The Balitsky Hierarchy and the BK Equation

Large logarithms, evolution equations, and small-x saturation

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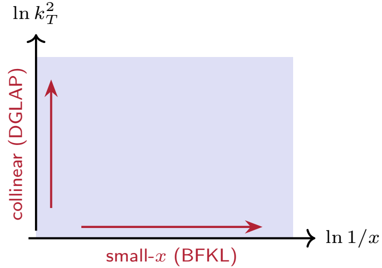
Large logarithms from one emission

A single gluon emission carries *two* potentially large logs:

$$dP \;\propto\; \alpha_s \frac{dk_T^2}{k_T^2}\,\frac{dx}{x} \;=\; \alpha_s\, d(\ln k_T^2)\, d(\ln \tfrac{1}{x}).$$

- **Collinear** $(k_T \to 0)$: $\displaystyle\int_{Q_0^2}^{Q^2} \frac{dk_T^2}{k_T^2} = \ln \frac{Q^2}{Q_0^2}$.
- **Small-x** $(x \to 0)$: $\displaystyle\int_x^1 \frac{dx'}{x'} = \ln \frac{1}{x}$.

Once $\alpha_s \ln \sim 1$, fixed-order perturbation theory fails and the logs must be **resummed**.



uniform dP in $(\ln 1/x,\ \ln k_T^2)$

Two big logs ⇒ two resummations: **DGLAP** (collinear, $\ln Q^2$) and **BFKL** (small-x, $\ln 1/x$).
DGLAP = Dokshitzer-Gribov-Lipatov-Altarelli-Parisi

BFKL = Balitsky-Fadin-Kuraev-Lipatov

Strong ordering $\Rightarrow$ exponentiation

Keep only **strongly ordered** ladders (each emission a small perturbation on the last, no interference):

$$Q^2 \gg k_{Tn}^2 \gg \cdots \gg k_{T1}^2 \gg Q_0^2.$$

The nested integral is an ordered simplex,

$$\int_{Q_0^2}^{Q^2} \frac{dk_{Tn}^2}{k_{Tn}^2} \int_{Q_0^2}^{k_{Tn}^2} \frac{dk_{T(n-1)}^2}{k_{T(n-1)}^2} \cdots = \frac{1}{n!}\left(\ln \frac{Q^2}{Q_0^2}\right)^n,$$

the $1/n!$ being the volume $\int_0^L dy_1 \int_{y_1}^L dy_2 \cdots = L^n/n!$. Identically for small-$x$, $1 \gg x_1 \gg \cdots \gg x_n \gg x$:

$$\frac{1}{n!}\left(\ln \frac{1}{x}\right)^n.$$

Leading-log approximation: each rung $\sim \alpha_s L$ with L the big log; the tower $\sum_n \frac{1}{n!}(\alpha_s L)^n$ must be resummed — DGLAP (collinear) and BFKL (small-x).

DGLAP: the idea of an evolution equation

Resumming the collinear ladder is equivalent to a differential equation in the scale. Differentiating the one-rung result in $\ln Q^2$,

$$\boxed{\frac{\partial q(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\, P_{qq}(z)\, q\Big(\frac{x}{z}, Q^2\Big)}$$

with $q(x,Q^2)$ the quark distribution at scale Q^2
and the splitting function $P_{qq}(z) = C_F\big[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\big]$.

- $P_{qq}(z)$ is the probability density to split keeping momentum fraction z.
- Solving the equation = exponentiating = resumming the whole ladder at once.
- Generally, we also have splittings of gluons to quarks (of different flavors) and can track the gluon distribution function, so the full set of DGLAP equations is a coupled set of equations

The template for everything that follows: an emission kernel ⇒ a closed evolution equation. BK will be this same story, with a *nonlinear* kernel.

BFKL: the small-x large log

At small x one instead resums $(\alpha_s \ln 1/x)^n$, evolving in rapidity $Y = \ln 1/x$:

$$\frac{\partial f(Y,k_T)}{\partial Y} = \bar{\alpha}_s \left[\text{emission kernel}\right] \otimes f, \qquad \bar{\alpha}_s = \frac{\alpha_s N_c}{\pi},$$

with f the (unintegrated) gluon distribution. In the dilute limit N is proportional to the gluon density. Schematically $\partial_Y N = \omega\bar{\alpha}_s\, N$, so the density/dipole amplitude grows as a power of the energy,

$$N \sim e^{\,\omega\bar{\alpha}_s Y} \sim \left(\frac{1}{x}\right)^{\lambda}, \qquad \lambda = \bar{\alpha}_s\, \omega.$$

Growth is fast because the *emitted gluons are themselves color sources* — each radiates the next.

Same idea as DGLAP — an evolution equation — but in $Y = \ln 1/x$, and now the gluon density *grows* like a power of $1/x$.

From growth to saturation

Power-law growth $N \sim (1/x)^\lambda$ cannot last: probabilities are bounded, $N \leq 1$. At high density the gluons (all of similar size $\sim 1/k_T$) overlap and **recombine**, $gg \to g$, adding a *nonlinear* loss term:

$$\partial_Y N = \underbrace{\alpha_s\,[\text{BFKL growth}]}_{\text{emission}} - \underbrace{\alpha_s\,[N^2]}_{\text{recombination}} .$$

This nonlinear evolution equation is **Balitsky–Kovchegov** — the subject of the rest of the talk.

Linear evolution (DGLAP/BFKL) → the nonlinear closure ⇒ BK. We now derive it from first principles.

Roadmap: the logical chain

1. Fast partons are **eikonal Wilson lines**; observables are traces
2. One soft* gluon per dY produced through the **eikonal emission current $\mathbf{\Gamma}_n$**.
3. Its square gives the **JIMWLK Hamiltonian**, $H \sim \int_{\boldsymbol{z}} \sum_{n,m} \mathbf{\Gamma}_n \cdot \mathbf{\Gamma}_m$.
4. H acting on the dipole $\Rightarrow$ exact **Balitsky equation**
5. The dipole couples to $\langle DD \rangle$, $\langle DD \rangle$ to $\langle DDD \rangle, \langle QD \rangle, \ldots$ — the **hierarchy**.
6. One large-N_c factorization closes the tower into the **BK equation**.

Strategy: get the *exact* Balitsky hierarchy first, approximate *once* at the end.

* soft here meaning with small k^+ for a right moving parton (sometime scalled slow)

The problem: small-x evolution

A fast $q\bar{q}$ dipole scatters off a target. Resum logs of $1/x$ with the rapidity

$$Y \equiv \ln \frac{1}{x}, \qquad dY = \frac{dk^+}{k^+}.$$

Each rapidity step opens phase space for **one extra soft gluon**. We want the evolution of the dipole S-matrix

$$S_{\boldsymbol{x}_1\boldsymbol{x}_2}(Y) = \langle D_{\boldsymbol{x}_1\boldsymbol{x}_2} \rangle_Y \,.$$

Two regimes:

- Dilute target: amplitude small, evolution *linear* → BFKL.
- Dense target: amplitude saturates, evolution *nonlinear* → BK.

Plan: derive the exact Balitsky hierarchy, then one large-N_c step closes it into BK.

Setup: eikonal Wilson lines

A parton crossing the target color field at fixed transverse $\boldsymbol{x}$ picks up a fundamental Wilson line

$$V(\boldsymbol{x}) = \mathcal{P}\exp\Big[-ig\int dx^+ A_a^-(x^+,\boldsymbol{x})\, t^a\Big].$$

An antiquark at $\boldsymbol{y}$ carries $V^\dagger(\boldsymbol{y})$. Normalizations:

$$\mathrm{tr}(t^a t^b) = \tfrac{1}{2}\delta^{ab}, \qquad t^a t^a = C_F\,\mathbf{1}, \qquad C_F = \frac{N_c^2-1}{2N_c}.$$

Eikonal = the parton's transverse position is frozen; it only acquires a color phase V.

Gauge-invariant observables: the dipole

Traces of Wilson lines. The two simplest:

$$\begin{aligned} \text{dipole:} \quad & D_{\boldsymbol{x}_1\boldsymbol{x}_2} = \tfrac{1}{N_c}\,\mathrm{tr}[V(\boldsymbol{x}_1)V^\dagger(\boldsymbol{x}_2)], \\ \text{quadrupole:} \quad & Q_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} = \tfrac{1}{N_c}\,\mathrm{tr}[V_1V_2^\dagger V_3V_4^\dagger]. \end{aligned}$$

The dipole average is the physical S-matrix; the amplitude is $N = 1 - S$:

$$S_{\boldsymbol{x}_1\boldsymbol{x}_2} = \langle D_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle, \qquad N_{\boldsymbol{x}_1\boldsymbol{x}_2} = 1 - S_{\boldsymbol{x}_1\boldsymbol{x}_2}.$$

Why coordinate space? The eikonal interaction is diagonal in transverse position, and branching is local: gluon at $\boldsymbol{z} \to$ daughter dipoles $(\boldsymbol{x}_1, \boldsymbol{z})$ and $(\boldsymbol{z}, \boldsymbol{x}_2)$.

Soft emission I

The LCPT wavefunction is $\dfrac{\text{vertex } \mathcal{M}}{\text{energy denominator } \Delta E}$. Kinematics: incoming $p = (p^+, 0, \mathbf{0})$; gluon $k^+ = \xi p^+$ ($\xi \ll 1$), $k^- = \boldsymbol{k}^2/2k^+$; light-cone gauge $\epsilon_\lambda^+ = 0$, transversality $\epsilon_\lambda^- = \boldsymbol{\epsilon}_\lambda \cdot \boldsymbol{k}/k^+$.

Vertex (using $\bar{u}\gamma^\mu u \to 2p^\mu$ - see next slide/board):

$$\mathcal{M}_\lambda^a = g\, t^a\, \bar{u}(q)\gamma^\mu u(p)\, \epsilon_{\lambda\mu}^* \;\simeq\; 2g\, t^a\, p\cdot\epsilon_\lambda^* \;=\; 2g\, t^a\, p^+ \epsilon_\lambda^{-*} \;=\; 2g\, t^a\, \frac{p^+}{k^+}\, \boldsymbol{\epsilon}_\lambda^* \cdot \boldsymbol{k}.$$

Energy denominator (then soft, $\xi \ll 1$):

$$\Delta E = p^- - q^- - k^- = -\frac{\boldsymbol{k}^2}{2(p^+ - k^+)} - \frac{\boldsymbol{k}^2}{2k^+} \simeq -\frac{\boldsymbol{k}^2}{2k^+}.$$

Combine:

$$\frac{\mathcal{M}_\lambda^a}{\Delta E} \;\simeq\; 2g\, t^a\, \frac{p^+}{k^+}\, \boldsymbol{\epsilon}_\lambda^* \cdot \boldsymbol{k} \cdot \left(-\frac{2k^+}{\boldsymbol{k}^2}\right) \;=\; -4g\, t^a\, p^+\, \frac{\boldsymbol{\epsilon}_\lambda^* \cdot \boldsymbol{k}}{\boldsymbol{k}^2} \;\propto\; t^a\, \frac{\boldsymbol{\epsilon}_\lambda^* \cdot \boldsymbol{k}}{\boldsymbol{k}^2}.$$

eikonal vertex $\boldsymbol{\epsilon}^* \cdot \boldsymbol{k}$, soft pole $1/\boldsymbol{k}^2$, overall constant dropped.

Soft emission II: the eikonal vertex

The transverse coupling comes from the fast-quark current. For an energetic quark,

$$\bar{u}(q)\gamma^{\mu}u(p) \xrightarrow{q\to p} 2p^{\mu}.$$

Contract $\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}$ with p_{ν} and use the Dirac equation:

$$\bar{u}(p)\{\gamma^{\mu},\not{p}\}u(p) = 2p^{\mu}\,\bar{u}u$$

$$2m\,\bar{u}\gamma^{\mu}u = 4mp^{\mu}$$

$$\Rightarrow\; \bar{u}\gamma^{\mu}u = 2p^{\mu}.$$

$\bar{u}\gamma^{\mu}u = 2p^{\mu}$: a structureless fermion's current is just its momentum — soft radiation is spin-blind, so the current is universal.

Soft emission III: the 2D Coulomb field

Fourier transform the momentum structure to coordinate space (show):

$$\int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, \frac{k^i}{\boldsymbol{k}^2} = \frac{i}{2\pi}\,\frac{r^i}{\boldsymbol{r}^2}, \qquad \boldsymbol{r} = \boldsymbol{z} - \boldsymbol{x},$$

so the emission profile is a **two-dimensional Coulomb field**,

$$\phi^a_\lambda(\boldsymbol{x};\boldsymbol{z}) \;\propto\; t^a\, \boldsymbol{\epsilon}^*_\lambda \cdot \frac{\boldsymbol{x}-\boldsymbol{z}}{(\boldsymbol{x}-\boldsymbol{z})^2}.$$

Later, when squaring, the polarization sum will lead to:

$$\sum_\lambda \left(\boldsymbol{\epsilon}^*_\lambda \cdot \tfrac{\boldsymbol{r}_n}{\boldsymbol{r}_n^2}\right)\left(\boldsymbol{\epsilon}_\lambda \cdot \tfrac{\boldsymbol{r}_m}{\boldsymbol{r}_m^2}\right) = \frac{\boldsymbol{r}_n\cdot\boldsymbol{r}_m}{\boldsymbol{r}_n^2\,\boldsymbol{r}_m^2}, \qquad \boldsymbol{r}_n \equiv \boldsymbol{x}_n - \boldsymbol{z}.$$

We wrote t^a for the *color* of the emission, but when the emitted can also interact with the shockwave, we should treat different cases — next.

Color

Define operators acting on the projectile line $V_n \equiv V(\boldsymbol{x}_n)$ that insert t^a on the left or right:

$$T^a_{L,n} V_n = t^a V_n, \qquad T^a_{R,n} V_n = V_n\, t^a.$$

$$T^a_{L,n} V_n^\dagger = -V_n^\dagger t^a, \qquad T^a_{R,n} V_n^\dagger = -t^a V_n^\dagger.$$

The soft gluon is itself a fast parton: if it crosses the target it contributes its own **adjoint** Wilson line

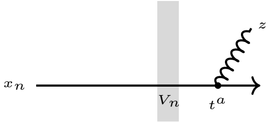
$$U^{ab}(\boldsymbol{z}) = 2\,\mathrm{tr}[t^a V_z\, t^b V_z^\dagger], \qquad V_z^\dagger\, t^a\, V_z = U^{ab}(\boldsymbol{z})\, t^b, \qquad U\, U^\top = \mathbf{1}.$$

T_L acts on the outgoing line, T_R on the incoming one; U appears only when the gluon itself crosses the shockwave.

Two time orderings ⇒ the eikonal current

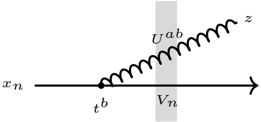
Same transverse vertex $\boldsymbol{r}_n/\boldsymbol{r}_n^2$, two orderings relative to the shockwave (gray band):

Emit after the target



No U: insertion on outgoing end, $t^a V_n = T^a_{L,n} V_n$.

Emit before the target



Gluon crosses ⇒ adjoint U^{ab}: $U^{ab} T^b_{R,n} V_n = U^{ab} V_n t^b$.

The current is their difference:

$$\boxed{\Gamma^{i,a}_n(\boldsymbol{z}) = \frac{r^i_n}{\boldsymbol{r}^2_n}\left(T^a_{L,n} - U^{ab}(\boldsymbol{z})\, T^b_{R,n}\right)} \quad \xrightarrow{V=U=\mathbf{1}} \quad t^a - t^a = 0 \ \checkmark$$

From the current to the Hamiltonian

The emission probability per dY is amplitude × conjugate summed over all possible gluons (all $\boldsymbol{z}$, polarizations, colors) and over which line emits in amplitude (n) and conjugate (m), so the evolution operator is the current squared

The phase space gives $\int dk^+/k^+ = dY$ and $\int d^2\boldsymbol{z}$; the polarization sum gives $\boldsymbol{r}_n \cdot \boldsymbol{r}_m / \boldsymbol{r}_n^2 \boldsymbol{r}_m^2$. Hence $\partial_Y \langle O \rangle = -\langle HO \rangle$ with

$$H = \mathcal{N} \int d^2\boldsymbol{z} \sum_{n,m} \boldsymbol{\Gamma}_n(\boldsymbol{z}) \cdot \boldsymbol{\Gamma}_m(\boldsymbol{z}) = \mathcal{N} \int d^2\boldsymbol{z} \sum_{n,m} \frac{\boldsymbol{r}_n \cdot \boldsymbol{r}_m}{\boldsymbol{r}_n^2 \boldsymbol{r}_m^2} \big(T_{L,n} - U T_{R,n}\big) \cdot \big(T_{L,m} - U T_{R,m}\big).$$

H is fixed up to a single overall constant $\mathcal{N}$ — the normalization we deferred.

Fixing the normalization, once

The structure fixes everything except $\mathcal{N}$. Fix it *physically*: acting on a single dipole, the **virtual** part of H must reproduce the known soft-gluon emission probability $\frac{\alpha_s C_F}{\pi^2}\int d^2\boldsymbol{z}\, K_{12\boldsymbol{z}}$. The virtual color weight is $2C_F$, so

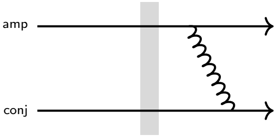
$$\mathcal{N}\cdot 2C_F = \frac{\alpha_s C_F}{\pi^2} \qquad \Longrightarrow \qquad \boxed{\mathcal{N} = \frac{\alpha_s}{2\pi^2}}.$$

Real and virtual

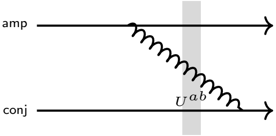
Using $U^{ab}U^{ac} = \delta^{bc}$, the squared current splits cleanly:

$$(T_L - UT_R)_n \cdot (T_L - UT_R)_m = \underbrace{T^a_{L,n}T^a_{L,m} + T^a_{R,n}T^a_{R,m}}_{\text{virtual (no } U)} - \underbrace{2\,T^a_{L,n}U^{ab}T^b_{R,m}}_{\text{real (gluon crosses)}} \,.$$

Virtual: same side $\Rightarrow$ no U



Real: opposite sides $\Rightarrow$ U



Acting on the dipole: the transverse kernel

Apply H to $D_{\boldsymbol{x}_1\boldsymbol{x}_2}$. Only the charges at $\boldsymbol{x}_1, \boldsymbol{x}_2$ act ($n, m \in \{1, 2\}$). The four index pairs combine, with the singlet's relative minus on the interference, into the **dipole kernel**:

$$\frac{1}{\boldsymbol{r}_1^2} + \frac{1}{\boldsymbol{r}_2^2} - 2\,\frac{\boldsymbol{r}_1\cdot\boldsymbol{r}_2}{\boldsymbol{r}_1^2\boldsymbol{r}_2^2} = \frac{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2}{(\boldsymbol{x}_1-\boldsymbol{z})^2(\boldsymbol{x}_2-\boldsymbol{z})^2} \equiv K_{12\boldsymbol{z}}.$$

The relative minus sign follows when computing the color factors of the different combinations ((1,1), (2,2), (1,2), (2,1)). The first two come with a plus, the last ones with a minus.

Finite as $\boldsymbol{z} \to \infty$ (color transparency, numerator $(\boldsymbol{x}_1-\boldsymbol{x}_2)^2$); singular only at the endpoints $\boldsymbol{z} \to \boldsymbol{x}_1, \boldsymbol{x}_2$.

Acting on the dipole: the color structure

The color action organizes as “virtual minus real”:

$$\underbrace{2C_F\ \mathrm{tr}[V_1V_2^\dagger]}_{\text{virtual}} - \underbrace{2\ \mathrm{tr}[t^aV_1\,t^bV_2^\dagger]\,U^{ab}(\boldsymbol{z})}_{\text{real}}.$$

- **Virtual:** $t^at^a = C_F$ on each line (left and right) $\Rightarrow 2C_F$, times $K_{12\boldsymbol{z}}$.
- **Real:** carries U^{ab} — the emitted gluon scatters. Convert and Fierz next.

Everything now rides on one Fierz of the two t^a in the real term.

Real term: convert U, then Fierz

Eliminate U with $U^{ab}t^b = V_z^\dagger t^a V_z$, then Fierz the two t^a:

$$\mathrm{tr}[t^a V_1\, t^b V_2^\dagger]\, U^{ab} = \mathrm{tr}[t^a V_1 V_z^\dagger\, t^a V_z V_2^\dagger] = \tfrac{1}{2}\underbrace{\mathrm{tr}[V_1 V_z^\dagger]\,\mathrm{tr}[V_z V_2^\dagger]}_{\text{double dipole}} - \tfrac{1}{2N_c}\quad\underbrace{\mathrm{tr}[V_1 V_2^\dagger]}_{\text{singlet subtraction}}\quad .$$

(Used $\sum_a (t^a)_{ij}(t^a)_{kl} = \frac{1}{2}(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl})$.)

The double dipole = two daughter dipoles; the $-1/N_c$ singlet subtraction reflects that the emitted gluon is a color *octet*, not a singlet.

The decisive identity: $2C_F + \frac{1}{N_c} = N_c$

Combine virtual and real:

$$2C_F\ \mathrm{tr}[V_1V_2^\dagger] - 2\left[\tfrac{1}{2}\,\mathrm{tr}[V_1V_z^\dagger]\,\mathrm{tr}[V_zV_2^\dagger] - \tfrac{1}{2N_c}\,\mathrm{tr}[V_1V_2^\dagger]\right] = \left(2C_F + \tfrac{1}{N_c}\right)\mathrm{tr}[V_1V_2^\dagger] - \mathrm{tr}[V_1V_z^\dagger]\,\mathrm{tr}[V_zV_2^\dagger].$$

$$\boxed{2C_F + \frac{1}{N_c} = \frac{N_c^2 - 1}{N_c} + \frac{1}{N_c} = N_c}$$

The virtual self-energy gives $2C_F = N_c - \frac{1}{N_c}$; the real singlet subtraction supplies the missing $+\frac{1}{N_c}$.

The two $1/N_c$ pieces cancel exactly — *no* $1/N_c^2$ remainder. Finite-N_c exact.

The Balitsky equation (exact, finite N_c)

$$\boxed{\frac{\partial D_{\boldsymbol{x}_1\boldsymbol{x}_2}}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\boldsymbol{z}\,\frac{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2}{(\boldsymbol{x}_1-\boldsymbol{z})^2(\boldsymbol{x}_2-\boldsymbol{z})^2}\left[D_{\boldsymbol{x}_1\boldsymbol{z}}D_{\boldsymbol{z}\boldsymbol{x}_2} - D_{\boldsymbol{x}_1\boldsymbol{x}_2}\right]}$$

- Sanity check: a non-interacting dipole ($V \equiv \mathbf{1}$, all $D = 1$) gives $1{\cdot}1 - 1 = 0$.
- Both terms share the coefficient $\propto N_c$
- An *operator* equation — still exact at finite N_c.

But the average does not close: $\partial_Y \langle D\rangle \propto \langle DD\rangle - \langle D\rangle$ contains the unknown two-dipole correlator.

The Balitsky hierarchy

Evolving $\langle DD \rangle$ with H produces two kinds of terms

$$\langle D \rangle \to \langle DD \rangle \to \langle DDD \rangle, \ \langle QD \rangle \to \cdots$$

Infinite coupled tower of single traces of $2, 4, 6, \ldots$ lines — exact, but never closes.

Closing the hierarchy: one large-N_c step

The only approximation — mean-field factorization of the two-dipole correlator:

$$\boxed{\langle D_{\boldsymbol{x}_1\boldsymbol{z}} D_{\boldsymbol{z}\boldsymbol{x}_2}\rangle = \langle D_{\boldsymbol{x}_1\boldsymbol{z}}\rangle\langle D_{\boldsymbol{z}\boldsymbol{x}_2}\rangle + O(1/N_c^2)}$$

't Hooft factorization: normalized single traces behave as classical c-numbers at large N_c. Correlating two traces needs a bridging gluon ($g^2 = \lambda/N_c$) plus a Fierz swap ($1/N_c$) $\Rightarrow O(1/N_c^2)$.

Same $1/N_c^2$ as the quadrupole: factorizing the correlator and dropping the inter-dipole exchange are *the same step*.

The Balitsky–Kovchegov equation

With $S = \langle D \rangle$:

$$\boxed{\frac{\partial S_{\boldsymbol{x}_1\boldsymbol{x}_2}}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\boldsymbol{z}\, K_{12\boldsymbol{z}} \left[S_{\boldsymbol{x}_1\boldsymbol{z}} S_{\boldsymbol{z}\boldsymbol{x}_2} - S_{\boldsymbol{x}_1\boldsymbol{x}_2}\right]}$$

A single closed, nonlinear integro-differential equation for the dipole S-matrix.

The entire large-N_c content of BK is concentrated in that one factorization.

The two limits of BK

In terms of the amplitude $N = 1 - S$:

$$\frac{\partial N_{\boldsymbol{x}_1\boldsymbol{x}_2}}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\boldsymbol{z}\, K_{12\boldsymbol{z}} \big[N_{\boldsymbol{x}_1\boldsymbol{z}} + N_{\boldsymbol{z}\boldsymbol{x}_2} - N_{\boldsymbol{x}_1\boldsymbol{x}_2} - N_{\boldsymbol{x}_1\boldsymbol{z}} N_{\boldsymbol{z}\boldsymbol{x}_2} \big].$$

Dilute ($N \ll 1$): drop the quadratic term,

$$\partial_Y N \simeq \tfrac{\alpha_s N_c}{2\pi^2} \int K \, [N_{1z} + N_{z2} - N_{12}].$$

Coordinate-space **BFKL** — linear, unbounded growth.

Dense ($N \to 1$): the $-N \cdot N$ term kicks in. $S = 0$ $(N = 1)$ is a fixed point. **Saturation**: $N \leq 1$, the black-disk limit.

BK = BFKL growth + unitarization, both from the *same* nonlinear term.

Summary of the logical chain

1. Eikonal partons → Wilson lines; observable $D_{\boldsymbol{x}_1\boldsymbol{x}_2} = \frac{1}{N_c}\,\mathrm{tr}[V_1 V_2^\dagger]$.
2. One soft gluon → eikonal current $\Gamma_n^{i,a} = \frac{r_n^i}{\boldsymbol{r}_n^2}(T^a_{L,n} - U^{ab} T^b_{R,n})$.
3. Its square → JIMWLK $H = \frac{\alpha_s}{2\pi^2}\int_{\boldsymbol{z}} \sum_{n,m} \boldsymbol{\Gamma}_n \cdot \boldsymbol{\Gamma}_m$; $\mathcal{N}$ fixed by matching the virtual part.
4. H on the dipole → exact Balitsky equation; $2C_F + \frac{1}{N_c} = N_c$, no $1/N_c^2$.
5. Hierarchy $\langle D \rangle \to \langle DD \rangle \to \langle DDD \rangle, \langle QD \rangle \to \cdots$.
6. One large-N_c factorization ⇒ BK.

The Balitsky–Kovchegov equation

$$\frac{\partial S_{\boldsymbol{x}_1\boldsymbol{x}_2}}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\boldsymbol{z}\, \frac{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2}{(\boldsymbol{x}_1 - \boldsymbol{z})^2(\boldsymbol{x}_2 - \boldsymbol{z})^2} \left[S_{\boldsymbol{x}_1\boldsymbol{z}} S_{\boldsymbol{z}\boldsymbol{x}_2} - S_{\boldsymbol{x}_1\boldsymbol{x}_2}\right]$$

exact hierarchy $+$ one large-N_c factorization

BACKUP: Deriving the Fierz / completeness identity

The $N_c^2 - 1$ traceless generators t^a together with $\mathbf{1}$ form a complete, trace-orthogonal basis of $N_c \times N_c$ matrices $\left(1 + (N_c^2 - 1) = N_c^2\right)$. So *any* matrix expands as

$$M = c_0\,\mathbf{1} + c_a\,t^a, \qquad c_0 = \tfrac{1}{N_c}\,\mathrm{tr}\,M, \quad c_a = 2\;\mathrm{tr}(t^a M),$$

using $\mathrm{tr}\,t^a = 0$, $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$, $\mathrm{tr}\,\mathbf{1} = N_c$. In components $(\mathrm{tr}\,M = M_{kk}$, $\mathrm{tr}(t^a M) = (t^a)_{kl} M_{lk})$,

$$M_{ij} = \tfrac{1}{N_c}\,M_{kk}\,\delta_{ij} + 2\,(t^a)_{ij}(t^a)_{kl}\,M_{lk}.$$

This holds for arbitrary M. Writing $M_{ij} = M_{lk}\delta_{il}\delta_{jk}$ and $M_{kk}\delta_{ij} = M_{lk}\delta_{kl}\delta_{ij}$ and matching the coefficient of M_{lk},

$$\delta_{il}\delta_{jk} = \tfrac{1}{N_c}\delta_{ij}\delta_{kl} + 2\sum_a (t^a)_{ij}(t^a)_{kl} \implies \boxed{\sum_a (t^a)_{ij}(t^a)_{kl} = \tfrac{1}{2}\left(\delta_{il}\delta_{jk} - \tfrac{1}{N_c}\delta_{ij}\delta_{kl}\right)}$$

The $-1/N_c$ enforces tracelessness (SU vs. U(N_c)): it subtracts the singlet $\mathbf{1}$ direction — the very same subtraction that becomes the $-\frac{1}{N_c}$ singlet term in the dipole's real emission. Check: contract $j = k \Rightarrow$ $t^a t^a = \frac{1}{2}\left(N_c - \frac{1}{N_c}\right)\mathbf{1} = C_F\,\mathbf{1}$.

EXTRA: From the hierarchy to one master equation

The Balitsky tower is the set of *moments* of a single weight functional $W_Y[V]$ over Wilson-line configurations,

$$\langle \hat{O} \rangle_Y = \int \mathcal{D}V \; \hat{O}[V]\, W_Y[V],$$

whose common source is the JIMWLK equation — a Fokker–Planck equation for W:

$$\boxed{\partial_Y W_Y[V] = H\, W_Y[V]}$$

- Differentiating $\langle \hat{O} \rangle_Y$ in Y regenerates the hierarchy line by line.
- $H = \mathcal{N} \int_z \sum_{n,m} \boldsymbol{\Gamma}_n \cdot \boldsymbol{\Gamma}_m$ is the *current squared*: second order in the color-rotation derivatives, positive kernel.
- A positive second-order operator on a probability ⇒ *diffusion* — the Wilson lines random-walk in color space as Y grows.

The hierarchy and JIMWLK are the same content — moments vs. the distribution they come from — and that distribution diffuses.

EXTRA: Langevin form: solve it by sampling

Every Fokker–Planck equation is equivalent to a Langevin equation: evolve *sample* Wilson lines with noise instead of the distribution,

$$V_x(Y+dY) = \exp\big(i\,\theta^a_{L,x}\,t^a\big)\;V_x(Y)\;\exp\big(-i\,\theta^a_{R,x}\,t^a\big),$$

$$\theta_{L,R} = \int_z K^i_{xz}\,\big[\,\text{bare}(L)\;/\;U_z\text{-dressed}(R)\,\big]\,\xi_i(z), \qquad K^i_{xz} = \sqrt{\tfrac{\alpha_s}{\pi}}\,\frac{(x-z)^i}{(x-z)^2},$$

with Gaussian white noise $\langle \xi^a_i(x)\xi^b_j(y)\rangle = \delta^{ab}\delta_{ij}\delta^2_{xy}$.

- Bare vs. U_z-dressed rotation = the $T_L - U\,T_R$ split of the emission current.
- Noise covariance = $\mathbf{\Gamma}\cdot\mathbf{\Gamma}$ = the JIMWLK kernel $r_n\cdot r_m/(r_n^2 r_m^2)$.
- One dY = emit one random soft gluon, rotate every line; average D, DD, $DDD, \ldots$ over histories.
- Solves the *full, untruncated* finite-N_c hierarchy — BK is the large-N_c shortcut.

Sample histories and average: no truncation, no large-N_c. [Weigert; Blaizot–Iancu–Weigert; Rummukainen–Weigert]