

Going beyond leading order: Leading logarithmic corrections

Probability for gluon emission

$$dP \propto \alpha_s \frac{dk_T^2}{k_T^2} \frac{dx}{x}$$

Integrate over phase space: Two types of singularities that lead to logarithmic enhancements

- Collinear enhancement: gluon emitted almost parallel to parent parton ($k_T \rightarrow 0$)

Integral $\int_{Q_0^2}^{Q^2} \frac{dk_T^2}{k_T^2}$ gives $\ln(Q^2/Q_0^2)$

- Soft enhancement: emitted gluon carries small fraction of the parent's momentum ($x \rightarrow 0$)

Integral $\int_x^1 \frac{dx'}{x'}$ gives $\ln(1/x)$

Gluon Emissions and Logarithmic Enhancements

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Strong ordering

If we have consecutive emissions with the following strong ordering

$$Q^2 \gg k_{Tn}^2 \gg \dots \gg k_{T1}^2 \gg Q_0^2$$

we are in a regime where for each emission, the logarithmic enhancement dominates the result

In this regime each emission can be treated as a perturbation on the previous state; we can ignore interference/coherence between different emissions and write

$$\int_{Q_0^2}^{Q^2} \frac{dk_{Tn}^2}{k_{Tn}^2} \int_{Q_0^2}^{k_{Tn}^2} \frac{dk_{Tn-1}^2}{k_{Tn-1}^2} \dots \int_{Q_0^2}^{k_{T3}^2} \frac{dk_{T2}^2}{k_{T2}^2} \int_{Q_0^2}^{k_{T2}^2} \frac{dk_{T1}^2}{k_{T1}^2} = \frac{1}{n!} (\ln(Q^2/Q_0^2))^n$$

Strong ordering

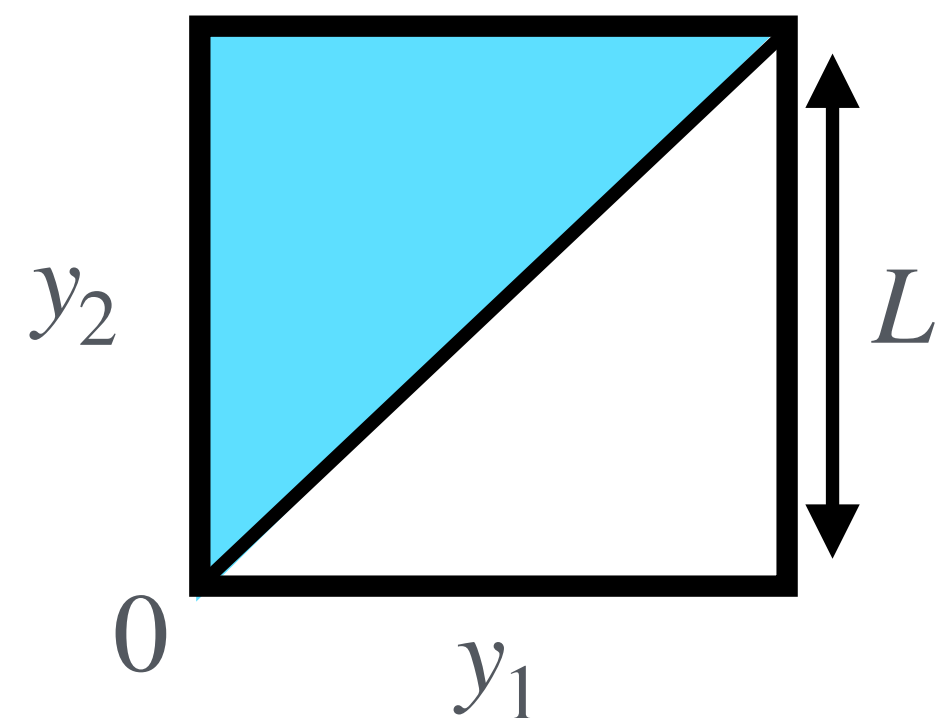
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$$0 < y_1 < y_2 < L$$

$$\int_0^L dy_1 \int_{y_1}^L dy_2 = L^2/2$$

Strong ordering

Same for the soft enhancement. If we have strongly ordered emissions:

$$1 \gg x_1 \gg \dots \gg x_n \gg x$$

we get a strong logarithmic enhancement:

$$\int_x^1 \frac{dx_n}{x_n} \int_{x_n}^1 \frac{dx_{n-1}}{x_{n-1}} \dots \int_{x_2}^1 \frac{dx_1}{x_1} = \frac{1}{n!} (\ln(1/x))^n$$

Taking these terms only will correspond to the leading logarithmic approximation.

As each emission comes with $\alpha_s L$, with L being the large logarithm, all these terms need to be resummed.

That is done via the DGLAP (for the collinear enhancement) and BFKL (for the small x) equations.

DGLAP equation

To start, let us write the precise splitting function for a near-collinear splitting of $q \rightarrow qg$

$$dP = \frac{\alpha_s C_F}{2\pi} \frac{dk_T^2}{k_T^2} \frac{1+z^2}{1-z} dz$$

follows from the Dirac trace algebra for the QCD vertex

where $1 - z$ is the gluon momentum fraction (we called it x earlier)

Number of quarks $q(x, Q^2)$ at a higher transverse resolution scale Q^2 comes from quarks that existed at an initial scale Q_0 with a higher momentum fraction x_0

$$q(x, Q^2) = q(x, Q_0^2) + \frac{\alpha_s}{2\pi} \ln \left(\frac{Q^2}{Q_0^2} \right) \int_x^1 \frac{dz}{z} \left[C_F \frac{1+z^2}{1-z} \right] q \left(\frac{x}{z}, Q_0^2 \right)$$

which comes from integrating over the transverse phase space.

DGLAP equation

When $\alpha_s \ln(Q^2/Q_0^2)$ is large (order 1), we must account for infinite chain of emissions.

This can be done by taking the derivative with respect to the scale $\ln Q^2$ (see later as to why):

$$\frac{\partial q(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z) q\left(\frac{x}{z}, Q^2\right)$$

The kernel $P_{qq}(z)$ is the Altarelli-Parisi splitting function.

The soft singularity is handled with the '+'-prescription

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right]$$

which follows from including virtual corrections (see next slide).

Virtual corrections

Must also consider virtual processes, such as a quark emitting and then reabsorbing a gluon (self-energy) or vertex corrections

Virtual corrections are proportional to $\delta(1 - z)$, meaning they contribute only to the case where the quark remains at its original momentum fraction

The infrared divergence from the "real" soft gluon emission ($z \rightarrow 1$) is exactly canceled by a corresponding divergence in the "virtual" corrections

The '+'-prescription is a way to write the combined finite result. It is defined such that for any smooth test function

$$\int_0^1 \frac{f(z)}{(1-z)_+} dz = \int_0^1 \frac{f(z) - f(1)}{1-z} dz$$

Splitting function with + prescription

Note that the term $\frac{3}{2}\delta(1 - z)$ also needs to be there to have the splitting function integrate to 0.

Take $\int_0^1 \frac{f(z)}{(1 - z)_+} dz = \int_0^1 \frac{f(z) - f(1)}{1 - z} dz$ and identify $f(z) = z^2 + 1$. So, $f(1) = 2$

$$\text{So } \int_0^1 \frac{(z^2 + 1) - 2}{(1 - z)_+} dz = \int_0^1 \frac{z^2 - 1}{1 - z} dz = - \int_0^1 (z + 1) dz = -3/2$$

Meaning that with the delta-function term the splitting function integrates to zero

$$P_{qq}(z) = C_F \left[\frac{1 + z^2}{(1 - z)_+} + \frac{3}{2}\delta(1 - z) \right]$$

as it should. It will also appear when carefully computing and adding the virtual corrections.

DGLAP equation and resummation

So, why does solving DGLAP

$$\frac{\partial q(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z) q\left(\frac{x}{z}, Q^2\right)$$

account for resumming all log enhanced terms?

Generally, a differential equation of this form $dN/dY = \omega N$ has a solution $N(Y) \propto e^{\omega Y}$.

Its Taylor expansion is

$$e^{\omega Y} = 1 + \omega Y + \frac{1}{2}(\omega Y)^2 + \dots + \frac{1}{n!}(\omega Y)^n + \dots$$

meaning the result resums terms with one emission, two emissions, ... n emissions ...

Full DGLAP equations

We have different splittings and can also track the gluon distribution function, so the full set of DGLAP equations is a coupled set of equations:

For quarks:

$$\frac{\partial q_f(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \left[P_{qq}(z) q_f\left(\frac{x}{z}, Q^2\right) + P_{qg}(z) G\left(\frac{x}{z}, Q^2\right) \right]$$

For gluons:

$$\frac{\partial G(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \left[P_{gg}(z) G\left(\frac{x}{z}, Q^2\right) + P_{gq}(z) \sum_{f, \bar{f}} q_f\left(\frac{x}{z}, Q^2\right) \right]$$

BFKL equation

Now for the equation that resums powers of $\alpha_s \ln(1/x)$.

It's physical origin comes from the time scales discussed before:

A gluon with longitudinal momentum $k^+ = xP^+$ and transverse momentum k_T has lifetime

$$\Delta x^+ = \frac{2k^+}{k_T^2} = \frac{2xP^+}{k_T^2}$$

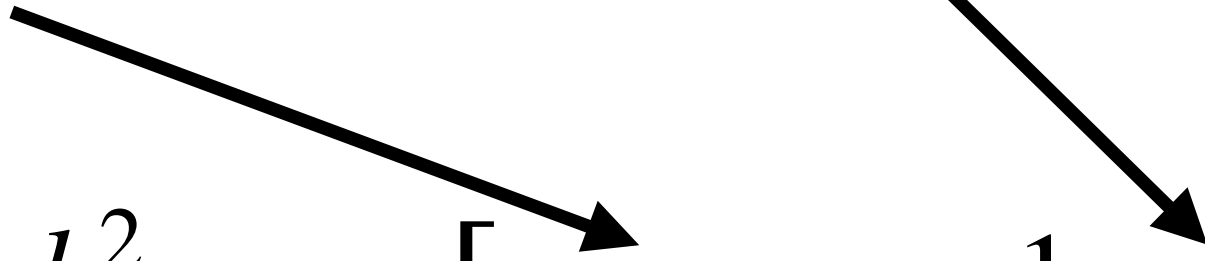
In the strongly ordered cascade, successively softer (smaller x) gluons have shorter lifetimes.

The emitters at $x' \gg x$ are effectively frozen during the dynamics at x and can be treated as static color sources (all k_T here are similar).

BFKL equation

Evolution in rapidity $Y = \ln(1/x)$. Equivalent to evolution to higher energy.

Like DGLAP, BFKL has a **real** emission and **virtual** emission part:

$$\frac{\partial f(Y, k_T^2)}{\partial Y} = \bar{\alpha}_s \int \frac{d^2 p_T}{\pi} \frac{k_T^2}{p_T^2 (k_T - p_T)^2} \left[f(Y, p_T^2) - \frac{1}{2} f(Y, k_T^2) \right]$$


where $f(Y, k_T^2)$ is the unintegrated (k_T -dependent) gluon distribution.

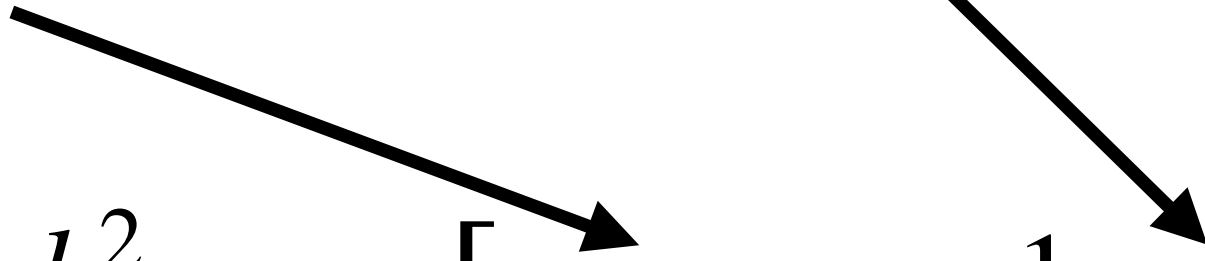
Real and virtual parts cancel the IR-divergence when $p_T \rightarrow k_T$

(note that the emitted gluon has $q_T = p_T - k_T$)

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- The real part is computed by summing all possible ways a gluon can be emitted. Summarized by the **Lipatov vertex**, an effective vertex that ensures gauge invariance in the high-energy limit
- In the BFKL context, summing virtual corrections to all orders turns a standard gluon into a **reggeized gluon**. This modified "quasi-particle" propagator accounts for the probability that the system remains in its current state

Growth in gluon number and towards saturation

Ignoring the complex transverse momentum dynamics, schematically the equation looks like

$$\frac{\partial N}{\partial Y} = \omega \bar{\alpha}_s N(Y)$$

which, again, is solved by

$$N(Y) \sim e^{\omega \bar{\alpha}_s Y}$$

which is equivalent to a power law in x , because $Y = \ln(1/x)$

$$N(x) \sim (1/x)^\lambda$$

The growth happens because emitted gluons are themselves color-charged sources

BFKL is a "high parton density machine": Because the transverse size of each emitted gluon is about the same (similar k_T), gluon number is growing exponentially but their size isn't shrinking.

They eventually begin to overlap. Saturation will happen when gluons are so packed they begin to recombine.

