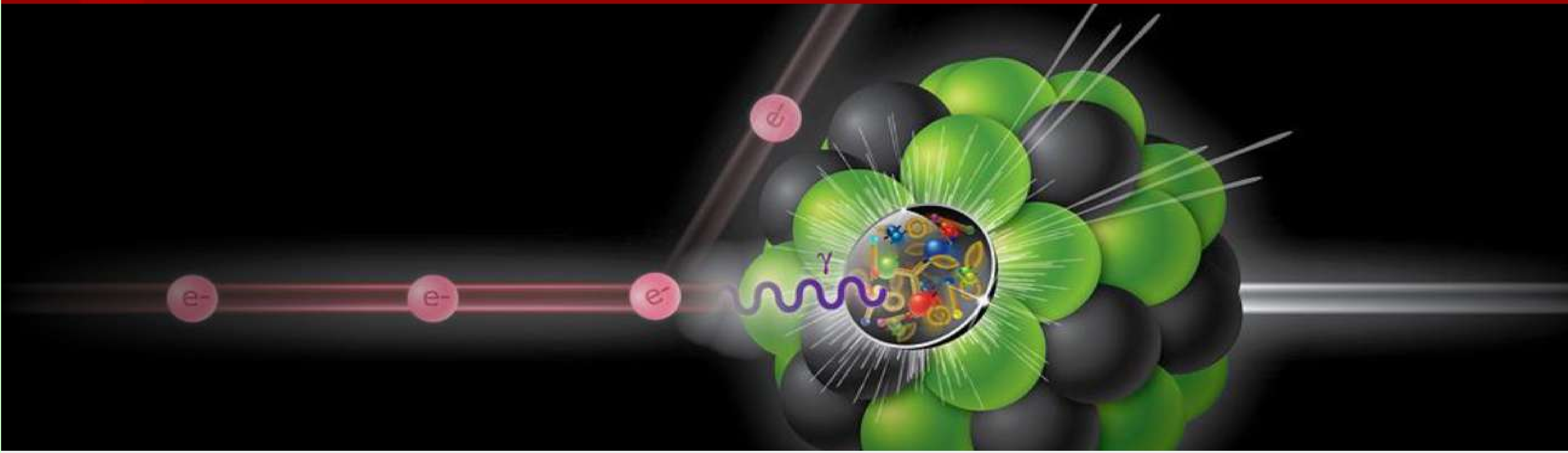


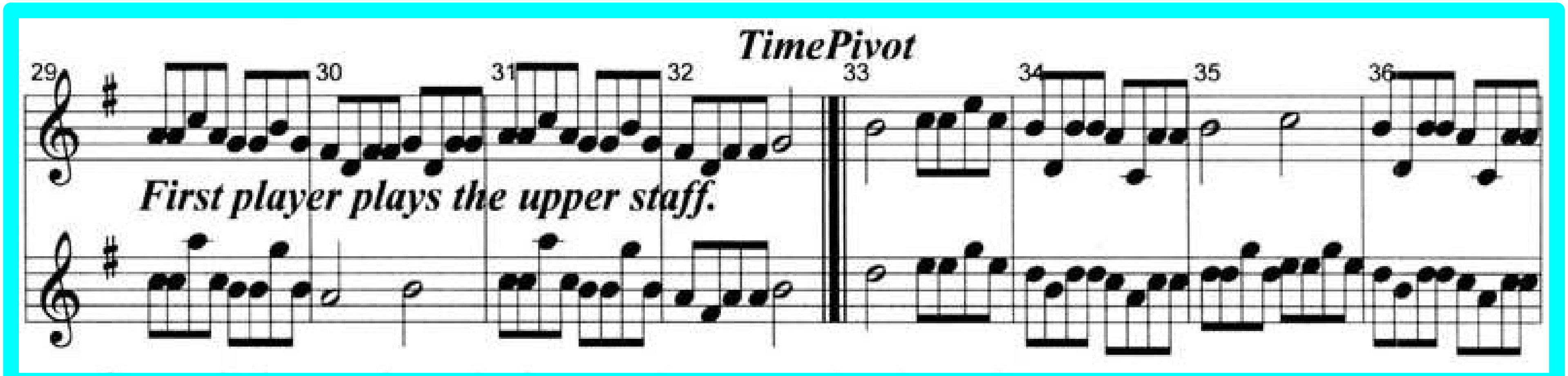
Introduction to the Parton Model and Perturbative QCD

Fred Olness (SMU)

2026 CFNS School
Stony Brook University
1-12 June 2026

Welcome to QCD:

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$
$$= \bar{\psi}_i (i\gamma^\mu \partial_\mu - m)\psi_i - g G^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a ,$$



Mozart: Inverted retrograde canon in G

Patterns, Symmetry (obvious & hidden), interpretation

Notes ⇒ themes ⇒ Melody/Harmony ⇒ interaction/counterpoint ⇒ structure

Welcome to QCD:

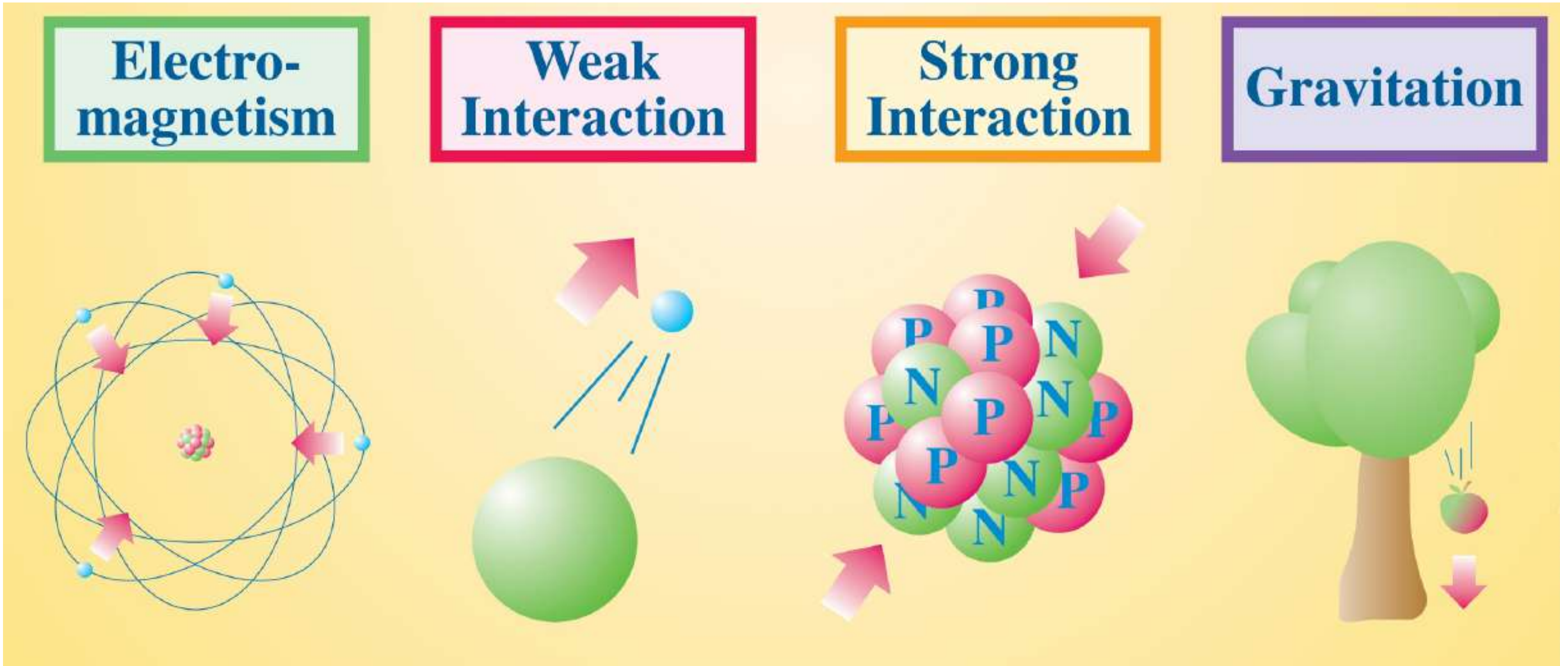
$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

$$= \bar{\psi}_i (i\gamma^\mu \partial_\mu - m)\psi_i - g G^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a ,$$

Mozart: Inverted retrograde canon in G

Patterns, Symmetry (obvious & hidden), interpretation

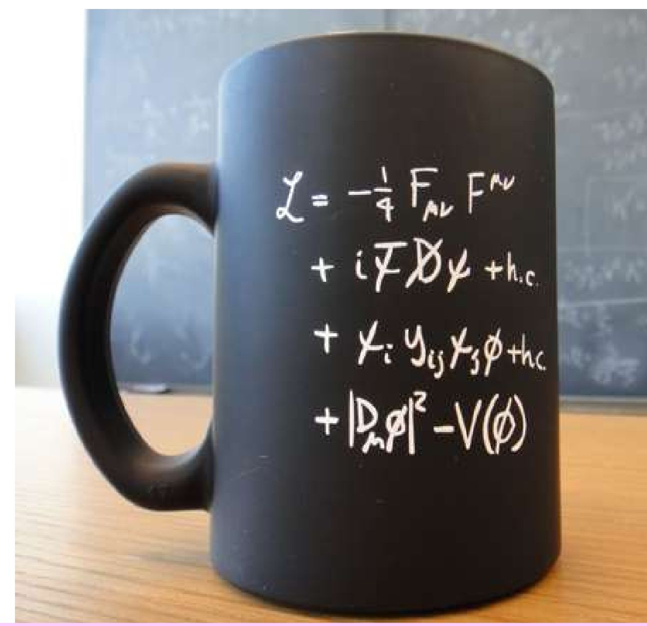
Notes $\Rightarrow$ themes $\Rightarrow$ Melody/Harmony $\Rightarrow$ interaction/counterpoint $\Rightarrow$ structure



QCD is our most perfect physical theory

What QCD Tells Us About Nature – and Why We Should Listen. *Frank Wilczek*

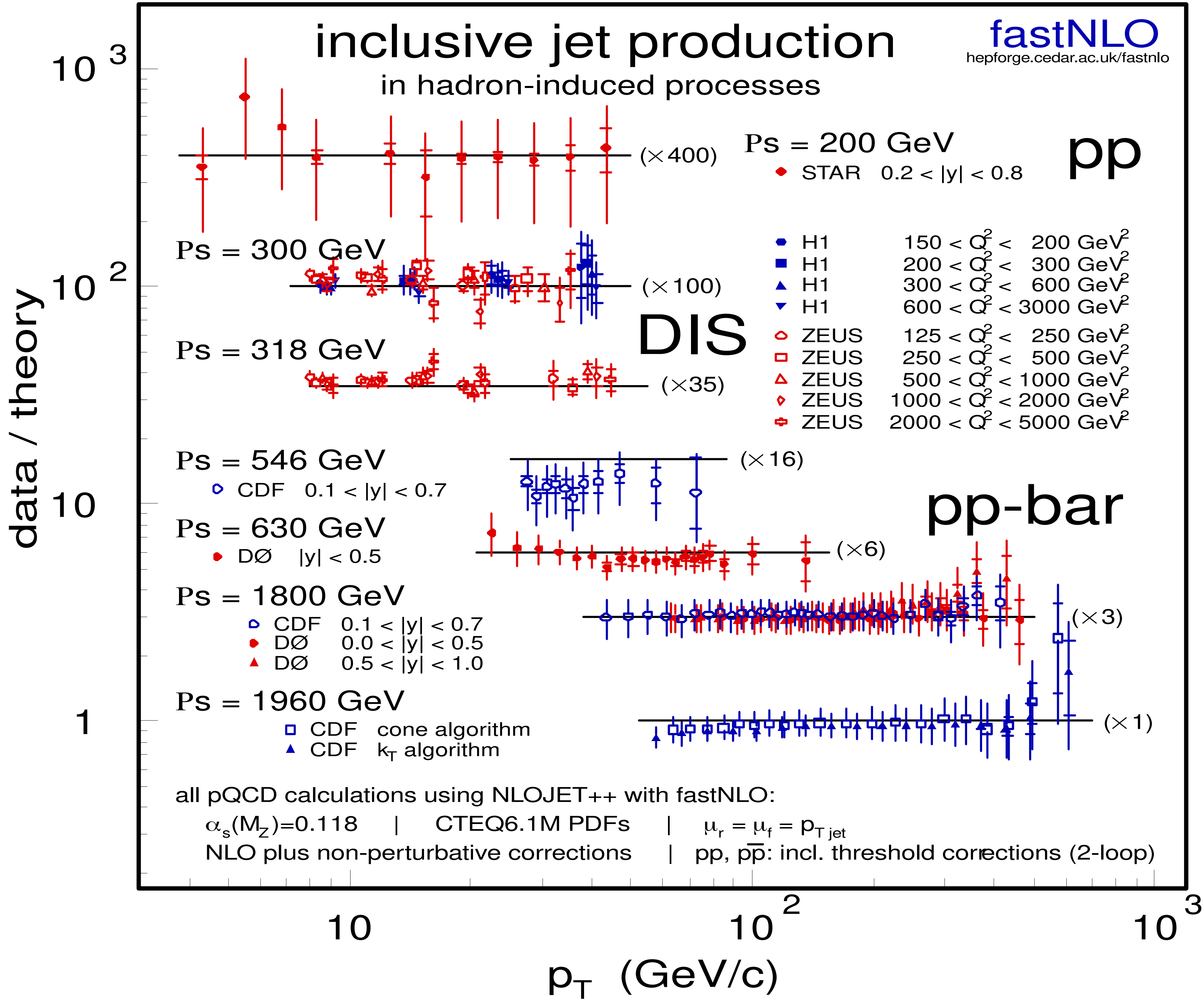
In many respects, our most complex
- asymptotic freedom
- strong color confinement
- … associated manifestations



The Lagrangian fits on a coffee cup …

How complex could it be???

QCD does a remarkable job!!!

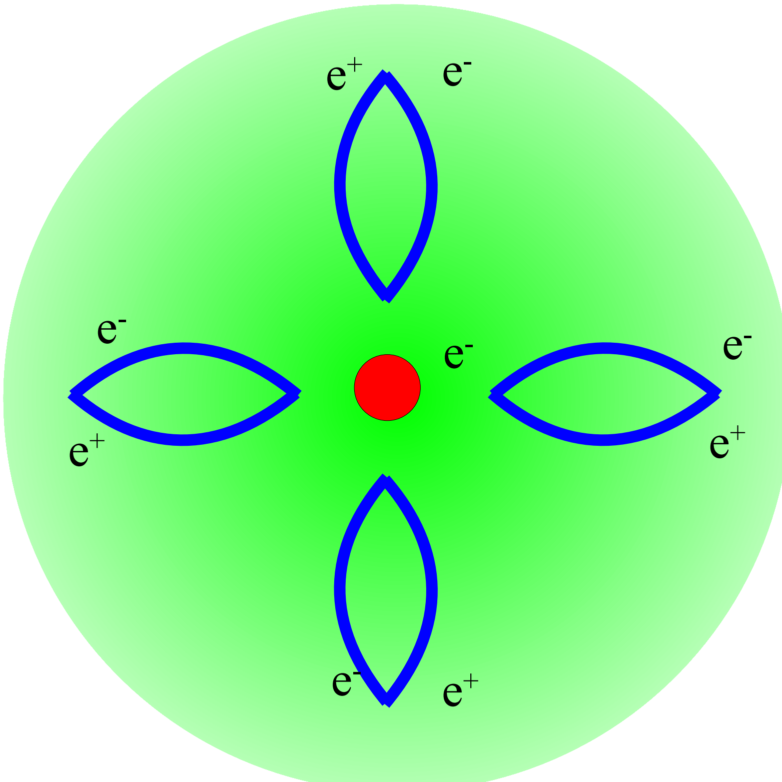


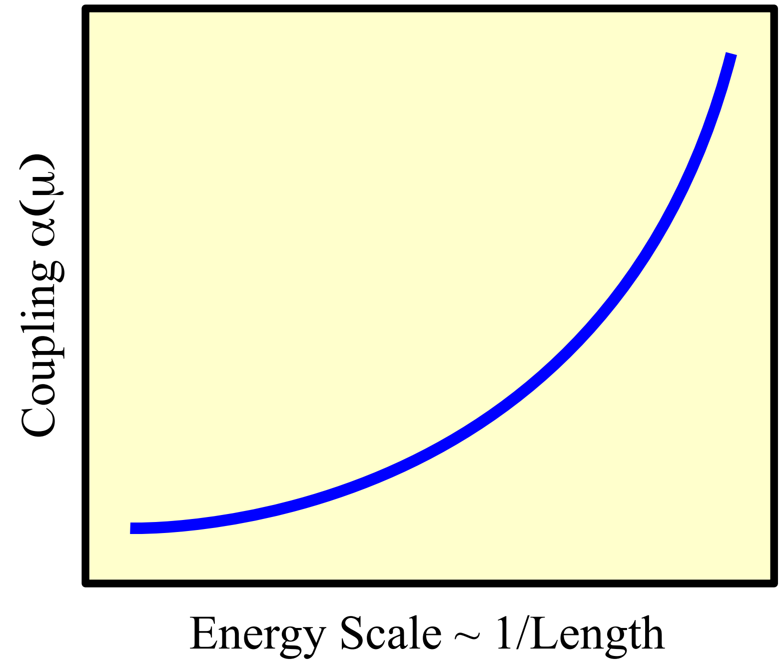
QCD is just like QED,

.... *only different*

QCD is just like QED, only different ...

QED: Abelian U(1) Symmetry





Perturbation theory at large distance is convergent

$$\alpha(\infty) \sim \frac{1}{137}$$

$$\alpha(M_Z) \sim \frac{1}{128}$$

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu} D_{\mu} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;,$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - 0$$

Abelian

α_{QCD} is good expansion parameter

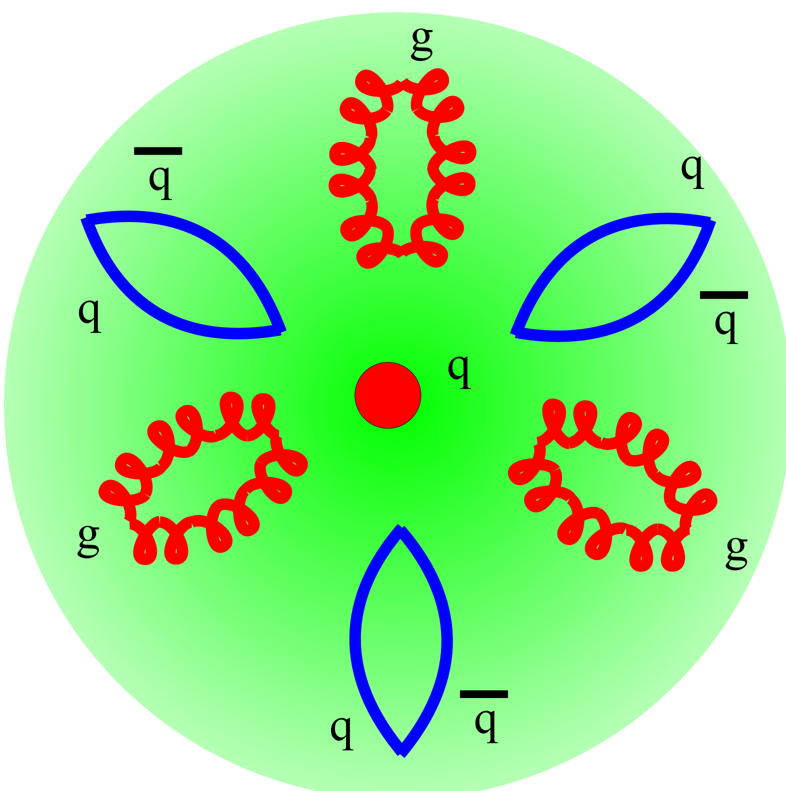
QCD is Non-Abelian SU(3) Symmetry; Quarks are Confined!!!

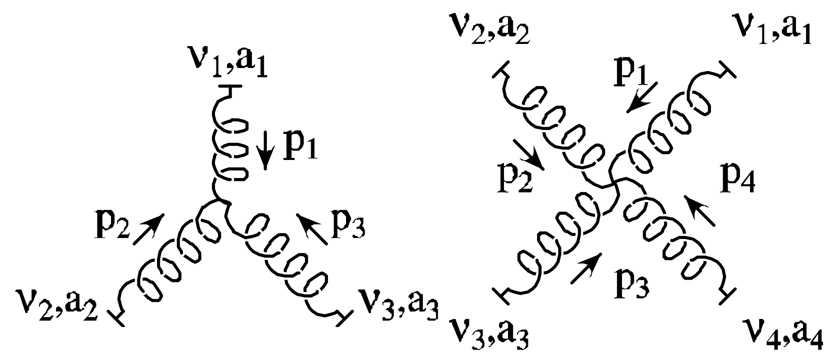
$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

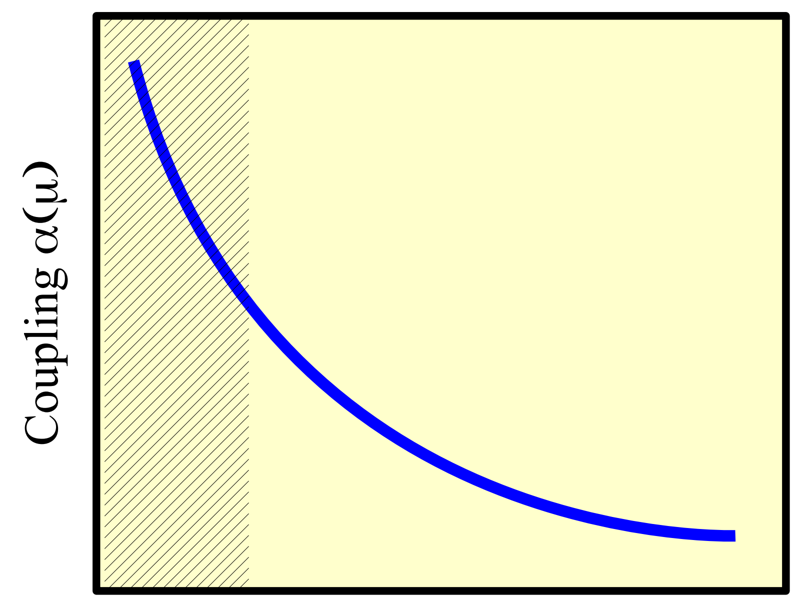
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

Non-Abelian

$$[T_a, T_b] = i \sum_{c=1}^{8} f_{abc} T_c$$





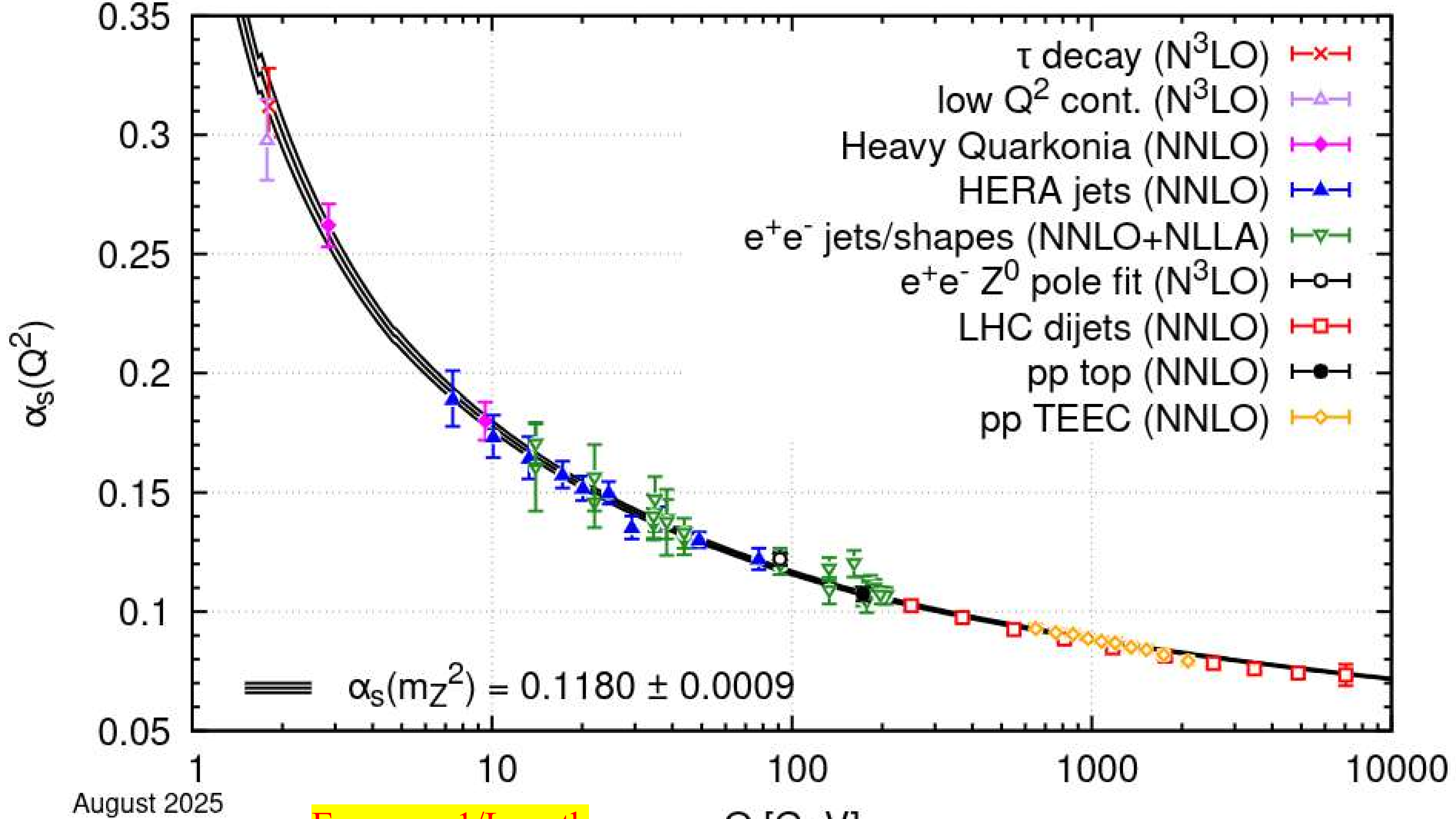


Energy Scale ~ 1/Length

$$\alpha_s(M_Z) \sim 0.118$$

Cannot perform Perturbation theory at low energy scales

Comparison with data: Strong Coupling Constant



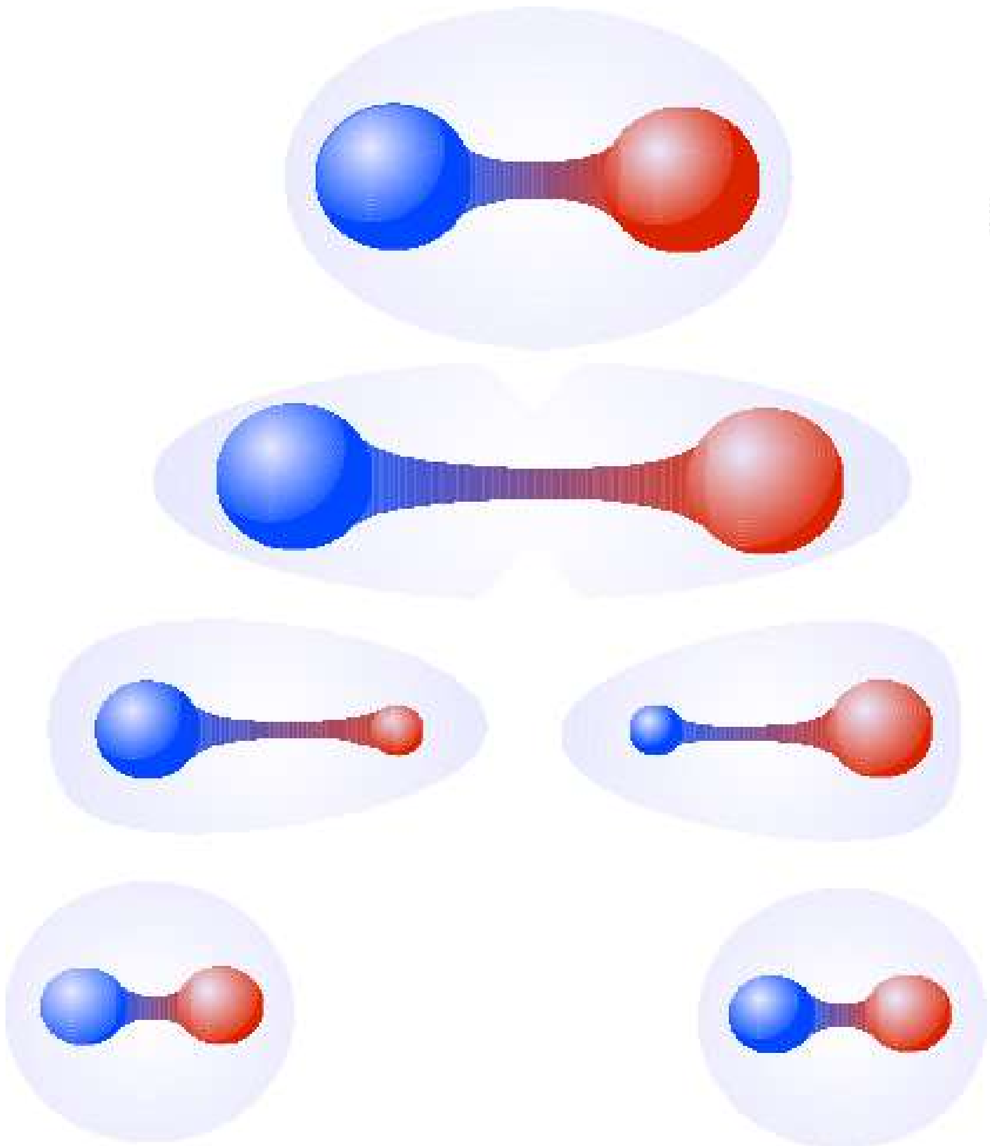
Energy ~ 1/Length

Low Q points have more discriminating power

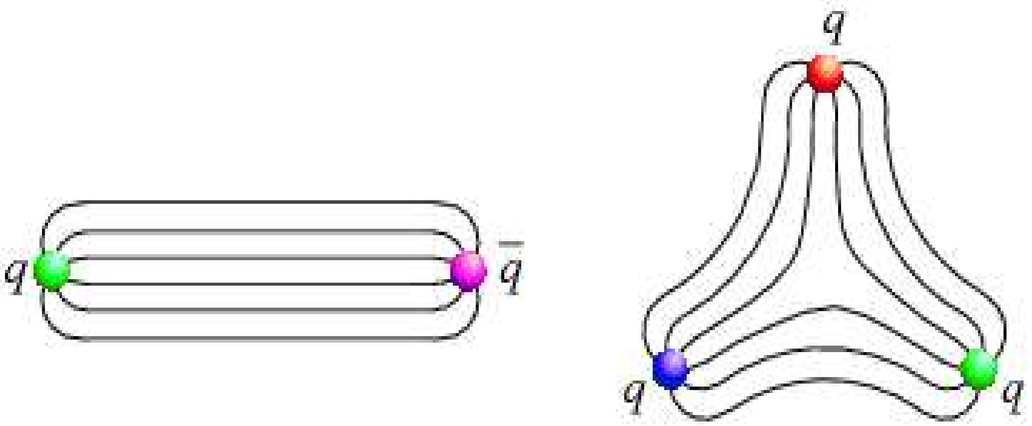
$$\alpha_s(M_Z) = 0.118$$

Caution: α_s is NOT a physical observable

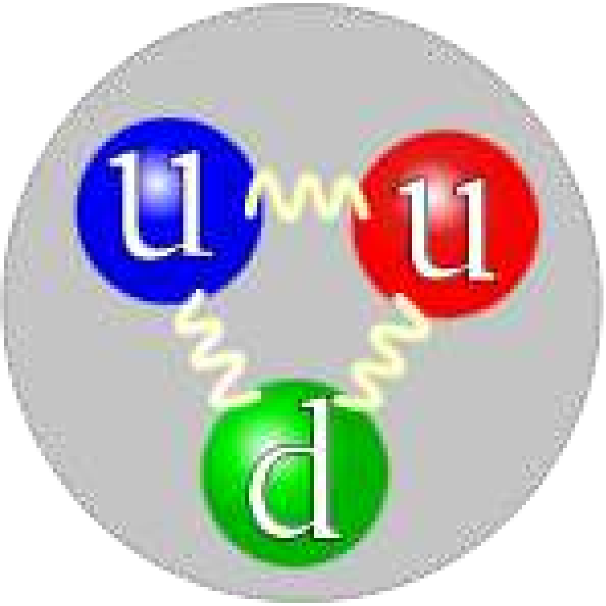
Quark Confinement & String Interpretation



Thomas Lippert, NIC-ZAM, Jülich, for the SESAM Collaboration)



http://www.scholarpedia.org/



http://en.wikipedia.org/

Quarks are confined

Statement of the problem

Theorist #1: The universe is completely described by the symmetry group SO(10)

Theorist #2: You're wrong; the correct answer is SuperSymmetric flipped SU(5)xU(1)

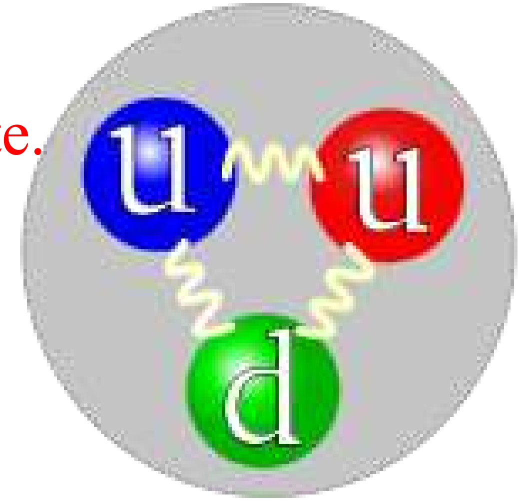
Theorist #3: You've flipped! The only rational choice is E8xE8 dictated by SuperString Theology.

Experimentalist: Enough of this speculative nonsense. I'm going to measure something to settle this question. What can you predict???

Theorist #1: We can predict the interactions between fundamental particles such as quarks and leptons.

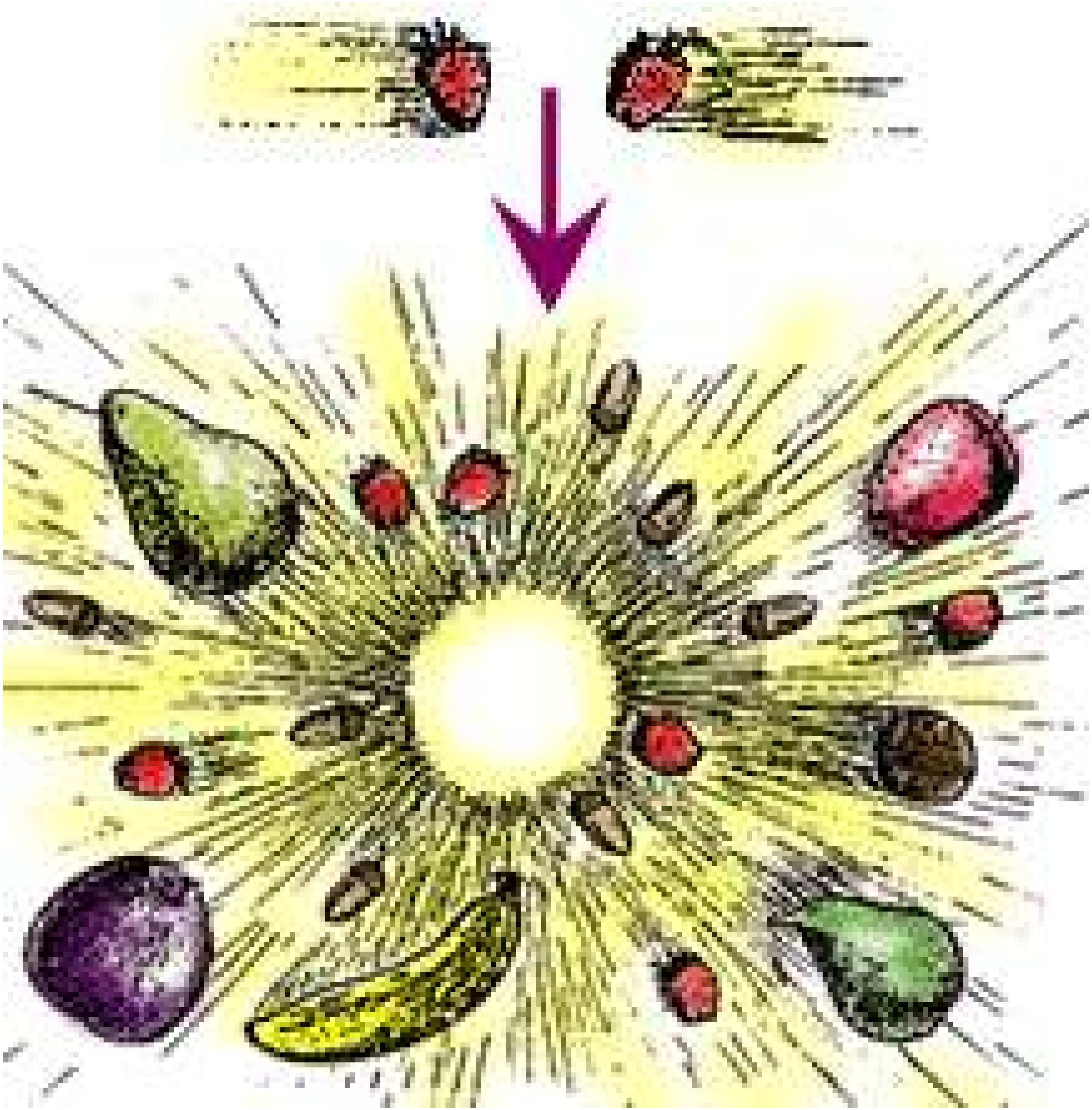
Experimentalist: Great! Give me a beam of quarks and leptons, and I can settle this debate.

Accelerator Operator: Sorry, quarks only come in a 3-pack and we can't break a set!



http://en.wikipedia.org/

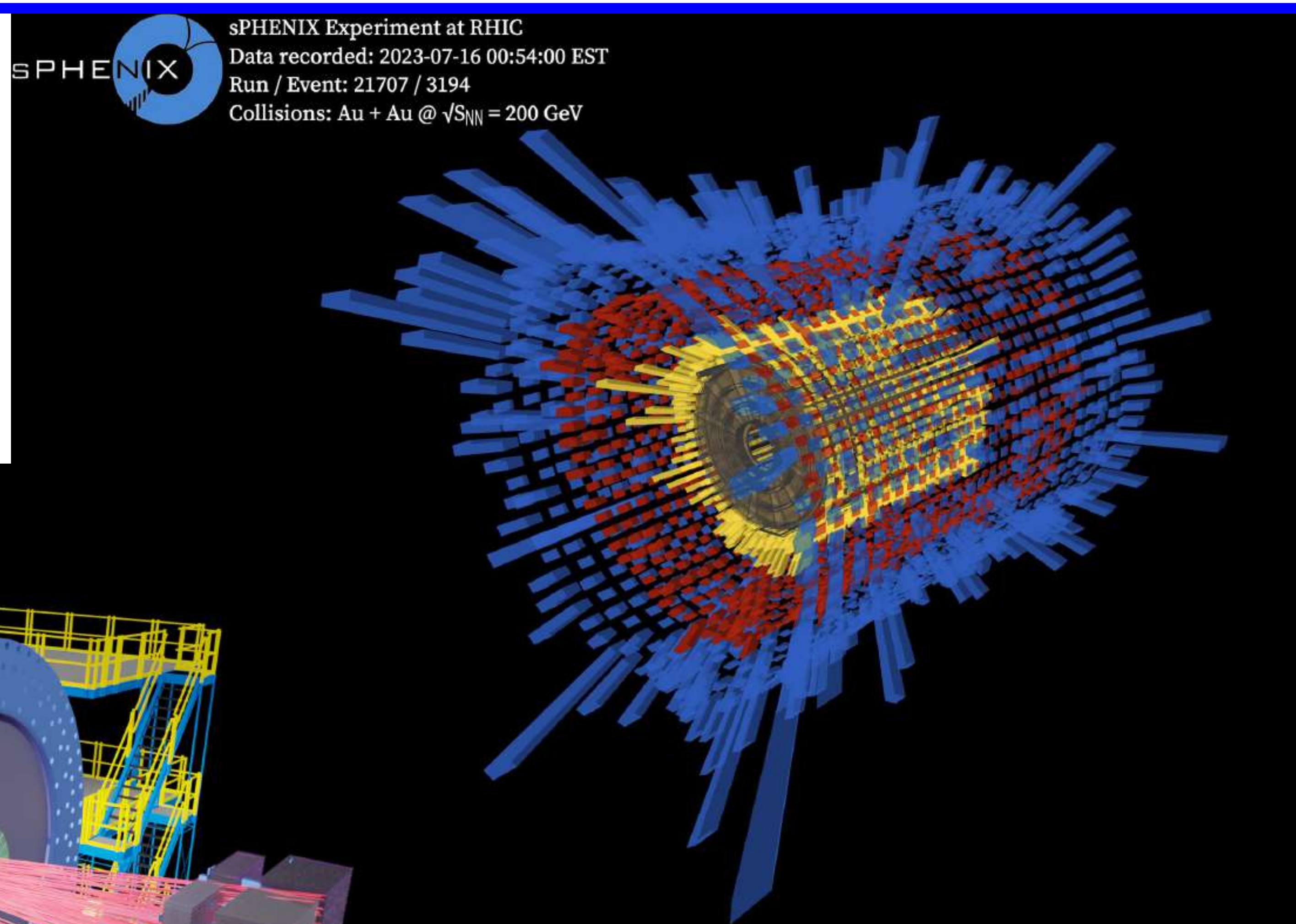
One interpretation of a hadron-hadron collision

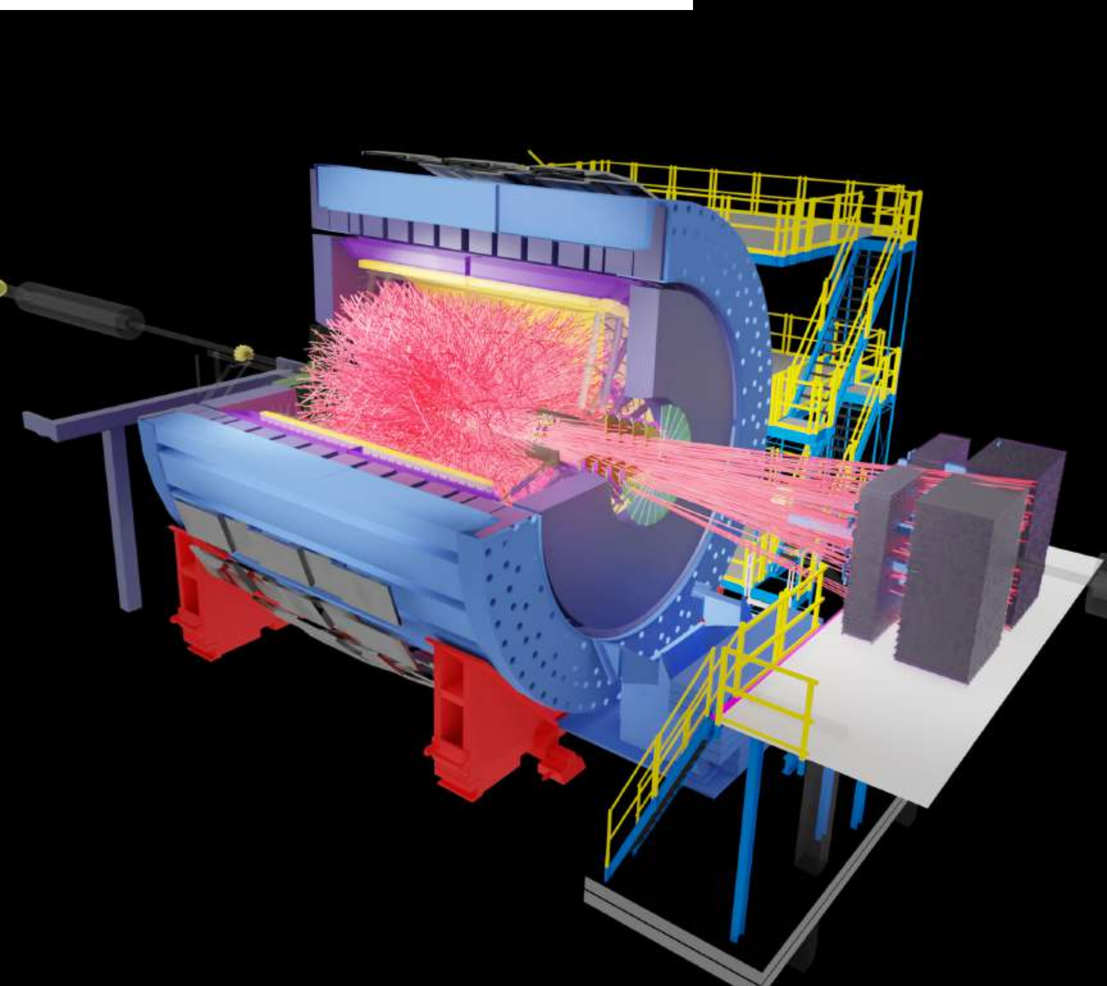


Did we find the Higgs?

A bit more realistic interpretation of a hadron-hadron collision

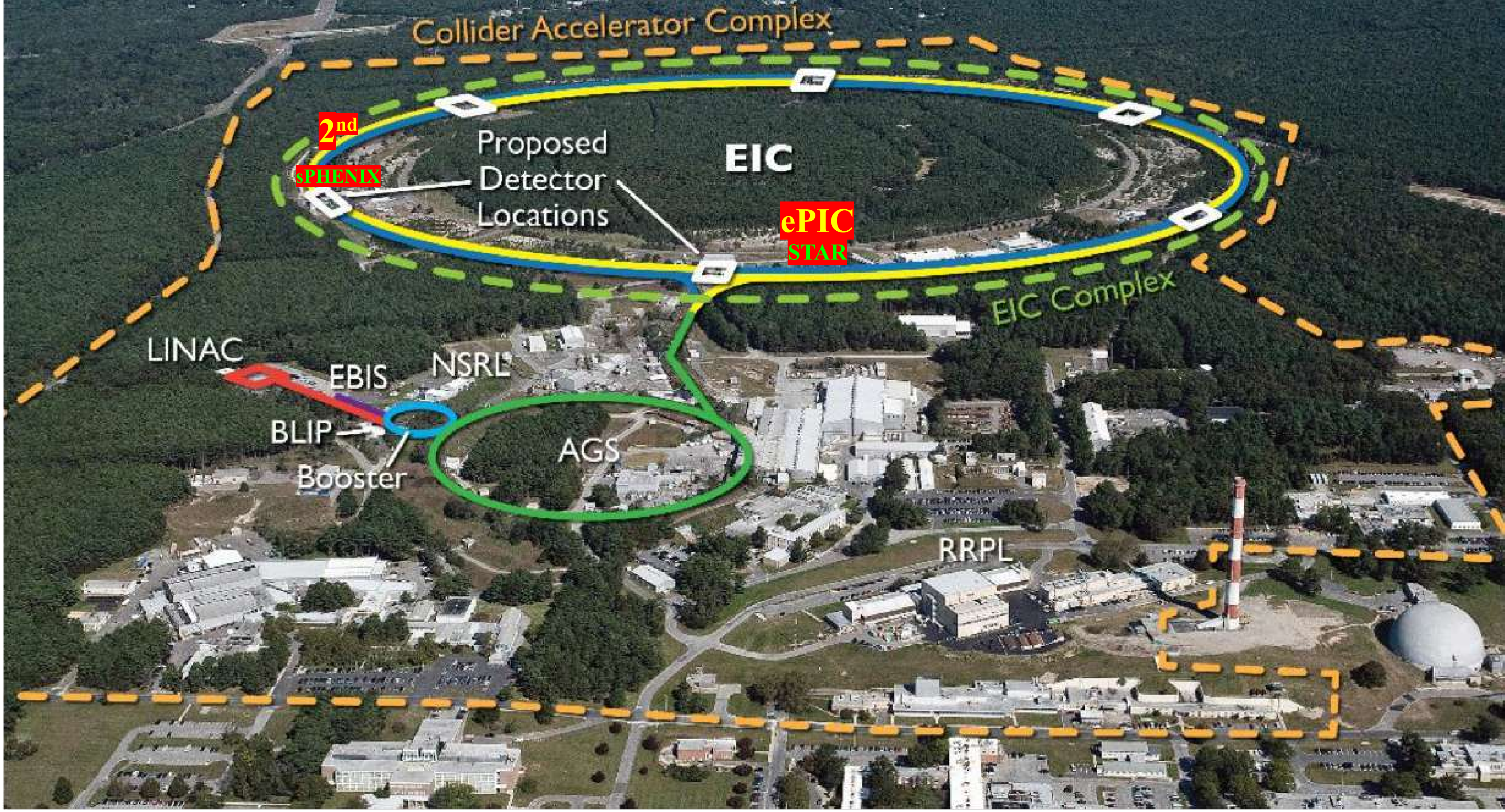
sPHENIX





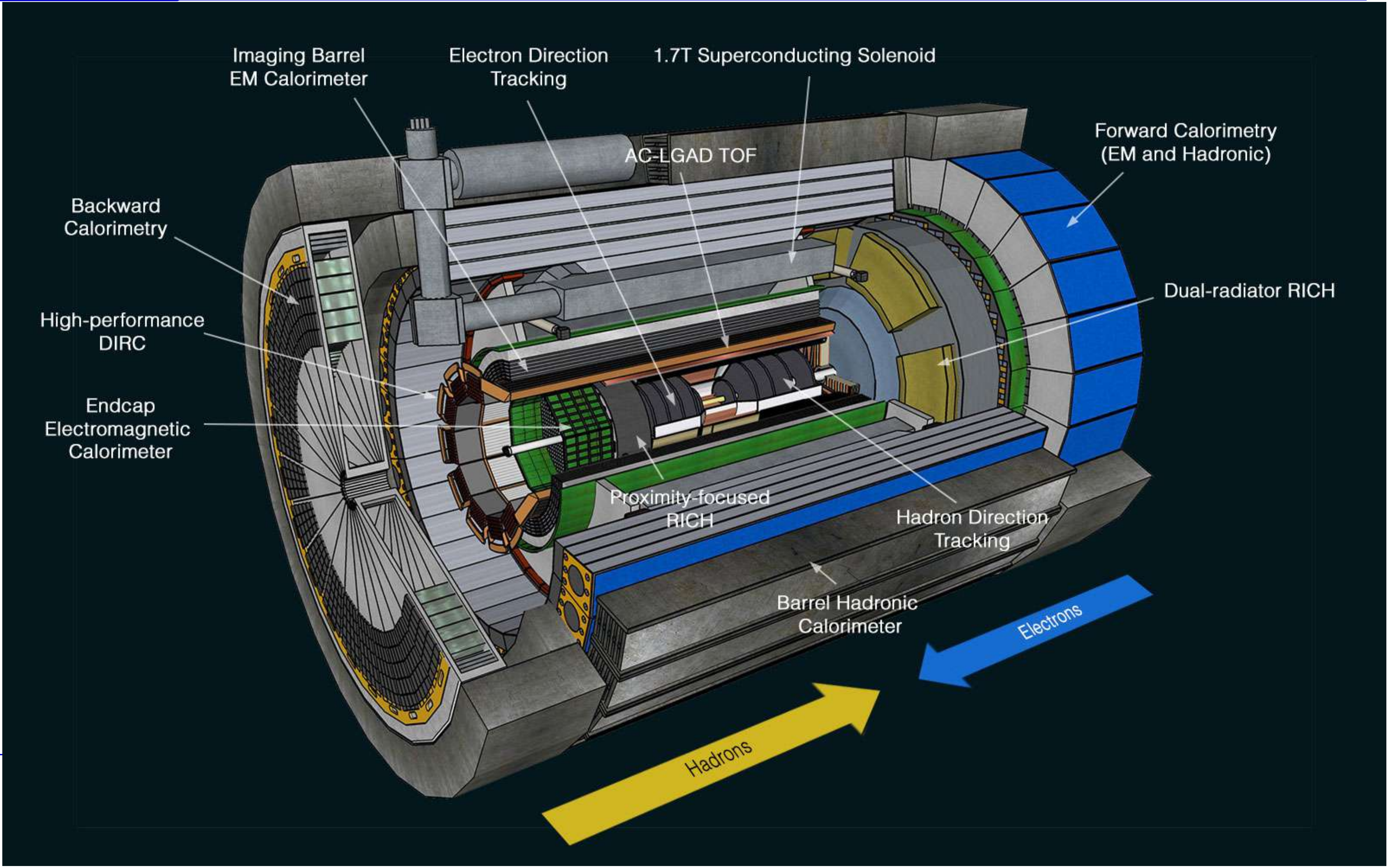
STAR Detector

Electron Ion Collider at BNL



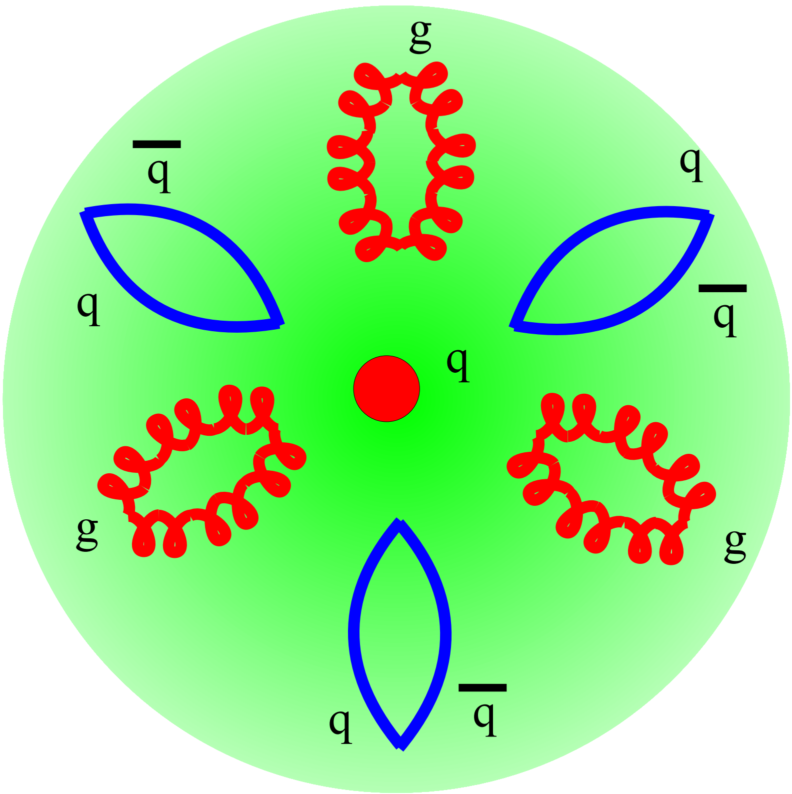
- EIC benefits from large investments at BNL and the highly successful RHIC program
- **RHIC concluded operations in 2026**
- EIC installation will begin after RHIC operation concludes

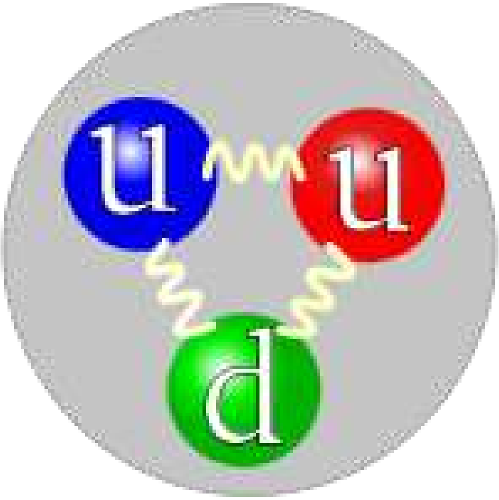
THE COMPLETE ePIC DETECTOR



Scaling, and the proton structure

How do we determine the proton structure





http://en.wikipedia.org/

Quarks confined, thus we must work with hadrons & mesons

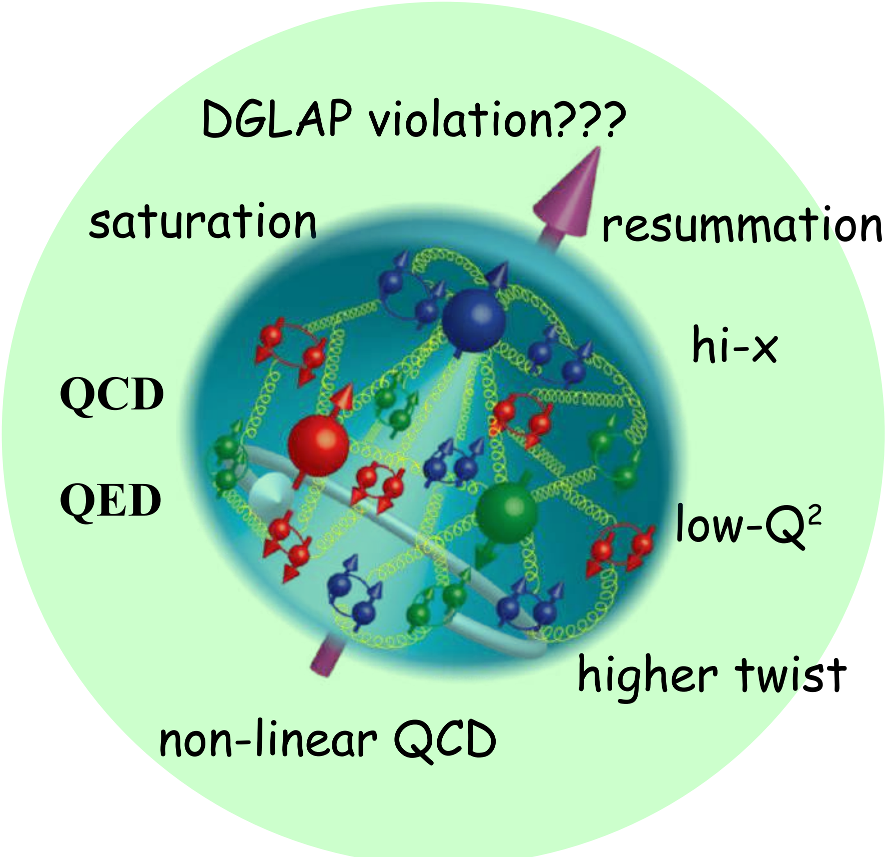
E.g, proton is a "minimal" unit

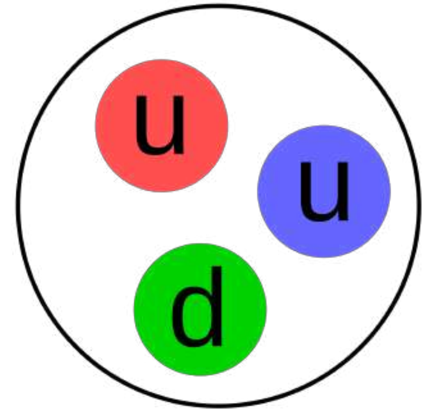
Highest energy (smallest distance) accelerators involve hadrons

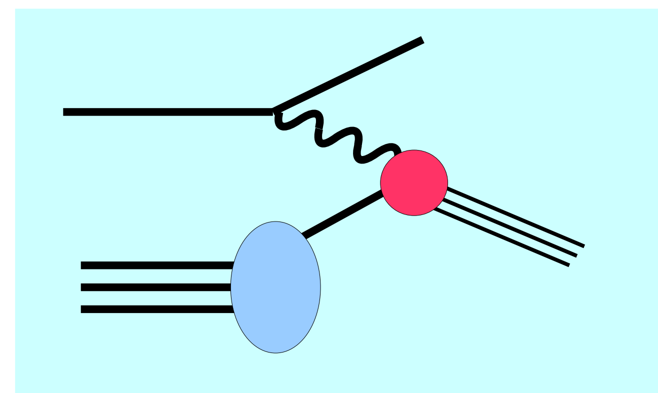
E.g., HERA, TEV, JLab, RHIC, LHC, EIC, ...

We'd better learn to work with proton

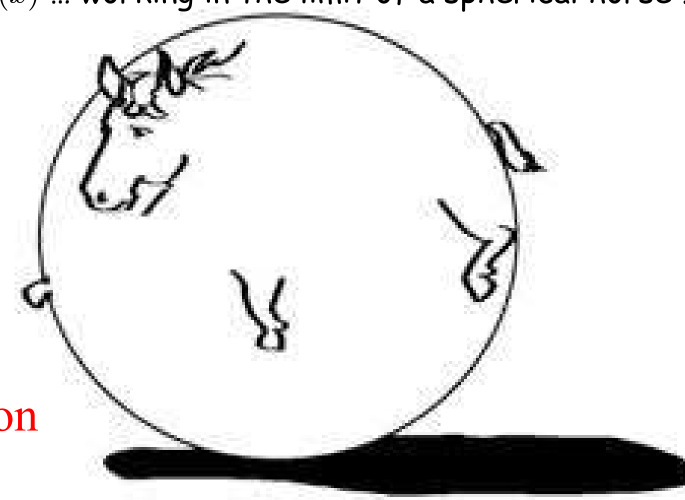
PDFs & the Parton Model … *a few simpifications*







$f_a(x)$ … working in the limit of a spherical horse …



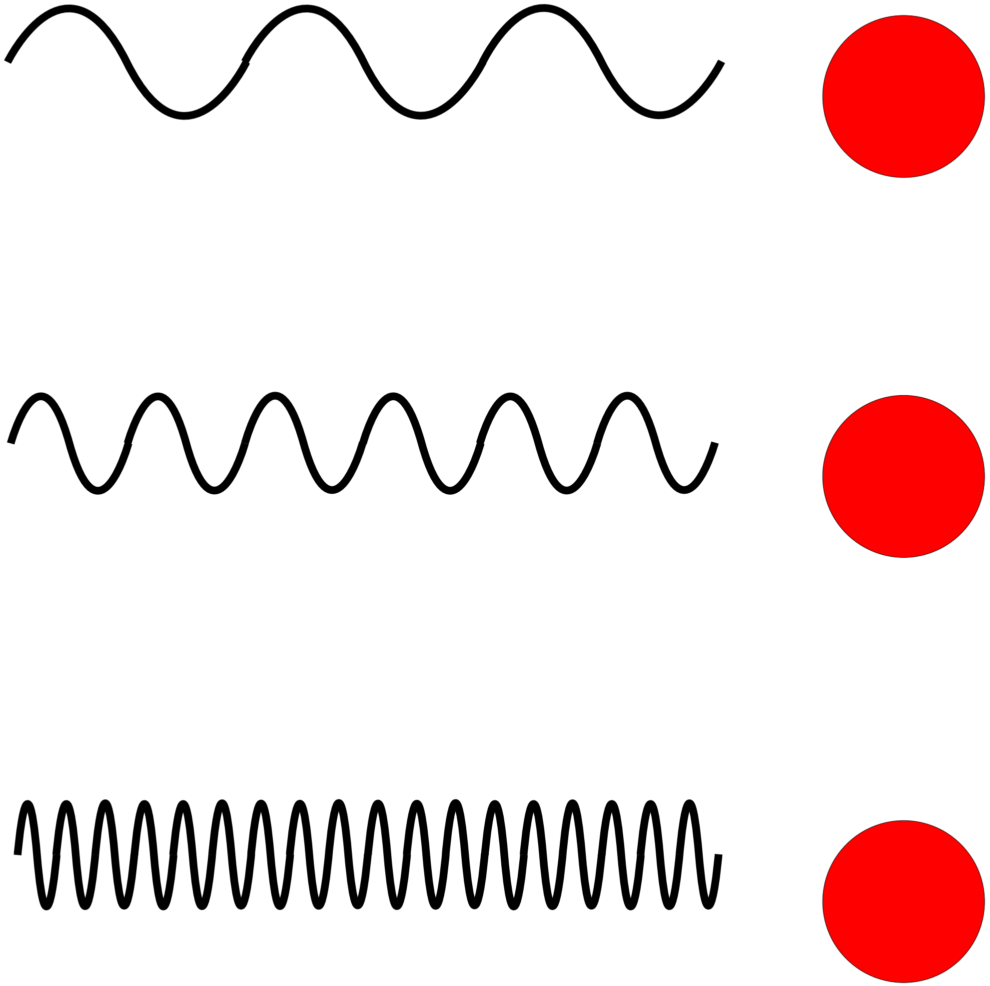
The QCD Parton Model

$$d\sigma = f_a(x) \otimes \widehat{\sigma}$$

Parameterized in terms of a single variable x, the momentum fraction

… use DGLAP to determine μ dependence

What do we expect for a point like particle



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$

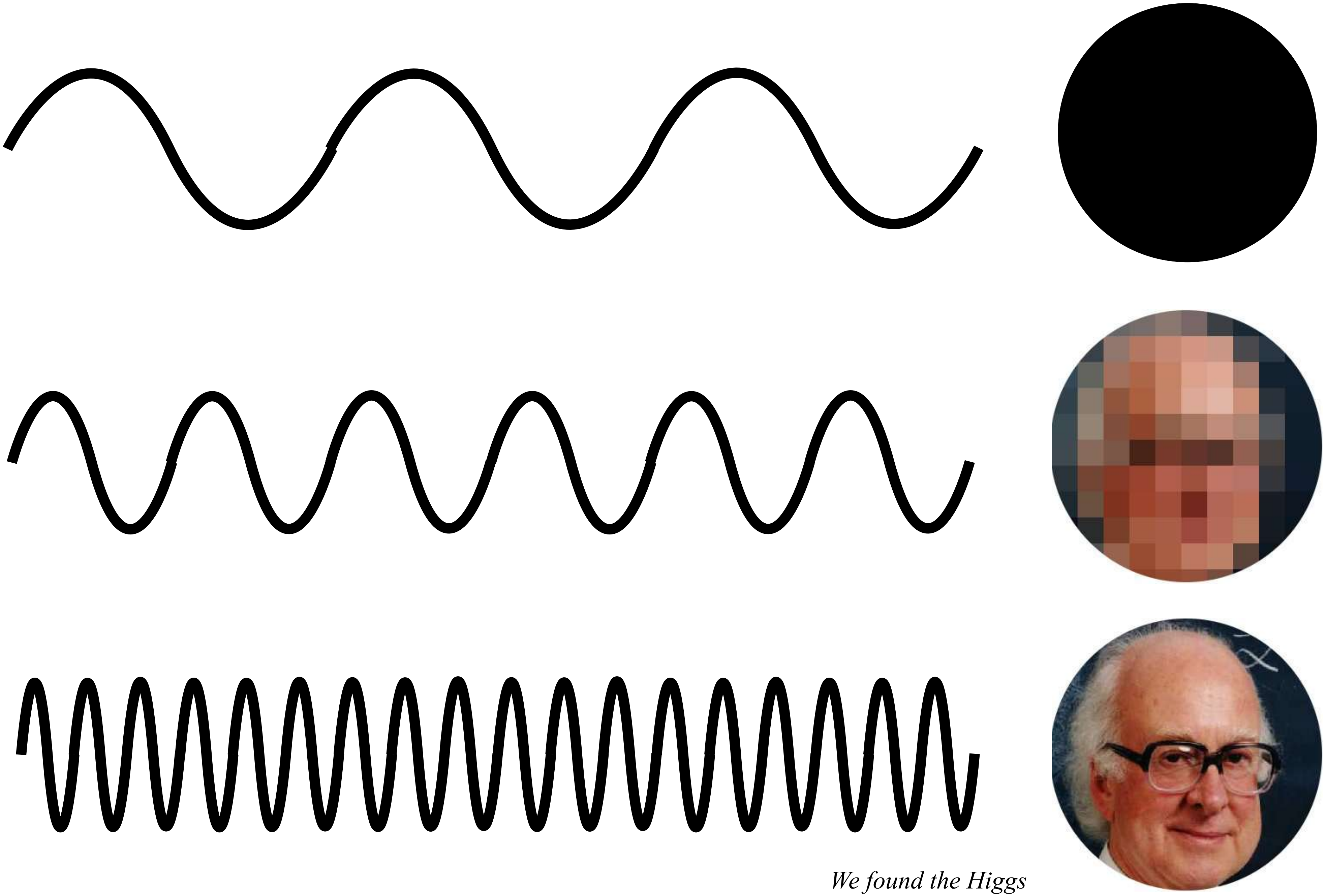
$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$

$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$

Dimensional considerations

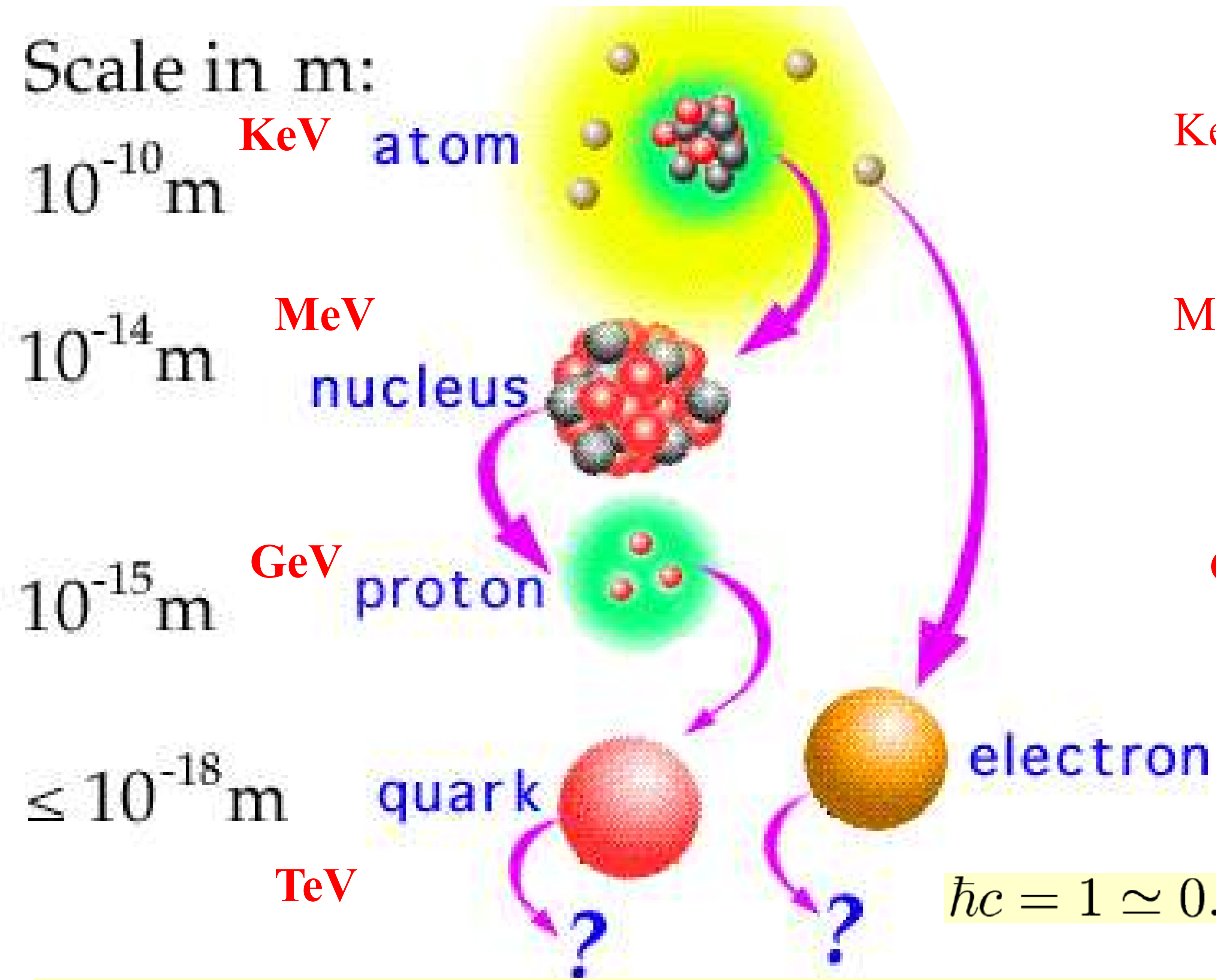
Structure Function

Is this a point like particle ???



We found the Higgs

Relative Sizes



KeV $\sim 10^{-9}$

MeV $\sim 10^{-12}$

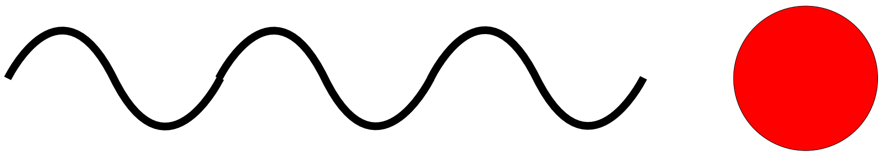
GeV $\sim 10^{-15}$

TeV $\sim 10^{-18}$

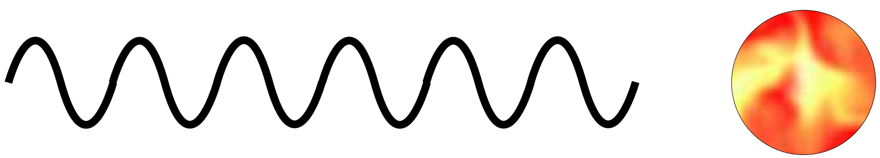
$$\hbar c = 1 \simeq 0.2 \text{ GeV fm}$$

Going to smaller scale, we get to <u>simpler</u>, more fundamental objects

Structure of the Proton

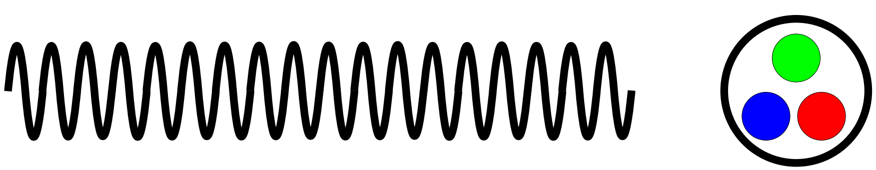


$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times 1$$



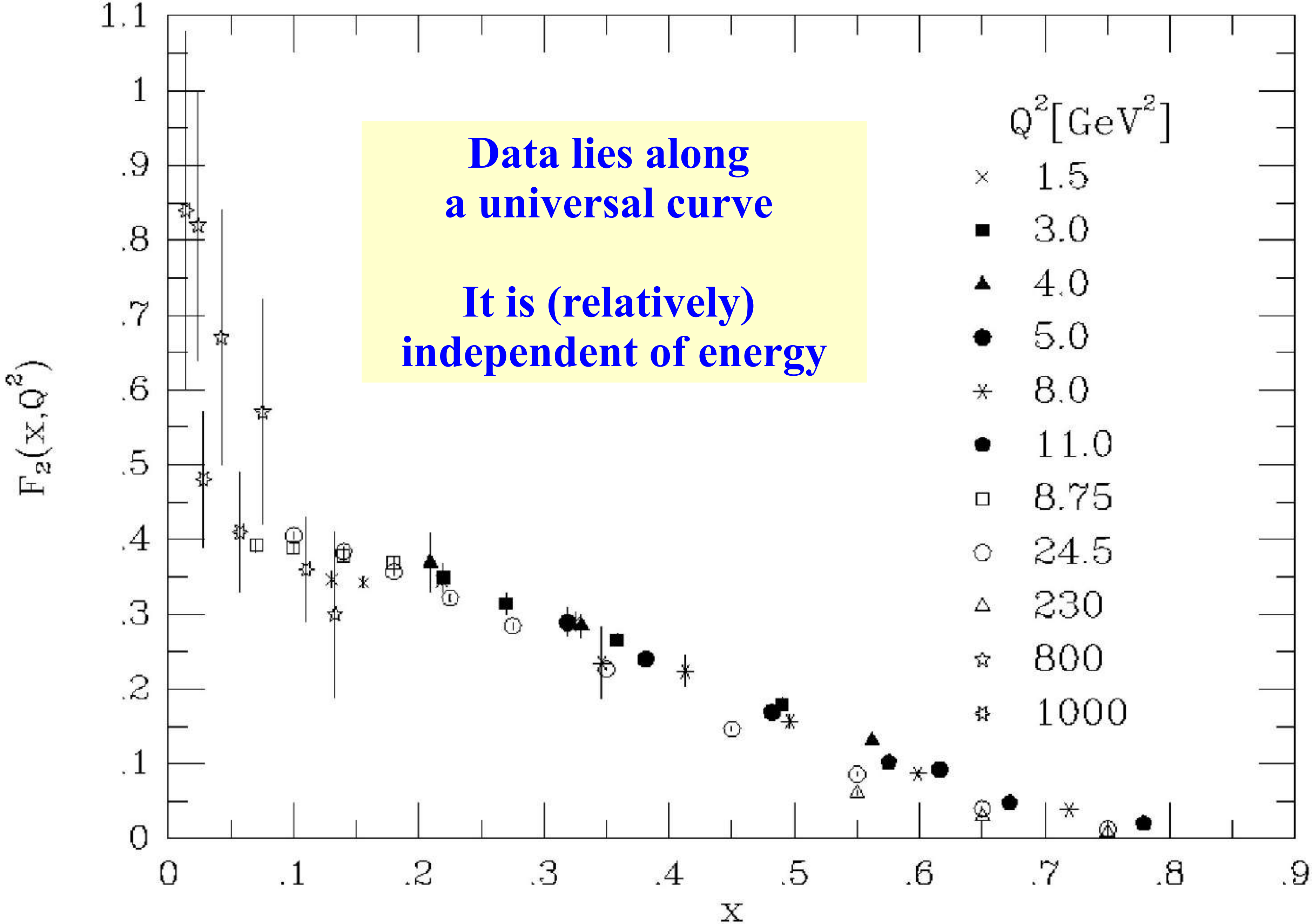
$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times F\left(\frac{Q^2}{\Lambda^2}\right)$$

Λ of order of the proton mass scale



$$d\sigma \sim \frac{4\pi\alpha^2}{Q^2} \times \sum_i e_i^2$$

The Scaling of the Proton Structure Function



Data lies along a universal curve

It is (relatively) independent of energy

Recap Part 1: ... so far

QCD is just like QED, ... only different

QCD is non-Abelian, Quarks are confined,

Running coupling $\alpha_s(\mu)$ tells how interaction changes with distance

β-function: logarithmic derivative of $\alpha_s(\mu)$

We can compute: Negative for QCD, positive for QED

$\alpha_s(\mu)$ is **not** a physical quanitity

Discontinous at NNLO

New physics can influence $\alpha_s(\mu)$

Unification of couplings at GUT scale

Running of $\alpha_s(\mu)$ can help us "resum" perturbation theory

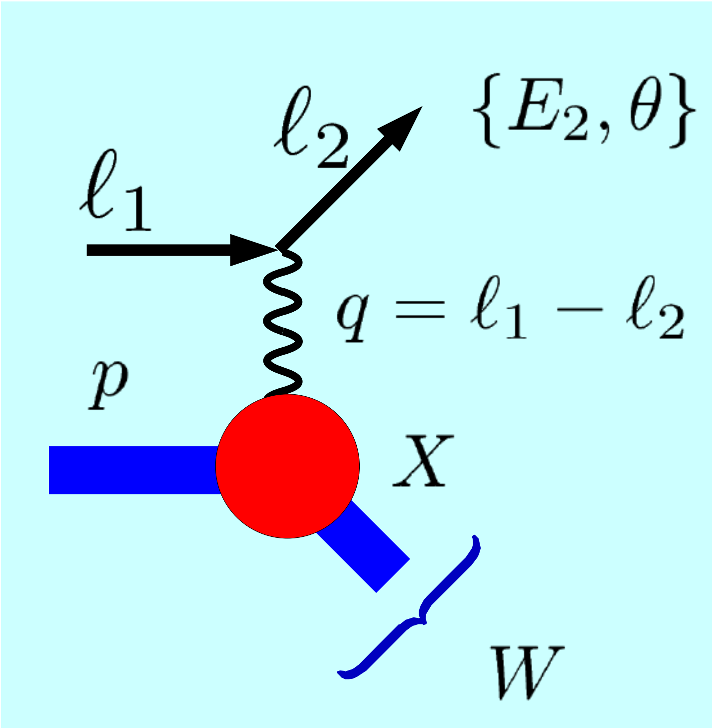
Scaling and Dimensional Analysis are useful tools

HOW TO CHARACTERIZE THE PROTON

Deeply Inelastic Scattering

(DIS)

Inclusive Deeply Inelastic Scattering (DIS)

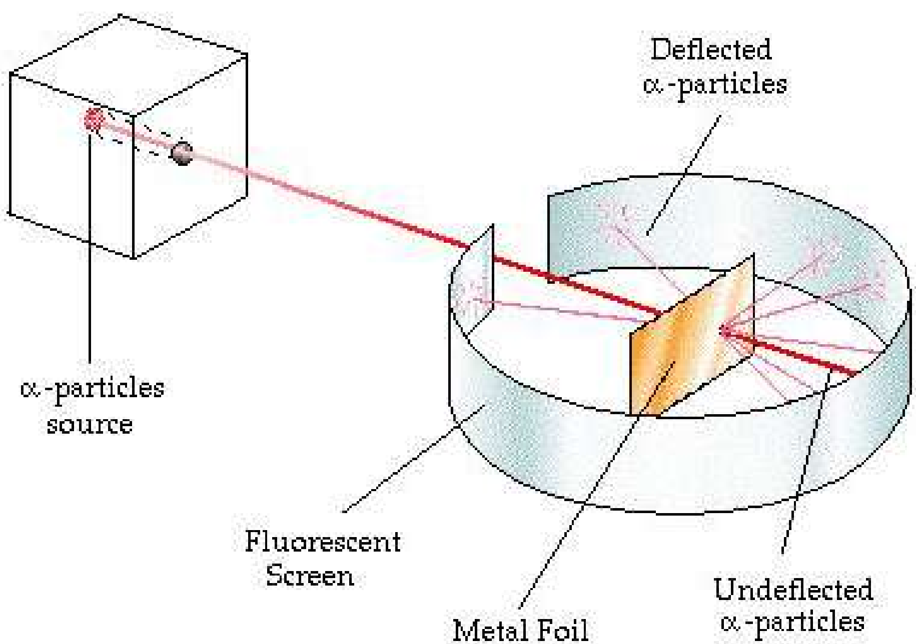


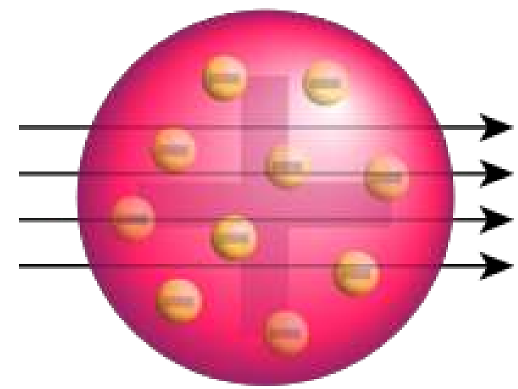
Measure $\{E_2, \theta\} \Leftrightarrow \{x, Q^2\}$ Inclusive

Deep: $Q^2 \geq 1 GeV^2$

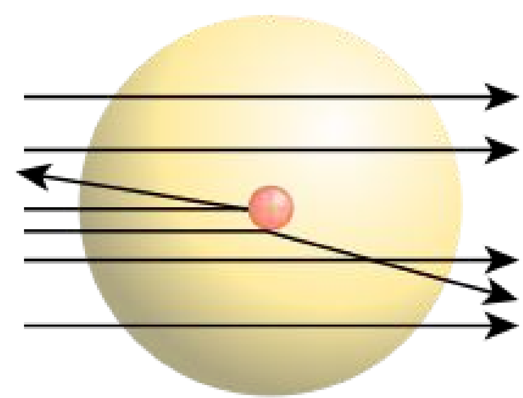
Inelastic: $W^2 \geq M_p^2$

Analogue of Rutherford scattering



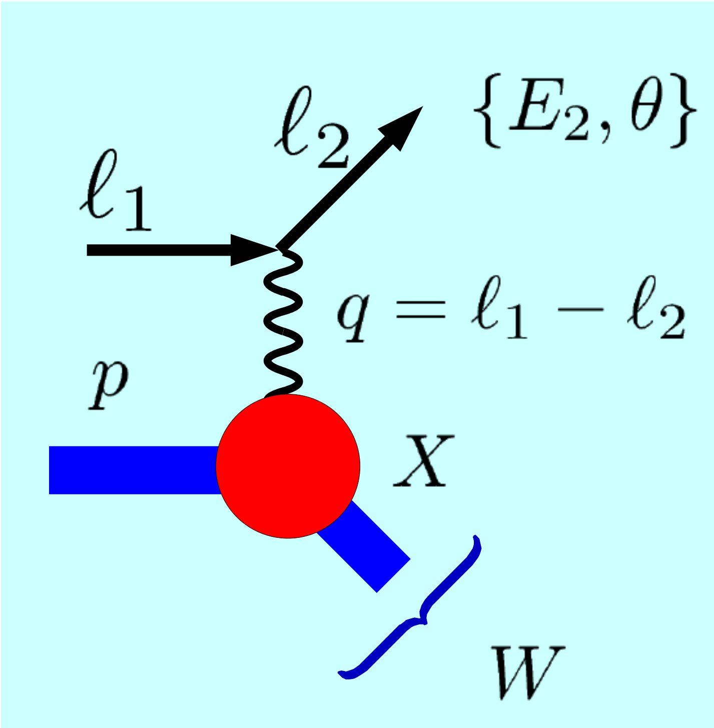


Hard center
Soft outside



Soft center
Hard outside

Inclusive Deeply Inelastic Scattering (DIS)



Measure $\{E_2, \theta\} \Leftrightarrow \{x, Q^2\}$

$$Q^2 = -q^2 = 4E_1 E_2 \sin^2(\theta/2)$$

$$x = \frac{Q^2}{2p \cdot q} = \frac{2E_1 E_2 \sin^2(\theta/2)}{M(E_1 - E_2)}$$

$$d\sigma \sim |A|^2$$

We only measure the outgoing lepton

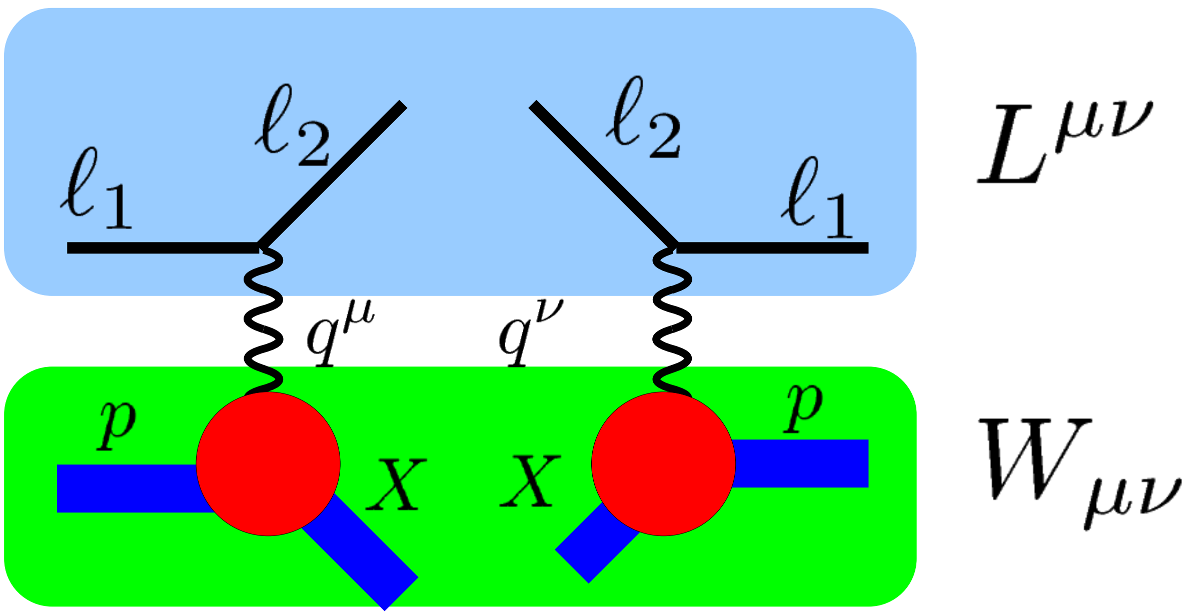
2 DOF

Other common DIS variables

$$\nu = \frac{p \cdot q}{p^2} = E_1 - E_2$$

$$y = \frac{\nu}{E_1} = \frac{Q^2}{2ME_2 x}$$

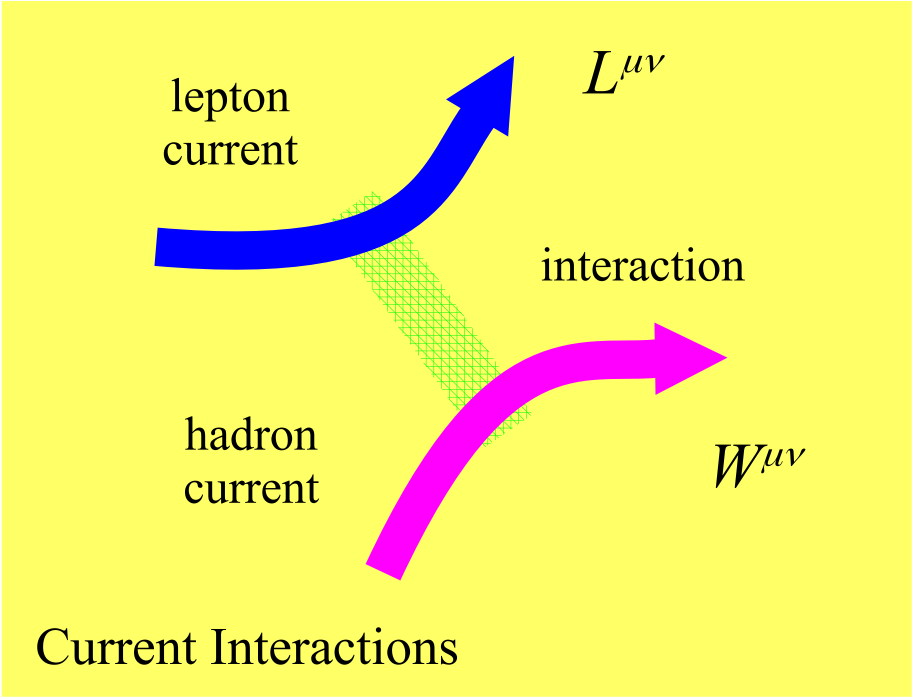
Lepton Tensor (L) and Hadronic Tensor (W)



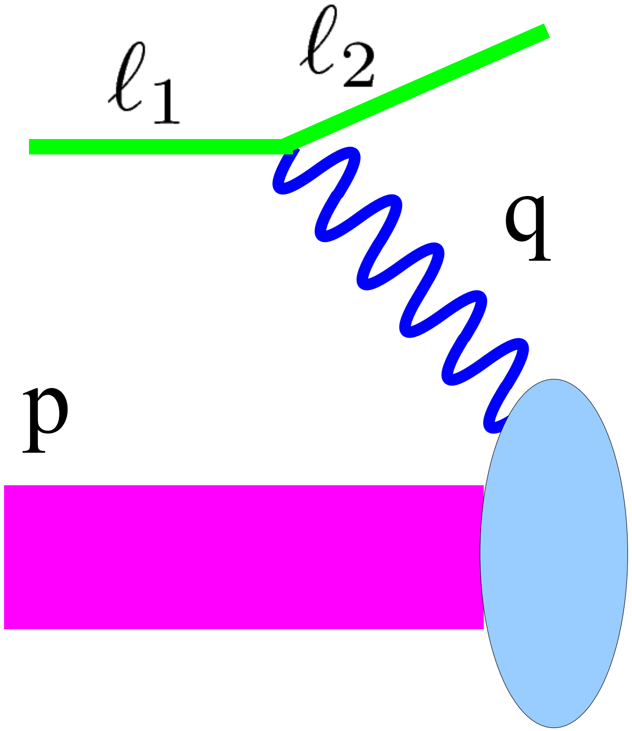
Leptonic Tensor

Hadronic Tensor

$$d\sigma \sim |A|^2 \sim L^{\mu\nu} W_{\mu\nu}$$



W and F Structure Functions



$$d\sigma \sim |A|^2 \sim L^{\mu\nu} W_{\mu\nu}$$

$$L^{\mu\nu} = L^{\mu\nu}(\ell_1, \ell_2)$$

$$W^{\mu\nu} = W^{\mu\nu}(p, q)$$

$$W^{\mu\nu} = -g^{\mu\nu} W_1 + \frac{p^\mu p^\nu}{M^2} W_2 - \frac{i\,\epsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma}{2M^2} W_3 + \ldots$$

Convert to “Scaling” Structure Functions

$$W_1 \to F_1 \qquad W_2 \to \tfrac{M}{\nu} F_2 \qquad W_3 \to \tfrac{M}{\nu} F_3$$

$$\frac{d\sigma}{dx\,dy} = N\left[xy^2 F_1 + (1 - y - \tfrac{Mxy}{2E_2})F_2 \pm y(1 - y/2)xF_3\right]$$

Use Helicity Basis

$$\frac{d\sigma}{dx\,dy} = N\left[xy^2F_1 + (1 - y - \tfrac{Mxy}{2E_2})F_2 \pm y(1 - y/2)xF_3\right]$$

Taking the limit $M_{proton} \rightarrow 0$ for neutrino DIS

$$\frac{d\sigma^\nu}{dx\,dy} = N\left[(1-y)^2F_+ + 2(1-y)F_0 + F_-\right]$$

For $\bar{\nu}$, $F_+ \Leftrightarrow F_-$

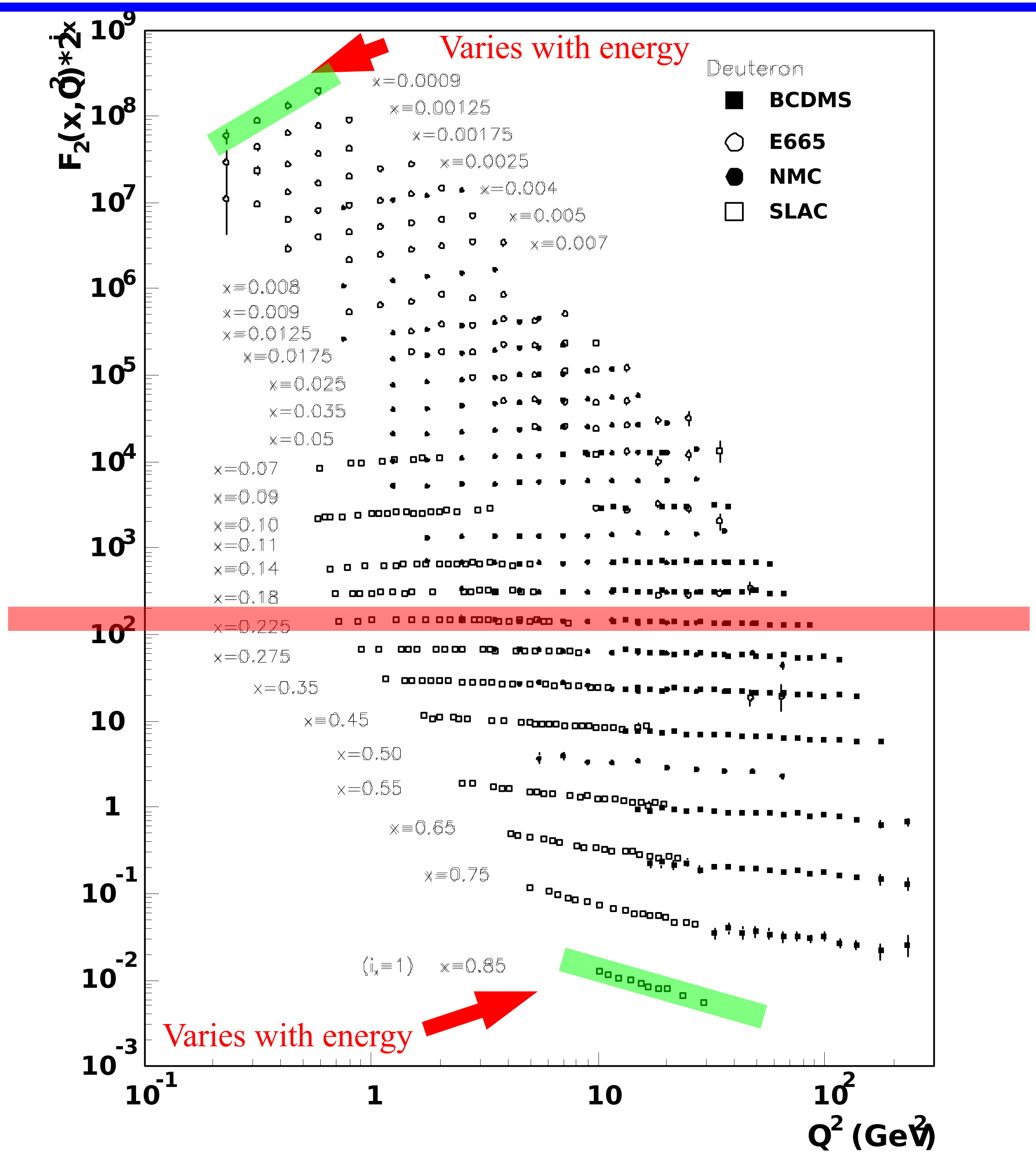
$$\begin{aligned} F_1 &= \tfrac{1}{2}(F_- + F_+) \\ F_2 &= x(F_- + F_+ + 2F_0) \\ F_3 &= (F_- - F_+) \end{aligned}$$

$$\begin{aligned} F_+ &= F_1 - \tfrac{1}{2}F_3 \\ F_- &= F_1 + \tfrac{1}{2}F_3 \\ F_0 &= \tfrac{1}{2x}F_2 - F_1 \end{aligned}$$

I have not yet mentioned the parton model!!!

A Review of Target Mass Corrections.
Ingo Schienbein et al.
J.Phys.G35:053101,2008.

The Scaling of the Proton Structure Function

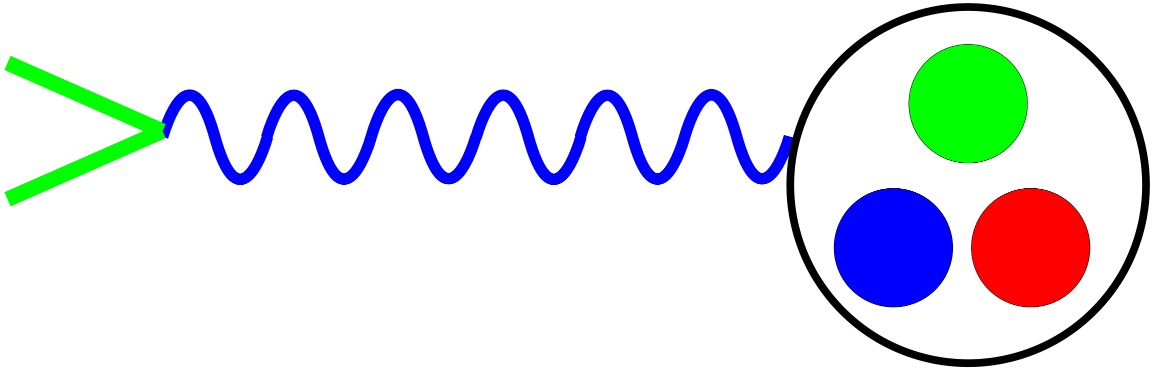


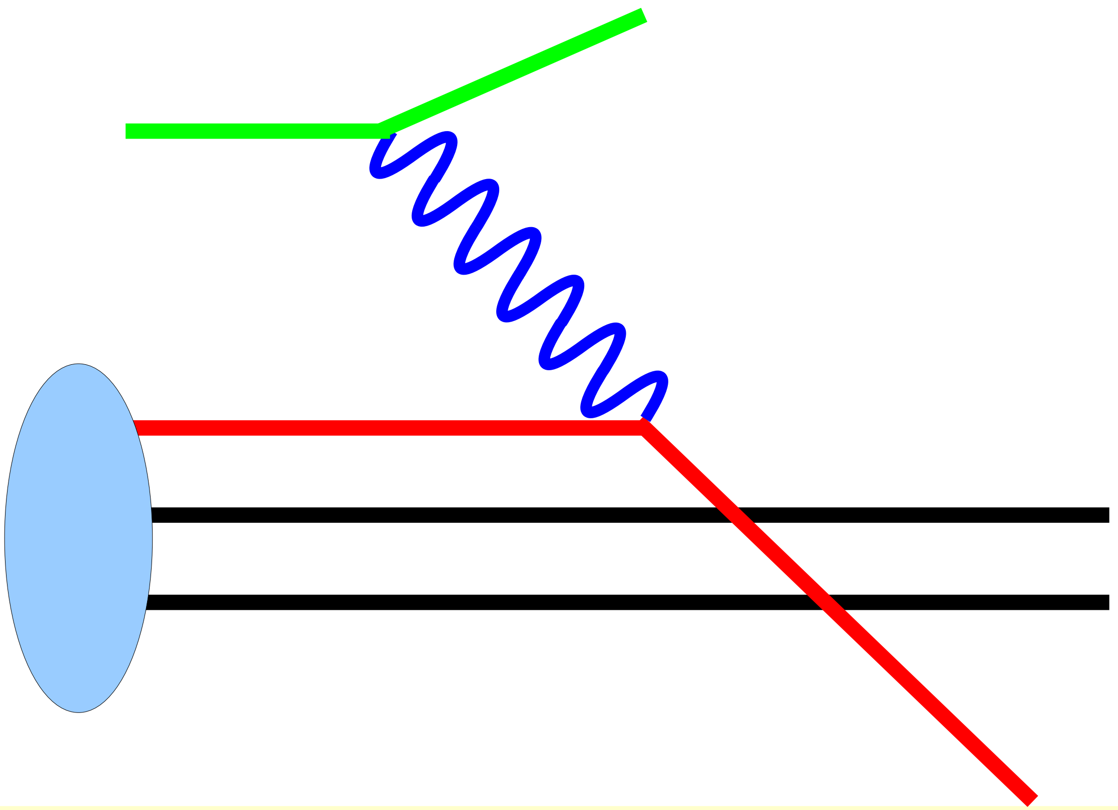
Data is (relatively) independent of energy

Scaling Violations observed at extreme x values

Parton Model

Proton as a bag of free Quarks

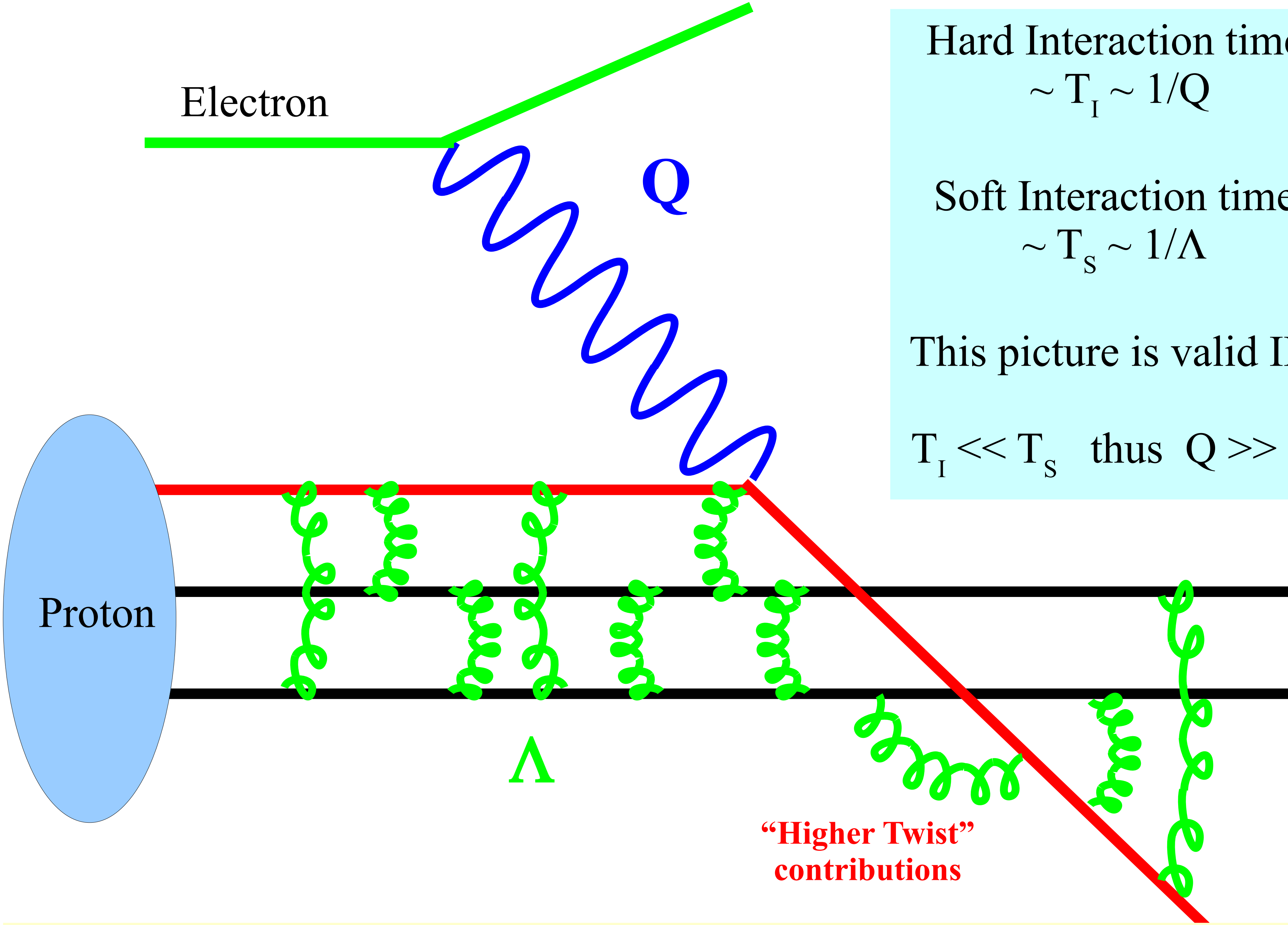




$$f(x,Q) = u(x,Q) + d(x,Q) = 2\ \delta(x - \tfrac{1}{3}) + 1\ \delta(x - \tfrac{1}{3})$$

Fred's PDFs

Quarks are not quite free



Hard Interaction time
$\sim T_I \sim 1/Q$

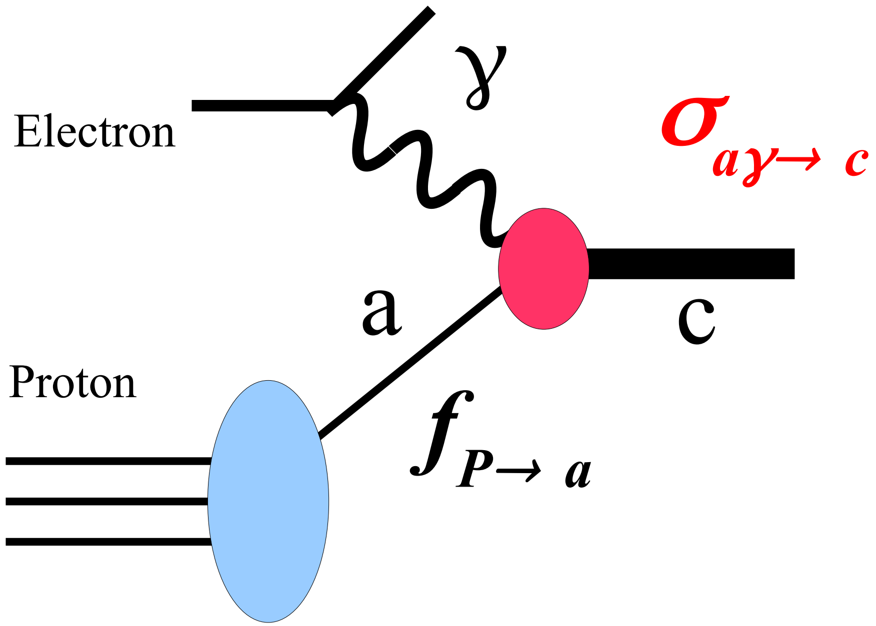
Soft Interaction time
$\sim T_S \sim 1/\Lambda$

This picture is valid IF:

$T_I \ll T_S$ thus $Q \gg \Lambda$

Corrections to this picture (non-factorizable/ higher twist) terms are suppressed by powers of Λ/Q

The Parton Model and Factorization



Parton Distribution Functions (PDFs) $f_{P\to a}$ are the key to calculations involving hadrons!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \hat{\sigma}_{a\gamma\to c}$$

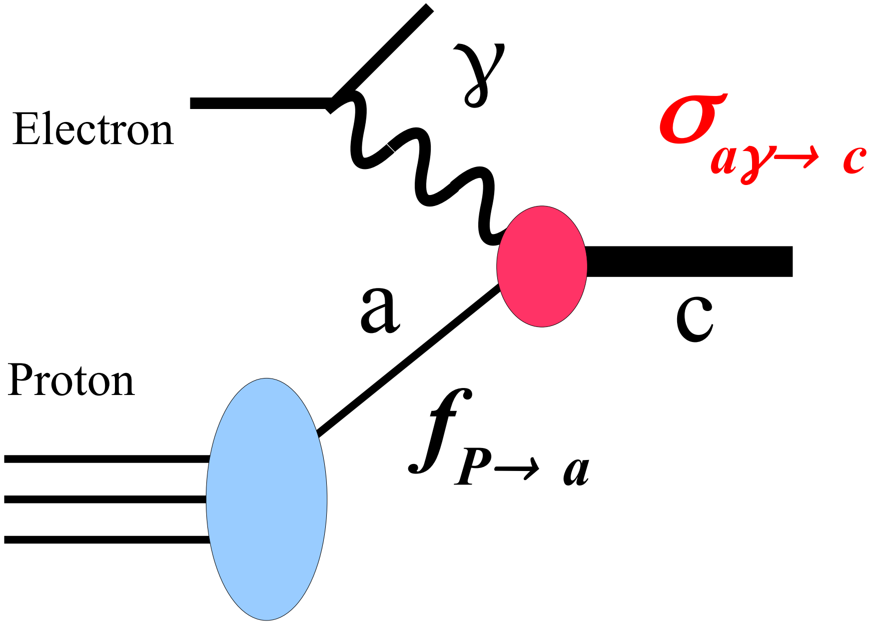
Corrections of order (Λ^2/Q^2)

must extract from experiment

calculable from theoretical model

Cross section is product of independent probabilities!!! *(Homework Assignment)*

The Parton Model and Factorization



Parton Distribution Functions (PDFs) $f_{P\to a}$ are the key to calculations involving hadrons!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \hat{\sigma}_{a\gamma\to c}$$

Scalar

$$f(x) = \sum q(x) + \bar{q}(x) + \phi(x) + \ldots = u(x) + d(x) + \ldots$$

Homework Problem: Convolutions

Part 1) Show these 3 definitions are equivalent; work out the limits of integration.

$$f\otimes g=\int_0^1\int_0^1 f(x)\,g(y)\,\delta(z-x*y)\,dx\,dy$$

$$f\otimes g=\int f(x)\,g\left(\frac{z}{x}\right)\frac{dx}{x}$$

$$f\otimes g=\int f\left(\frac{z}{y}\right)g(y)\frac{dy}{y}$$

Part 2) Show convolutions are the ``natural" way to multiply probabilities.

If f represents the heads/tails probability distribution for a single coin flip, show that the distribution of 2 coins is $f \oplus f$ and 3 coins is: $f \oplus f \oplus f$.

$$f\oplus g=\int f(x)\,g(y)\,\delta(z-(x+y))\,dx\,dy$$

$$f(x)=\frac{1}{2}(\delta(1-x)+\delta(1+x))$$

Careful: convolutions involve + and *

***BONUS**: How many processes can you think of that don't factorize?*

Structure Function & PDF Correspondence at Leading Order

$$\frac{d\sigma^{\nu}}{dx\,dy} = N\left[(1-y)^2 F_+ + 2(1-y)F_0 + F_-\right]$$

Compute with Hadronic Tensor

$$\frac{d\sigma^{\nu}}{dx\,dy} = N\left[(1-y)^2(2\bar{q}) + 2(1-y)(\phi) + (2q)\right]$$

Compute in Parton Model

Scalar

$$F_+ = 2\bar{q} \qquad F_+ = F_1 - \tfrac{1}{2}F_3$$
$$F_- = 2q \qquad F_- = F_1 + \tfrac{1}{2}F_3$$
$$F_0 = \phi \qquad F_0 = \tfrac{1}{2x}F_2 - F_1$$

Scalar

$$F_L = 0 = F_0$$
$$F_2 = 2xF_1$$

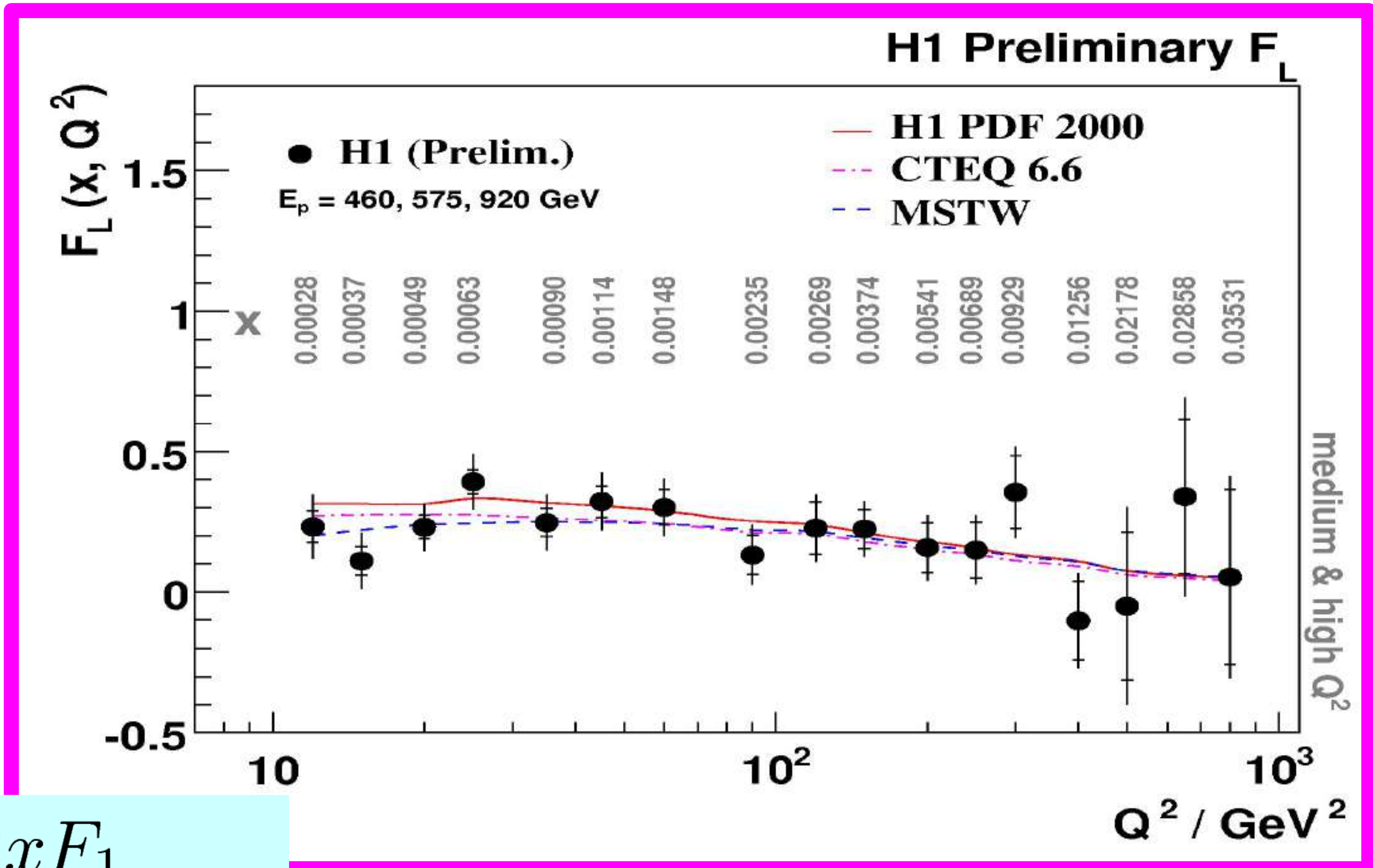
Callan-Gross Relation

$$F_L = 2xF_0$$

F_L

Longitudinal Structure Function

Why is F_L special ???



H1 Collaboration and ZEUS Collaboration (S. Glazov for the collaboration). Nucl.Phys.Proc.Suppl.191:16-24,2009.

$$F_L = 2xF_0 = F_2 - 2xF_1$$

$$F_L = 0 \quad \Longrightarrow \quad F_2 = 2xF_1$$

Callan-Gross

$$F_L \sim \frac{m^2}{Q^2} q(x) + \alpha_S \{c_g \otimes g(x) + c_q \otimes q(x)\}$$

Masses are important

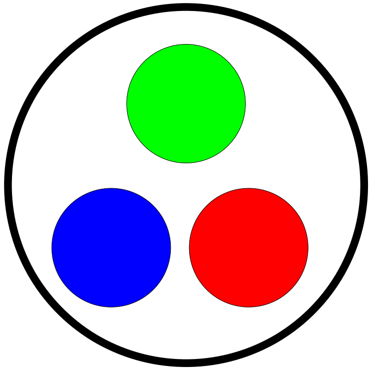
Higher orders are important

TOY

PDFs

Proton as a bag of free Quarks: Part 2

$$f(x,Q) = u(x,Q) + d(x,Q) = 2\ \delta(x - \tfrac{1}{3}) + 1\ \delta(x - \tfrac{1}{3})$$



$$u(x,Q) = 2\ \delta(x - \tfrac{1}{3})$$

$$d(x,Q) = 1\ \delta(x - \tfrac{1}{3})$$

Perfect Scaling PDFs
Q independent

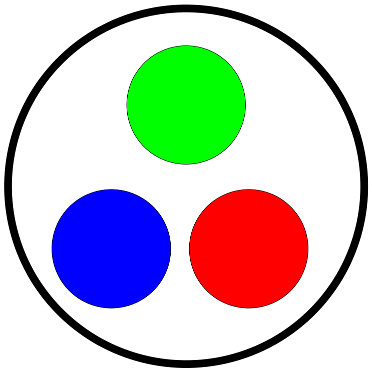
Quark Number Sum Rule

$$\langle q \rangle = \int_0^1 dx\, q(x) \qquad \langle u \rangle = 2 \quad \langle d \rangle = 1 \quad \langle s \rangle = 0$$

Quark Momentum Sum Rule

$$\langle x\, q \rangle = \int_0^1 dx\, x\, q(x) \qquad \langle x\, u \rangle = \tfrac{2}{3} \quad \langle x\, d \rangle = \tfrac{1}{3}$$

Problem #1: The proton does not add up???



$$
\begin{aligned}
F_+ &= 2\bar{q} \\
F_- &= 2q \\
F_L &= \phi
\end{aligned}
$$

$$q + \bar{q} = \frac{F_+ + F_-}{2}$$

Momentum Sum Rule

$$\sum_i \langle x\, q_i \rangle = \int_0^1 dx \sum x\,[q_i(x) + \bar{q}_i(x)] = 50\% \neq 100\%$$

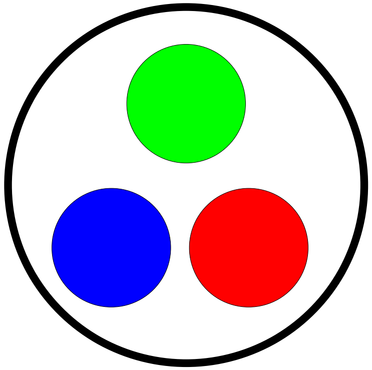
Substitute F

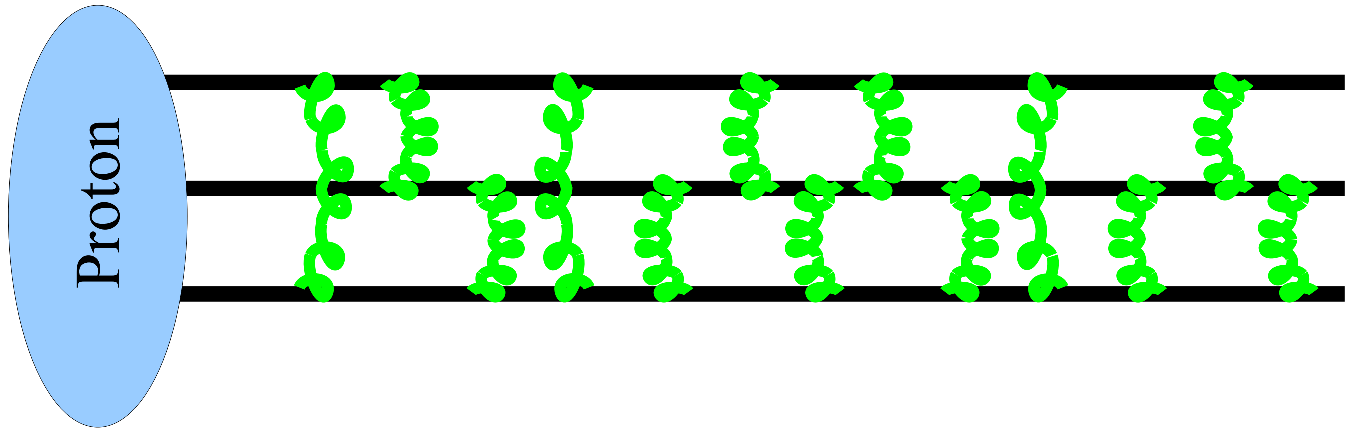
SOLUTION:

Gluons carry half the momentum,

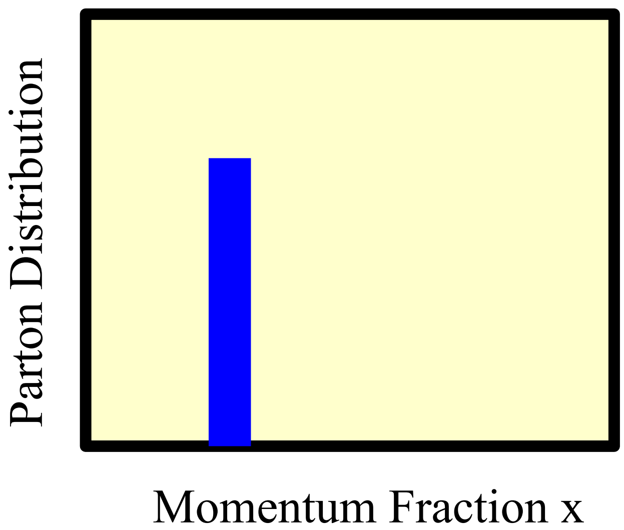
but do NOT couple to the photons

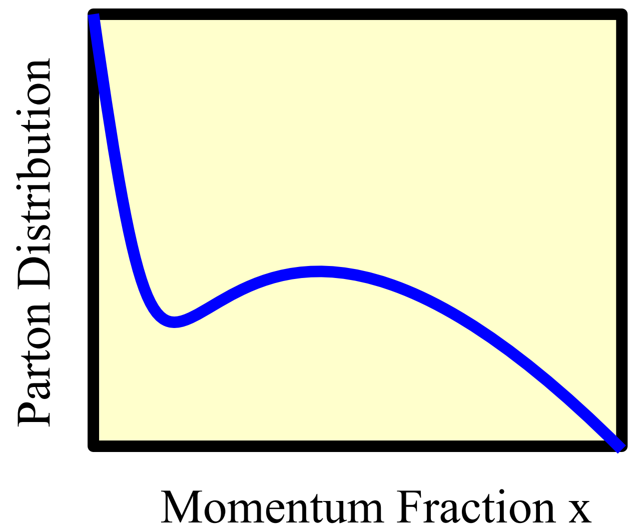
Gluons smear out PDF momentum





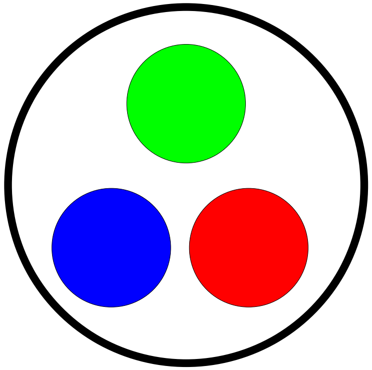
Gluons allow partons to exchange momentum fraction





α_s is large at low Q, so it is easy to emit soft gluons

Problem #2: Infinitely many quarks

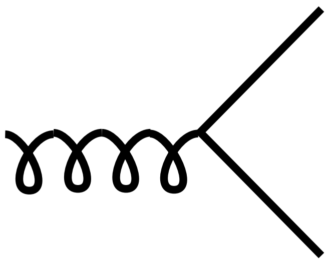


Reconsider the Quark Number Sum Rule

$$\langle u, d \rangle = \infty \qquad\qquad \langle q \rangle = \int_0^1 dx\, q(x)$$

Quark Number Sum Rule: More Precisely

$$q(x) \sim 1/x^{1.5}$$



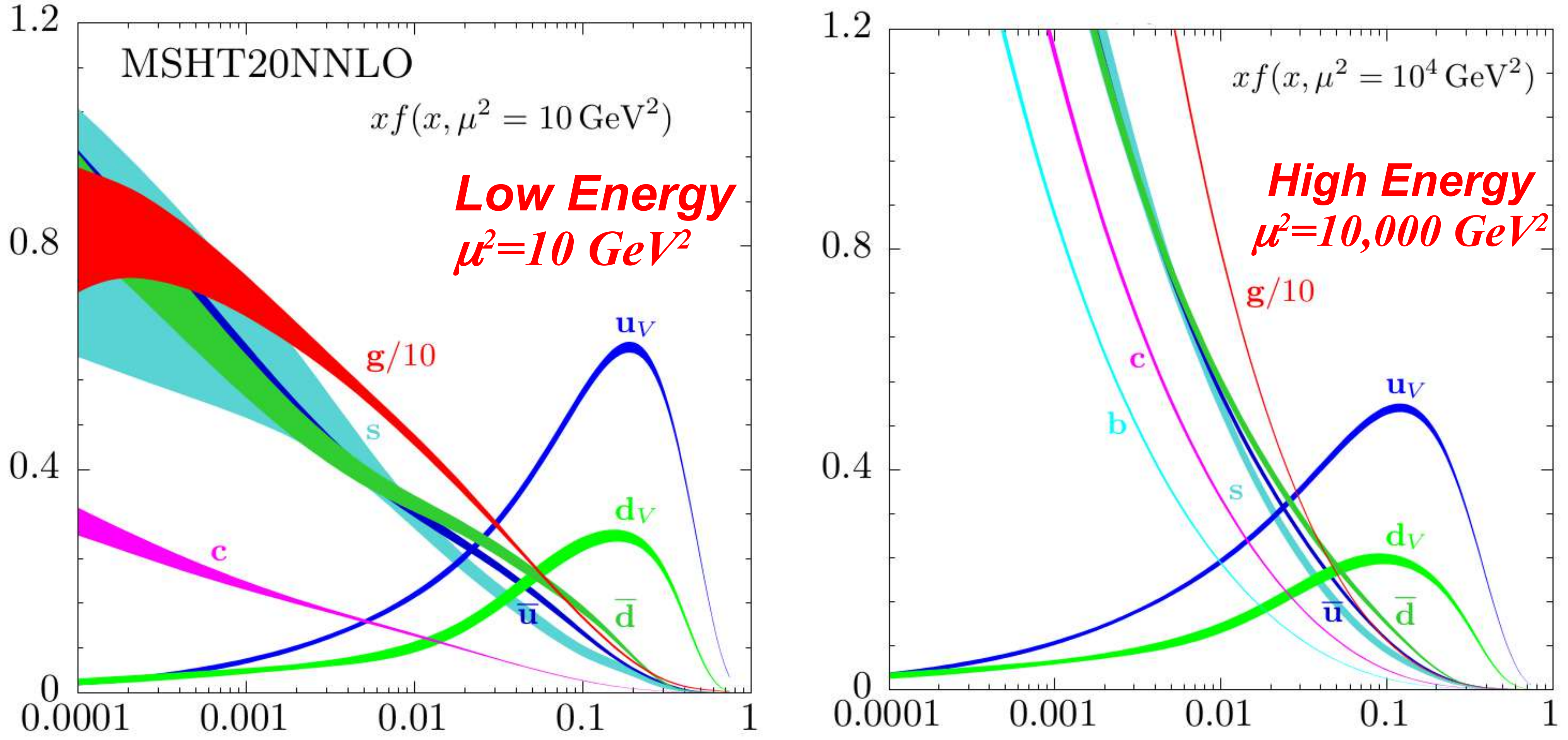
$$\langle u - \bar{u} \rangle = 2 \qquad \langle d - \bar{d} \rangle = 1 \qquad \langle s - \bar{s} \rangle = 0$$

SOLUTION: Infinite number of u quarks in proton, because they can be pair produced:
(We neglect saturation ....)

PDFs

cf., lectures by Pavel Nadolsky

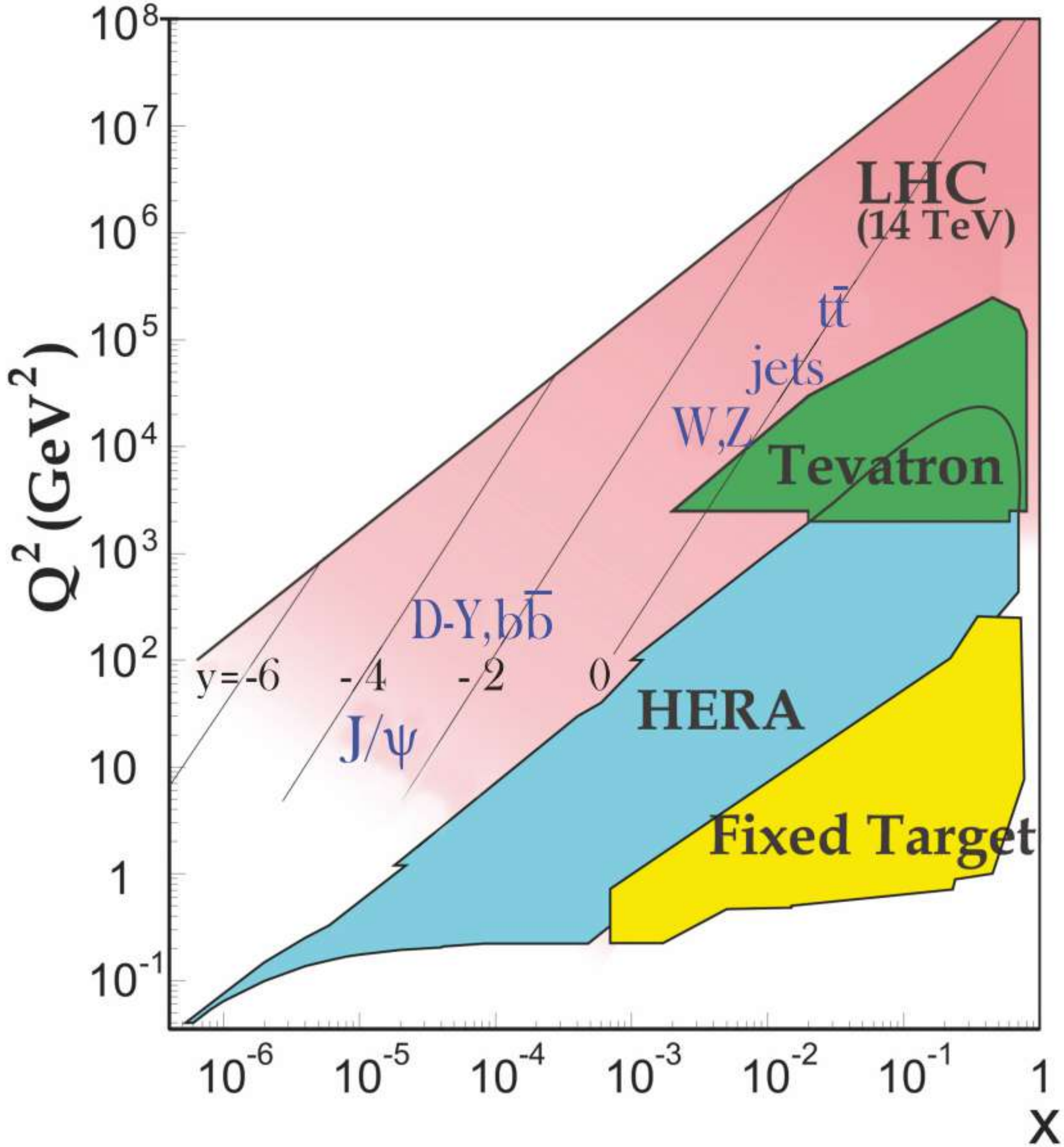
Sample PDFs: The rich structure of the proton



Scaling violations are essential feature of PDFs

Where do PDFs come from???? Universality!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \sigma_{a\gamma\to c}$$



Must extract from experiment

Calculable from theoretical model

Note we can combine different experiments.
FACTORIZATION!!!

HOMEWORK

Sum Rules
&
Structure Functions

$$
\begin{aligned}
F_2^{ep} &= \tfrac{4}{9}x\left[u+\bar{u}+c+\bar{c}\right] \\
&+ \tfrac{1}{9}x\left[d+\bar{d}+s+\bar{s}\right] \\
F_2^{en} &= \tfrac{4}{9}x\left[d+\bar{d}+c+\bar{c}\right] \\
&+ \tfrac{1}{9}x\left[u+\bar{u}+s+\bar{s}\right] \\
F_2^{\nu p} &= 2x\left[d+s+\bar{u}+\bar{c}\right] \\
F_2^{\nu n} &= 2x\left[u+s+\bar{d}+\bar{c}\right] \\
F_2^{\bar{\nu} p} &= 2x\left[u+c+\bar{d}+\bar{s}\right] \\
F_2^{\bar{\nu} n} &= 2x\left[d+c+\bar{u}+\bar{s}\right] \\
F_3^{\nu p} &= 2\left[d+s-\bar{u}-\bar{c}\right] \\
F_3^{\nu n} &= 2\left[u+s-\bar{d}-\bar{c}\right] \\
F_3^{\bar{\nu} p} &= 2\left[u+c-\bar{d}-\bar{s}\right] \\
F_3^{\bar{\nu} n} &= 2\left[d+c-\bar{u}-\bar{s}\right]
\end{aligned}
$$

Verify:

i.e., Check for typos ...

We use these different observables to dis-entangle the flavor structure of the PDFs

In the limit

$\theta_{Cabibbo} = 0$

$m_c = 0$

Homework: Part 2 Sum Rules

Verify:

i.e., Check for typos ...

Adler (1966)

$$\int_0^1 \frac{dx}{2x}\left[F_2^{\nu n} - F_2^{\nu p}\right] = 1$$

Bjorken (1967)

$$\int_0^1 \frac{dx}{2x}\left[F_2^{\bar{\nu} p} - F_2^{\nu p}\right] = 1$$

Gross Llewellyn-Smith (1969)

$$\int_0^1 dx\left[F_3^{\nu p} + F_3^{\bar{\nu} p}\right] = 6$$

Gottfried (1967)

if $\bar{u} = \bar{d}$ $$\int_0^1 dx\left[F_2^{ep} - F_2^{en}\right] = \frac{1}{3}$$

Homework (19??)

$$\frac{5}{18}F_2^{\nu N} - F_2^{eN} = ?$$

Before the parton model was invented, these relations were observed. Can you understand them in the context of the parton model?

This one has been particularly important/controversial

Evolution

What does the proton look like???

The answer is dependent upon the question

`Cheshire Puss,' ...

`Would you tell me, please, which way I ought to go from here?'

`That depends a good deal on where you want to get to,' said the Cat.

`I don't much care where--' said Alice.

`Then it doesn't matter which way you go,' said the Cat.

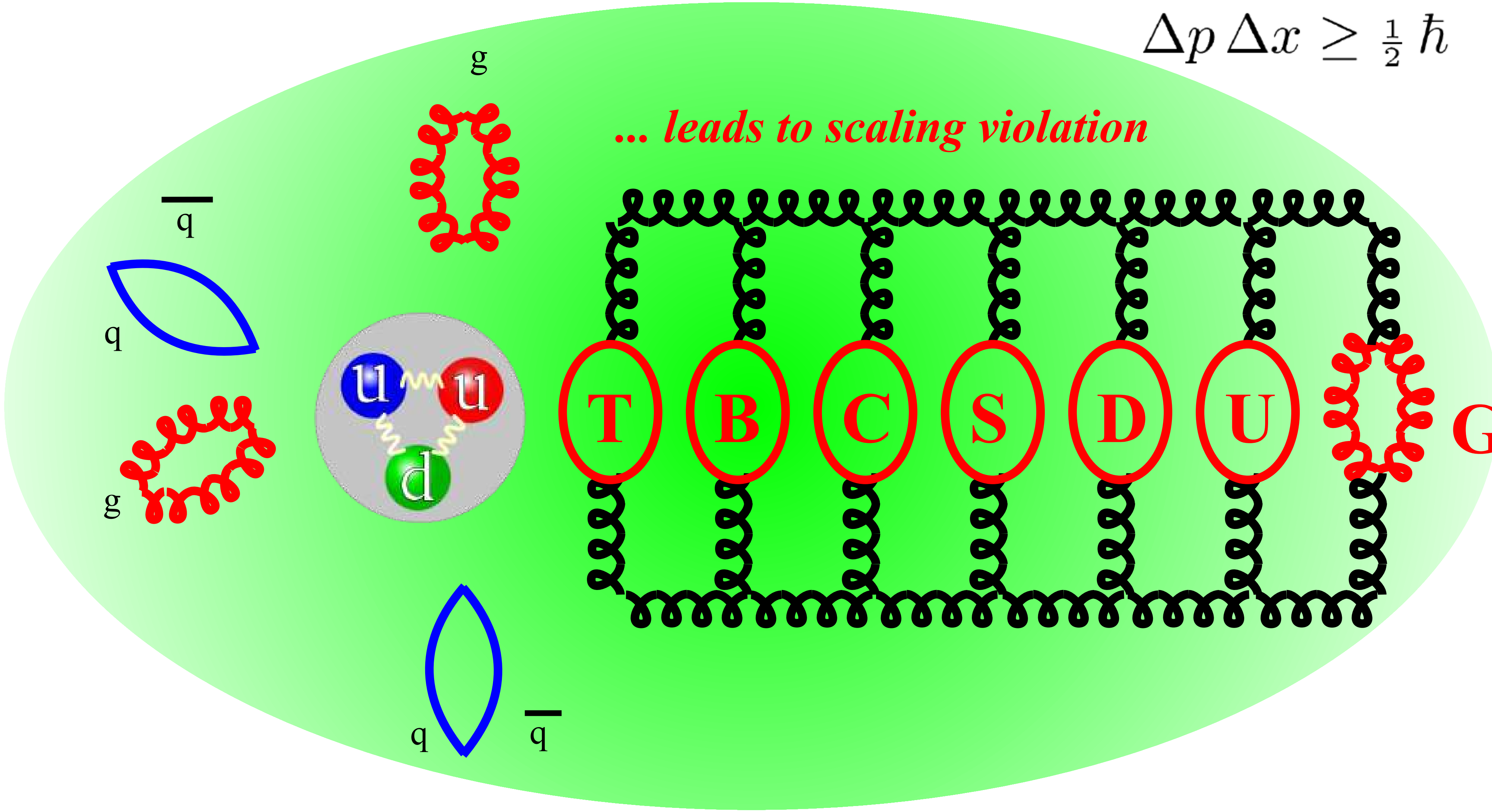
`--so long as I get somewhere,' Alice added as an explanation.

`Oh, you're sure to do that,' said the Cat,
`if you only walk long enough.'

Evolution: What you see depends upon what you ask

Proton is a complex object

$$\Delta E\,\Delta t \geq \tfrac{1}{2}\,\hbar$$
$$\Delta p\,\Delta x \geq \tfrac{1}{2}\,\hbar$$



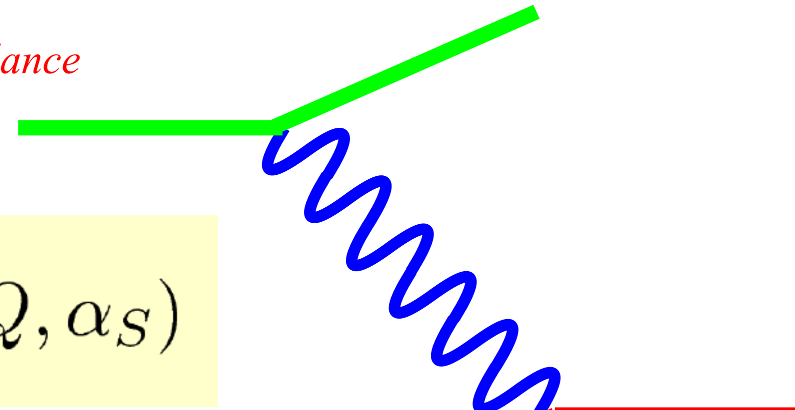
m_t	m_b	m_c	m_s	m_d	m_u	m_g
175	4.5	1.3	0.3	0.00?	0.00?	0

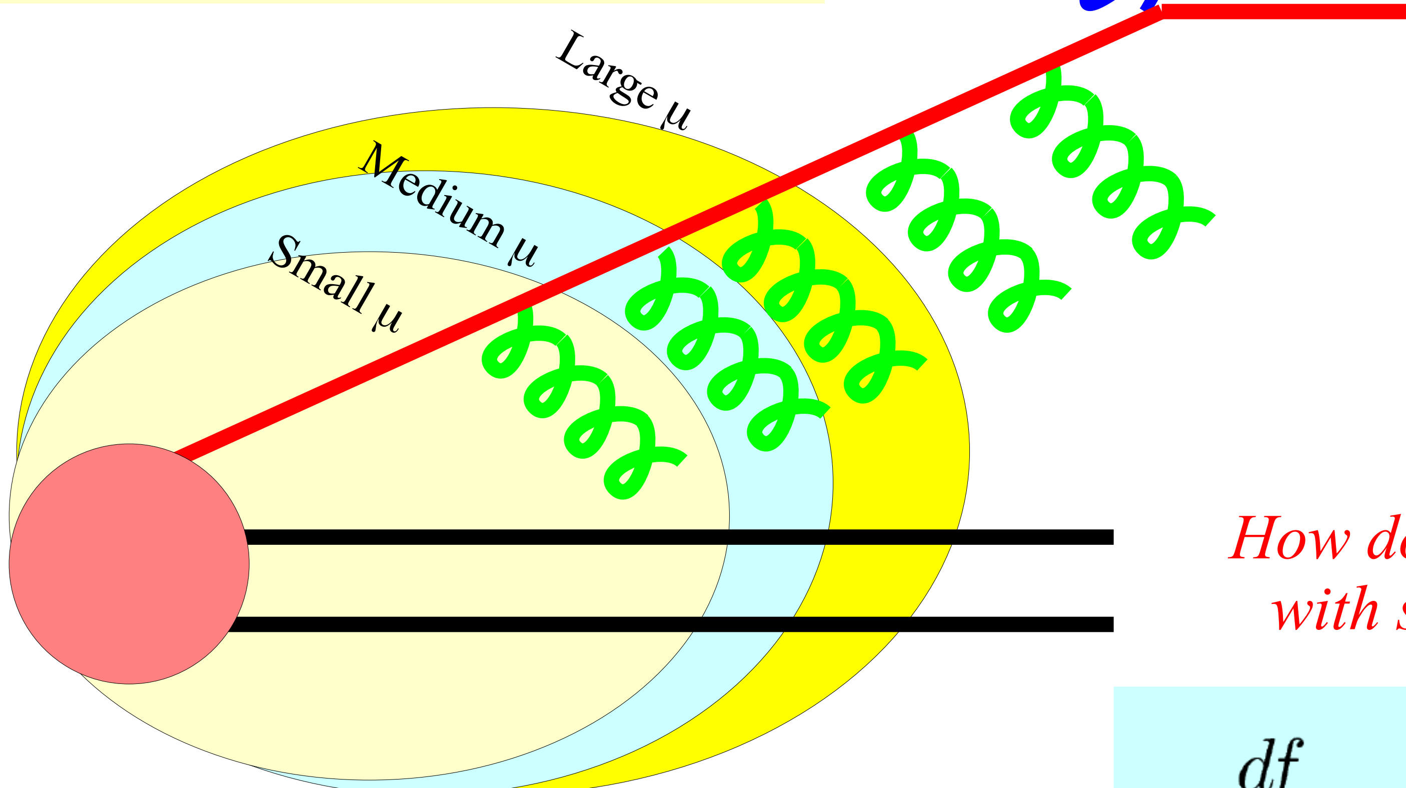
$$\Lambda_{QCD} \sim 200\,\mathrm{MeV}$$

Evolution of the PDFs

μ dependence must balance

$$\sigma(Q,x) = f(x,\mu) \otimes \widehat{\sigma}(\mu, Q, \alpha_S)$$





How does f change with scale μ???

$$\frac{df}{d\,\ln[\mu]} = ???$$

Renormalization Group Equation

Parton Model

$$\sigma = f \otimes \omega$$

ω or $\hat{\sigma}$

Not physical!
Poor notation

Renormalization Group Equation

$$\frac{d\sigma}{d\mu} = 0 = \frac{d\tilde{f}}{d\mu}\,\tilde{\omega} + \tilde{f}\,\frac{d\tilde{\omega}}{d\mu}$$

Take Mellin Transform

Separation of variables

$$\frac{1}{\tilde{f}}\,\frac{d\tilde{f}}{d\ln[\mu]} = -\gamma = -\frac{1}{\tilde{\omega}}\,\frac{d\tilde{\omega}}{d\ln[\mu]}$$

DGLAP Equation

DGLAP

$$\frac{d\tilde{f}}{d\ln[\mu]} = -\tilde{f}\,\gamma$$

$$\frac{df}{d\ln[\mu]} = P \otimes f$$

$$\tilde{f} \sim \mu^{-\gamma}$$

Anomalous Dimension

If "f" scaled,
γ would vanish

It is the dimension
of the mass scaling

Homework: Mellin Transform

$$\widetilde{f}(n) = \int_0^1 dx\, x^{n-1}\, f(x)$$

$$\sigma = f \otimes \omega$$

$$f(x) = \frac{1}{2\pi i} \int_C dn\, x^{-n}\, \widetilde{f}(n)$$

$$\widetilde{\sigma} = \widetilde{f}\, \widetilde{\omega}$$

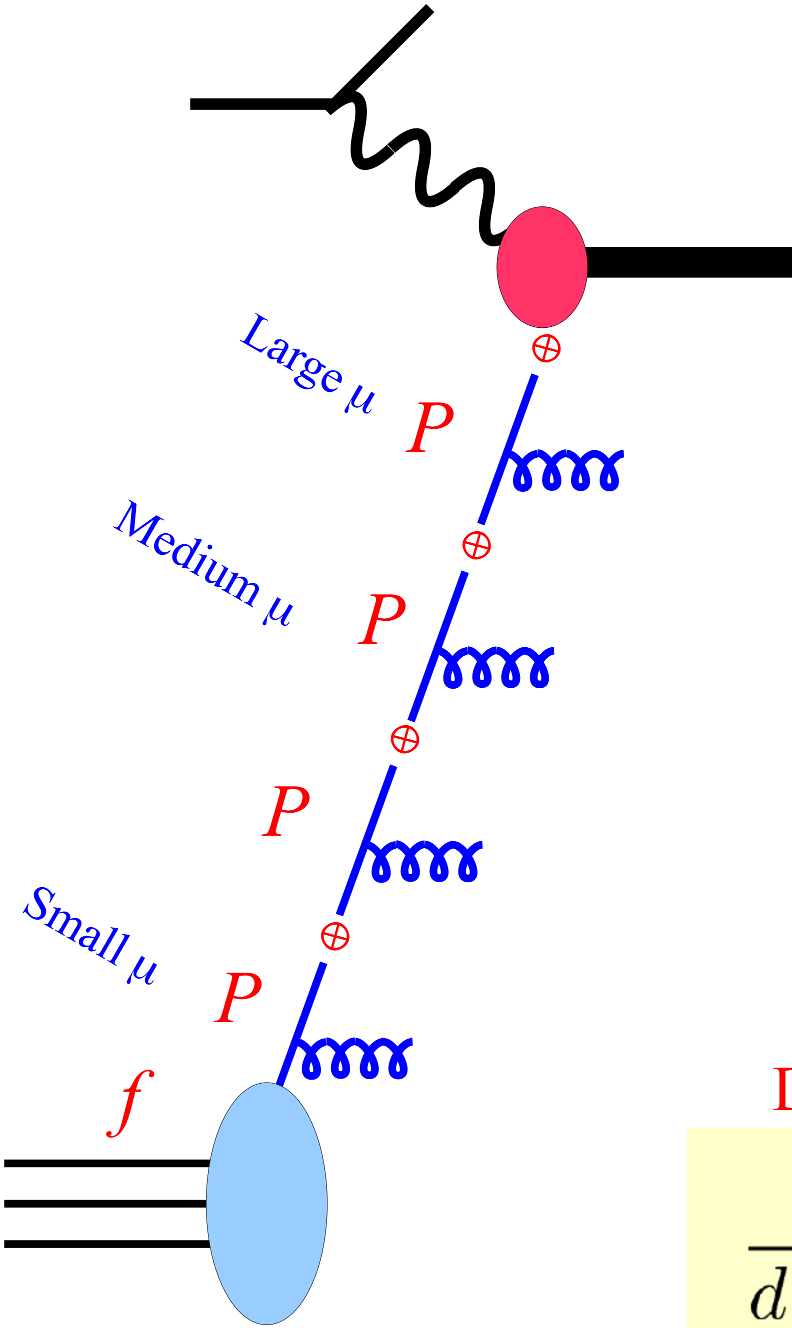
C is parallel to the imaginary axis, and to the right of all singularities

1) Take the Mellin transform of $f(x) = \sum_{m=1}^{\infty} a_m x^m$, and verify the inverse transform of $\widetilde{f}$ regenerates $f(x)$

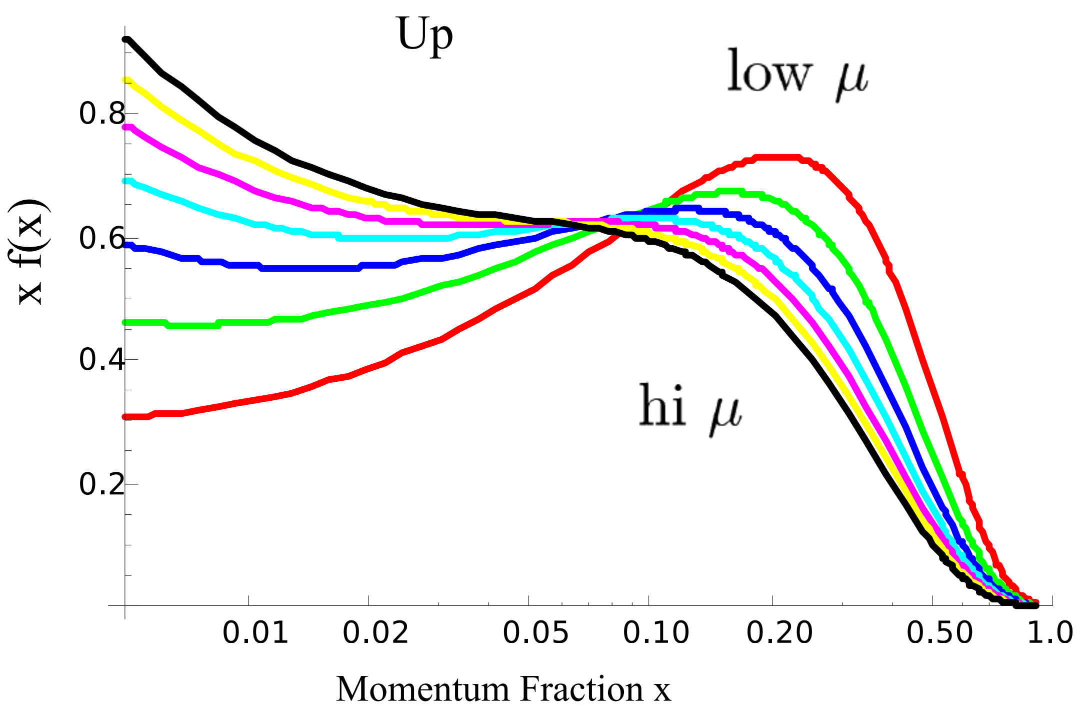
2) Take the Mellin transform of $\sigma = f \otimes \omega$ to demonstrate that the Mellin transform separates a convolution yields $\widetilde{\sigma} = \widetilde{f}\, \widetilde{\omega}$.

A useful reference:
Courant, Richard and Hilbert, David. Methods of Mathematical Physics, Vol. 1. New York: Wiley, 1989. 561 p.

Evolution of the PDFs

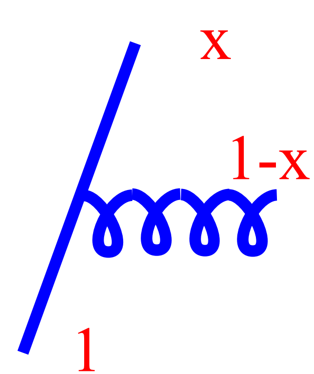


Evolution (generally) shifts partons from hi-x to low-x

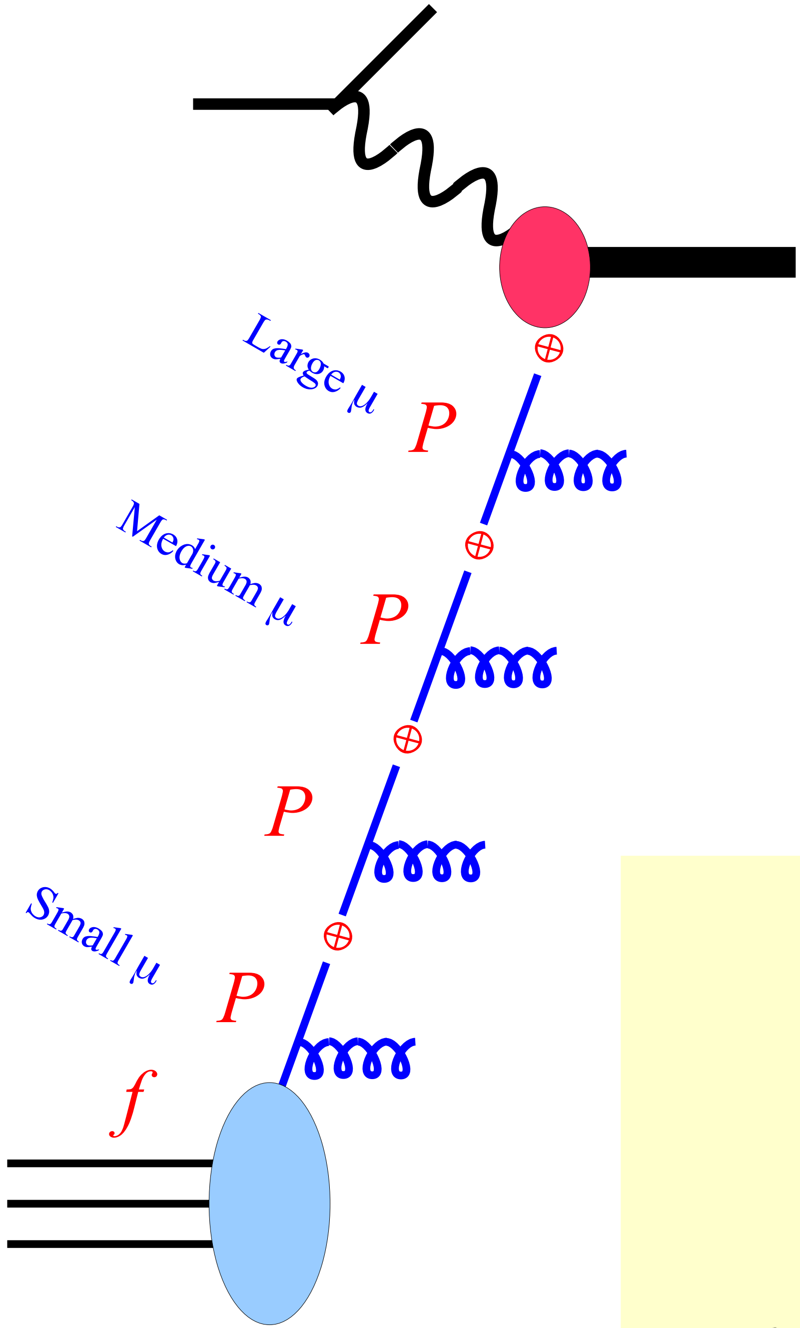


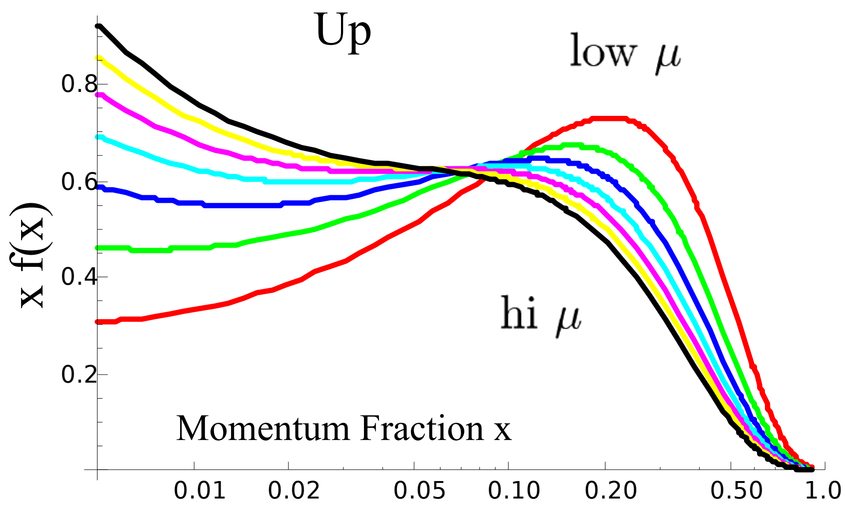
DGLAP Equation

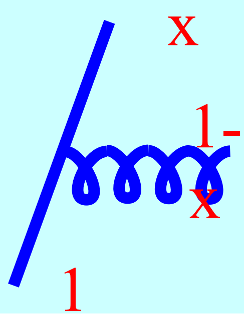
$$\frac{d\tilde{f}}{d\ln[\mu]} = P \otimes f$$



Evolution of the PDFs







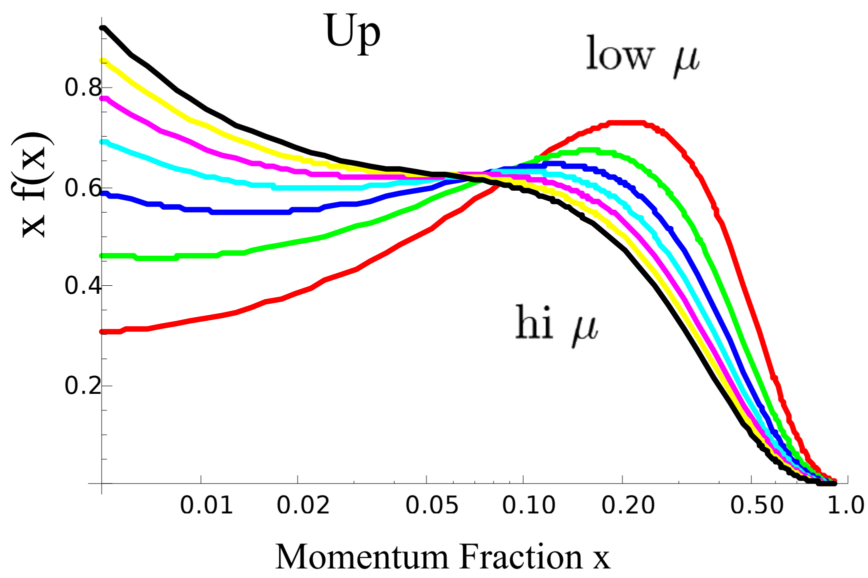
$$P^{(1)}_{qq}(x) = C_F \left[\frac{1+x^2}{1-x} \right]_+$$

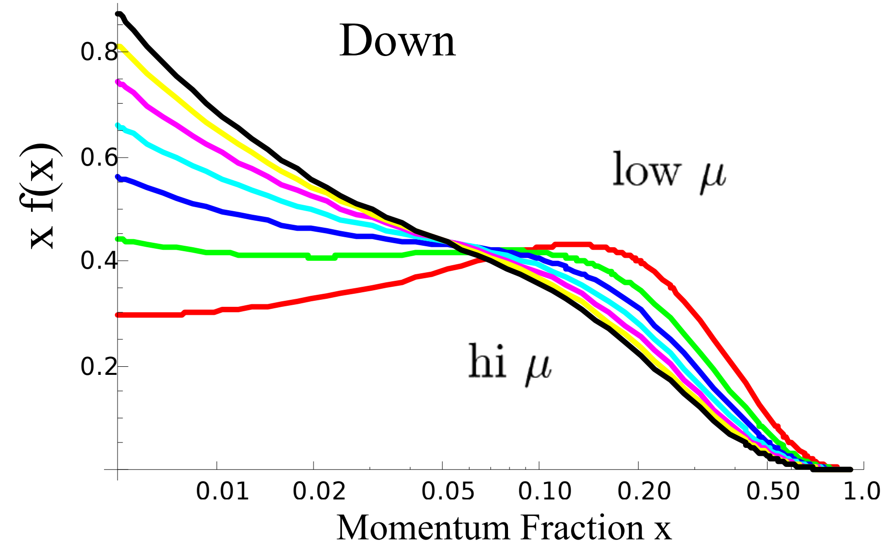
$$\frac{df}{d\ \ln[\mu]} = P \otimes f \ \simeq \ \frac{\alpha_S}{2\pi} P^{(1)} \otimes f$$

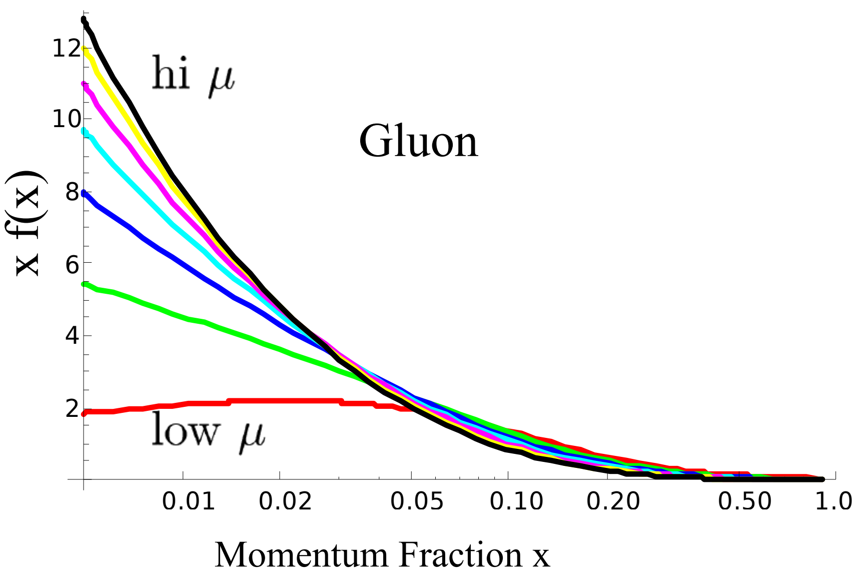
$$P \simeq \delta + \tfrac{\alpha_s}{2\pi} P^{(1)} + \left(\tfrac{\alpha_s}{2\pi}\right)^2 P^{(2)} + \ldots$$

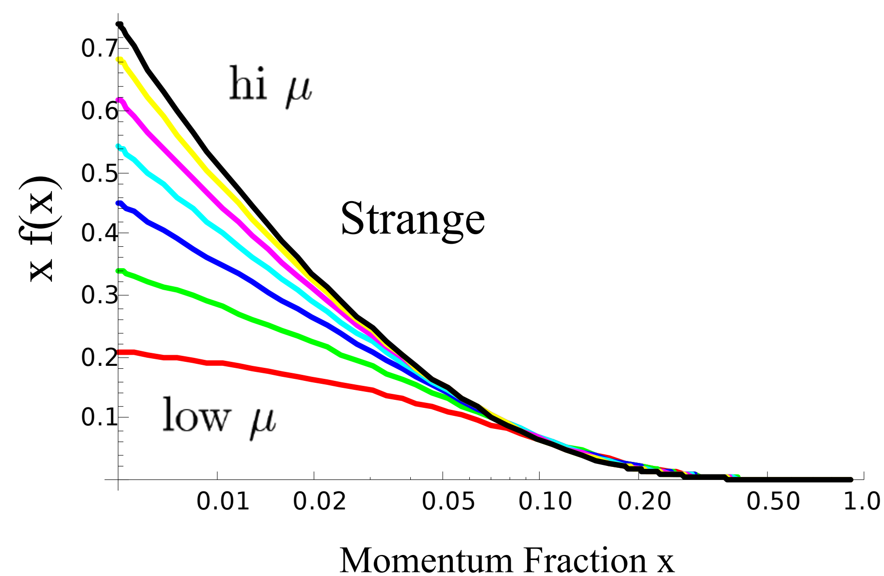
$$f_a(x,\mu_1) \sim f_a(x,\mu_0) + \frac{\alpha_S}{2\pi}\ P^{(1)}_{ab} \otimes f_b \ln\left(\frac{\mu_1^2}{\mu_2^0}\right)$$

Evolution of the PDFs

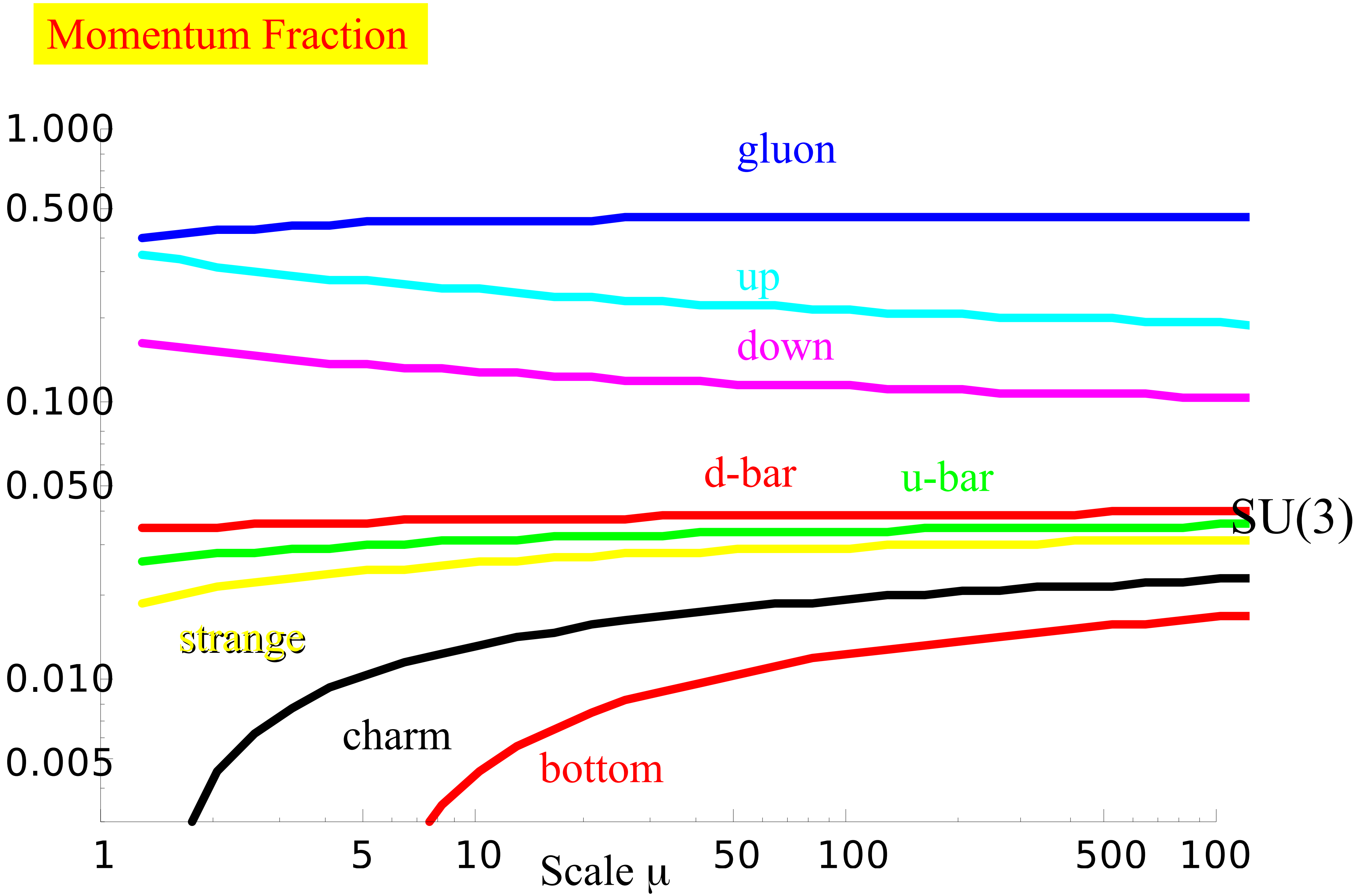








PDF Momentum Fractions vs. scale μ



Scaling violations are essential feature of PDFs

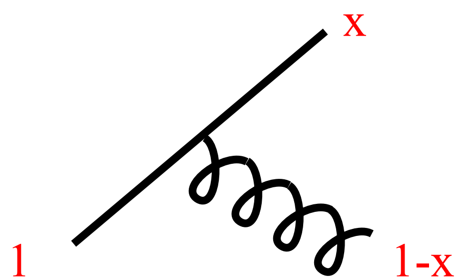
HOMEWORK

Splitting Functions
&
Conservation Relations

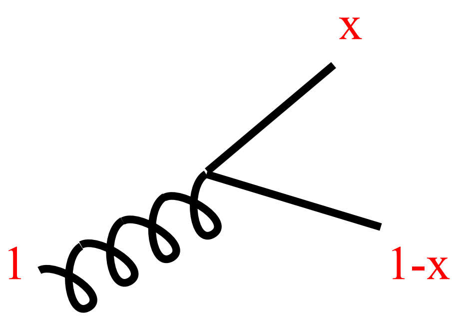
The Splitting Functions:

Read backwards

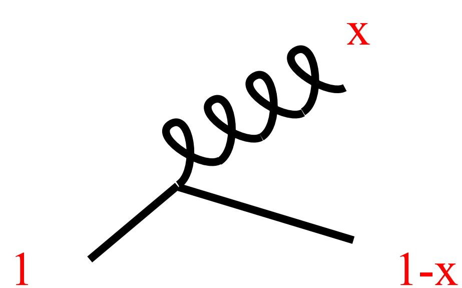
Note singularities



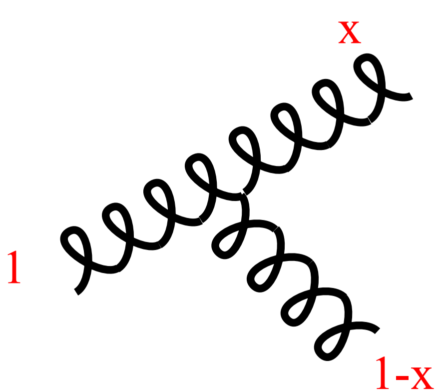
$$P_{qq}^{(1)}(x) = C_F \left[\frac{1+x^2}{1-x} \right]_+$$



$$P_{qg}^{(1)}(x) = T_F \left[(1-x)^2 + x^2 \right]$$



$$P_{gq}^{(1)}(x) = C_F \left[\frac{(1-x)^2+1}{x} \right]$$



$$P_{gg}^{(1)}(x) = 2C_F \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right] + \left[\frac{11}{6} C_A - \frac{2}{3} T_F N_F \right] \delta(1-x)$$

Homework: Part 1 The Plus Function

Definition of the Plus prescription:

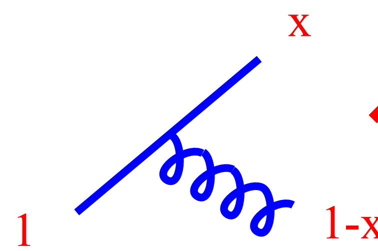
$$\int_0^1 dx \, \frac{f(x)}{(1-x)_+} = \int_0^1 dx \, \frac{f(x) - f(1)}{(1-x)}$$

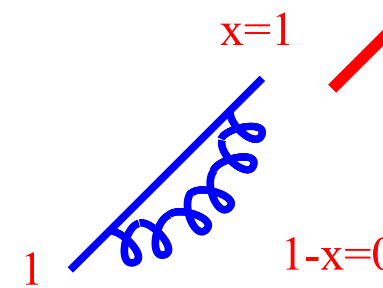
1) Compute: $$\int_a^1 dx \, \frac{f(x)}{(1-x)_+} = ???$$

Homework: Part 2 P(q←q)

2) Verify:

$$P_{qq}^{(1)}(x) = C_F \left[\frac{1+x^2}{1-x}\right]_+ \equiv C_F \left[(1+x^2)\left[\frac{1}{1-x}\right]_+ + \frac{3}{2}\delta(1-x)\right]$$

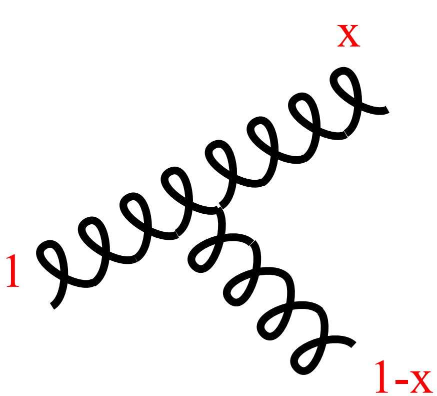


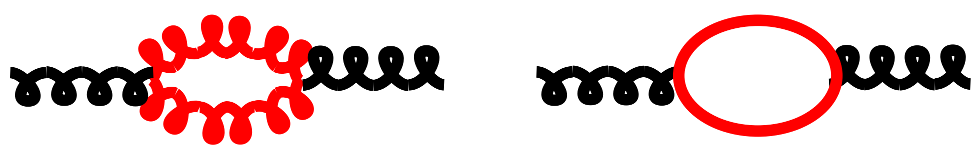


Observe

$$P_{gg}^{(1)}(x) = 2C_F \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right]$$

$$+ \left[\frac{11}{6}C_A - \frac{2}{3}T_F N_F\right] \delta(1-x)$$



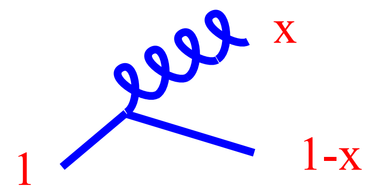


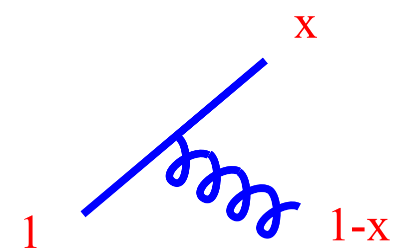
HOMEWORK: Part 3: Symmetries & Limits

Verify the following relation among the regular parts (from the real graphs)

For the regular part show:

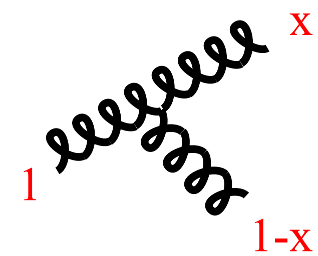
$$P_{gq}^{(1)}(x) = P_{qq}^{(1)}(1-x)$$





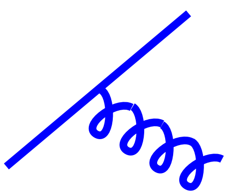
For the regular part show:

$$P_{gg}^{(1)}(x) = P_{gg}^{(1)}(1-x)$$

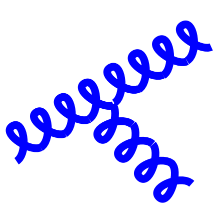


Verify, in the soft limit:

$$P_{qq}^{(1)}(x) \underset{x \to 1}{\longrightarrow} 2C_F \frac{1}{(1-x)_+}$$



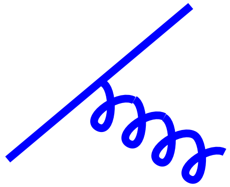
$$P_{gg}^{(1)}(x) \underset{x \to 1}{\longrightarrow} 2C_F \frac{1}{(1-x)_+}$$

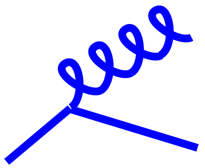


HOMEWORK: Part 4: Conservation Rules

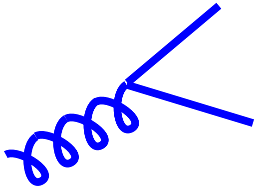
Verify conservation of momentum fraction

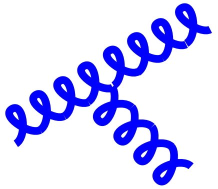
$$\int_0^1 dx\, x\, \left[P_{qq}(x) + P_{gq}(x)\right] = 0$$





$$\int_0^1 dx\, x\, \left[P_{qg}(x) + P_{gg}(x)\right] = 0$$





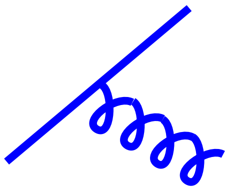
Verify conservation of fermion number

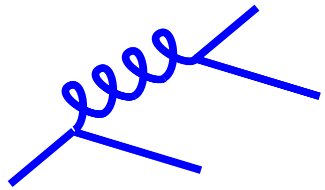
$$\int_0^1 dx \quad \left[P_{qq}(x) - P_{q\bar{q}}(x)\right] = 0$$

Homework: Part 5: Using the Real to guess the Virtual

Use conservation of fermion number to compute the delta function term in P(q←q)

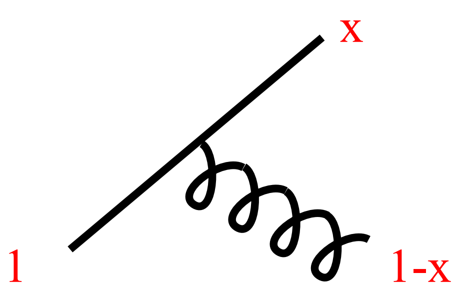
$$\int_0^1 dx \quad [P_{qq}(x) - P_{q\bar{q}}(x)] = 0$$

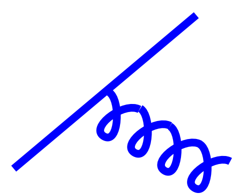


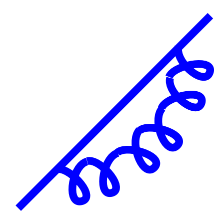


This term only starts at NNLO

$$P_{qq}^{(1)}(x) = C_F \left[\frac{1+x^2}{1-x}\right]_+ \equiv C_F \left[(1+x^2)\left[\frac{1}{1-x}\right]_+ + \frac{3}{2}\,\delta(1-x)\right]$$







Powerful tool: Since we know real and virtual must balance, we can use to our advantage!!!

End of lecture: Part 2 Recap

Rutherford Scattering $\Rightarrow$ Deeply Inelastic Scattering (DIS)

- Works for protons as well as nuclei

Compute Lepton-Hadron Scattering 2 ways

- Use Leptonic/Hadronic Tensors to extract Structure Functions
- Use Parton Model; relate PDFs to F_{123}

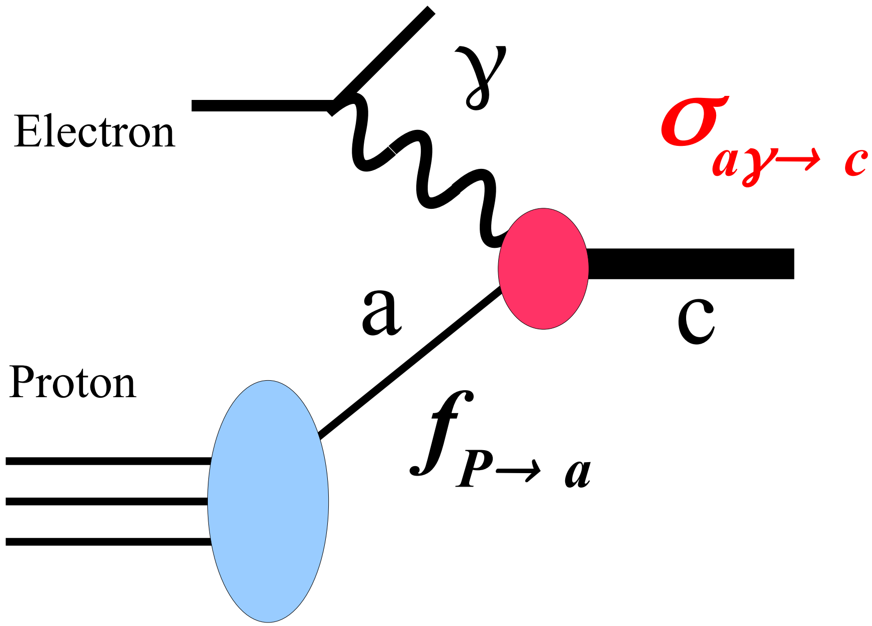
Parton Model Factorizes Problem:

- PDFs are independent of process
- Thus, we can combine different experiments. ESSENTIAL!!!

PDFs are not truly scale invariant; they evolve

- We use evolution to "resum" an important set of graphs

The Parton Model and Factorization



Parton Distribution Functions

(PDFs) $f_{P\to a}$

are the key to calculations

involving hadrons!!!

$$\sigma_{P\gamma\to c} = f_{P\to a} \otimes \hat{\sigma}_{a\gamma\to c}$$

Corrections of order (Λ^2/Q^2)

must extract from experiment

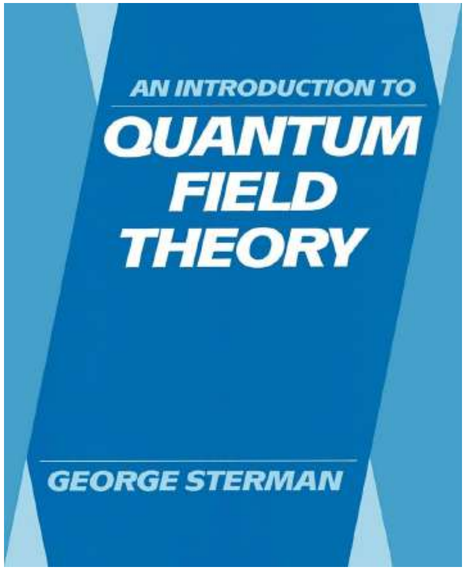
calculable from theoretical model

Cross section is product of independent probabilities!!! *(Homework Assignment)*

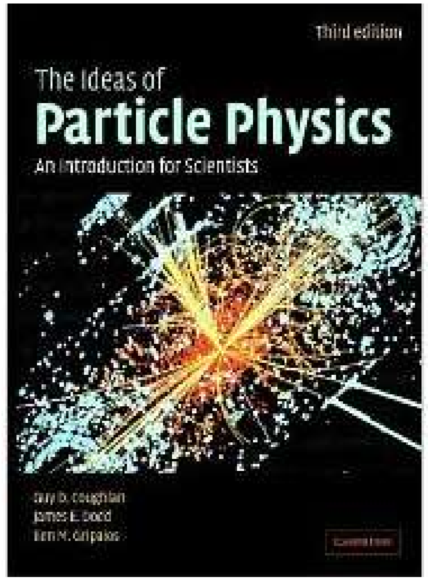
END OF LECTURE

Useful References & Thanks:

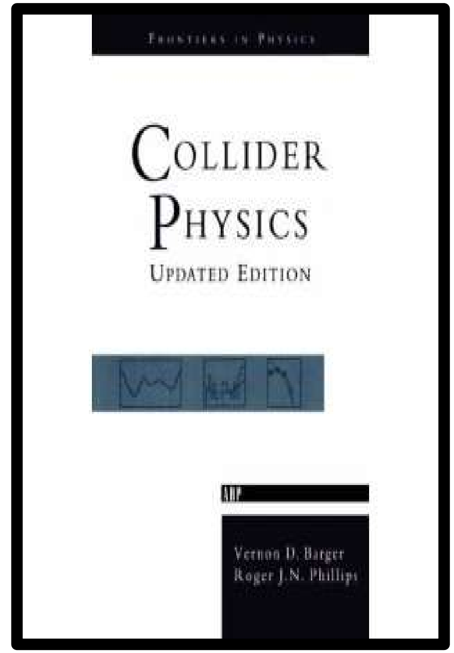
Useful References



George Sterman



Coughlan, Dodd, Gripaios



Barger & Phillips

CTEQ Handbook of
Perturbative QCD

https://inspirehep.net/files/a07787f3653b2a41945961ad44fbe926

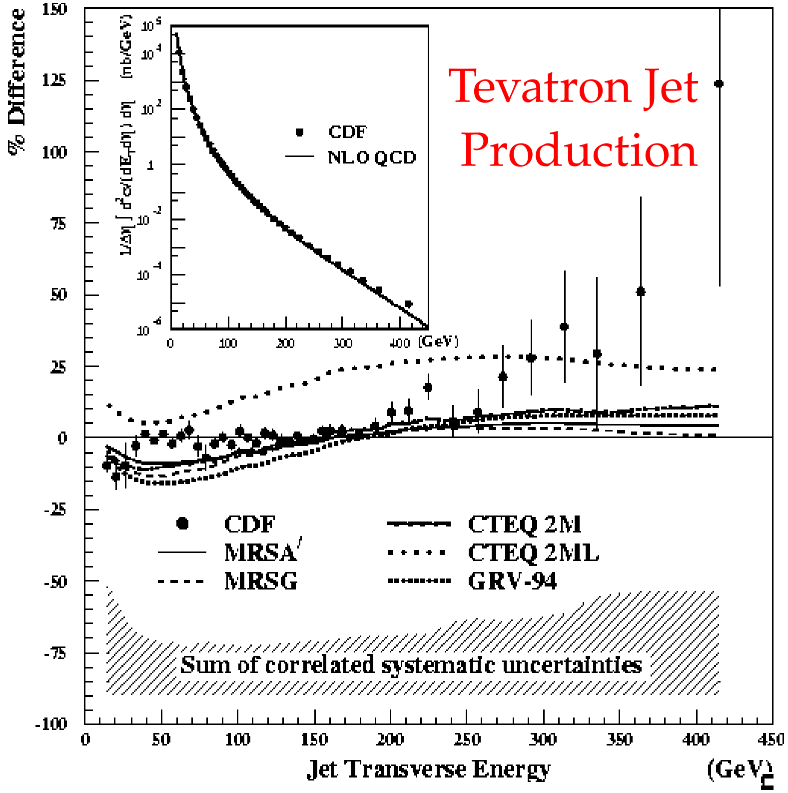
An Introduction to QFT
Peskin & Schroeder

QCD and Collider Physics
Ellis, Stirling, Webber

Particle Data Group
http://pdg.lbl.gov

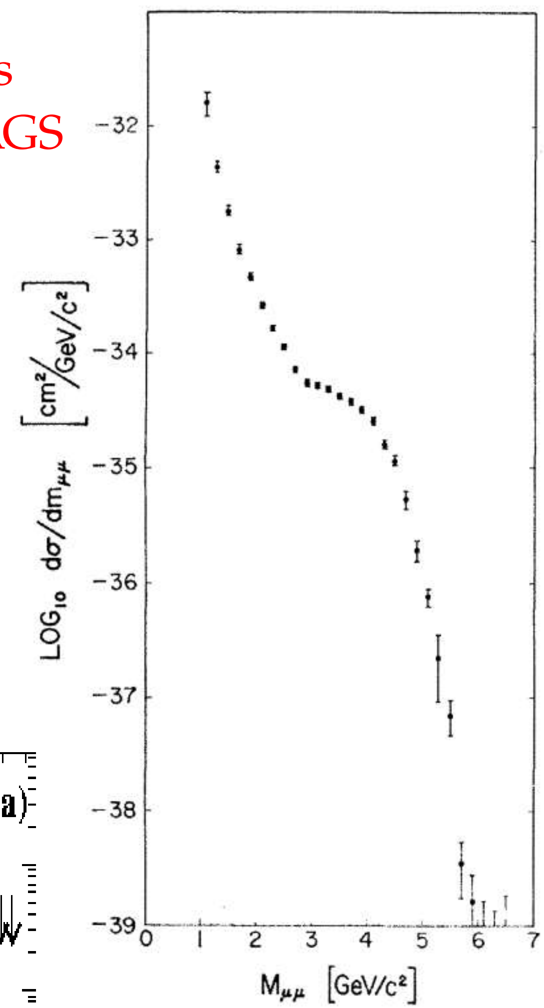
Leftovers

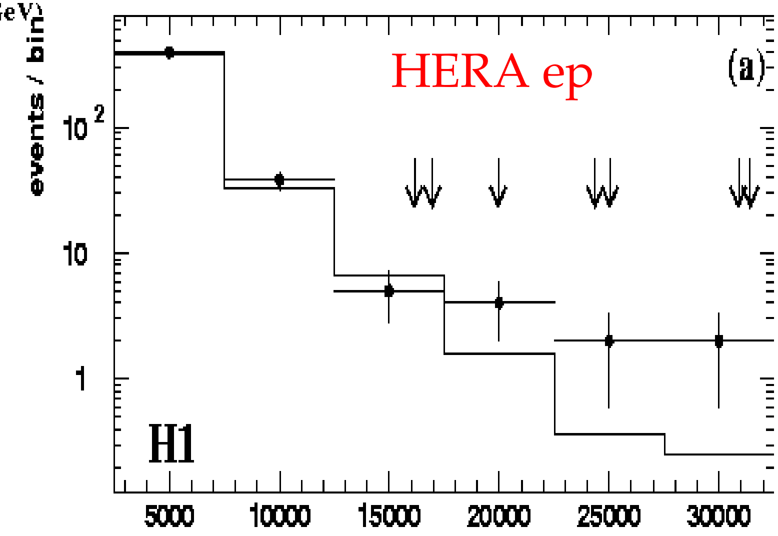
Can you find the Nobel Prize?



CDF Collaboration, PRL 77, 438 (1996)

Muon pairs
at the BNL AGS



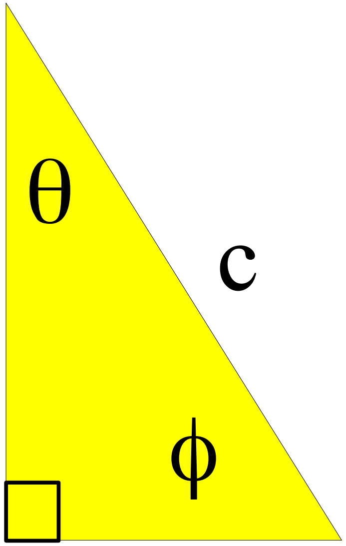


H1 Collaboration, ZPC74, 191 (1997)
ZEUS Collaboration, ZPC74, 207 (1997)

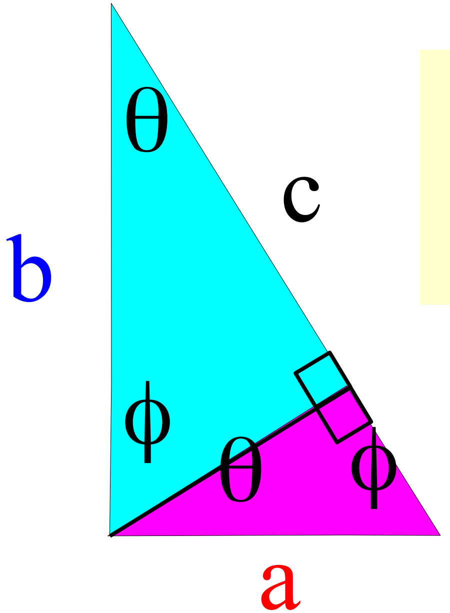
Warm up: Dimensional Analysis: Pythagorean Theorem

GOAL:
Pythagorean Theorem

METHOD:
Dimensional Analysis



$$A_c = c^2 f(\theta, \phi)$$



$$A_a + A_b = a^2 f(\theta, \phi) + b^2 f(\theta, \phi)$$

$$A_a + A_b = A_c$$

$$a^2 + b^2 = c^2$$

Two examples to come: 1) Resummation, and 2) Scaling