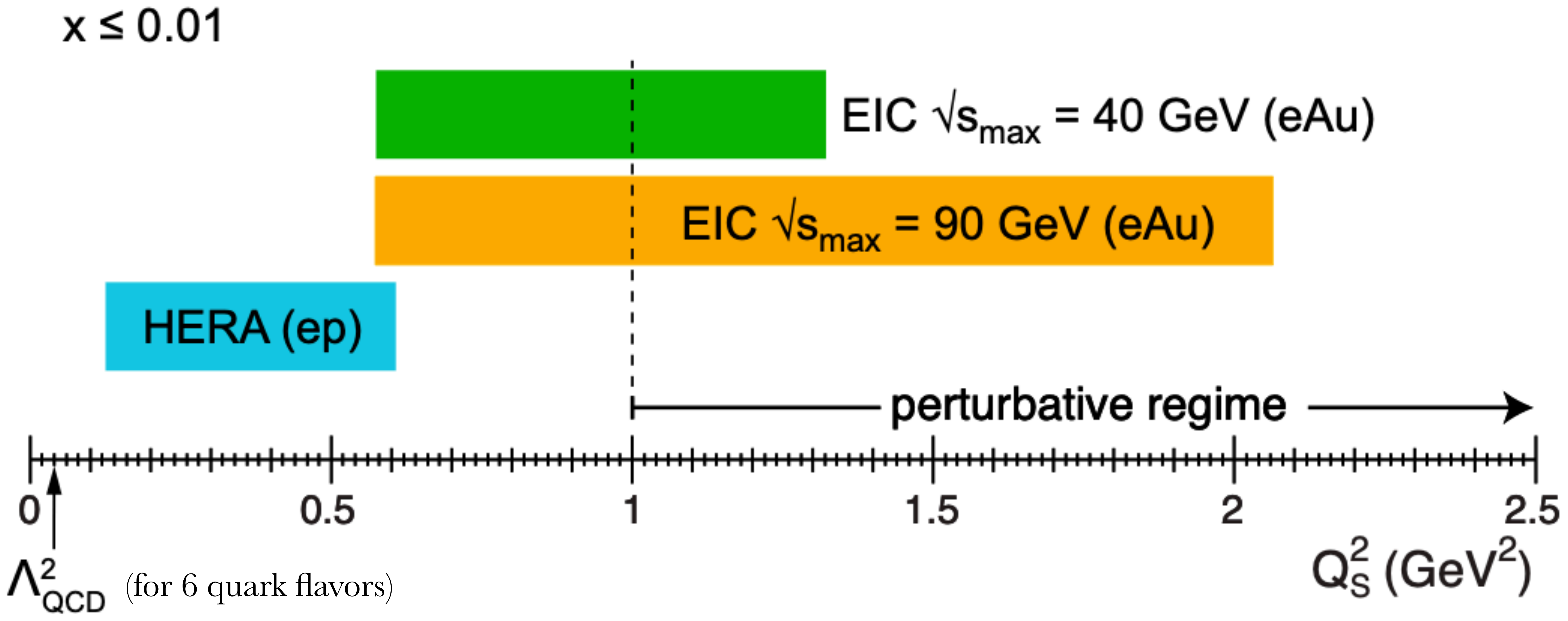


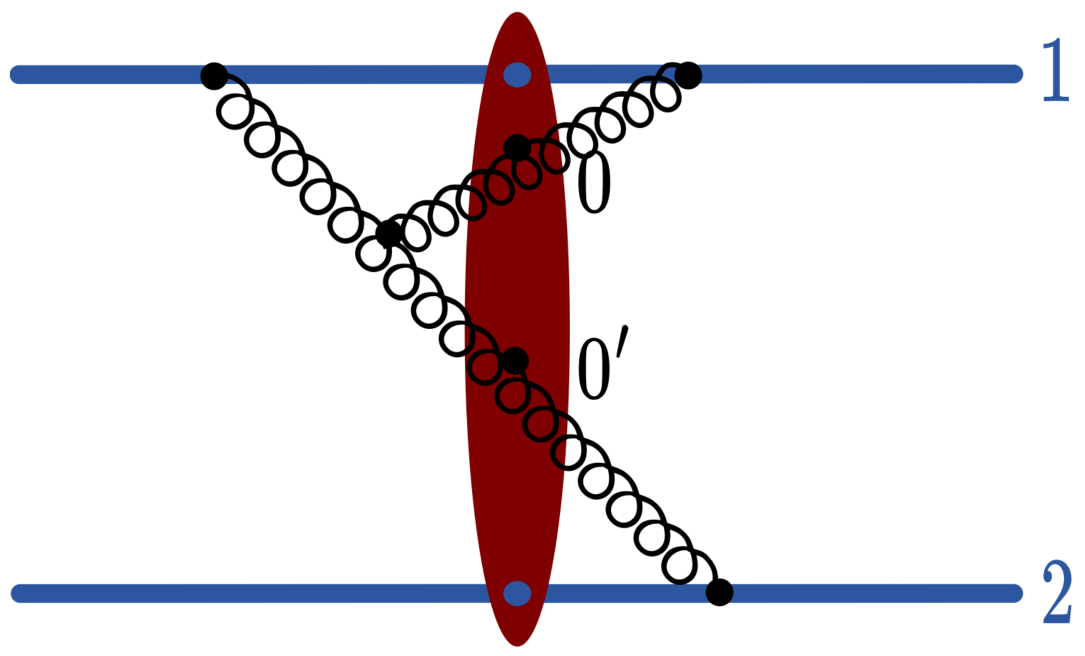
Figure 6: Accessible values of the saturation scale Q_s^2 at an EIC in e+A collisions assuming two different maximal center-of-mass energies. The reach in Q_s^2 for e+p collisions at HERA is shown for comparison.

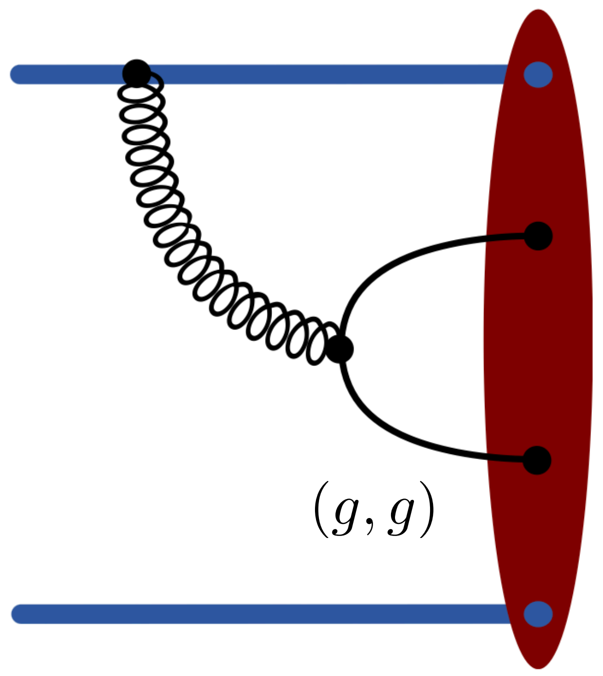
HERA kinematic range is *outside* saturation perturbative regime:

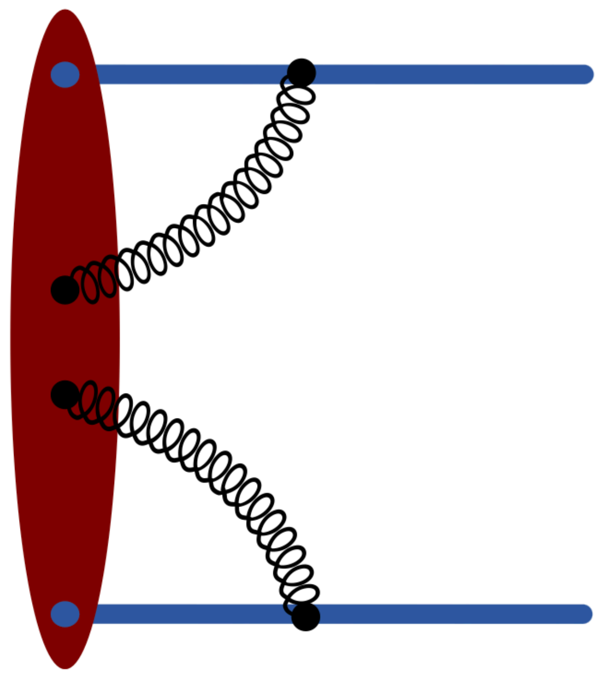
$Q_s^2(x_{\rm min.}^{\rm HERA}) \sim 0.6\,{\rm GeV}^2$ is only marginally above the non-pertubarive scale

Review in 1708.01527

Example diagram: what are the expected integrals?

$H^{(2)}_{\mathrm{BK}} \supset$ 

$=$ 

$\times$ 
$\times\,\mathcal{U}_{10}\,\mathcal{U}_{00'}\,\mathcal{U}_{0'2}$

There are 5 virtuality integrals $\mathrm{d}k^-$: done trivially by residues

There are 4 Fourier transforms $\mathrm{d}^{2-2\varepsilon}k_0^\perp\,,\mathrm{d}^{2-2\varepsilon}k_{0'}^\perp$: **require some work**

(two on each side of the target)

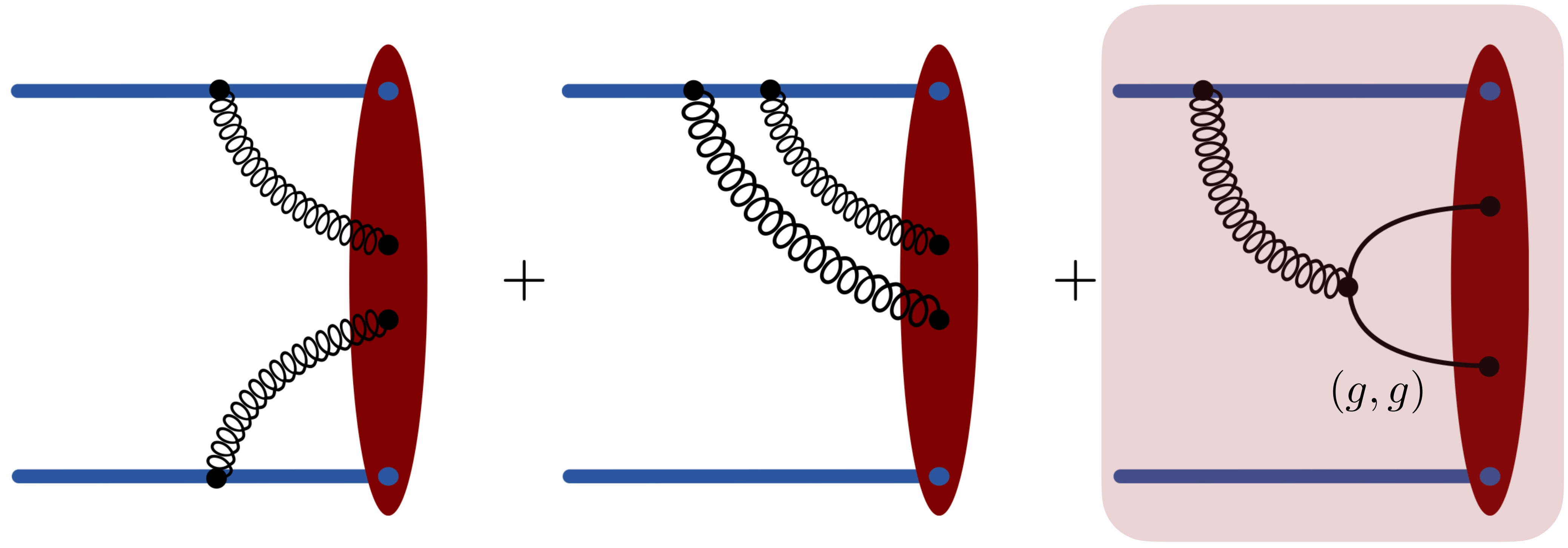
Two 'energy' integrals $\mathrm{d}\log(k_0^+)\,\mathrm{d}\log(k_{0'}^+)$: **require some work**

Variables extracting rapidity divergence: overall (geometric mean) $\nu_{[102]}$ and energy fraction τ

$$(k_0^+, k_{0'}^+) \xrightarrow{\text{c.v.}} \left(\nu_{[102]} = \sqrt{k_0^+ k_{0'}^+}, \tau = k_0^+/k_{0'}^+\right)$$

Integration over $\nu_{[102]}$ gives a rapidity log!
Only integrate the energy fraction @ amplitude-level

Example diagram: before the hard work apply Feynman rules, some numerator algebra and residues in k^- to find



+ perms

The k here are perpendicular ones!

Also the ν dependence fully factors out

$$\mathcal{A}^{(gg,0)ij}_{[100'2]}(\tau) = 4\int_{k_1,k_2} \mathrm{e}^{ik_1\cdot z_0 + ik_2\cdot z_{0'}}$$

$$\times\Bigg[\mathrm{e}^{-ik_1\cdot z_1 - ik_2\cdot z_2}\frac{k_1^i k_2^j}{k_1^2 k_2^2} - \frac{k_1^i k_2^j}{k_1^2+\tau k_2^2}\left(\frac{\mathrm{e}^{-i(k_1+k_2)\cdot z_1}}{k_2^2} + \tau\frac{\mathrm{e}^{-i(k_1+k_2)\cdot z_2}}{k_1^2}\right)$$

$$+\left(\mathrm{e}^{-i(k_1+k_2)\cdot z_1} - \mathrm{e}^{-i(k_1+k_2)\cdot z_2}\right)\frac{k_1^i(k_1+k_2)^j - \tau(k_1+k_2)^i k_2^j + \frac{\tau}{2(1+\tau)}(k_2^2 - k_1^2)\delta^{ij}}{(k_1^2+\tau k_2^2)(k_1+k_2)^2}\Bigg]$$

We want to do the integrals over momenta (Fourier transforms) and then those over the energy fraction

The two-loop Fourier transforms on the previous slide is a linear combination of two *master integrals*

$$\frac{1}{{C_\varepsilon}^2}\int_{k_1,k_2}\frac{\mathrm{e}^{ik_1\cdot z_{01}+ik_2\cdot z_{0'1}}}{(k_1^2+\tau k_2^2)k_2^2}=\frac{\tau^\varepsilon(z_{10}^2)^{2\varepsilon}}{32\varepsilon^2}\tilde{F}\left[\frac{z_{10'}^2}{\tau z_{10}^2}\right],$$

$$\frac{1}{{C_\varepsilon}^2}\int_{k_1,k_2}\frac{\mathrm{e}^{ik_1\cdot z_{01}+ik_2\cdot z_{0'1}}}{(k_1^2+\tau k_2^2)(k_1+k_2)^2}=\frac{\tau^\varepsilon(z_{00'}^2)^{2\varepsilon}}{32\varepsilon^2(1+\tau)^{1+2\varepsilon}}\tilde{F}\left[\frac{(\tau z_{10}+z_{10'})^2}{\tau z_{00'}^2}\right]$$

which can be expressed in terms of only one hypergeometric function

$$\tilde{F}[x]=F_1[x]-2F_2[x] \qquad F_1[x]=\frac{\Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon)}(1+x)^{2\varepsilon}, \qquad F_2[x]=\frac{\varepsilon x\Gamma(1-2\varepsilon)}{(1-\varepsilon)\Gamma^2(1-\varepsilon)}\,{}_2F_1(1-2\varepsilon,1-\varepsilon;2-\varepsilon;-x)$$

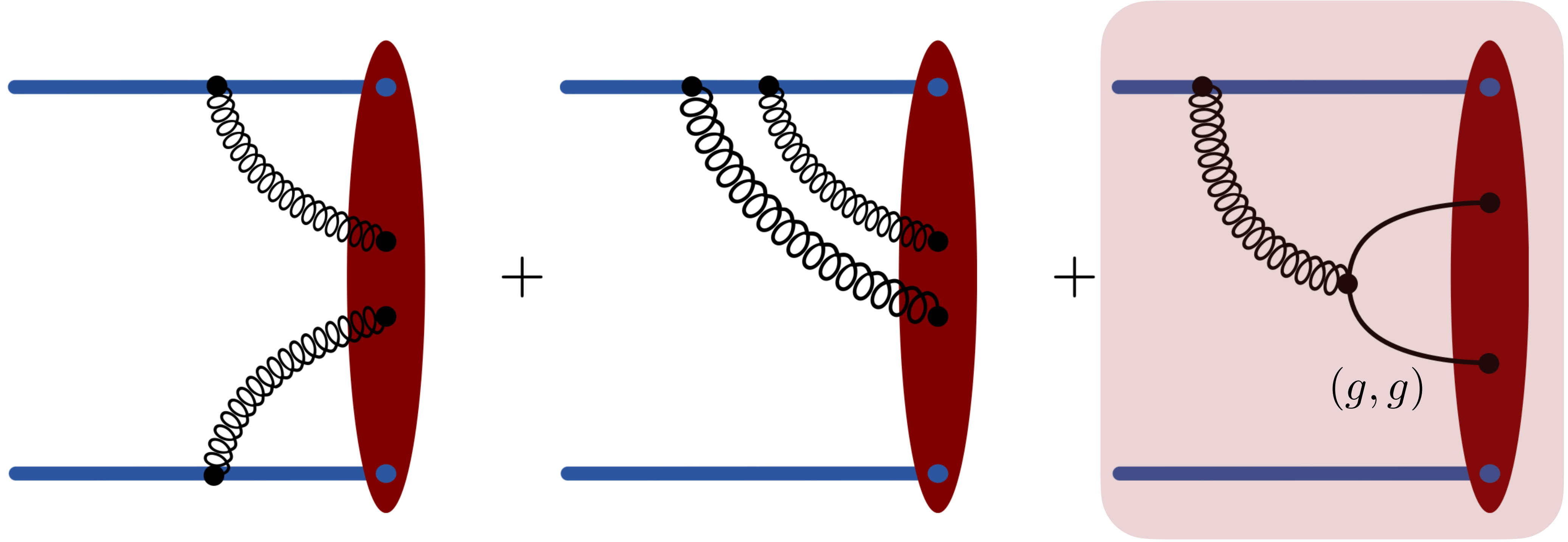
First shown using intersection theory and differential equations few years ago

[w/ Brunello, Crisanti, Mastrolia, Smith]

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix}F_1\\F_2\end{bmatrix}=\varepsilon\begin{bmatrix}\frac{2}{1+x} & 0\\ \frac{1}{1+x} & \frac{1}{x}\end{bmatrix}\cdot\begin{bmatrix}F_1\\F_2\end{bmatrix}\implies \begin{aligned}F_1[x]&=1+2\varepsilon\log(1+x)+\varepsilon^2(2\log^2(1+x)+\tfrac{\pi^2}{6})+\mathcal{O}(\varepsilon^3)\\F_2[x]&=\varepsilon\log(1+x)+\varepsilon^2(\log^2(1+x)-\mathrm{Li}_2(-x))+\mathcal{O}(\varepsilon^3)\end{aligned}$$

More systematic than Integrate[] in Schwinger parameters [now superseded by SubTropica, 2604.20954]

Example diagram: post Fourier transform, the amplitude is thus



$$\frac{1}{C_\varepsilon{}^2}\mathcal{A}^{(gg,0)ij}_{[100'2]}(\tau) = (1+\mathcal{P})\Bigg\{-\frac{z_{10}^i z_{20'}^j}{2(z_{10}^2 z_{20'}^2)^{1-\varepsilon}} + \frac{\tau^\varepsilon z_{10}^i z_{10'}^j}{(\tau z_{10}^2 + z_{10'}^2)(z_{10}^2)^{1-2\varepsilon}} G\left[\frac{z_{10'}^2}{\tau z_{10}^2}\right]$$

$$+\frac{\tau^\varepsilon}{(1+\tau)^{2\varepsilon}}\frac{z_{00'}^i z_{10'}^j + \tau z_{10}^i z_{00'}^j + \frac{\tau}{2(1+\tau)}(z_{10}^2 - z_{10'}^2)\delta^{ij}}{(\tau z_{10}^2 + z_{10'}^2)(z_{00'}^2)^{1-2\varepsilon}} G\left[\frac{(\tau z_{10} + z_{10'})^2}{\tau z_{00'}^2}\right]$$

$$+\frac{(1-\tau)\tau^{1+\varepsilon}\delta^{ij}}{2(1+\tau)^{1+2\varepsilon}}\frac{(z_{00'}^2)^{2\varepsilon}}{(\tau z_{10} + z_{10'})^2} F_2\left[\frac{(\tau z_{10} + z_{10'})^2}{\tau z_{00'}^2}\right]\Bigg\},$$

$$G[x] = F_1[x] - \frac{1+x}{x}F_2[x]$$

$$\mathcal{P}: (z_1, z_0, i, \tau) \leftrightarrow (z_2, z_{0'}, j, \tau^{-1})$$

Next, we want to do the integrals over the energy fraction

A sad fact of life: hypergeometric functions are hard to integrate directly…

The strategy is to first expand the integrand

$$\frac{1}{{C_\varepsilon}^2}\mathcal{A}^{(gg,0)ij}_{[100'2]}(\tau) = \;\ldots + \frac{\tau^\varepsilon}{(1+\tau)^{2\varepsilon}}\frac{z^i_{00'}z^j_{10'} + \tau z^i_{10}z^j_{00'} + \frac{\tau}{2(1+\tau)}(z^2_{10}-z^2_{10'})\delta^{ij}}{(\tau z^2_{10}+z^2_{10'})(z^2_{00'})^{1-2\varepsilon}} G\left[\frac{(\tau z_{10}+z_{10'})^2}{\tau z^2_{00'}}\right] + \ldots$$

in ε and then carry out the τ integral

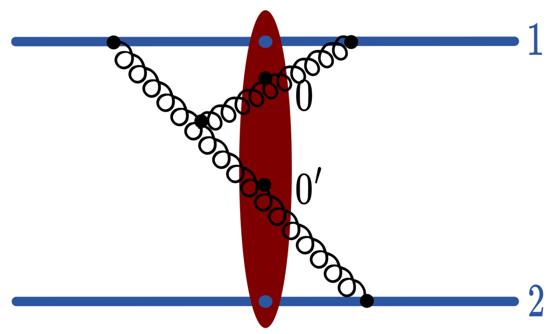
This is only possible ***if*** the integral is finite for all values of $0 \leq \tau \leq \infty$!

In other words, in this subtraction scheme, we write the two-loop double-real kernel as

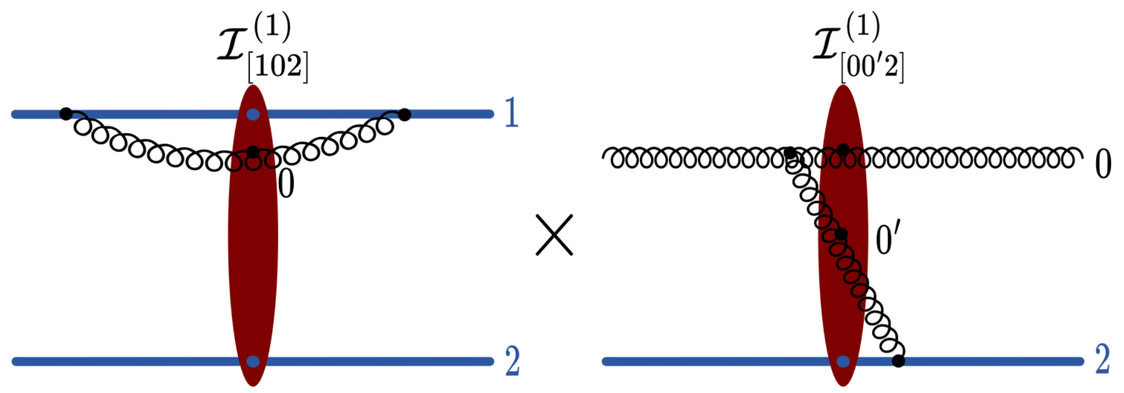
$$H^{(2)}_{[100'2]} = \frac{\mathcal{N}_\varepsilon^2}{C_\varepsilon{}^2} \int_0^\infty \frac{\mathrm{d}\tau}{\tau} \left[\mathcal{I}^{(2)}_{[100'2]}(\tau) \quad \begin{array}{l} -\mathcal{I}^{(1)}_{[10'2]}\mathcal{I}^{(1)}_{[100']}\theta(\nu_{[10'2]} > \nu_{[100']}) \\ -\mathcal{I}^{(1)}_{[102]}\mathcal{I}^{(1)}_{[00'2]}\theta(\nu_{[102]} > \nu_{[00'2]}) \end{array} \right]$$

$\mathcal{N}_\varepsilon \equiv 2\pi\mu^{2\varepsilon}$

$C_\varepsilon \equiv \frac{\Gamma(1-\varepsilon)}{\pi^{1-\varepsilon}}$



$+\ldots$



$+\ldots$

and the integral is now finite ⇒ do the integral above order-by-over in ε

`e.g., [IBPs, HyperInt, SubTropica…]`

Comments for the experts in the room

1. The scheme depends on a *choice* of geometric means: $\nu_{[i\,a_1 \ldots a_L\,j]} \equiv \lambda_L \times (\ell^+_{a_1} \cdots \ell^+_{a_L})^{1/L}$

2. Make a choice ensuring that all breaking of conformal symmetry comes from *running of coupling*:

$$\lambda_L \equiv \left[\frac{z^2_{ia_1}\, z^2_{a_1a_2} \cdots z^2_{a_Lj}}{z^2_{ij}} \right]^{\frac{1}{2L}} \qquad \text{(conformal scheme)} \qquad \Longrightarrow \quad \lambda_1 \equiv \left[\frac{z^2_{10}\, z^2_{02}}{z^2_{12}} \right]^{\frac{1}{2}}$$

Collecting the double-real (DR) ingredients we get

$$H_{\mathrm{BK}}^{(3,0)} \simeq \boxed{b_0 H_{\mathrm{BK}}^{(2,1)}} + b_1 H_{\mathrm{BK}}^{(1,1)} - b_0^2 H_{\mathrm{BK}}^{(1,2)} + H_{\mathrm{NGL}}^{\overline{\mathrm{MS}}(3)}$$

$$H_{[100'2]}^{(2)} = \left(\frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2}\right)^{1-\varepsilon} \tilde{H}_{[100'2]}^{(2)}$$

$$H_{\mathrm{BK}}^{(2,1)}\Big|_{\mathrm{DR}} = \int \frac{\mathrm{d}^2 z_0\, \mathrm{d}^2 z_{0'}}{\pi^2} \frac{z_{12}^2}{z_{10}^2 z_{00'}^2 z_{0'2}^2} \left[(\mathcal{U}_{10}\mathcal{U}_{00'}\mathcal{U}_{0'2} - \mathcal{U}_{12})\tilde{H}_{[100'2]}^{(2,1)}\right]$$

Result for the DR BK kernel *naturally* organizes nicely in terms of SUSY combinations

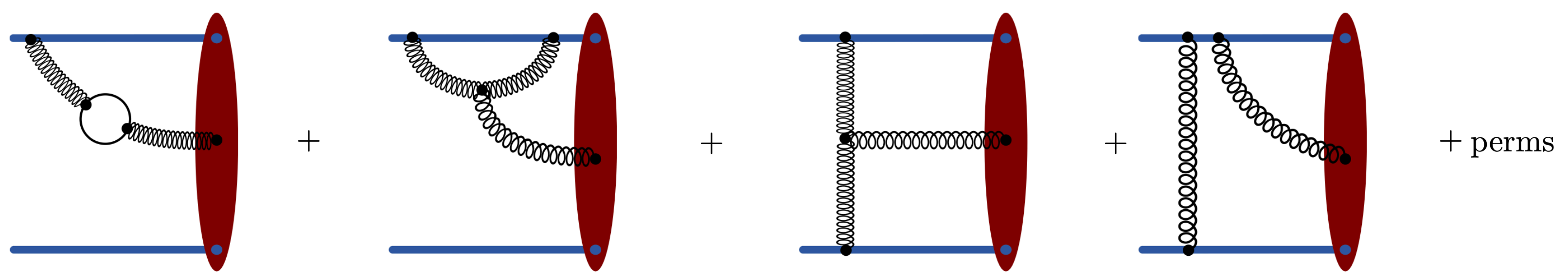
$$\tilde{H}_{[100'2]}^{(2)} \equiv \left[\begin{array}{l} \quad (n_{\mathrm{adj}}^s - 2n_{\mathrm{adj}}^f + 2 - 2\varepsilon)\tilde{H}_{[100'2]}^{(2)\mathcal{N}=0} \\ +(n_{\mathrm{adj}}^f - 4)\tilde{H}_{[100'2]}^{(2)\mathcal{N}=1} \\ +\tilde{H}_{[100'2]}^{(2)\mathcal{N}=4} \end{array}\right]$$

What we did on previous slides: determined the $\tilde{H}$ in terms of explicit transverse functions

More integrals: similar calculations for real-virtual

$$H^{(2)}_{[102]} = \frac{\mathcal{N}_\varepsilon^2}{C_\varepsilon}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\left[\mathcal{I}^{(2)}_{[102]}(\tau) - \mathcal{S}^{(2)}_{[102]}(\tau)\right] - \frac{b_0}{\varepsilon} H^{(1)}_{[102]}$$

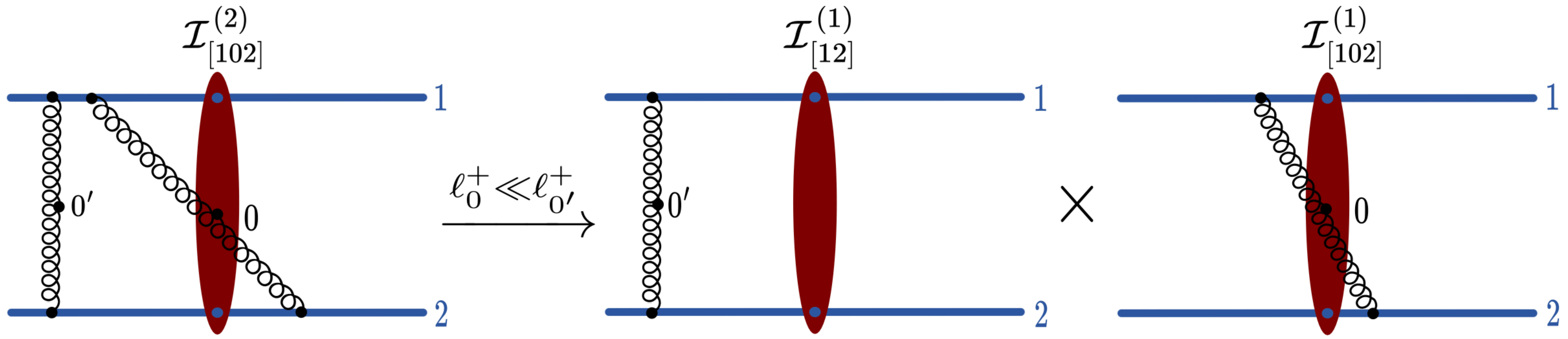
square of



counterterm

subdivergence subtractions

$$\mathcal{S}^{(2)}_{[102]}(\tau) = \int_{z_{0'}} \mathcal{I}_{[102]}\left(\mathcal{I}^{(1)}_{[10'2]}\theta\left(1 > \frac{\nu_{[102]}}{\nu_{[10'2]}}\right) + \mathcal{I}^{(1)}_{[00'2]}\theta\left(\frac{\nu_{[102]}}{\nu_{[00'2]}} > 1\right) + \mathcal{I}^{(1)}_{[10'0]}\theta\left(\frac{\nu_{[102]}}{\nu_{[10'0]}} > 1\right)\right)$$



By inspection, the integrals are finite everywhere but at the integration boundary

In these limits, diagrams (integrands) nicely factorize as "iterations of one-loop diagrams"

These factorization channels are thus what we need to subtract from the original expression before expanding in ε