Lattice QCD: From Quarks and Gluons to Hadrons, Nucleons and Nuclei

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- Day 1 : Introduction to Lattice QCD methods

  Highlights of some Lattice QCD results

  Euclidean QFT via path integration

  Discretization of action: scalar, fermion, Yang-Mills gauge boson

  Basic algorithms

  Hadron Properties from Lattice

  What are the Challenges?

- Day 2 : Highlights of Important Lattice QCD Areas and Results

  Structure : Form Factors, Spin, Mass, Parton Disributions

  Nuclear Matter Sensitivity to DM and Symmetry Violations

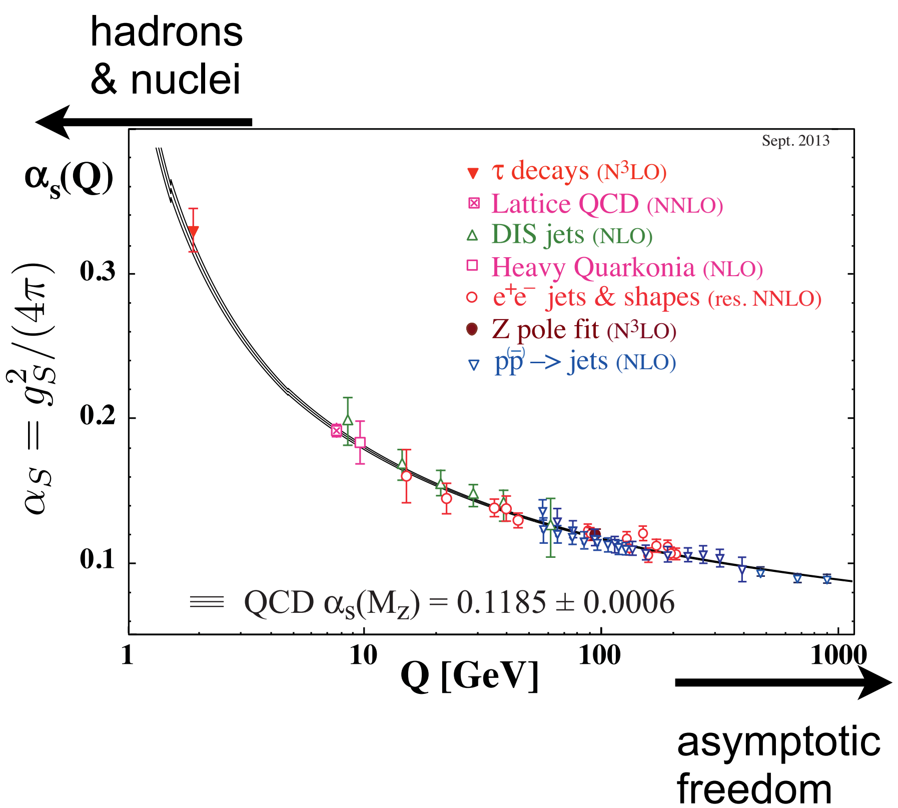
  Spectrum & Interactions

  (if time allows) Confinement, Entanglement and Quantum Information

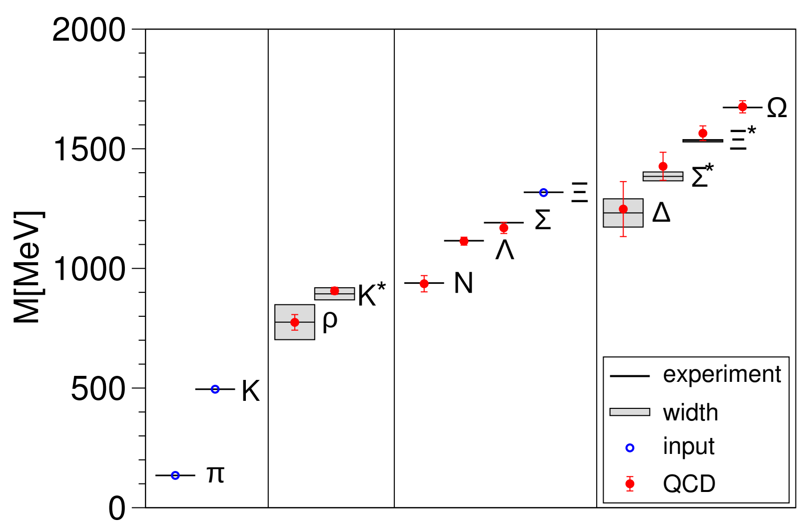
Why Need Numerical Calculations in QCD?

QCD has unique features in the low-scale (< 1 GeV) / long range (>0.2 fm)

- Nonperturbative with $\alpha_S \sim 1$

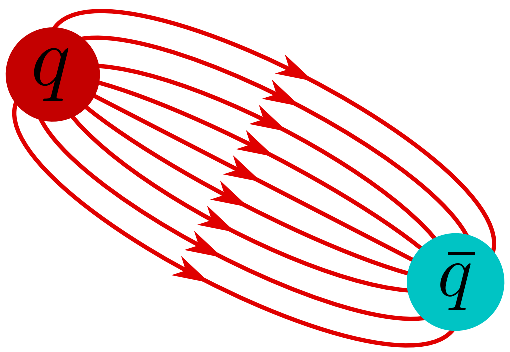


- Dynamically generated mass, most from the gluon energy
  $m_{p,n} >> m_{u,d,s}$

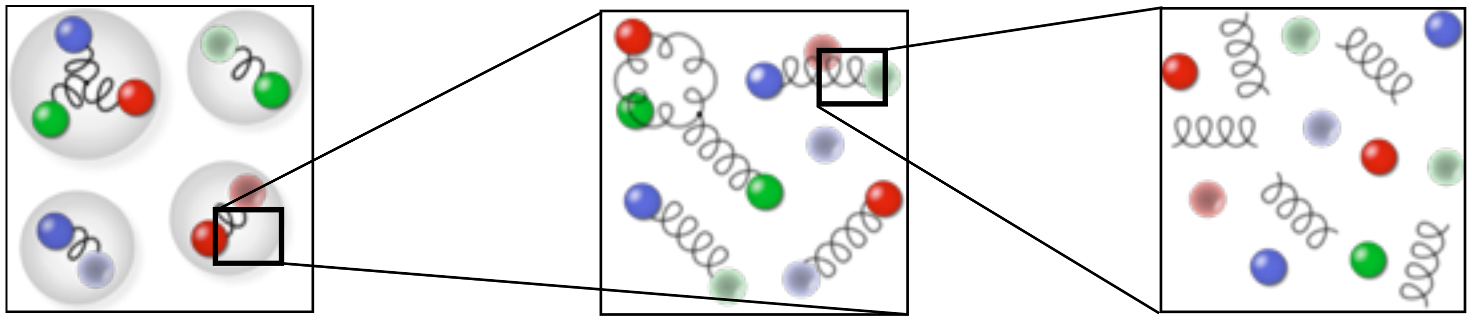


- Spontaneous breaking of chiral symmetry
  $\Longrightarrow$ Nambu-Goldstone π, K, ...

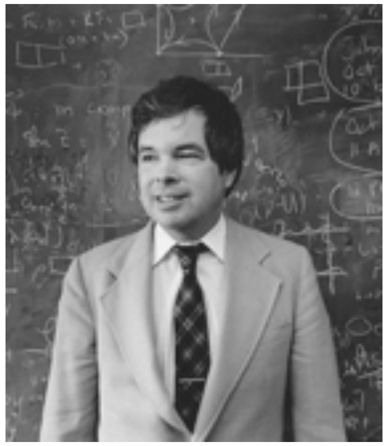
- Confinement & flux tube



- Hadron Spectrum? Interactions?
- Hadron Structure? Spin?
- Hadron Sensitivity to new physics (dark matter, symmetry breaking, etc)
- Mechanism for binding quarks & gluons?

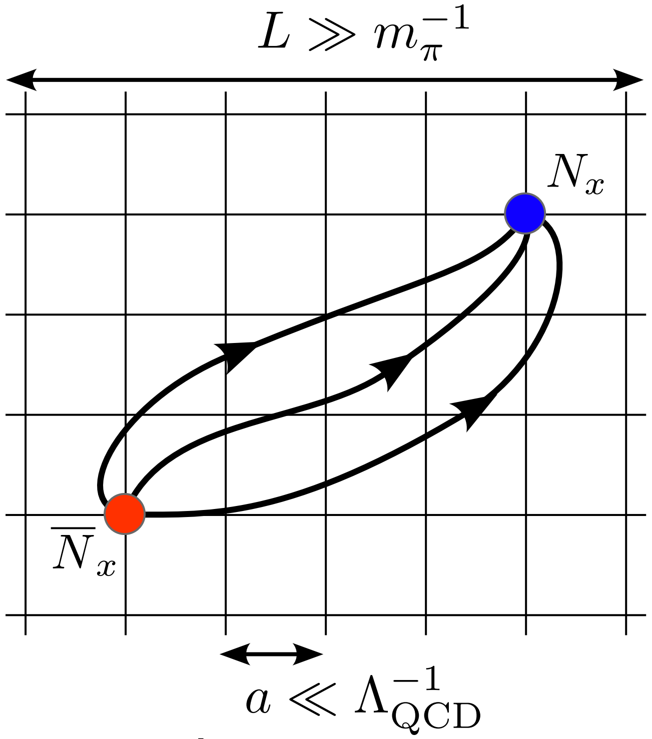


Lattice QCD: Numerical QFT on a Space-Time Grid



Kenneth Wilson '76 : formulation of QCD on 4D Euclidean lattice

- Originally intended as a way to **regularize QCD in a gauge-invariant way**
- ... (*5 decades of developing theory, algorithms, computing*) ...
- calculations with physical parameters, most systematic effects controlled
  (*) *Your Uncertainty May Vary depending on the observable*



At present:

The only way to work with QCD

- at long distance ≳ 5 fm
  dictated by the lightest d.o.f : pion
- low energy-scale $a^{-1} \approx$ 1–4 GeV
  dictated by discretization errors,
  scale window for renormalization / matching

Large community:

- Annual symposium with 500++ attendance
- Open community software, datasets
- Biennial report (latest [FLAG Review 2024, Aoki et al PRD113:014508 (2026)]
- Synergies with phenomenology (e.g. PDF fits)

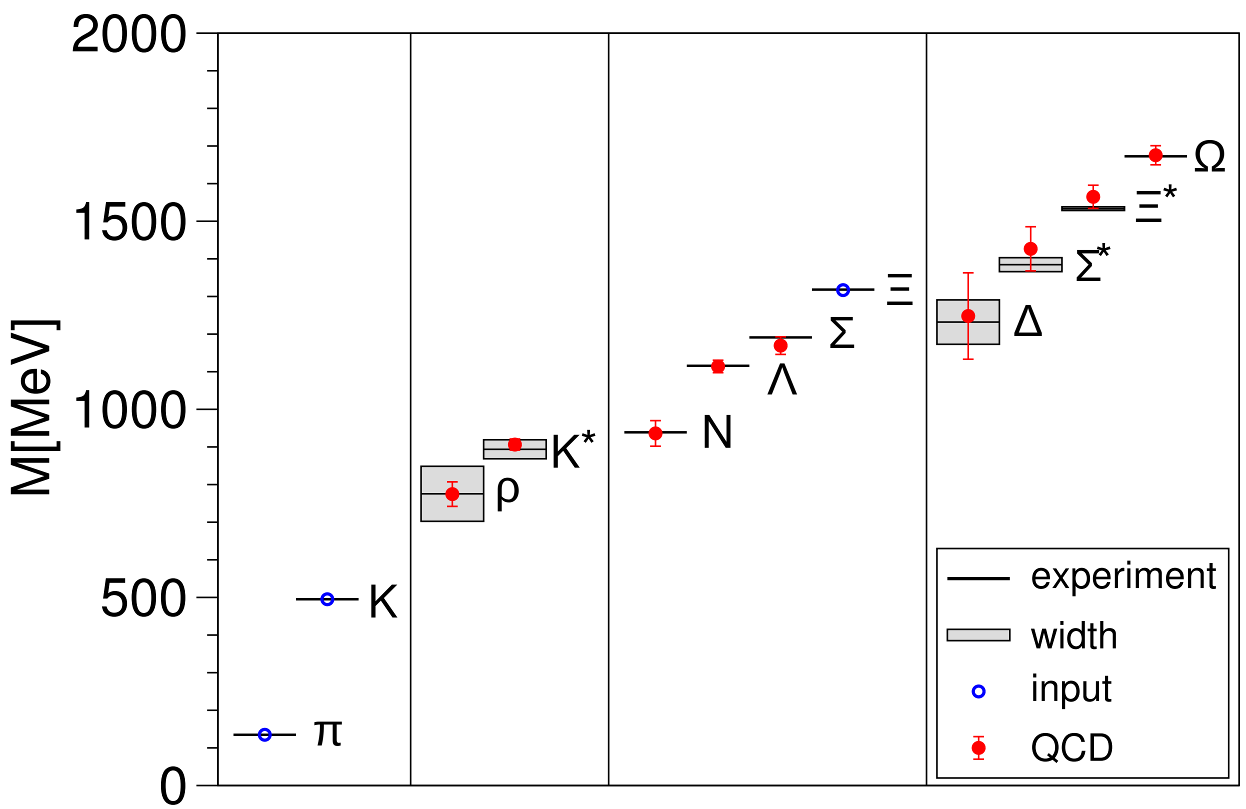
Successes of Lattice QCD: Hadron spectrum

- 2008 : Light hadron masses

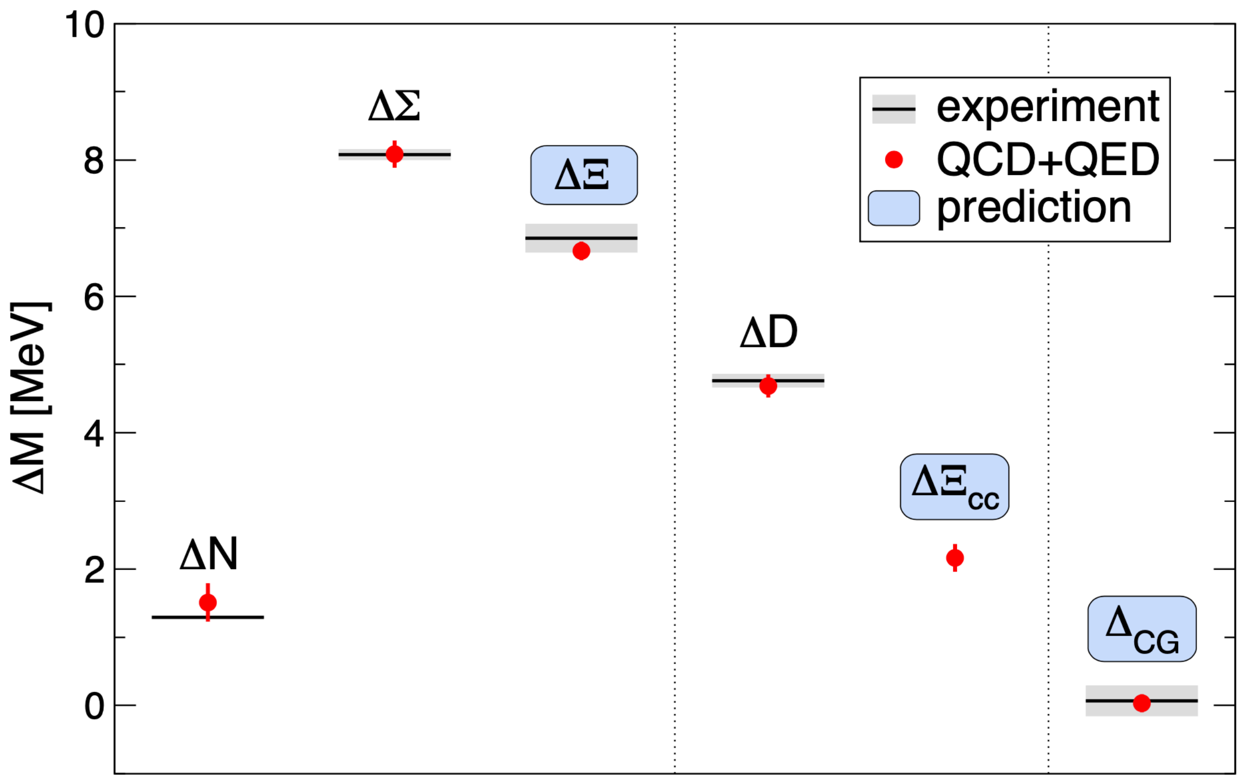
  QCD with 2 light ($m_u = m_d$) + strange quarks

- 2013 : Isospin + QED corrections to baryons

  QCD + QED, ($m_u - m_d$) effects



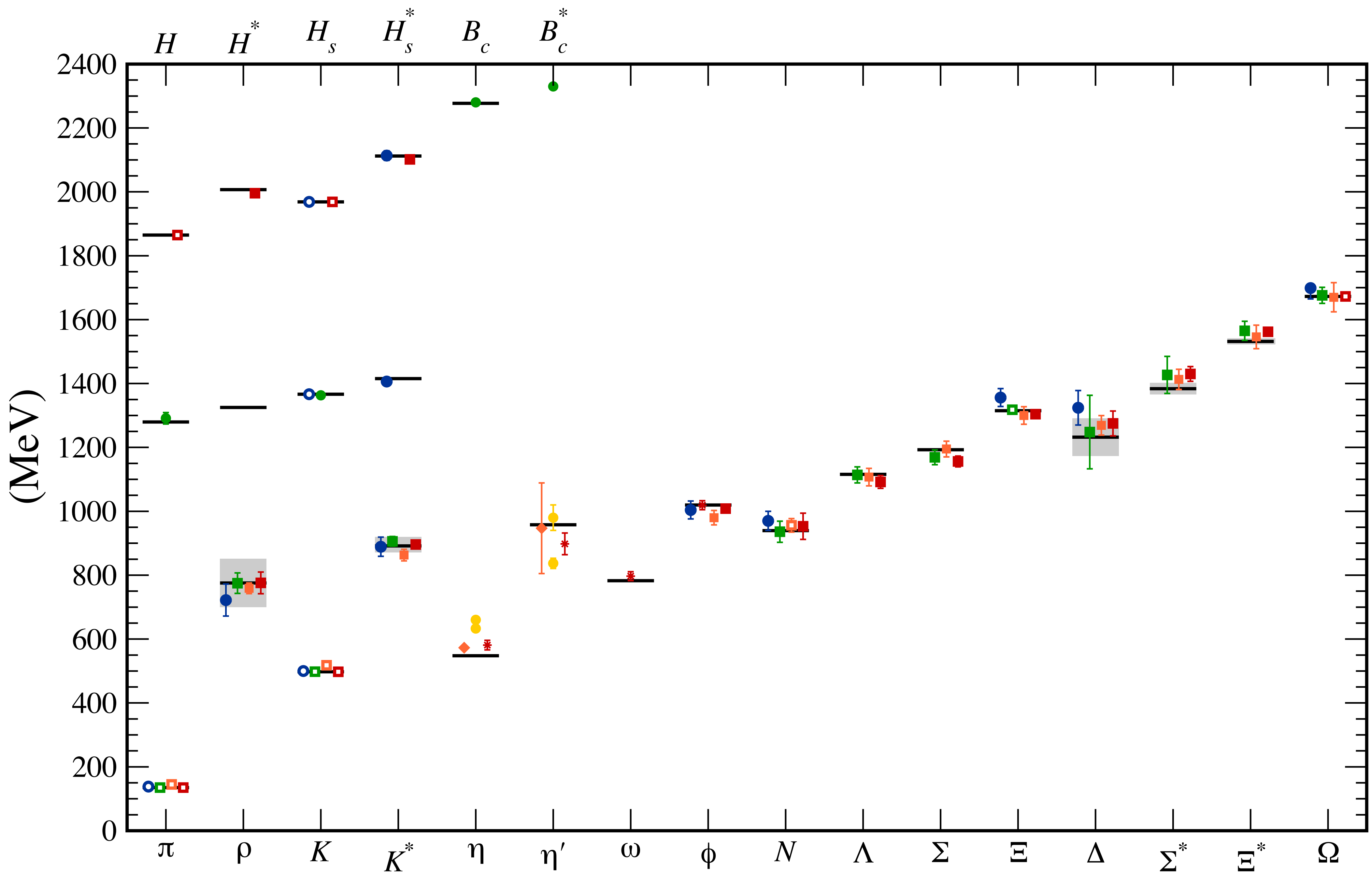
[S.Durr et al (BMW collaboration), "Ab-Initio Determination of Light Hadron Masses" **Science** 322 (2008) 1224]



[S.Borsanyi et al (BMW collaboration), "Ab initio calculation of the neutron-proton mass difference" **Science** 347 (2015) 1452]

Successes of Lattice QCD: Hadron spectrum (2)

- Some lattice mass determinations are more precise than experiment
- States without experimental value

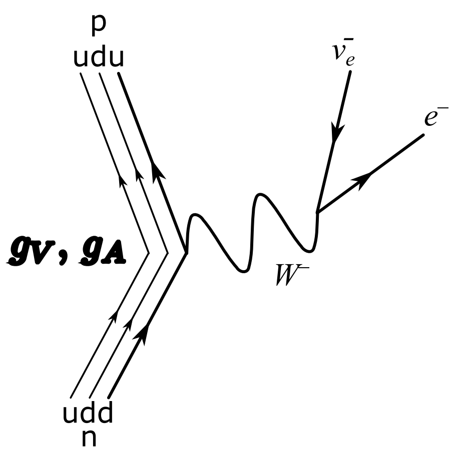


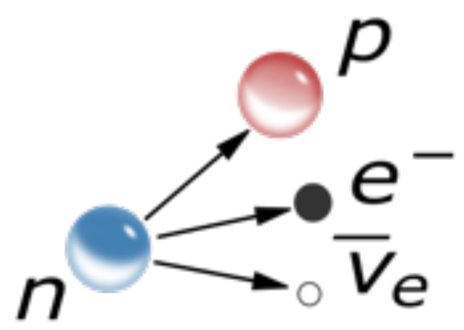
[A. Kronfeld, Ann.Rev.Nucl.Part.Sci 62(2012) 265; arXiv:1209.3468]

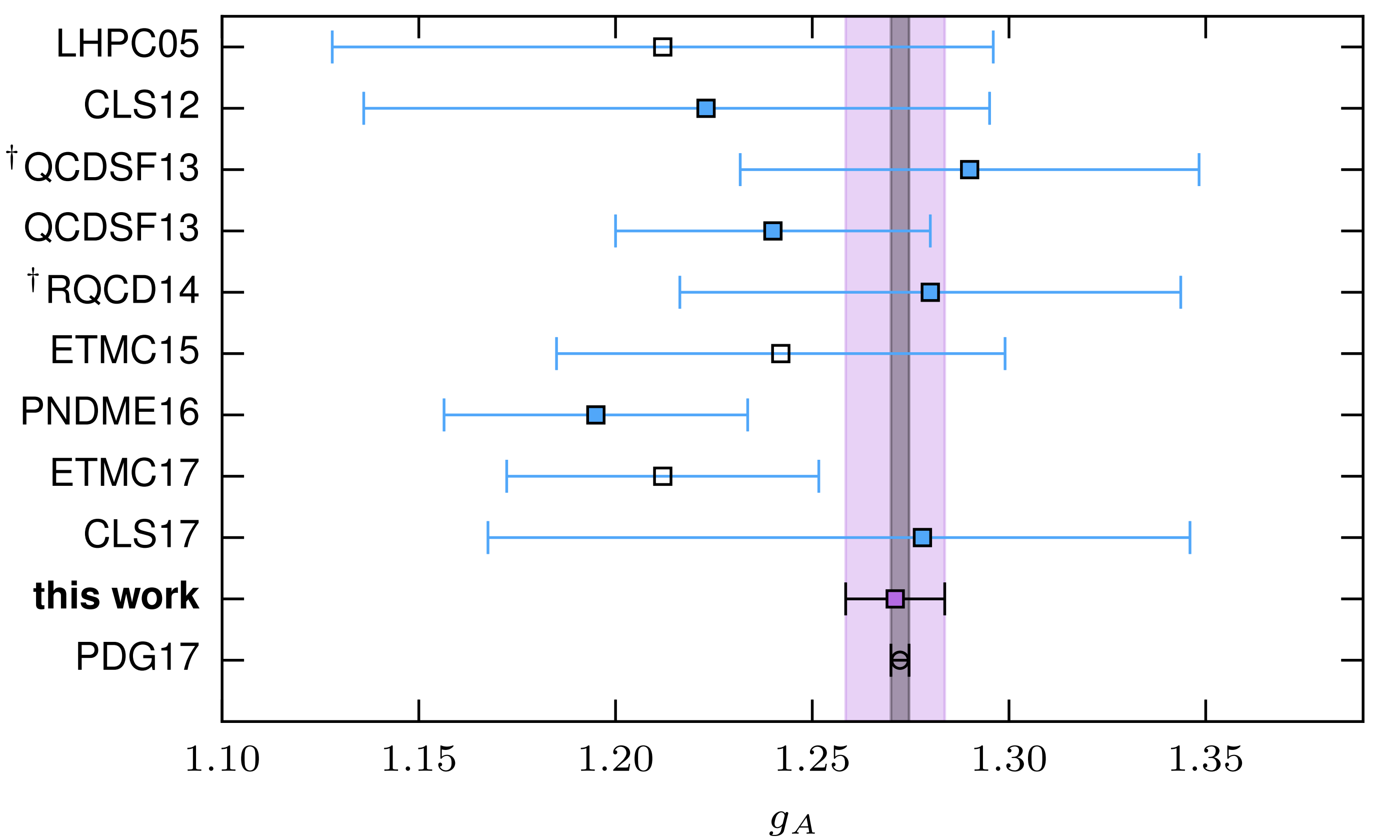
Successes of Lattice QCD: Axial charge

- Axial charge g_A: β-decay & Neutron lifetime

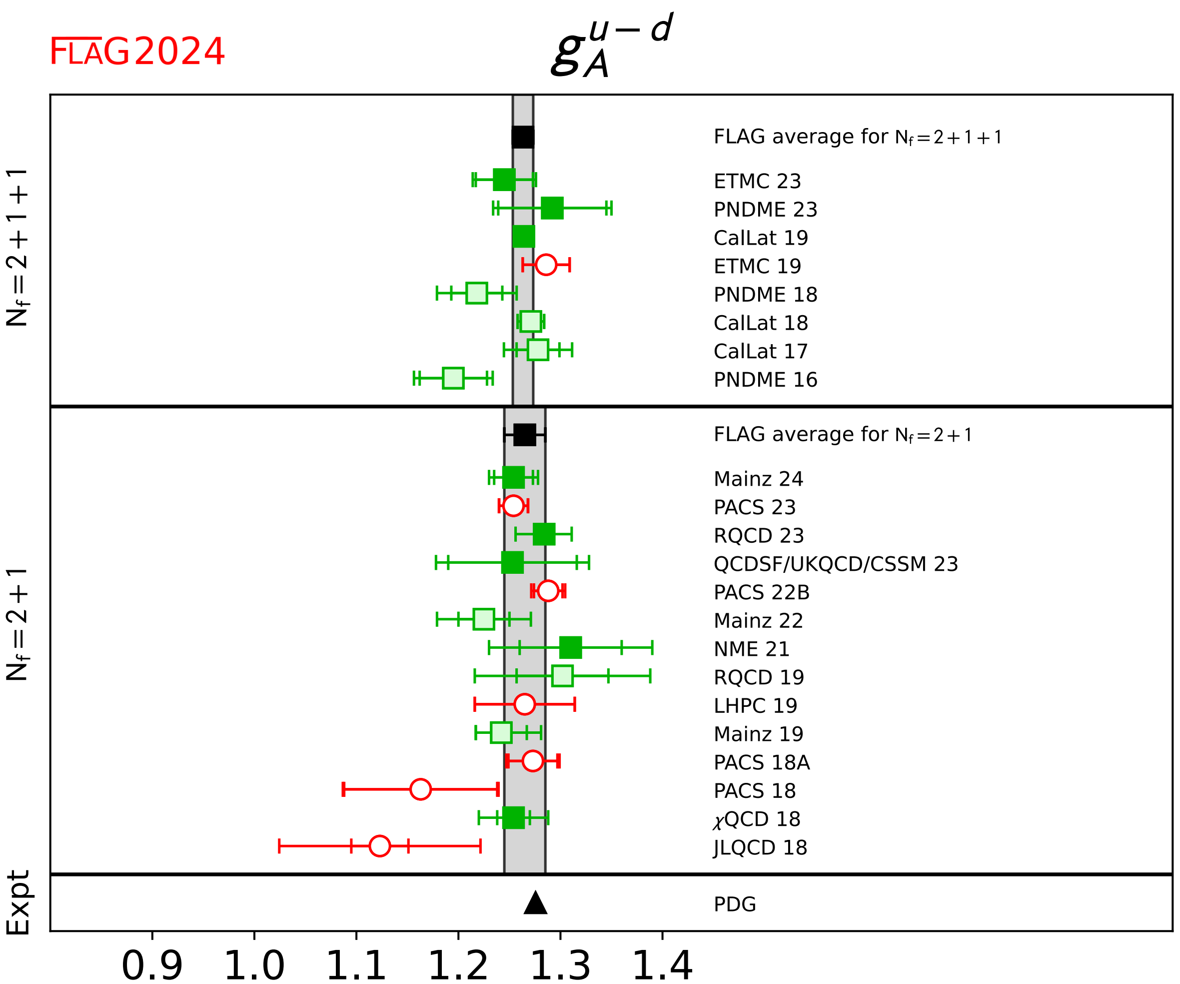
$$\frac{1}{\tau_n} = \frac{f}{2\pi^3} G_F^2 m_e^5 V_{ud}^2 (g_V^2 + 3g_A^2)$$







[Chang et al, Nature 558:7708 (2018)]



[Flavour Lattice Averaging Group(FLAG) Review 2024, Phys.Rev.D. 113:014508]

Successes of Lattice QCD: Nucleon Form Factors

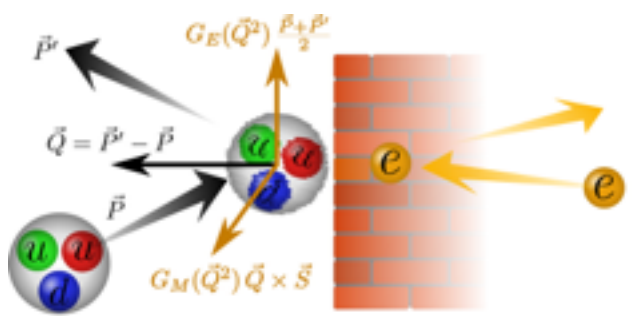
- Vector form factors: charge & magnetization structure

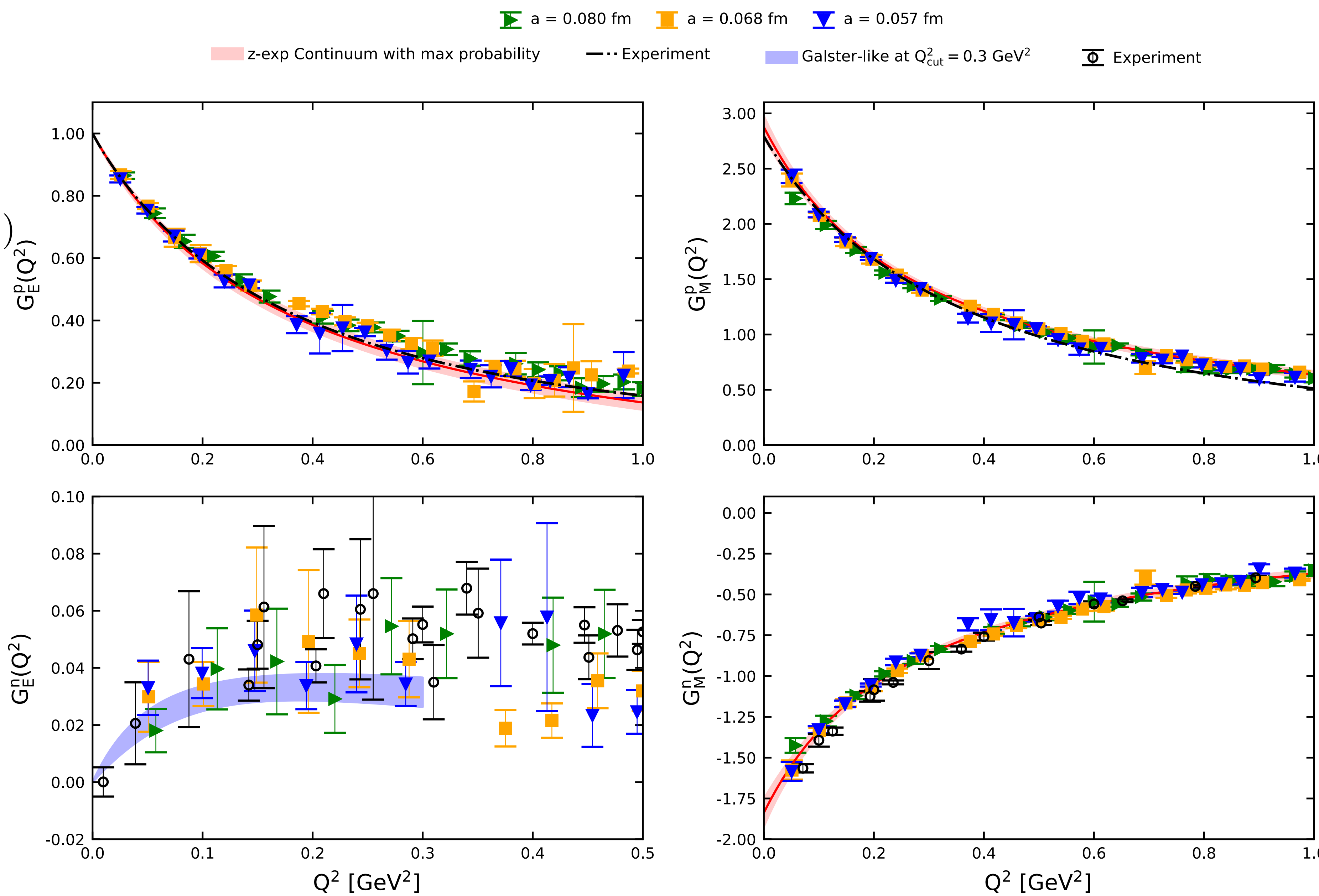
$$\langle P+q|\,\bar{q}\gamma^\mu q\,|P\rangle = \bar{U}_{P+q}\Big[F_1(Q^2)\gamma^\mu + F_2(Q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M_N}\Big]U_P$$

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2}F_2(Q^2)$$

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

(charge & current density in *"brick-wall" reference frame)*





[Alexandrou et al (ETM collaboration), arXiv:2507.20910]

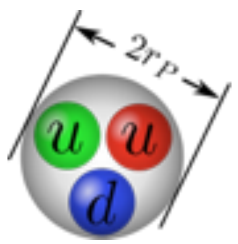
From QCD Lagrangian to Hadronic Matter

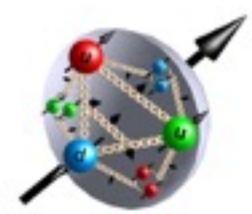
TASK: Starting from the Lagrangian ...

$$\mathcal{L}_{QCD} = \underbrace{-\frac{1}{4}(G^a_{\mu\nu})^2}_{\text{color gluon field}} + \underbrace{\bar\psi}_{\text{quark fields}} \left[i\gamma^\mu\partial_\mu - m_q\right] \underbrace{\psi}_{\text{quark fields}} - g_S \overbrace{\bar\psi\, A^\mu\, \gamma_\mu \psi}^{\text{quark–gluon interaction}}$$

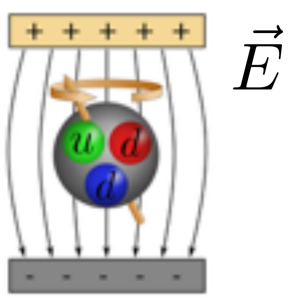
... Determine

- Hadron spectrum and interactions/scattering
- Hadron structure,
  mass & spin contributions





- Low-energy theory for nuclear forces
- Coupling to
  Dark Matter,
  beyond-SM particles&interactions,
  symmetry violations

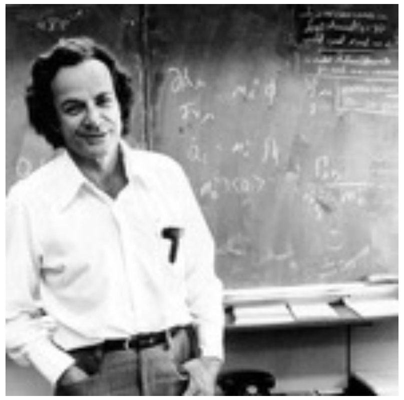


- Equation of State, phase transition,
  chiral symmetry restoration

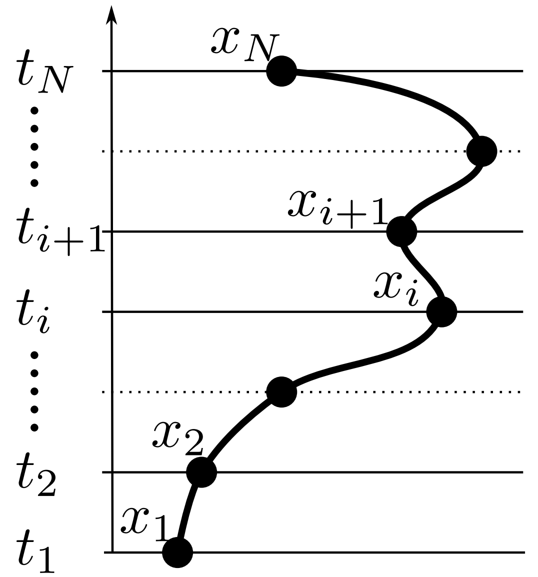
... with all systematic effects controlled (ultimately)

- Gauge-invariant by construction
- Symmetries : chiral, Lorenz
- Discretization $a \to 0$
- Finite volume $L \ggg (m_\pi)^{-1}$
- All parameters to physical values, include charm, bottom, ...
- Incorporate QED, ($m_u \neq m_d$), neutrinos, ...

Quantum Field Theory via Path Integral



R. Feynman
Path integrals in Quantum Mechanics



$$\cdot \int dx_{i+1} \quad K(x_{i+1}, \quad x_i)$$
$$\cdot \int dx_i \quad K(x_i, \quad x_{i-1})$$
$$\cdot \int dx_2 \quad K(x_2, \quad x_1)$$

Quantum Mechanics

$$\langle x(t_N)|x(t_1)\rangle \propto \prod_{i=2}^{N-1} \int dx_i \underbrace{K(x_i, t_i;\, x_{i-1}, t_{i-1})}_{\langle x(t_i)|x(t_{i-1})\rangle} \xrightarrow{N\to\infty} \int_{x(t_1)}^{x(t_N)} \mathcal{D}x(t) e^{i\int dt \mathcal{L}[x(t)]}$$

"path" — classical Lagrangian

Quantum Fields

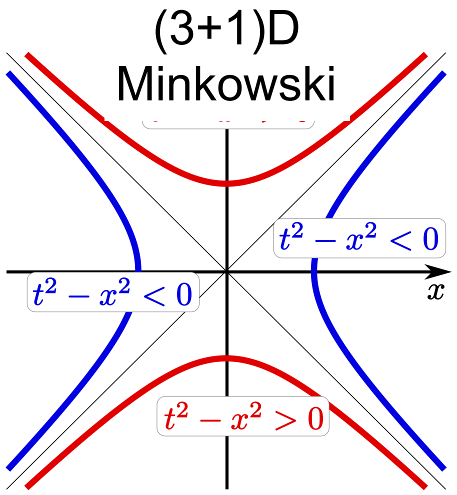
$$\langle \mathcal{O} \rangle \propto \int \mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\bar\psi \, e^{i\int d^4x\, \mathcal{L}[A,\psi,\bar\psi]} \, \mathcal{O}[A,\psi,\bar\psi]$$

all fields at each 4D spacetime point — Lagrangian — any observable

From formal expression to practical (numerical) calculation :

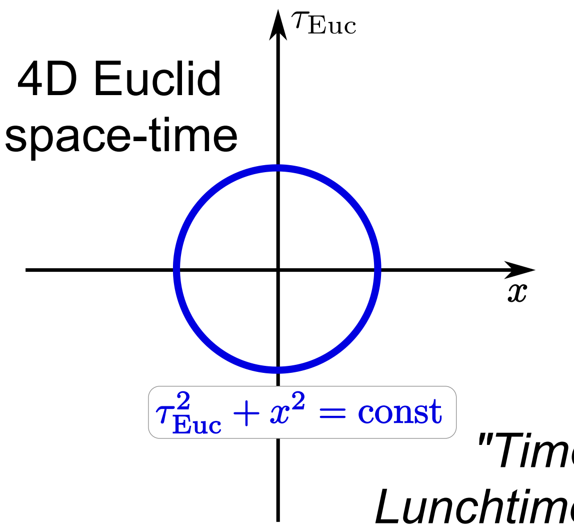
1. Convert (3+1)D Minkowski ➔ 4D Euclidean
2. Discretize fields and action on 4D grid
3. Sample fields with Monte Carlo
4. Evaluate observables

From Minkowski to Euclidean Space



Wick rotation

$$t \longrightarrow (-i\,\tau_{\rm Euc})$$



"Time is an illusion. Lunchtime – doubly so."
[D.Adams, "Hitchhiker's guide to Galaxy"]

$$e^{\,i\int dt\, d^3x\, \mathcal{L}[A,\psi,\bar\psi]} \longrightarrow e^{-\int d\tau_{\rm Euc}\, d^3x\, \mathcal{L}_{\rm Euc}[A,\psi,\bar\psi]}$$

oscillating complex phase — *real-valued probability*

Example : scalar theory $\quad \mathcal{L} = |\dot\phi|^2 - |\nabla\phi|^2 - V(\phi) \qquad \mathcal{L}_{\rm Euc} = |\partial_\tau\phi|^2 + |\nabla\phi|^2 + V(\phi)$

State evolution: $\quad |\phi(t)\rangle = e^{-iEt}\,|\phi(0)\rangle \qquad |\phi(\tau)\rangle = e^{-E\tau}\,|\phi(0)\rangle$

Field correlators: $\quad \langle\phi(t)\,\phi(0)\rangle \propto e^{-iEt} \qquad \langle\phi(\tau)\,\phi(0)\rangle \propto e^{-E\tau}$

- All separations are ≈ "spacelike"; all particles are ≈ "virtual"
- Euclidean *space-time* is O(4)-symmetric ⟹ easy to discretize
- Cannot study real-time-dependent processes
- No asymptotic states before/after interaction: NO scattering (directly)
- ONLY spectra (energies) , matrix elements

Euclidean QCD → Correlation Functions

The rest is very similar to standard QFT ...

- Partition function

$$Z = \int \mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\bar\psi\, e^{-S_g[A] - \bar\psi(\not{D}+m)\psi} \quad \longrightarrow \quad \int \mathcal{D}A_\mu\, e^{-S_g[A]} \left[\det(\not{D}+m)\right]^{\#\text{ flavors}}$$

(Quarks are Grassman fields, integrate out)

- Quark correlators

$$\langle \psi_x \bar\psi_y \ldots \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\bar\psi\, e^{-S_g[A] - \bar\psi(\not{D}+m)\psi} \left[\psi_x \bar\psi_y \ldots\right]$$

$$= \langle (\not{D}+m)_{xy} \ldots \rangle_{e^{-S_g[A]}\det(\not{D}+m)}$$

(all Wick contractions,
work out by hand)

... Except integration over GAUGE field A_μ: done by Markov-Chain Monte Carlo (MC) :

Either

- **Metrpololis/Heatbath** for local updates: only gluons, no dynamical fermions ("quenched")
- **Hybrid Monte Carlo (HMC)** for nonlocal updates (e.g. with dynamical-fermion determinant)

$$\{A_\mu(x)\}_n \to \{A_\mu(x)\}_{n+1} \to \{A_\mu(x)\}_{n+2} \to \{A_\mu(x)\}_{n+3}$$

$$\langle \mathcal{O} \ldots \rangle = \frac{1}{N} \sum_n^N \left(\mathcal{O}[\{A_\mu\}_n] \ldots\right)_{Wick}$$

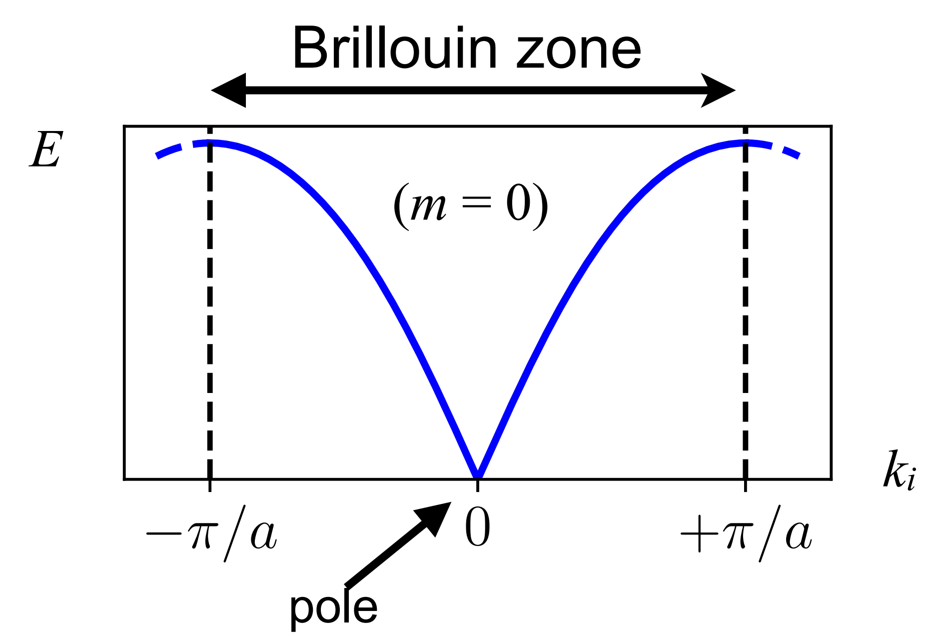
Discretization : Scalar Field

- derivative $$\int d^4x \sum_\mu |\partial_\mu \phi_x|^2 \quad \longrightarrow \quad \frac{1}{a^2}\sum_{x,\mu} |\phi_{x+\mu} - \phi_x|^2$$
- free EoM $$\partial_\tau^2 \phi + \nabla^2 \phi + m^2 \phi_x = 0 \quad \longrightarrow \quad \frac{1}{a^2}\sum_\mu (\phi_{x+\hat\mu} - 2\phi_x + \phi_{x-\hat\mu}) + m^2\phi = 0$$
- free dispersion relation
  for plane wave $\phi \sim e^{-E\tau + i\vec{k}\vec{x}}$ $$\frac{4}{a^2}\sinh^2\frac{aE}{2} = \frac{4}{a^2}\sum_i \sin^2\frac{ak_i}{2} + m^2$$

  $\text{lattice } \vec{k}^2$
- Energy is periodic in momentum (like in crystals)

  1 pole per Brillouin zone $-\pi/a < k_i < +\pi/a$
- free propagator for $k_\mu = (\vec{p}, iE)$

$$\langle \phi_x \phi_y^\dagger \rangle = (\Delta + m^2)_{xy} \,,$$
$$\langle \phi_k \phi_{k'}^\dagger \rangle = \frac{\delta_{kk'}}{m^2 + \frac{4}{a^2}\sum_\mu \sin^2\frac{1}{2}ak_\mu}$$



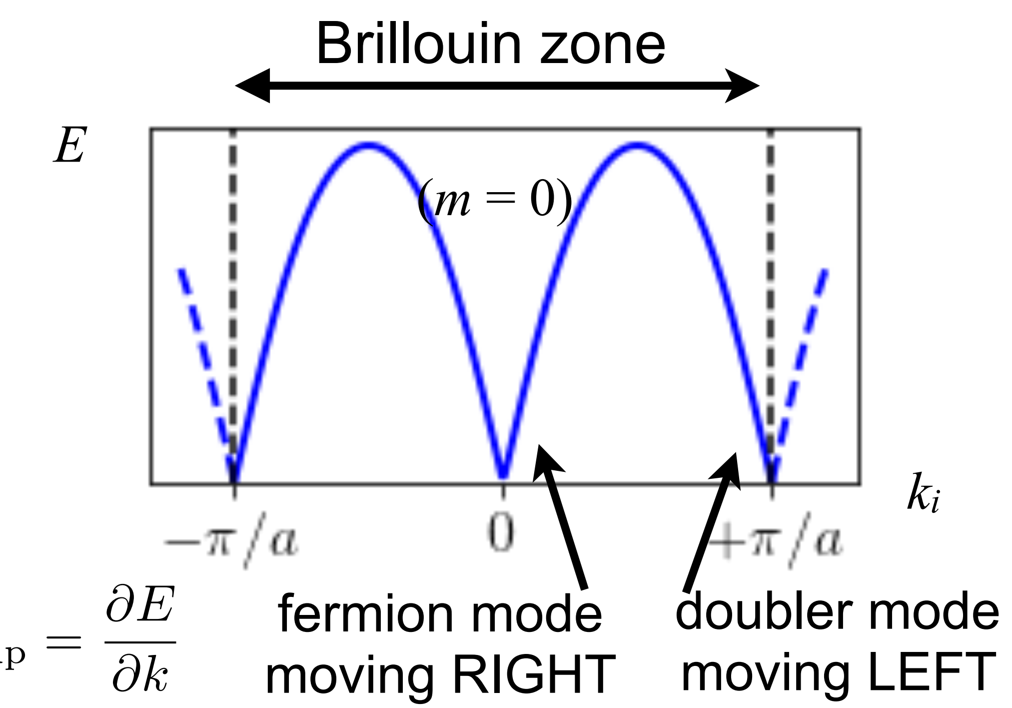
Discretization : Fermions, Naive

- free Dirac EoM $$\gamma_\mu(\partial_\mu\psi)_x + m\psi_x = 0 \quad\longrightarrow\quad \frac{1}{2a}\sum_\mu \gamma_\mu(\psi_{x+\hat\mu} - \psi_{x-\hat\mu}) + m\psi_x = 0$$
- free dispersion relation for plane wave $\psi \sim e^{-E\tau + i\vec{k}\vec{x}}$ $$\frac{1}{a^2}\sinh^2(aE) = \frac{1}{a^2}\sum_i \sin^2(ak_i) + m^2$$

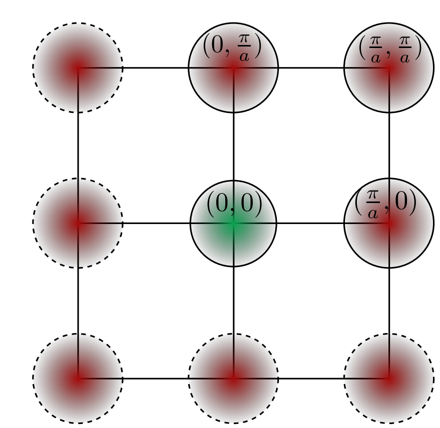
- Additional pole for every periodic dimension ; **16 total species** on momentum 4D-torus
- equal number of left & right Weyl species $N_L = N_R$ **at least one doubler** of opposite chirality (topology)
- wrong number of "flavors" in $\det\left(\not{D} + m\right)$
- doublers cancel any chiral effects, e.g. ABJ anomaly

$$\partial_\mu J^5_\mu \equiv 0 \neq \frac{1}{16\pi^2} G_{\mu\nu}\widetilde{G}_{\mu\nu} \qquad \text{(m=0)}$$

(axial current $J^5_\mu = \bar\psi_R\gamma_\mu\psi_R - \bar\psi_L\gamma_\mu\psi_L$)



$$v_{\rm group} = \frac{\partial E}{\partial k}$$



NO-GO Theorem [Nielsen, Ninomiya '81]: choose three out of four

- regularized chiral fermions
- gauge invariance
- local action
- no doubler(s)

Discretization : Fermions, Wilson

- Solution to the doublers problem: add irrelevant scalar-like term (irrelevant, vanishes with a→0)

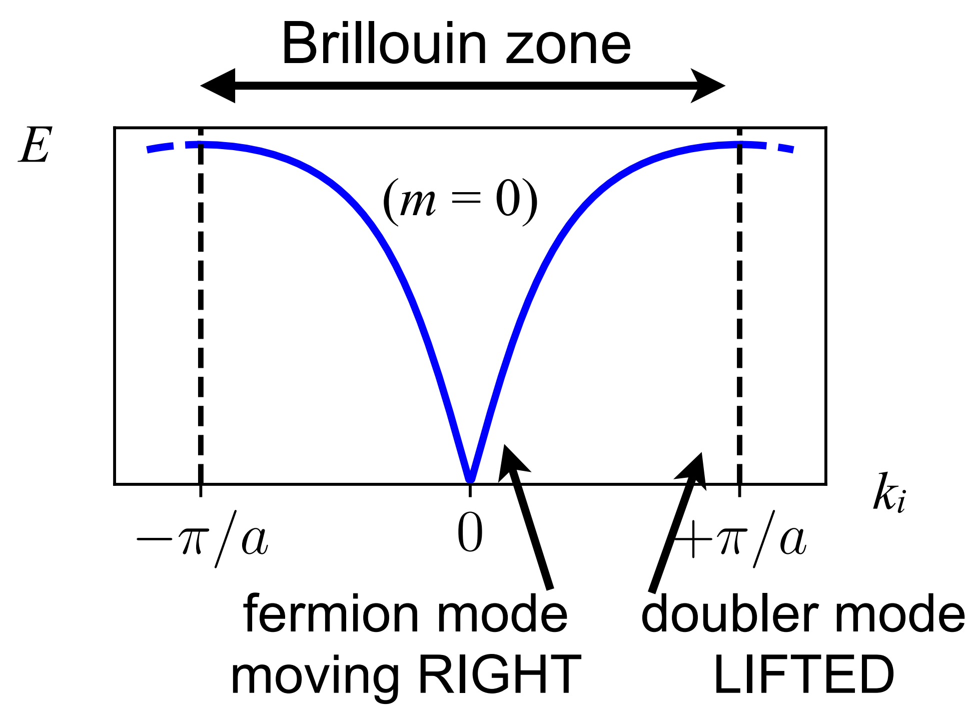
$$\frac{a}{2}[-\bar{\psi}_R\,\partial_\mu^2\psi_L - \bar{\psi}_L\,\partial_\mu^2\psi_R]$$

(Note left-right fermion mixing)

- the Laplacian lifts doubler-fermion poles:

for plane wave $\psi \sim e^{-E\tau + i\vec{k}\vec{x}}$

Dirac operator $\not{D} \quad\longrightarrow\quad \not{D} + \frac{2}{a}\sum_\mu \sin^2\frac{ak}{2}$



- restores ABJ but breaks chiral symmetry HARD
- L and R modes mix via divergent additive mass correction

$$\delta m \quad \propto \quad a^{-1}$$

- fine tuning of bare mass required to cancel δm

Most frequently used fermion action

Common variations

- "Clover"-improved discretization to O(a^2) order, faster continuum limit
- Twisted-mass fermion: automatic O(a^2) improvement

Alternative: "staggered" fermions
- eliminate some doublers
- economical as "sea" quarks
- cumbersome as "valence" quarks

Discretization : Fermions, Ginsparg-Wilson

Modified chiral symmetry on the lattice [Ginsparg, Wilson '82]

$$\{\gamma_5, \not{D}\} = a\not{D}\gamma_5\not{D}$$

- RHS appears due to fermion field blocking (continuum → lattice)
- vanishes in the continuum a→0

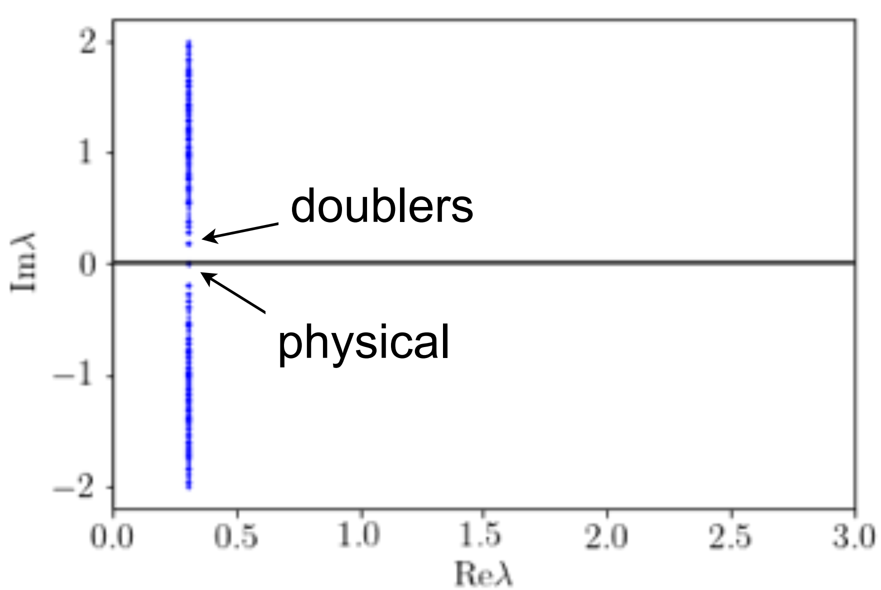
- all eigenvalues on a complex circle
- index (N_L–N_R) for Atiyah-Singer thm

$$\lambda = \frac{1}{a}\left(1 + e^{-i\alpha}\right)$$

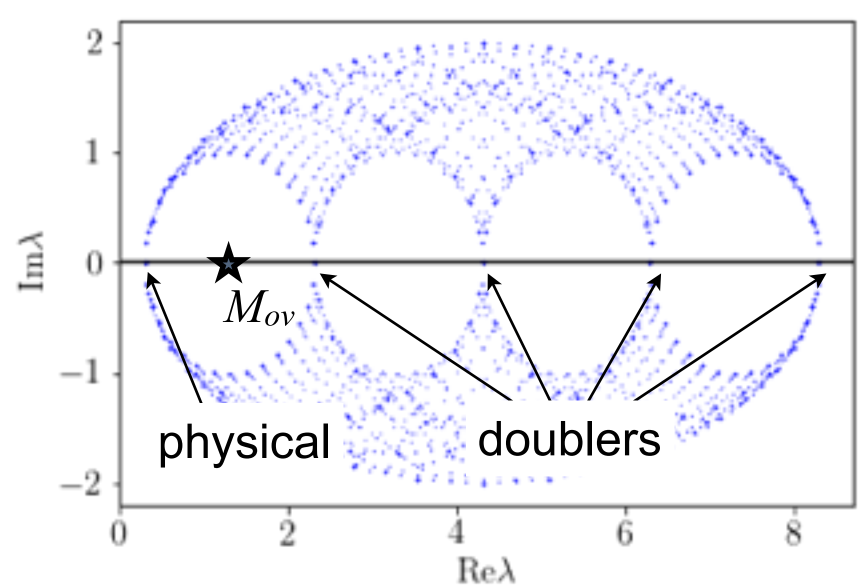
continuum → lattice:
maps Dirac eigenvalues
from a line to a circle

Spectra of lattice Dirac operators

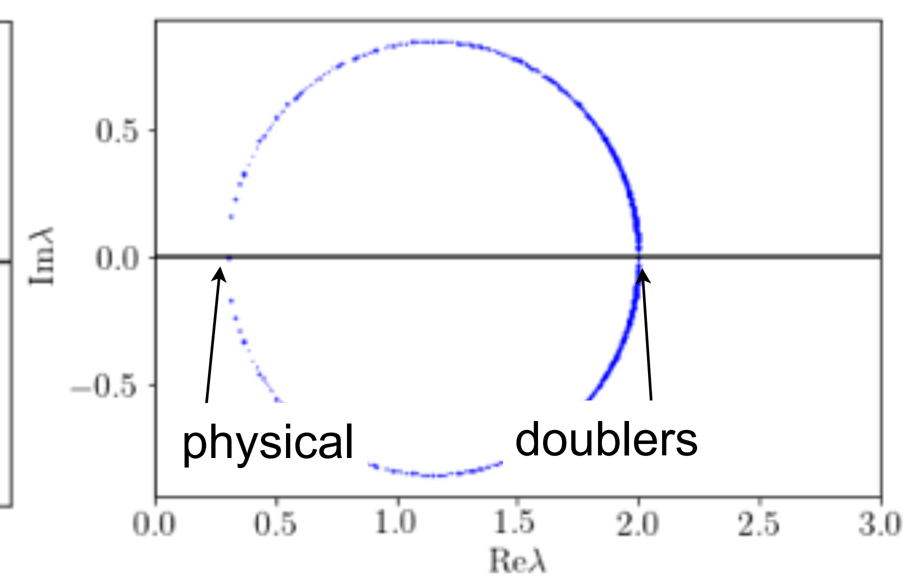
- Naive



- Wilson



- Ginsparg-Wilson



- "Overlap" fermions [Neuberger '98]

$$\not{D}_{ov} = 1 + \gamma_5\, \mathrm{sign}\left[\gamma_5(D_{\mathrm{Wilson}} - M_{\mathrm{ov}})\right]$$

- "Domain Wall" fermions [D.Kaplan ; Y. Shamir]
physical modes on a "defect" in 5th dimension
(equivalent to overlap)

Discretization : Gluons & Gauge Invariance

- Gauge action $\quad S_g = \frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2 \qquad [A^{\mathrm{lat}}_\mu = g A^{\mathrm{pert}}_\mu]$

- Gauge potentials : on links connecting "matter" sites

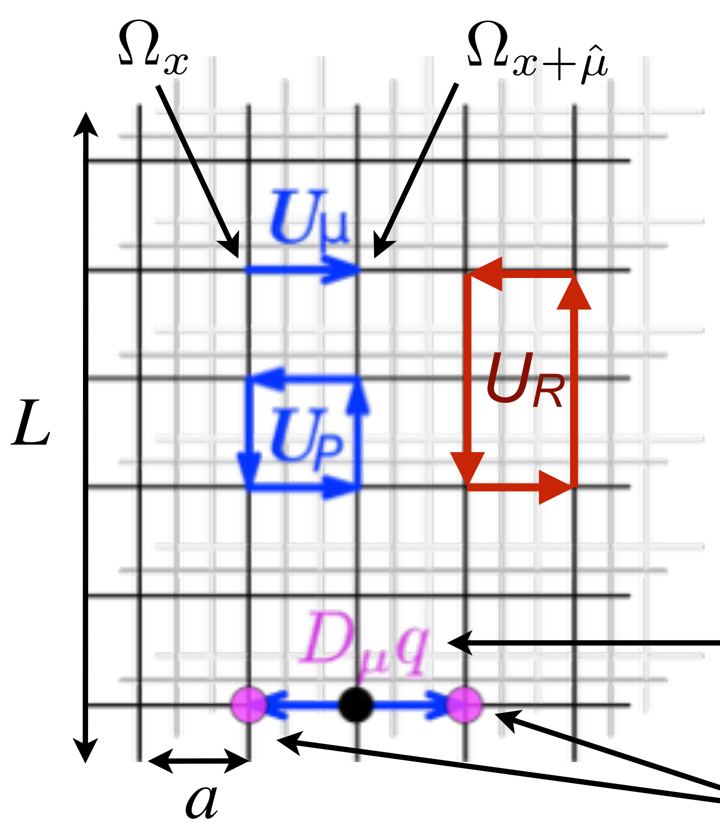
$$(D_\mu\phi)_x = \frac{1}{2a}\left(U_{x,\mu}\phi_{x+a\hat\mu} - U_{x,-\mu}\phi_{x-a\hat\mu}\right)$$

- gauge link $U_{x,\mu}$ = covariant transporter

$$U_{x,\pm\mu} = \mathrm{P}\exp\left\{ i\int_x^{x\pm a\hat\mu} [\lambda^a A^a_\mu] dx'_\mu \right\}$$

- Gauge covariant by construction

$$U_{x,\mu} \to \Omega_x U_{x,\mu} \Omega^\dagger_{x+a\hat\mu}$$
$$\phi_x \to \Omega_x \phi_x$$
$$(D_\mu\phi)_x \to \Omega_x (D_\mu\phi)_x$$



$$U_\mu \approx e^{ig\,aA_\mu}$$
gluon field

$$U_P \approx e^{ig\,a^2 F_{\mu\nu}}$$
$(\text{"curl"}\, A_\mu)$

covariant derivative

matter field (fermion, scalar)

- Plaquette = square loop transporter (curvature)

$$U_{P\mu\nu} = U_1 U_2 U_3^\dagger U_4^\dagger$$

- Lattice [Wilson] action

$$S^{\mathrm{lat}}_g = \sum_{x,\mu<\nu} \frac{\beta}{N_c} \underbrace{\mathrm{Re}\,\mathrm{Tr}\left(1 - U_{P\mu\nu}\right)}_{\sim\frac{1}{4g^2}(F^a_{\mu\nu})^2 + O(a^2)}$$

- Typically improved to O(a^4) with 1x2 rectangles U_R [Symanzik]

Path Integration : Metropolis&Heatbath

- *How to* ***EFFICIENTLY*** *sample contributions to the path integral?*

  Relevant *{A}* field configurations at narrow balance of entropy / action

- Strategy: create Markov Chain

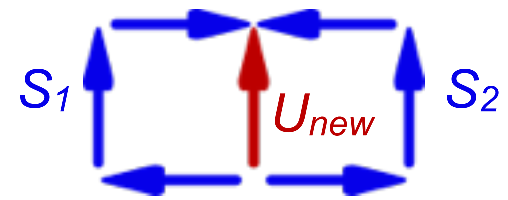
$$\{A_\mu\}_n \longrightarrow \{A_\mu\}_{n+1}$$

  such that the probability distribution converges to

$$P(A_\mu) \sim e^{-S[A_\mu]}$$

- Metropolis algorithm :
  - construct candidate for $\{A\}_{n+1}$
  - calculate change of action $\Delta S = S[\{A\}_{n+1}] - S[\{A\}_n]$
  - accept with probability $\min\{1, e^{-\Delta S}\}$
    (always accept if $\Delta S<0$)
- works well if easy to construct candidates, calculate ΔS

- Example: local link updates
  in quenched (no dynamical quarks) QCD



$$\Delta S = \frac{\beta}{N_c} \mathrm{ReTr}\left[(S_1 + S_2)^\dagger\,(U_{\mathrm{old}} - U_{\mathrm{new}})\right]$$

- What if the change in action nonlocal / difficult to calculate?

$$\exp\left[-S_g(A_\mu)\right]\det(\not{D}+m) \quad\longrightarrow\quad \exp\left[-S_g(A_\mu) + \underbrace{\log\det(\not{D}+m)}_{\text{nonlocal}}\right]$$

Path Integration : Molecular Dynamics/Hybrid MC

- For global updates: need to find candidate with ΔS=O(1)
- Example: Simulate fermion determinant using BOSONIC pseudo-fermions

$$\det(\not{D}+m) \quad\longrightarrow\quad \int \mathcal{D}\eta\,\mathcal{D}\eta^\dagger\;\exp\big[-\eta^\dagger(\not{D}+m)^{-1}\eta\big]$$

- Convert Canonical ensemble with action S

  to Micro-canonical with Hamiltonian

$$H[\pi_x,\phi_x] = \frac{1}{2}\sum_x \pi_x^2 + S[\phi]$$

conjugate momenta to $\phi_x = \{A_\mu, \eta\}_x$,
pick randomly for each update

- No change for the averages

$$\langle\mathcal{O}[\phi]\rangle = \frac{1}{Z}\int\mathcal{D}\phi\,e^{-S[\phi]}\,\mathcal{O}[\phi]$$
$$= \frac{1}{Z'}\int\mathcal{D}\pi\,\mathcal{D}\phi\,e^{-\frac{1}{2}\sum_x\pi_x^2-S[\phi]}\,\mathcal{O}[\phi]$$

- Conservative evolution with MC "time":

  integrate $\{\pi, \phi\}$ $\rightarrow$ new MC sample $\{\pi', \phi'\}$

$$\begin{cases}\dot\phi_x &= \pi_x\\ \dot\pi_x &= -\frac{\delta S}{\delta\phi_x}\end{cases} \quad\Rightarrow\quad H[\pi_x,\phi_x] = H[\pi'_x,\phi'_x]$$

- must be reverisible : $\{-\pi', \phi'\} \rightarrow \{-\pi, \phi\}$

  (otherwise non-ergodic)

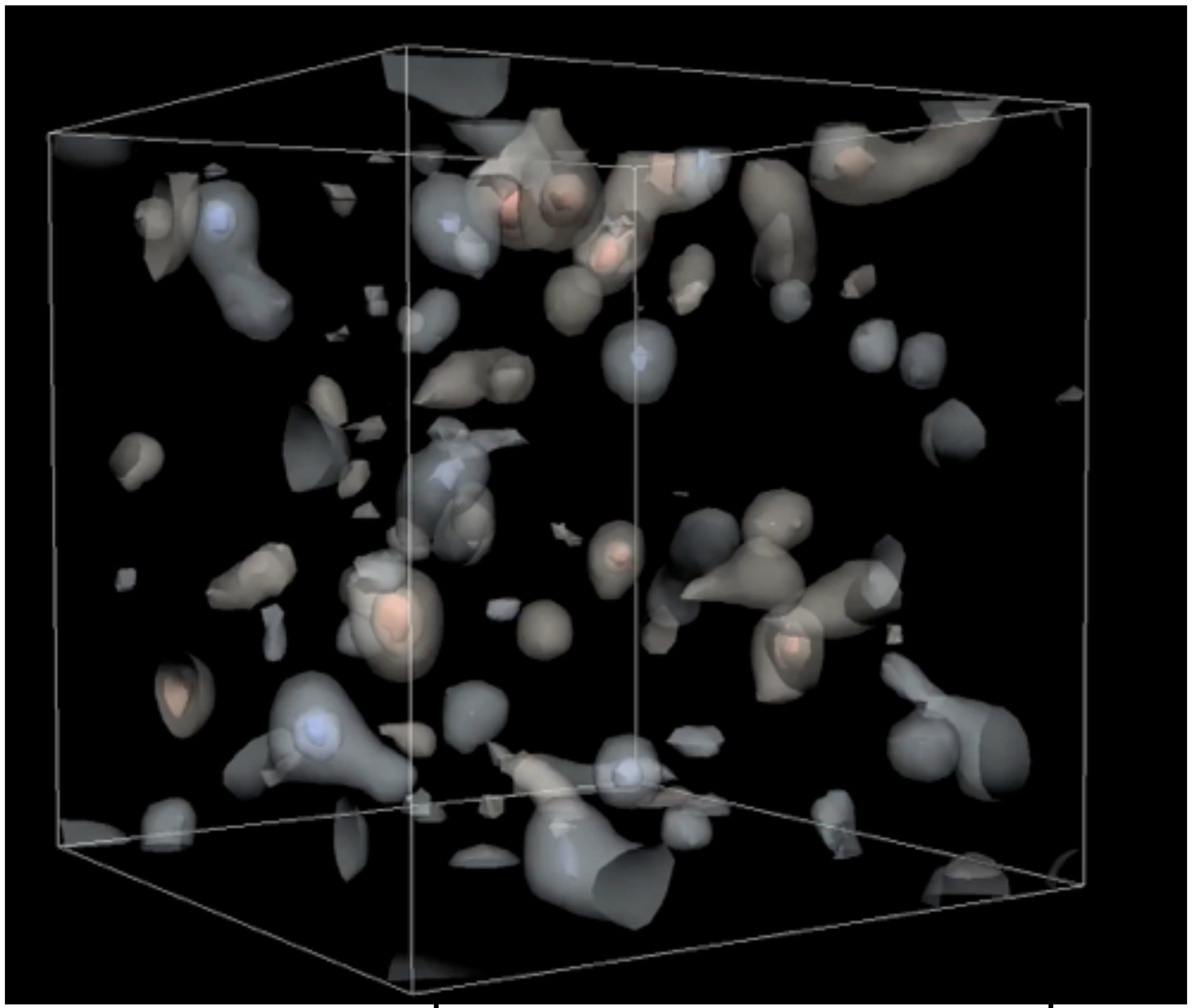
- correct integration error with Metropolis:

  accept $\quad [\pi,\phi] \longrightarrow [\pi',\phi']$

  with prob. $\min\{1, \exp(H[\pi,\phi]-H[\pi',\phi'])\}$

Topology of Gluon Fields

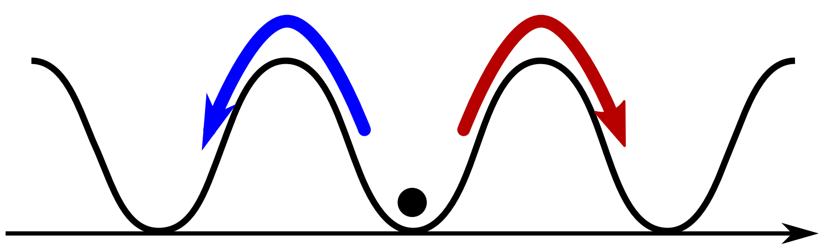
Density of $(\mathbf{E}\cdot\mathbf{H})_{\text{color}}$



5 fm=5·10^{-15}m

6 s video = 5 fm / c = 1.7·10^{-23} s physical time
[Lattice QCD simulations with physical parameters]

Quantum tunneling of color fields:
instantons and anti-instantons



$$|vac\rangle_\theta = \sum_Q e^{i\theta Q}\,|Q\rangle$$

$$Q = \int d^4x\,\frac{1}{32\pi^2}\,F^a_{\mu\nu}\,\widetilde{F}^a_{\mu\nu}$$

$$\propto \int d^4x\,(\mathbf{E}^a\cdot\mathbf{H}^a)$$

= (winding number $SU(N_c) \to S_3$)

- Strong CP problem: is Θ angle zero?

$$\mathcal{L}_\theta = \theta Q$$

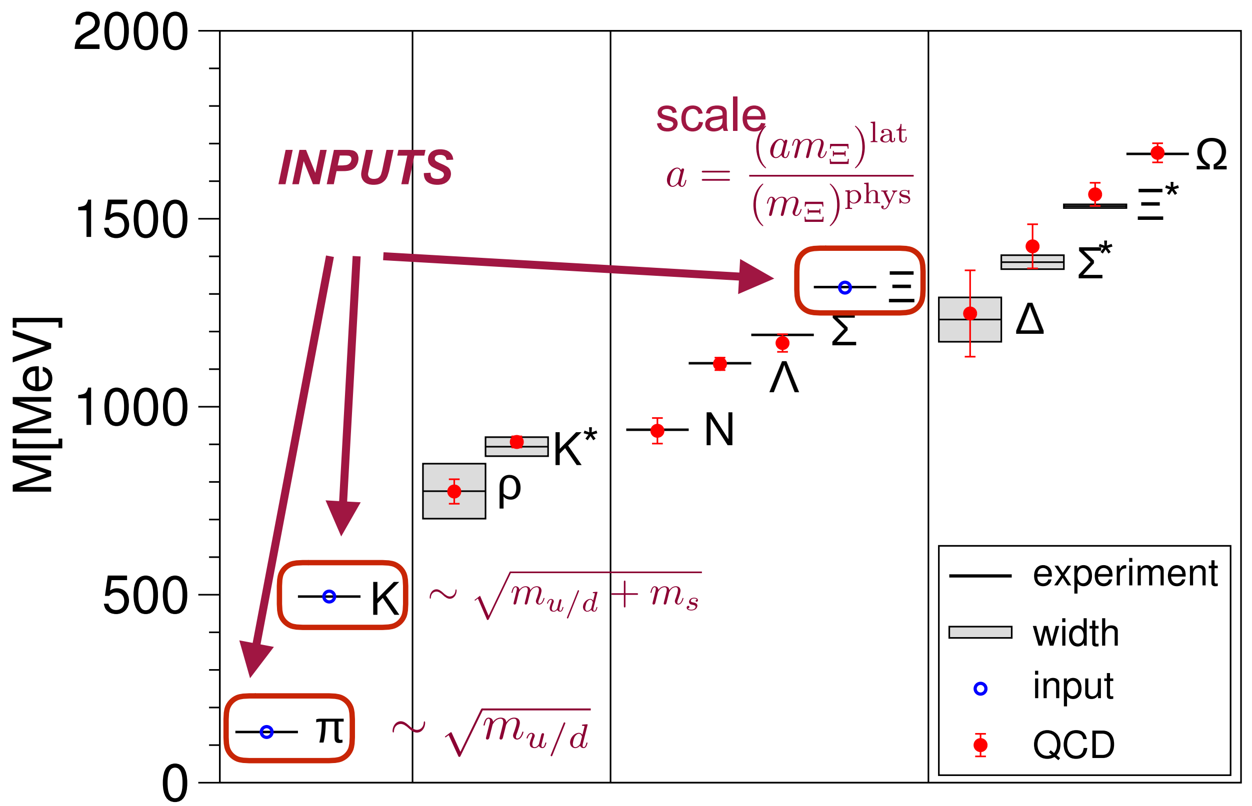
Θ-term violates CP symmetry

Setting Parameters in Lattice QCD

$$\mathcal{L}^{\mathrm{lat}} = \sum_{\mu<\nu} \frac{\beta}{N_c} \mathrm{Re\,Tr}\,(1 - U_{P_{\mu\nu}}) + \sum_q \bar{q}(\not{D} + m_q)q$$

Typical "2+1 flavor" calculation: 2 light u/d + strange dynamical quarks

- QCD coupling β = varied to change lattice scale(spacing) a
- quark masses $m_{u/d}$ and m_s = tuned to reproduce pion, kaon masses (requires chiral perturbation theory)



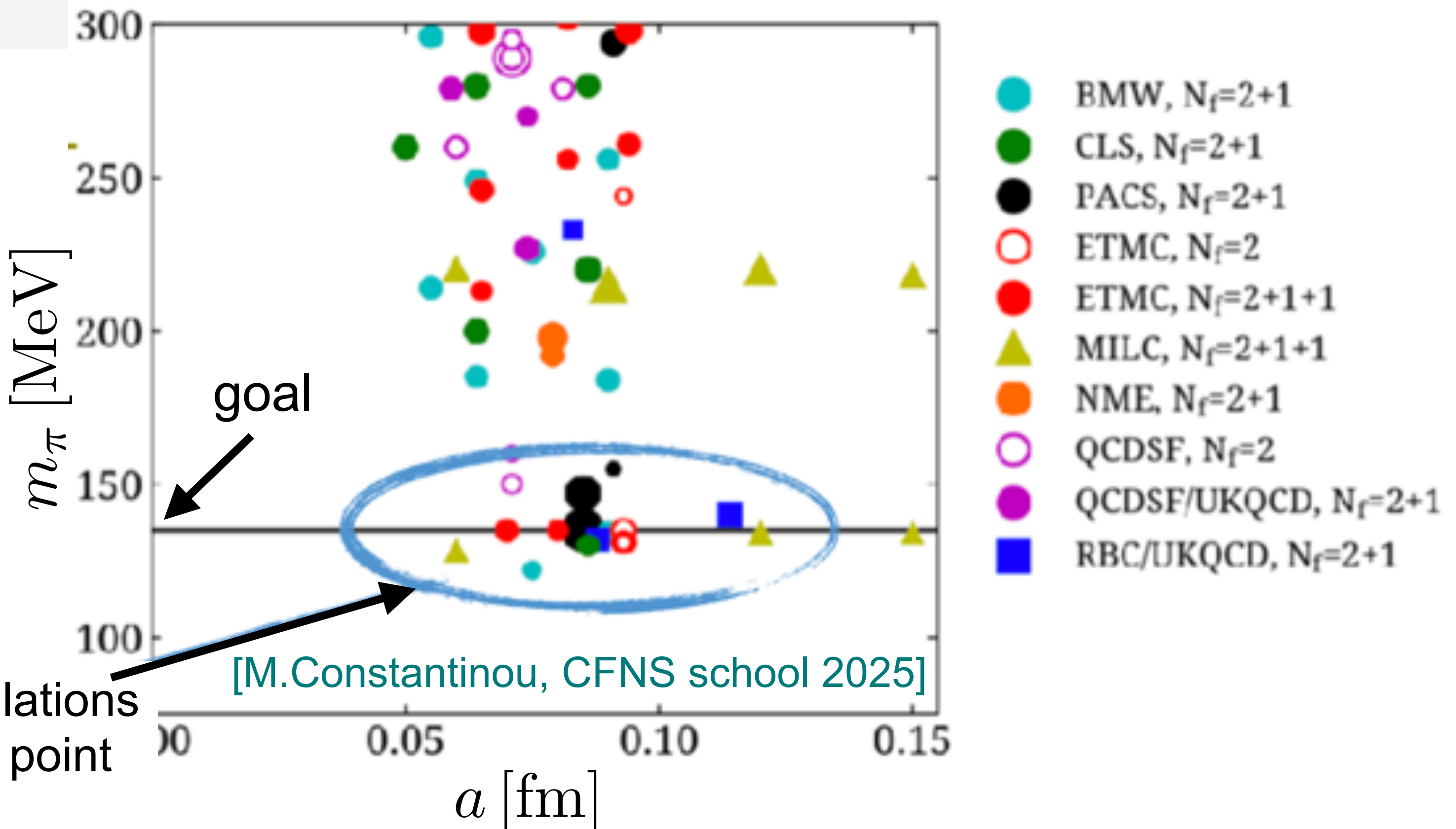
[S.Durr et al, **Science** 322 (2008) 1224]

- Everything else are pure QCD predictions!

It Does Get Expensive!

Brute-force'able challenges

- typical modern lattice size $64^3 \times 128$
  more towards continuum limit, large volume
- light-quark Dirac operator inversion
  for propagators , HMC evolution
- statistical noise, especially
  for gluonic observables



Tough challenges

- critical slowing down: non-ergodic MC
  due to topological sector barrier $\Rightarrow$ freezing

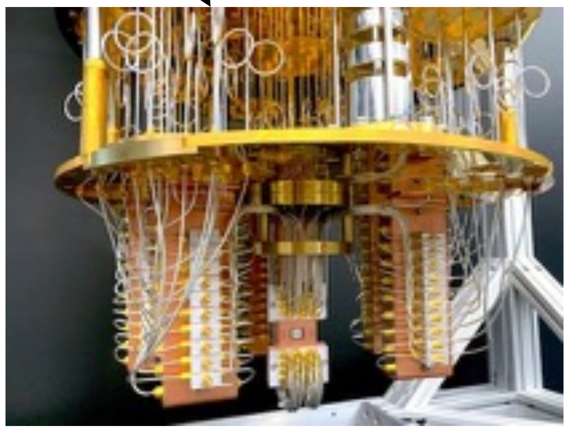
  Multiscale evolution, ML-based sampling algorithms

- sign problem with baryon chem.potential $\mu_B \neq 0$
  $\Rightarrow$ QCD Equation of State $P(\mu_B, T)$ is hard

  Quantum computing?

Now

Future?

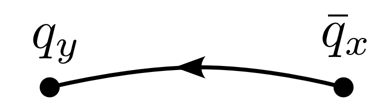


Hadron Correlators on Lattice

- Quark propagator

$$Q_{yx} = \langle q_y \, \bar{q}_x \rangle = (\not{D} + m)^{-1}_{yx}$$

solve lattice Dirac equation ×N_s ×N_c
for chosen source point

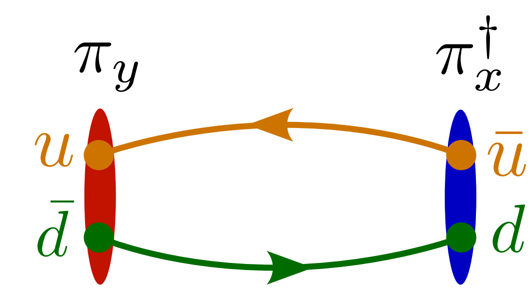


- Pion correlator $\pi = \bar{d}\,\gamma_5\, u$

$$\langle [\bar{d}\,\gamma_5\, u]_y \, [\bar{u}\,\gamma_5\, d]_x \rangle = \mathrm{Tr}[Q_{yx} Q_{xy}] \xrightarrow{Q_{xy} = \gamma_5 Q^\dagger_{yx} \gamma_5} \mathrm{Tr}|Q_{yx}|^2$$

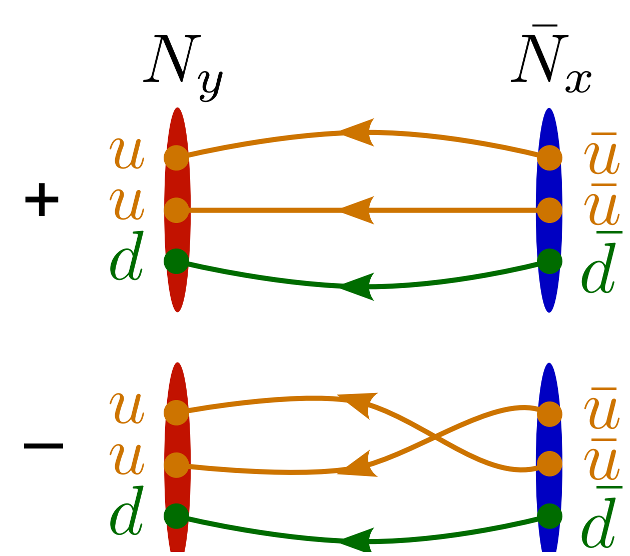
"sink" "source"

no cancellations,
the lightest hadron



- Nucleon correlator $N = \epsilon^{abc}\,[u^{aT}\, C\gamma_5\, d^b]\, u^c$

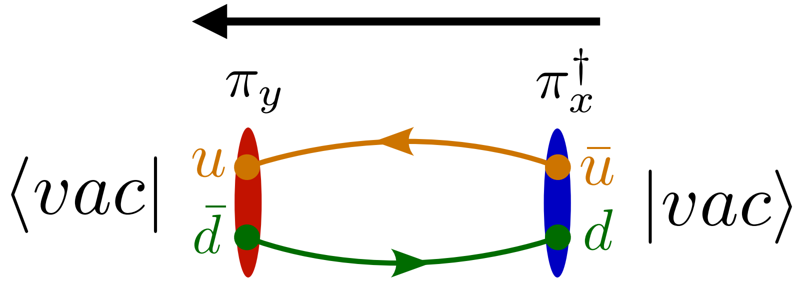
$$\langle [(u\, C\gamma_5\, d)\, u]_y \, [\bar{u}\, (\bar{d}\, C\gamma_5\, \bar{u})]_x \rangle$$



Hadron Masses and Energies

- Example : pion correlator

Euclidean time (y_4–x_4)



$$\mathbf{1} = \sum |n\rangle \langle n|$$

Complete set of states

(*) only if transfer matrix $e^{-a\hat{H}}$ exists, strictly proven only for some actions

Sum over **π**-like states E_0<E_1<...

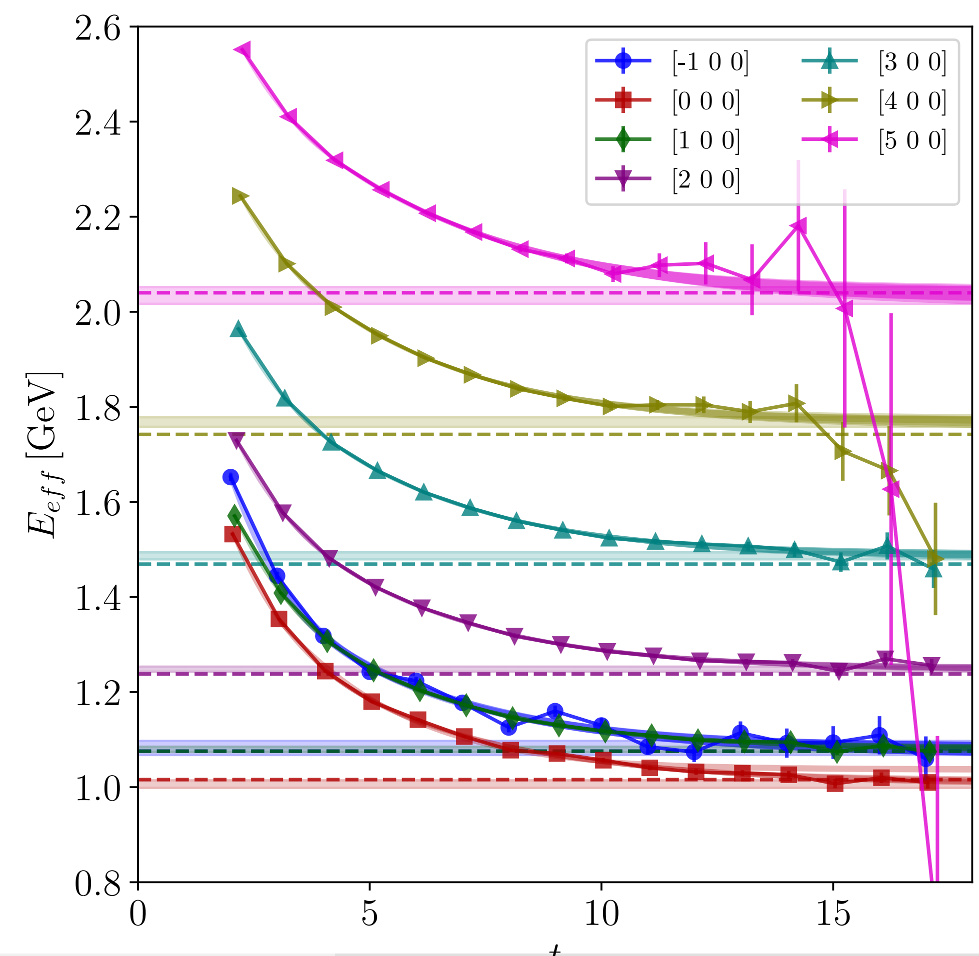
$$= \sum_n \langle vac|\pi|\pi_n\rangle \, e^{-E_n^\pi (y_4 - x_4)} \, \langle \pi_n|\pi^\dagger|vac\rangle$$

Hadron operator Overlaps $\boldsymbol{Z_n}$ with **π**-like states

$$\longrightarrow \; |Z_0|^2 \, e^{-E_0^\pi (y_4 - x_4)}$$ [(y_4–x_4) → large]

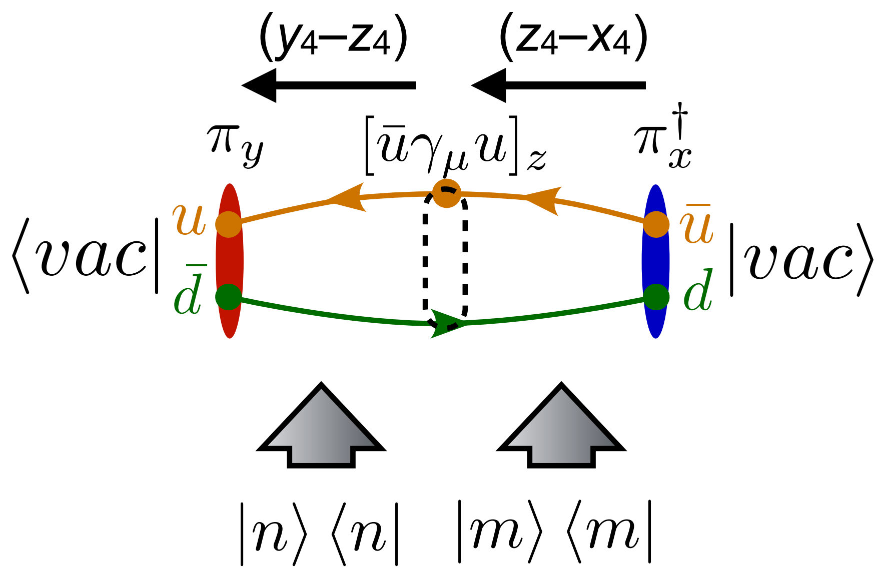
- Effective mass (energy)

$$m^{\rm eff}(t) = \log \frac{\langle \pi(t+1)\, \pi^\dagger(0)\rangle}{\langle \pi(t)\, \pi^\dagger(0)\rangle}$$



Hadron Matrix Elements

- Example : Pion-current correlator



Complete set of states

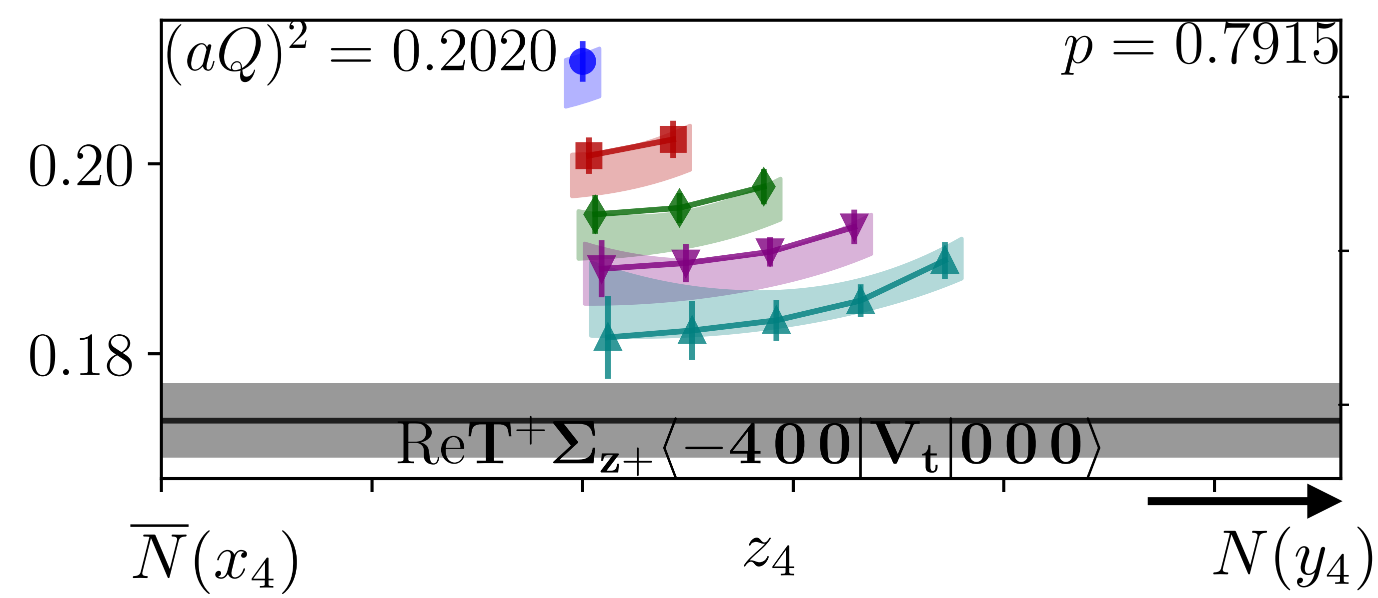
$$= \sum_{m,n} Z_n \cdot e^{-E_n^\pi (y_4 - z_4)} \cdot \langle n|\bar{u}\gamma_\mu u|m\rangle \cdot e^{-E_m^\pi (y_4 - z_4)} \cdot Z_m^*$$

$$\longrightarrow \; |Z_0|^2 \, \langle 0|u\gamma_\mu u|0\rangle \, e^{-E_0^\pi (y_4 - x_4)}$$

[(z4–x4), (y4–z4) → large]

- Matrix element from correlator ratio

$$\frac{\langle \pi(t)\; [\bar{u}\gamma_4 u]_\tau \; \pi^\dagger(0)\rangle}{\langle \pi(t)\, \pi^\dagger(0)\rangle} \xrightarrow{t,\tau\to\infty} \langle 0|u\gamma_\mu u|0\rangle$$



Hadron Excited States & Fits

- Pion correlator, Multi-state fits

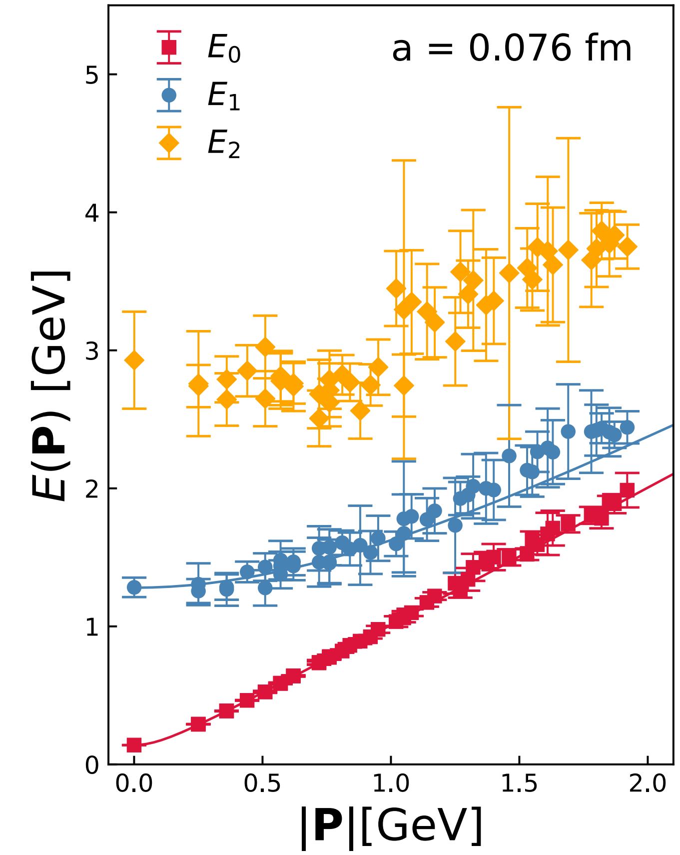
$$\langle \pi(t)\,\pi^\dagger(0)\rangle \quad \sim \quad C_0 e^{-E_0 t} + C_1 e^{-E_1 t} + \ldots$$

Can pick out excited states, but better with variational analysis (multiple $\boldsymbol{\pi}$ *operators + diagonalization)*

2nd excited

1st excited

ground



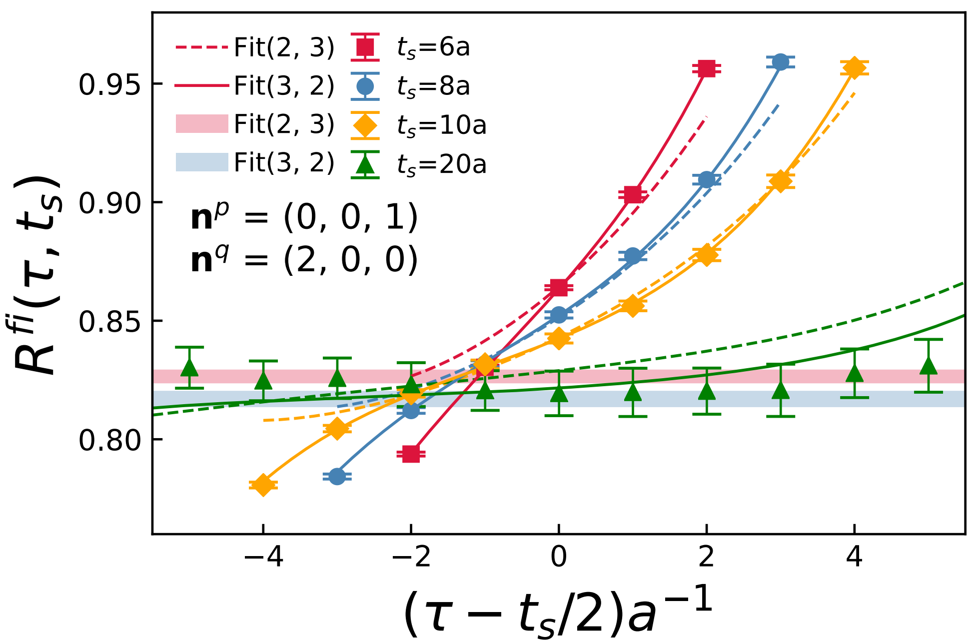
- Pion-current correlator, two-state fit

$$\langle \pi(t_s)\,[\mathcal{O}]_\tau \pi^\dagger(0)\rangle \quad \sim \quad e^{-E_0 t_s}\Big(\mathcal{A}_{00} + \mathcal{A}_{01} e^{-\Delta E_1 \tau} + \mathcal{A}_{10} e^{-\Delta E_1 (t_s-\tau)} + \mathcal{A}_{11} e^{-\Delta E_1 t_s}\Big)$$

Excited states $\Longrightarrow$ *hardest systematic to conrol if ΔE is small*

$$\langle \pi(p')|\mathcal{O}|\pi(p)\rangle = \frac{\mathcal{A}_{00}}{C_0}$$

matrix elements ➞ observables
form factors, couplings, etc



[X.Gao et al, arXiv: 2102.06047]

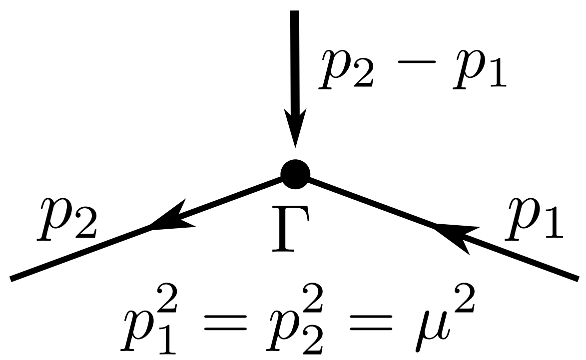
Lattice Operator Renormalization

Operators are expressed in lattice fields, need renormalization (e.g. matching to MSbar)

- Lattice perturbation theory – cumbersome and limited by order

- RI-MOM regulator-independent scheme [Martinelli et al (1995)]

  operator vertices with external 4D plane-wave fields, all momenta ~ μ



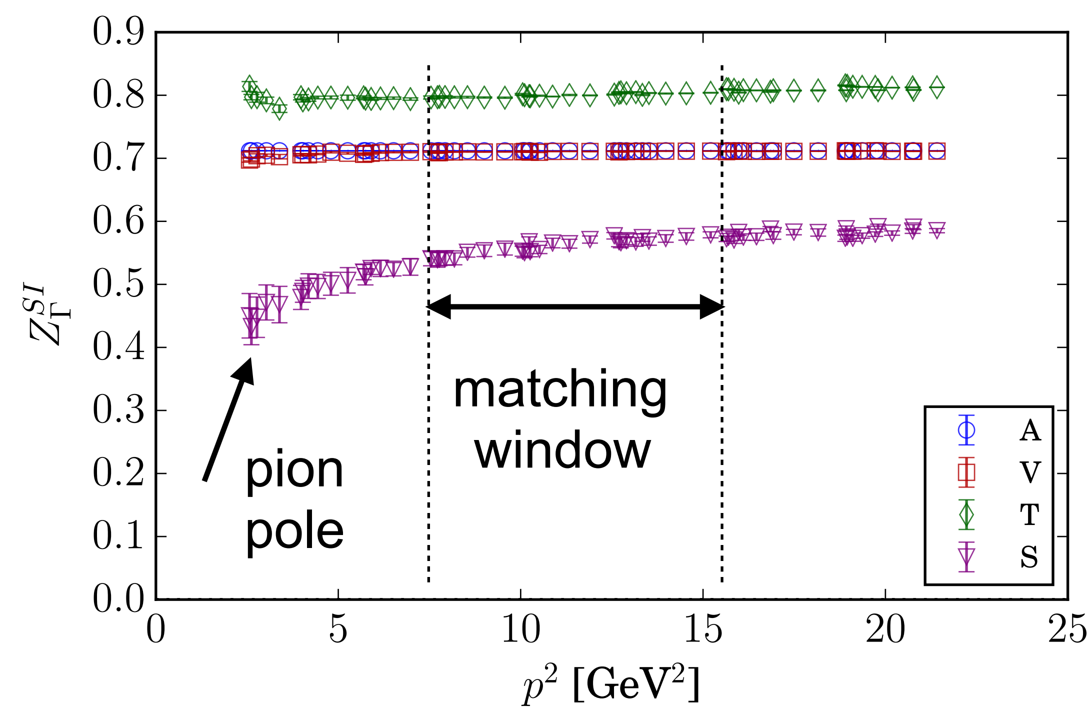
$$\langle [\bar\psi\Gamma\psi]_0 \, \psi(p_1)\bar\psi(p_2) \rangle = \sum_{x,y} e^{-ip_1 x + ip_2 y} \langle [\bar\psi\Gamma\psi]_0 \; \psi_x \bar\psi_y \rangle$$

$$Z_\Gamma^{lat}(\mu^2) \langle [\bar\psi\Gamma\psi]_0 \, \psi(p_2)\bar\psi(p_1) \rangle = \langle [\bar\psi\Gamma\psi]_0 \, \psi(p_2)\bar\psi(p_1) \rangle^{\text{tree}}$$

Scale-independent ratio

$$Z_\Gamma^{SI}(\mu^2) = \frac{Z_\Gamma^{\text{lat}}(\mu^2)}{Z_\Gamma^{\overline{MS}}(\mu^2)} \longrightarrow$$

≈ constant
lattice to MSbar
matching factor



- choose matching window $\Lambda_{QCD} \ll \mu \ll a^{-1}$
- symmetric scheme (SMOM) with $(p_1 - p_2)^2 = \mu^2$

  to avoid pion pole

Systematic Effects

- Discretization effects
  most calculations performed with O(a^2) action
  ⟹ extrapolation to a→0

$$X(a) \sim X|_{\mathrm{cont}} + \cancel{c_1 a} + c_2 a^2$$

- Finite volume effects

  ⟹ become small for L > 4 $(m_\pi)^{-1}$

$$X(L) \sim X(\infty) + C e^{-m_\pi L}$$

- Light u,d quark masses : (used to be) unphysically heavy
  ⟹ After ~2010, enough compute for physical-point calculations
  ⟹ use low-energy (chiral) effective theory

$$X(m_\pi) = X_0 + B m_\pi^2 \log m_\pi^2 + \ldots$$

- Excited states contamination
  Can be a major problem, e.g. Nπ states in g_A (axial charge)
  [O. Bar, PRD(2019), 1812.09191]
  ⟹ Multi-state fits, variational analysis

$$X(t_{\mathrm{sep}}) = X + A e^{-\Delta E t_{\mathrm{sep}}/2}$$

Calculations with multiple a, L, mπ are required

- Lattice Applications to Nucleon Structure and Beyond

  Structure : Form Factors, Spin, Mass, Parton Disributions

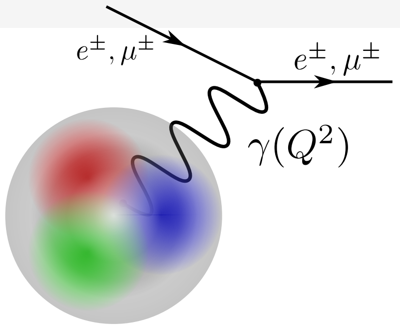
  Spectrum & Interactions

  Nuclear Matter Sensitivity to DM and Symmetry Violations

  (if time allows) Confinement, Entanglement and Quantum Information

Nucleon Vector Form Factors and Radii

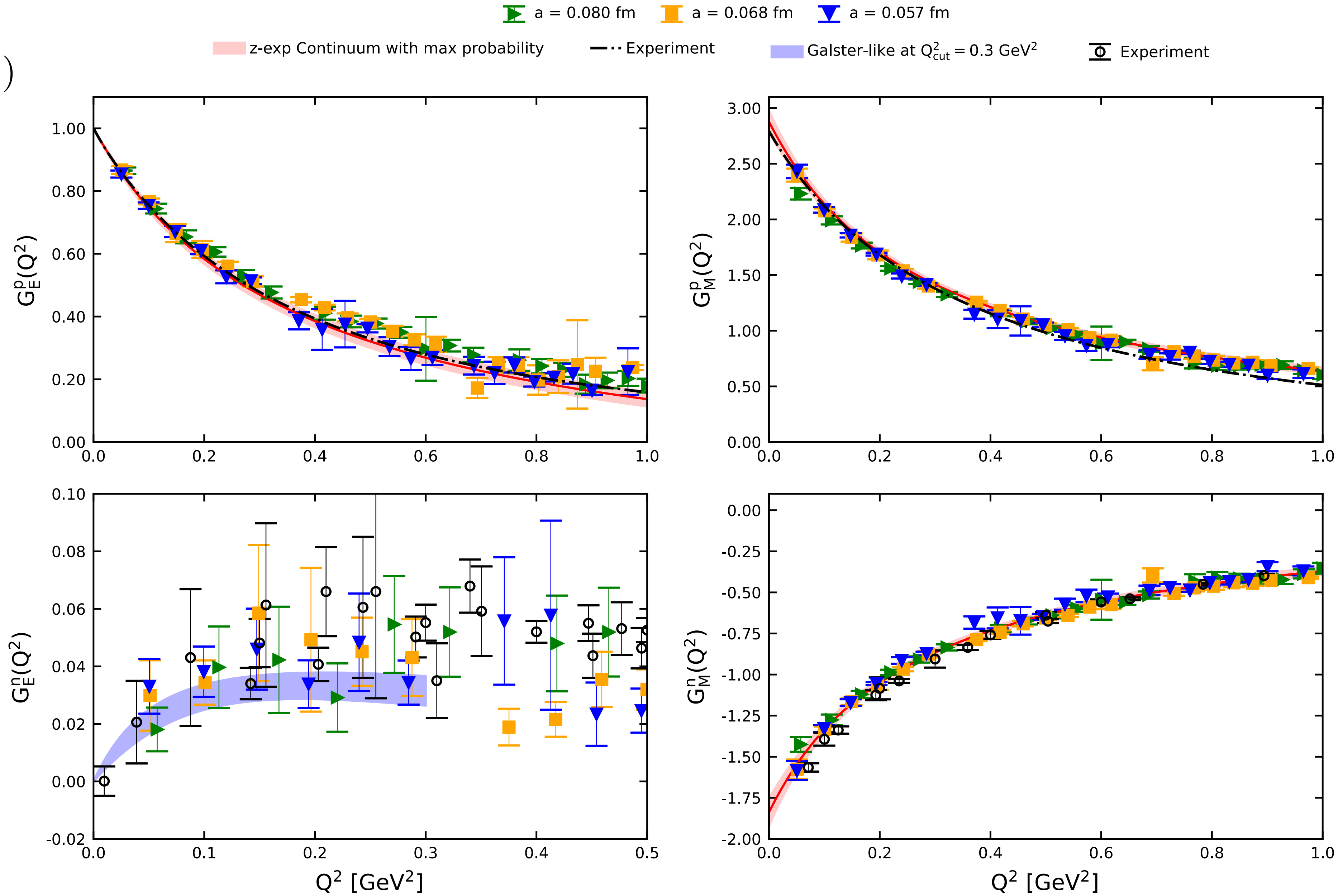
- Vector form factors: charge & magnetization structure



$$\langle P+q|\,\bar{q}\gamma^\mu q\,|P\rangle = \bar{U}_{P+q}\left[F_1(Q^2)\gamma^\mu + F_2(Q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M_N}\right]U_P$$

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2}F_2(Q^2)$$

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

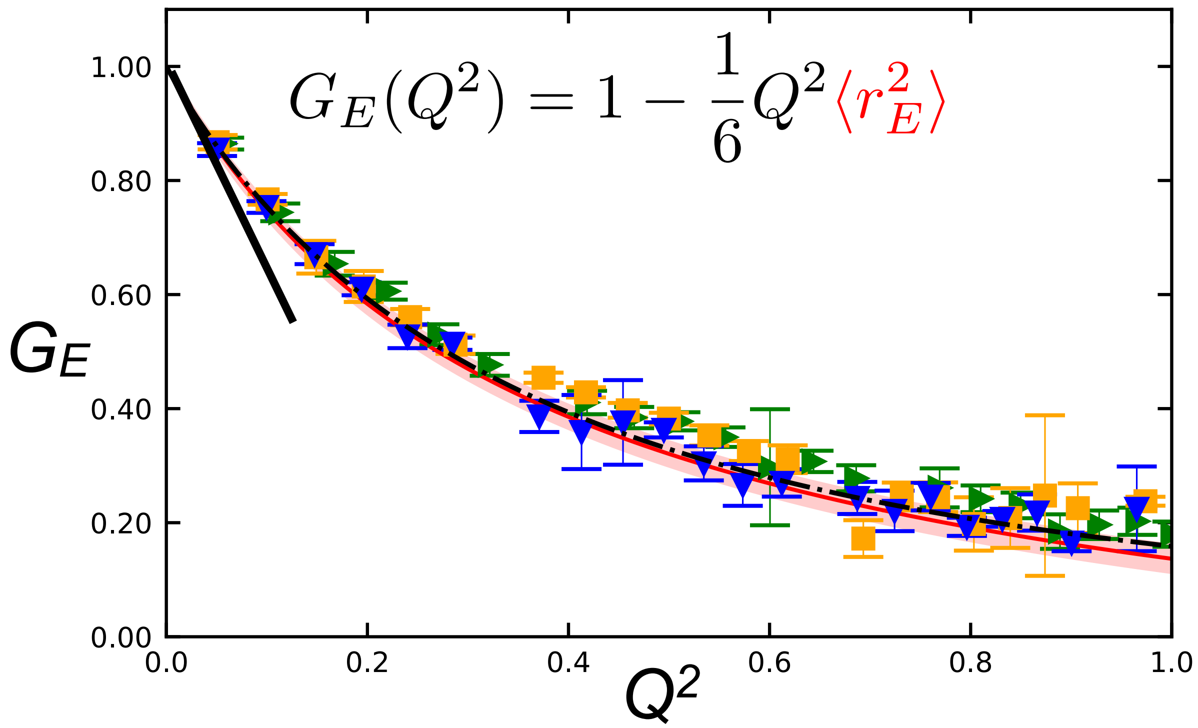


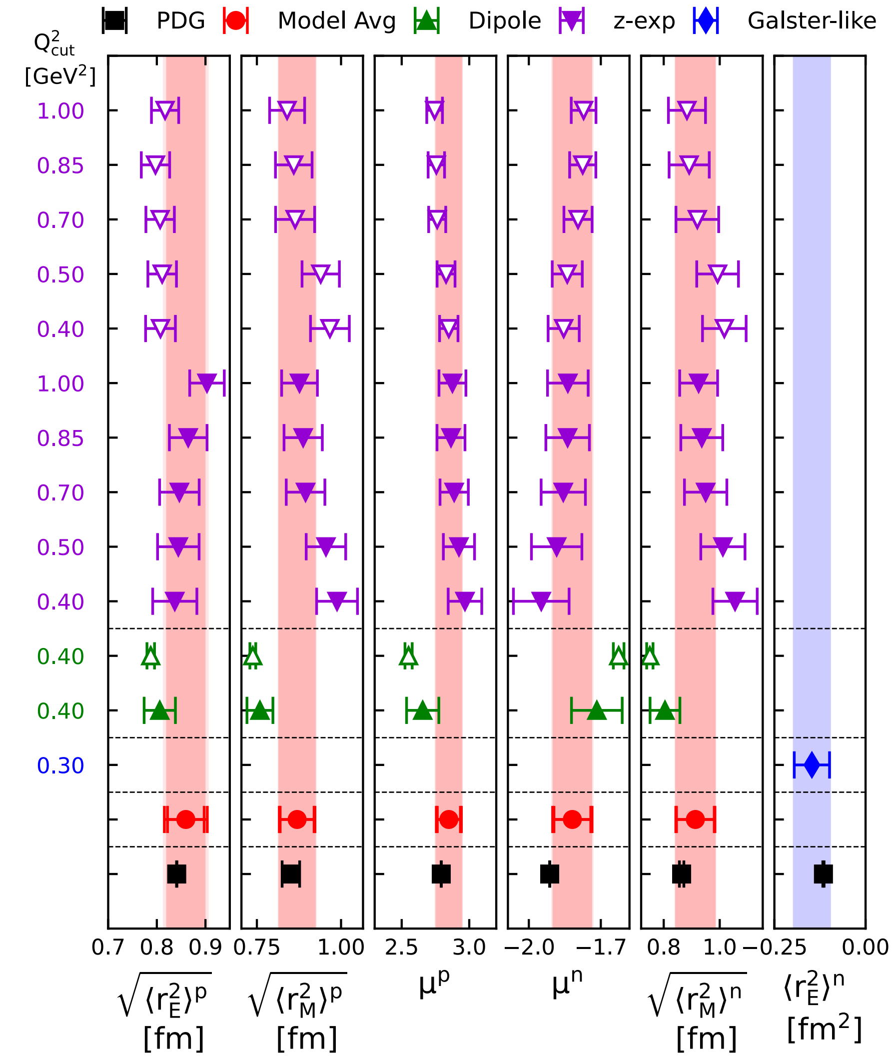
[Alexandrou et al (ETM collaboration), arXiv:2507.20910]

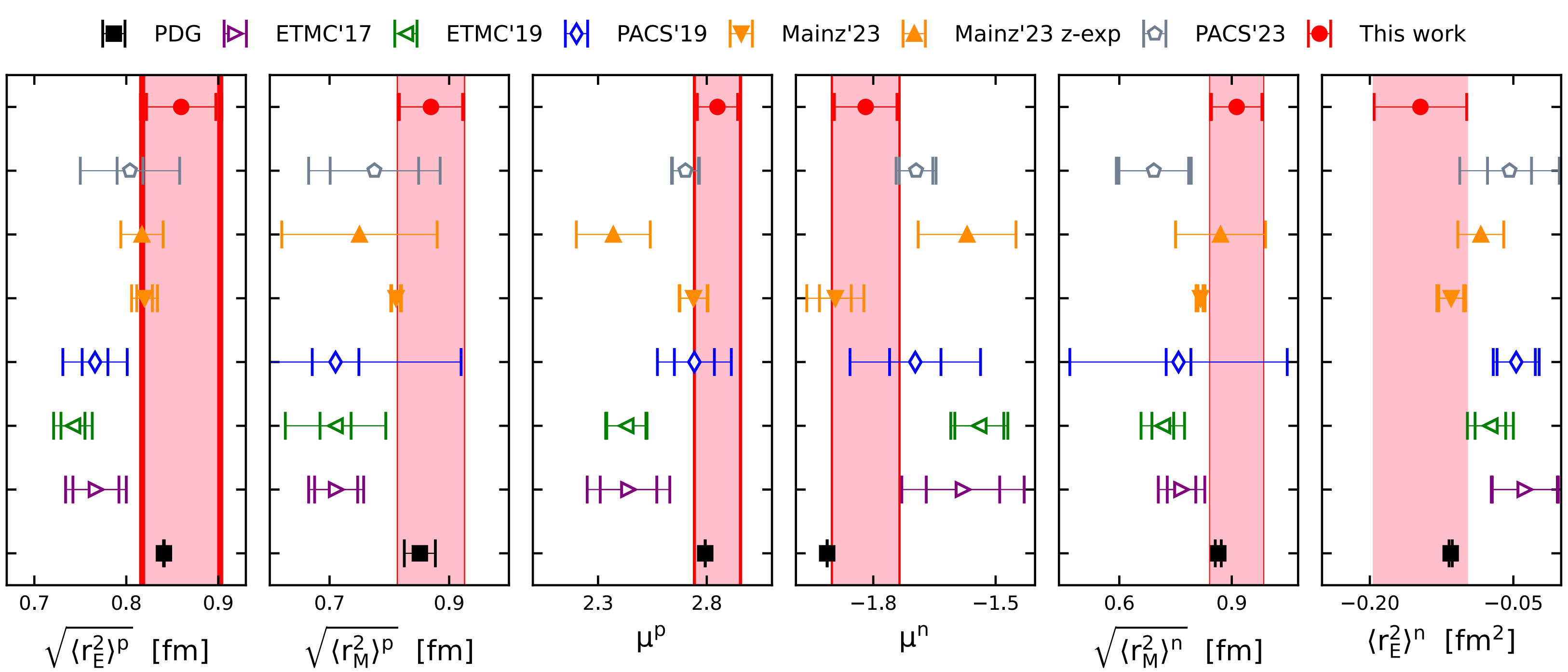
Nucleon Vector Form Factors and Radii (2)

- Electric and Magnetic Radii

$$G_{E,M}(Q^2) = G_{E,M}(0)\left[1 - \frac{1}{6}Q^2\langle r_E^2\rangle + O(Q^4)\right]$$



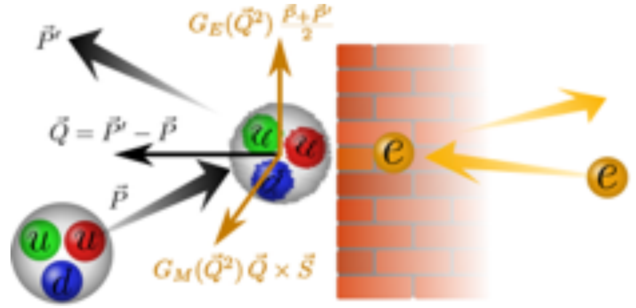


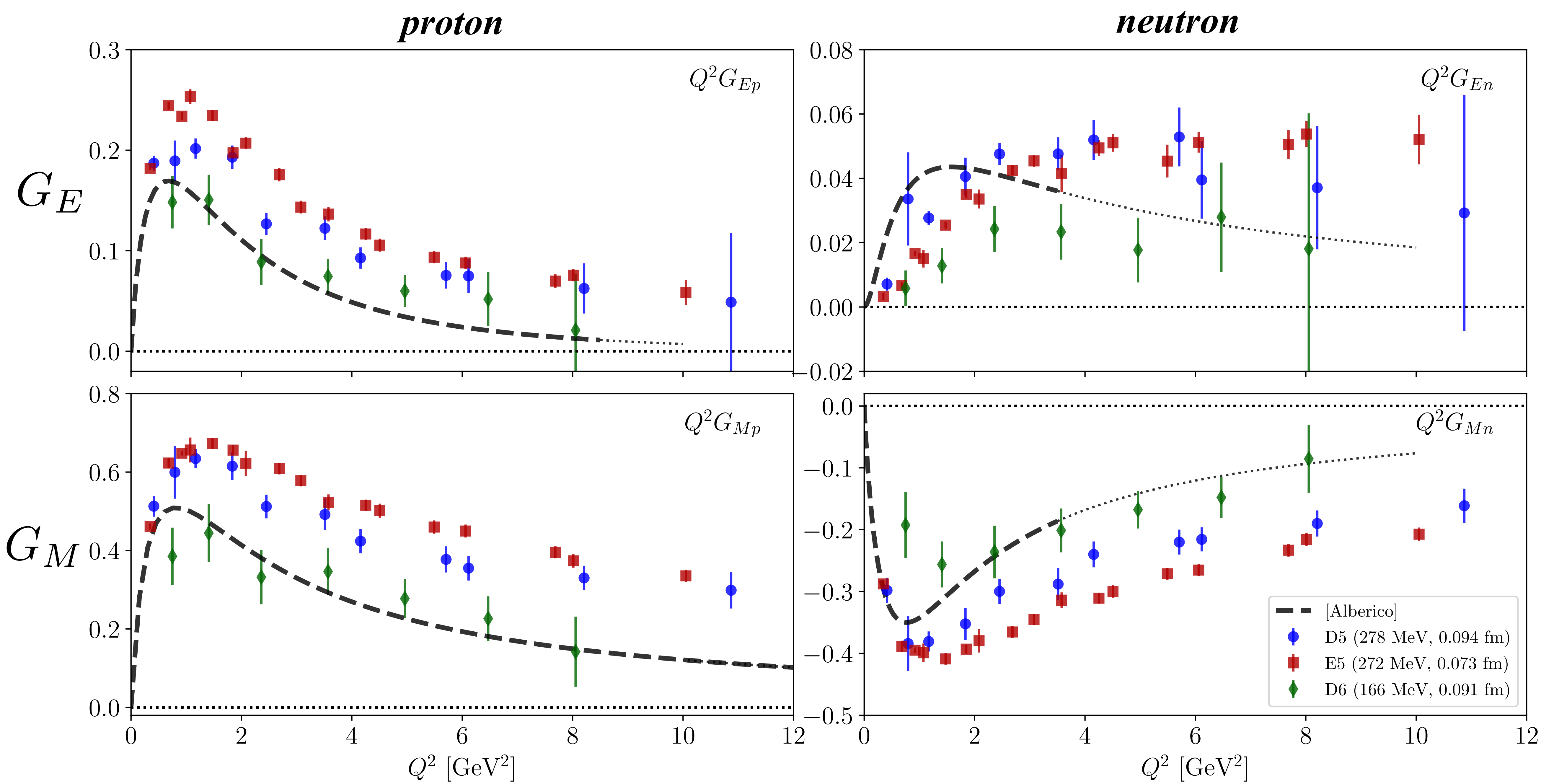


[Alexandrou et al (ETM collaboration), arXiv:2507.20910]

Nucleon Vector Form Factors : High Momentum

$$\langle P+q|\,\bar{q}\gamma^\mu q\,|P\rangle = \bar{U}_{P+q}\Big[F_1(Q^2)\gamma^\mu + F_2(Q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M_N}\Big]U_P$$

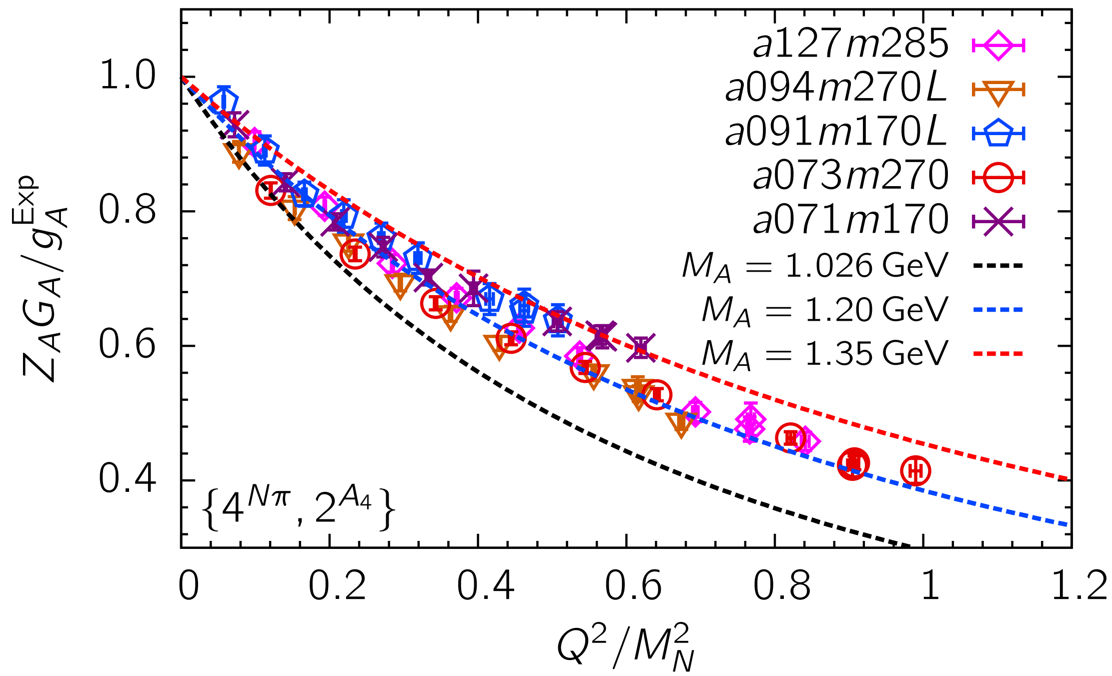


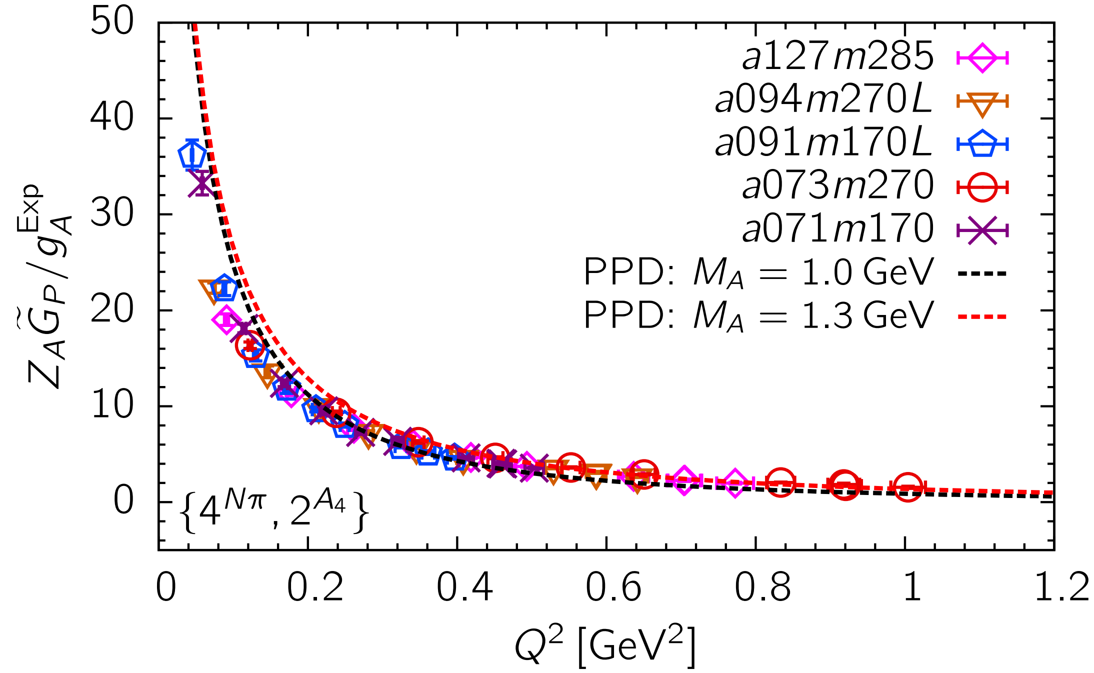


Nucleon Axial Form Factors

- Axial form factors: neutrino scattering , muon capture

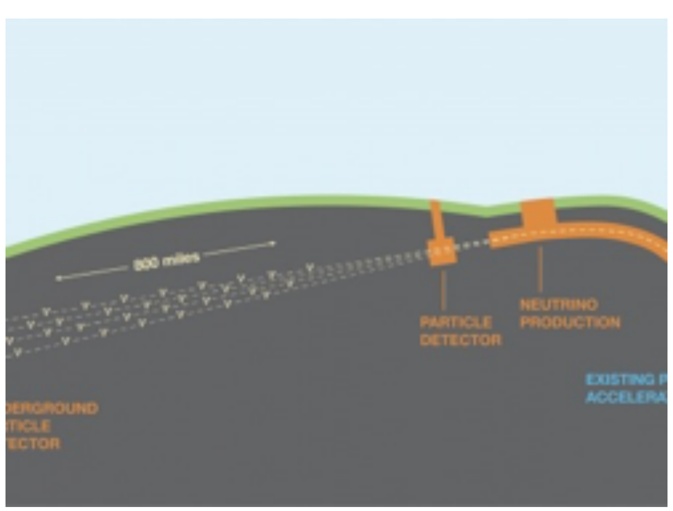
$$\langle P+q|\,\bar{q}\gamma^\mu\gamma^5 q\,|P\rangle \;=\; \bar{U}_{P+q}\Big[G_A(Q^2)\,\gamma^\mu\gamma^5 + G_P(Q^2)\frac{\gamma^5 q^\mu}{2M_N}\Big]U_P$$

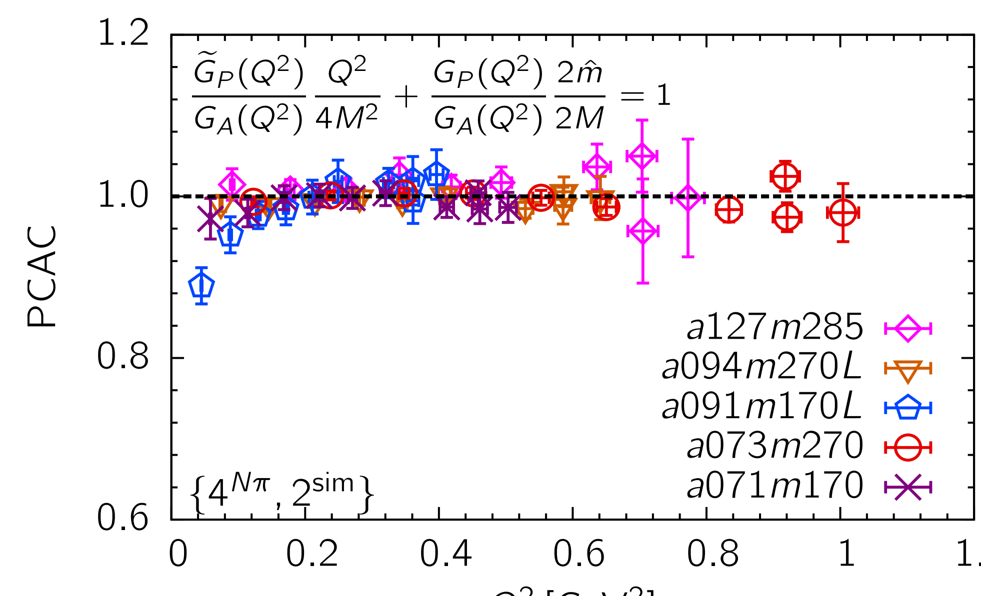




G_A : Neutrino Flux measurement at DUNE

G_P : Muon capture $p+\mu \rightarrow n+\nu_\mu$
Partial axial current conservation (PCAC)





Proton Spin Decomposition and Sum Rule

Deeply-Virtual Compton Scattering(DVCS)

DGLAP ;
J/ψ - production
at Electron-Ion Collider

polarized p-p collisions at RHIC
scaling violations in g_1

$$\frac{1}{2} = J^{quarks} + J^{Glue}$$

$$(S^q + L^q)$$

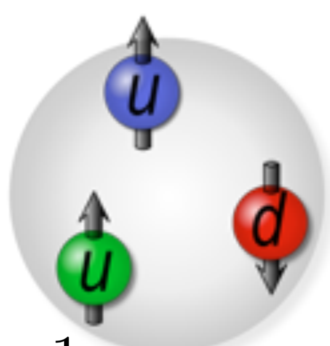
$$(S^G + L^G + L^{qG\text{-int}})$$

No local QFT definitions for S^G, L^G, L^{qG};
interpretation depends on experiment

polarized parton
distributions $\Delta q(x)$
from DIS (EMC exp)

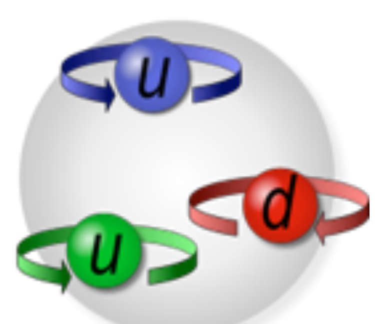
Wigner-like parton
distributions (GTMDs)?
Twist-3 corrections to DVCS?

[K.F.Liu and C.Lorce, contribution to EPJA
"3D Structure of the Nucleon", 1508.00911]



$$\frac{1}{2}\bar{q}\Sigma^{0i}q$$

+



$$\vec{L}^q = \vec{r} \times \vec{p}$$

=

$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x \left[x^j\, T^{0k} - x^k\, T^{0j}\right]$$

Proton Spin Decomposition

- Energy-Momentum distribution in nucleon

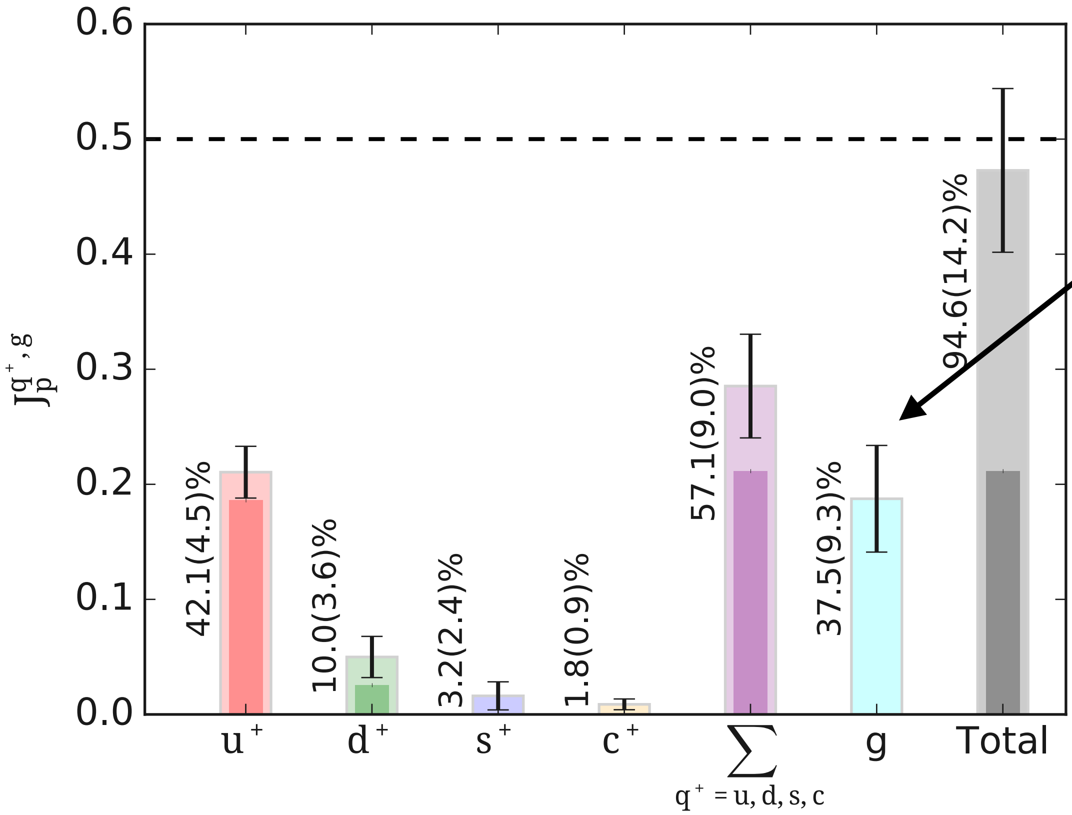
$$\langle N(p+q)|T^{\mathrm{q,\,g}}_{\mu\nu}|N(p)\rangle \longrightarrow \{A_{20},\, B_{20},\, C_{20}\}^{q,g}\,(q^2)$$

$$T^{q}_{\mu\nu} = \bar{q}\,\gamma_{\{\mu}\overleftrightarrow{D}_{\nu\}}\,q$$

$$T^{\mathrm{glue}}_{\mu\nu} = G^a_{\mu\lambda}G^a_{\nu\lambda} - \frac{1}{4}\delta_{\mu\nu}(G_{\mu\nu})^2$$

- Ji's angular momentum decomposition
[X.Ji, PRL78:610(1997)]

$$J_{q,g} = \frac{1}{2}\left[A^{q,g}_{20}(0) + B^{q,g}_{20}(0)\right]$$

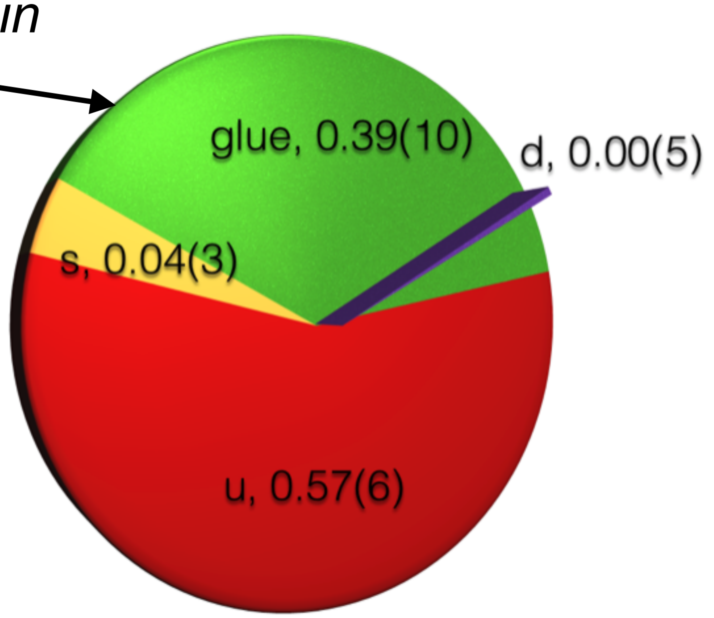


Nf=2+1+1 physical light quarks
[C.Alexandrou et al (ETMC) PRD101:094513 (2020)]

J_g = 38% of nucleon spin

Spin sum rule

$$\frac{1}{2} = J_q + J_g$$

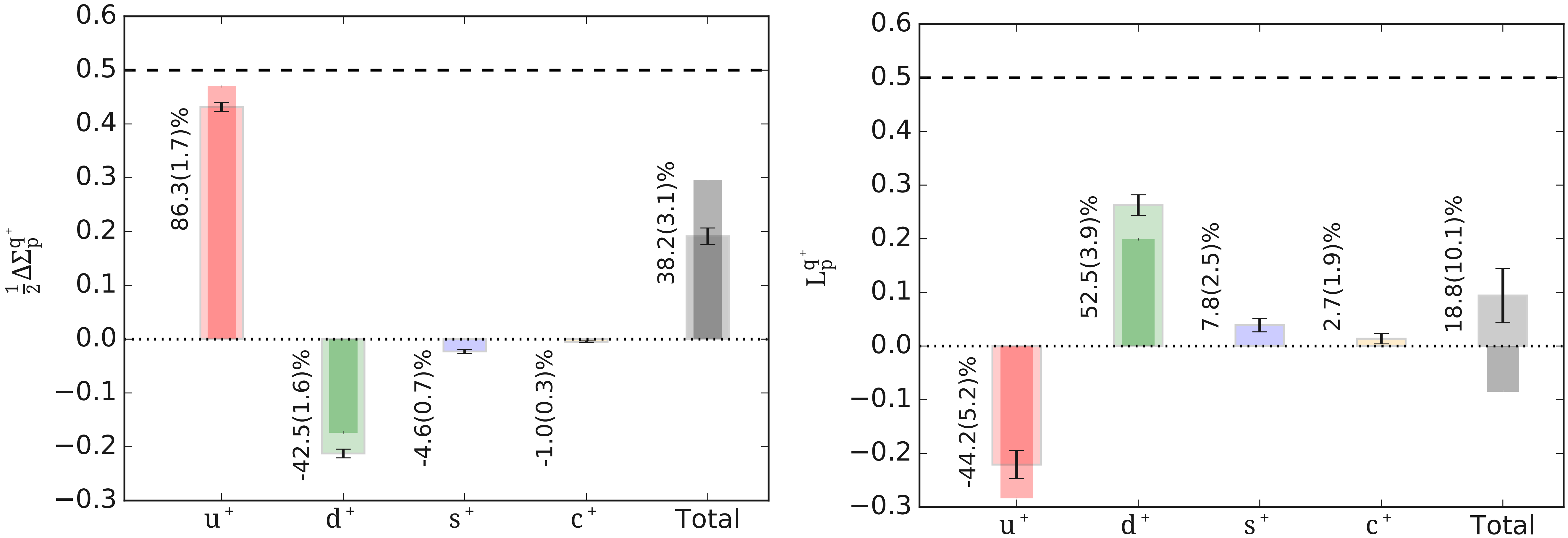


Nf=2+1 physical light+strange quarks
[Y.B.Yang, LATTICE 2018; arXiv:1904.04138]

Proton Spin Decomposition

- Quark Orbital angular momentum and Spin

$$L^{u,d,s} = J^{u,d,s} - \frac{1}{2}\Sigma^{u,d,s}$$



Nf=2+1+1 physical light quarks
[C.Alexandrou et al (ETMC) PRD101:094513 (2020)]

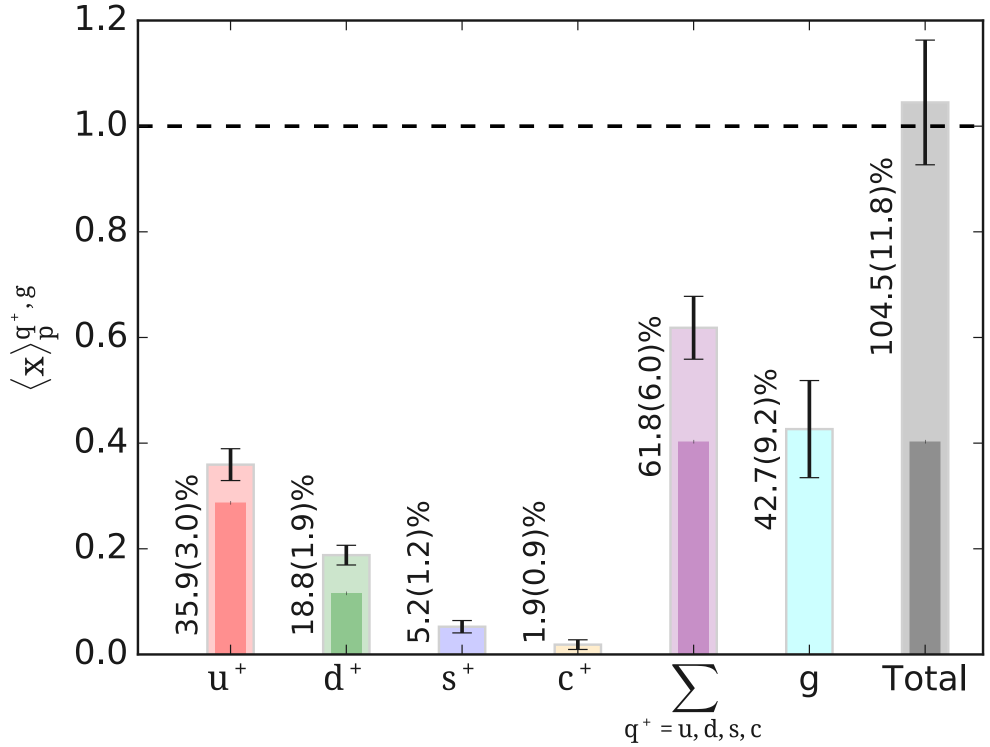
Nucleon Momentum and Mass Decomposition

- Momentum fraction
  (2nd Mellin moment of parton distribution)

$$\langle x\rangle^{q,g} = \int dx\, f_{q,g}(x) = A_{20}^{q,g}(0)$$

- Momentum sum rule

$$1 = \sum_{quarks} \langle x\rangle^q + \langle x\rangle^{\text{gluon}}$$



[C.Alexandrou et al (ETMC) PRD101:094513 (2020)]

- Proton mass decomposition:

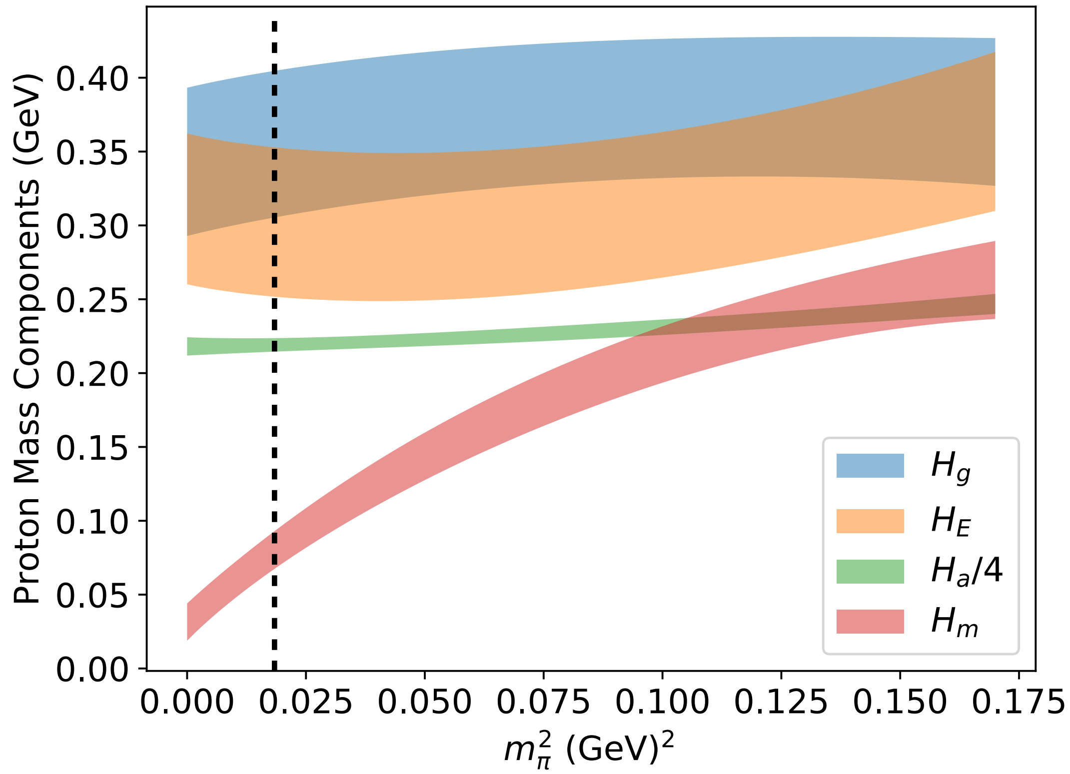
$$M = -\langle T_{44}\rangle = \langle H_m\rangle + \langle H_E\rangle(\mu) + \langle H_g\rangle(\mu) + \frac{1}{4}\langle H_a\rangle$$

	quark condensate	quark kin.energy	glue field energy	EM trace anomaly
At the physical point:	9(2)(1)%	32(4)(4)%	36(5)(4)%	23(1)(1)%

Quark and gluon energy contriubtions:

$$\langle H_E^R\rangle = \frac{3}{4}\langle x\rangle_q^R M - \frac{3}{4}\langle H_m\rangle$$

$$\langle H_g^R\rangle = \frac{3}{4}\langle x\rangle_g^R M$$



[Yang et al (χQCD), PRL121:212001(2018)]