Center for Frontiers in Nuclear Science

# Gravitational form factors on the light front

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**In collaboration with**
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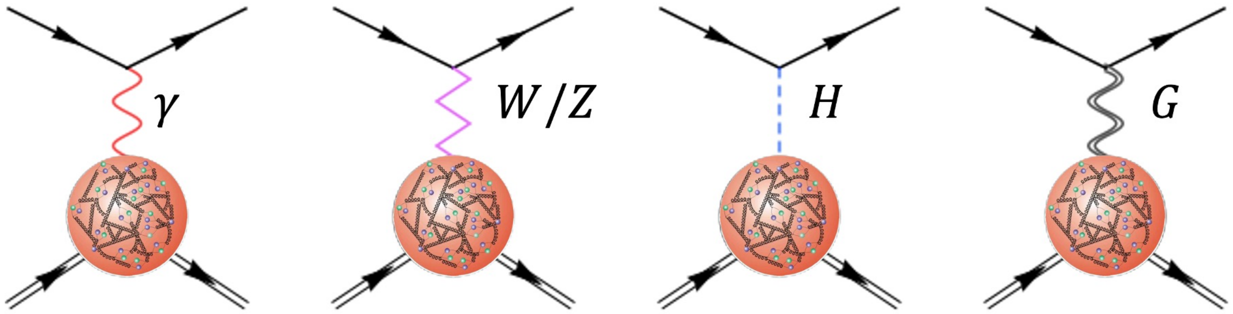
2026 CFNS Summer School on the Physics of the Electron-Ion Collider, New York, June 5, 2026

# The last global unknown



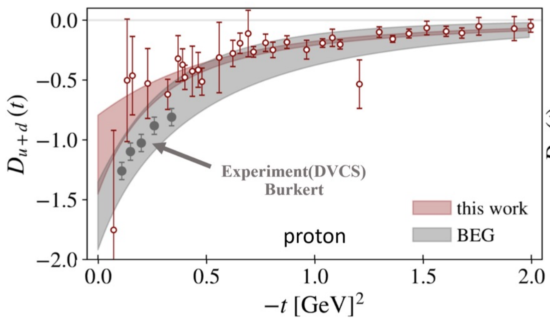

Energy-momentum tensor (EMT) of spin-1/2 hadrons: [Kobzarev:1962wt, Pagels: 1966zza]

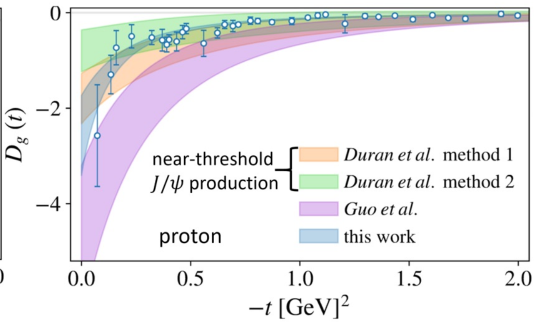
$$\langle p', s'|\hat{T}_i^{\mu\nu}(0)|p, s\rangle =$$

$$\frac{1}{2M}\bar{u}_{s'}(p')\left[2P^\mu P^\nu A_i(q^2) + iP^{\{\mu}\sigma^{\nu\}\rho}q_\rho J_i(q^2) + \frac{1}{2}(q^\mu q^\nu - g^{\mu\nu}q^2)D_i(q^2) + 2g^{\mu\nu}\bar{c}_i(q^2)\right]u_s(p)$$

Conservation laws constrain gravitational form factors except *D* [Polyakov:2018zvc, Hackett:2023rif]

| | | | | | | |
|---|---|---|---|---|---|---|
| **em:** | $\partial_\mu J^\mu_{\text{em}} = 0$ | $\langle N'\|J^\mu_{\textbf{em}}\|N\rangle$ | $\longrightarrow$ | $Q$ | $=$ | $1.602176487(40)\times 10^{-19}\text{C}$ |
| | | | | $\mu$ | $=$ | $2.792847356(23)\mu_N$ |
| **weak:** | PCAC | $\langle N'\|J^\mu_{\textbf{weak}}\|N\rangle$ | $\longrightarrow$ | $g_A$ | $=$ | $1.2694(28)$ |
| | | | | $g_p$ | $=$ | $8.06(55)$ |
| **gravity:** | $\partial_\mu T^{\mu\nu}_{\textbf{grav}} = 0$ | $\langle N'\|T^{\mu\nu}_{\textbf{grav}}\|N\rangle$ | $\longrightarrow$ | $m$ | $=$ | $938.272013(23)\ \text{MeV}/c^2$ |
| | | | | $J$ | $=$ | $\frac{1}{2}$ |
| | | | | $\boldsymbol{D}$ | $=$ | **?** |



# Mechanical properties and confinement

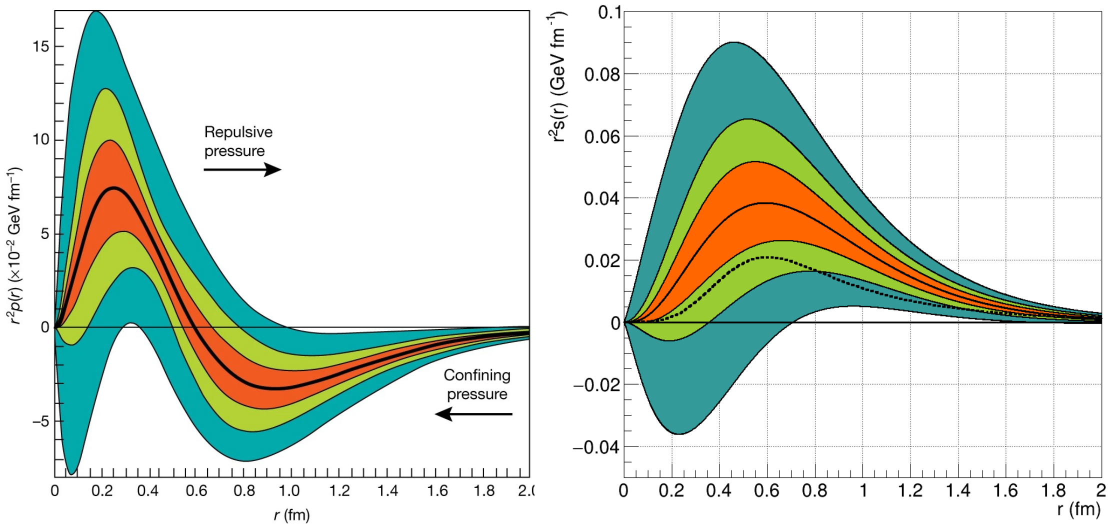
The $ij$ components of EMT define the stress tensor: [Burkert:2018bqq, Burkert:2021ith]

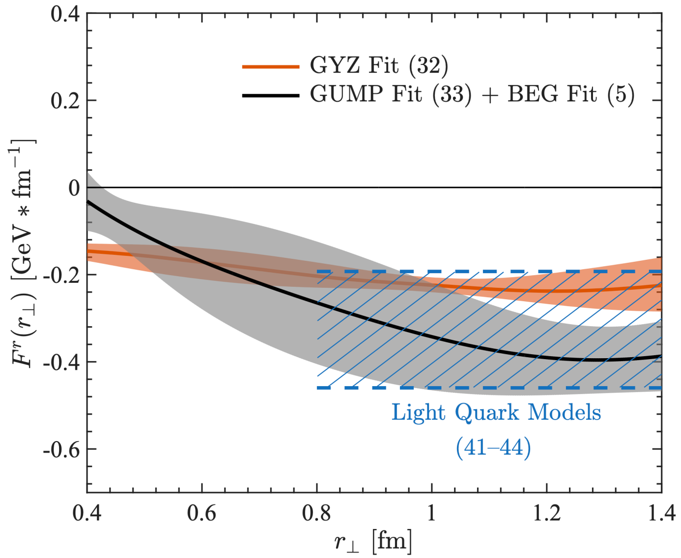
$$T^{ij}(\boldsymbol{r}) = \Big(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\Big)s(r) + \delta^{ij}p(r)$$

The divergence of EMT defines the force on quarks [Ji:2025qax, Ji:2026lyj]

$$\mathcal{F}_q^i \equiv \partial_\mu T_q^{\mu i}$$

Confinement induced by a constant force?



# Microscopic interpolation of form factors

Drell-Yan-West formula for charge form factor: [Drell:1969km, West:1970av, Brodsky:1980zm]

$$\rho_{\mathrm{ch}}(r_\perp) = \int [\mathrm{d}x_i \mathrm{d}^2 r_{i\perp}]_n |\tilde{\psi}_n(\{x_i, \vec{r}_{i\perp}\})|^2 \sum_j e_j \delta^{(2)}(r_\perp - r_{j\perp}) \equiv \langle \sum_j e_j \delta^{(2)}(r_\perp - r_{j\perp}) \rangle$$

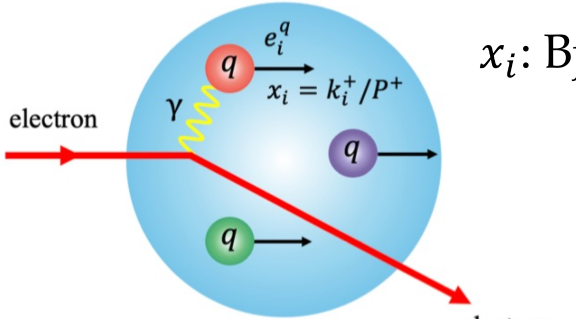
light-front wave functions

Brodsky-Hwang-Ma-Schmidt formula for gravitational form factor $A$: [Brodsky:2000ii]

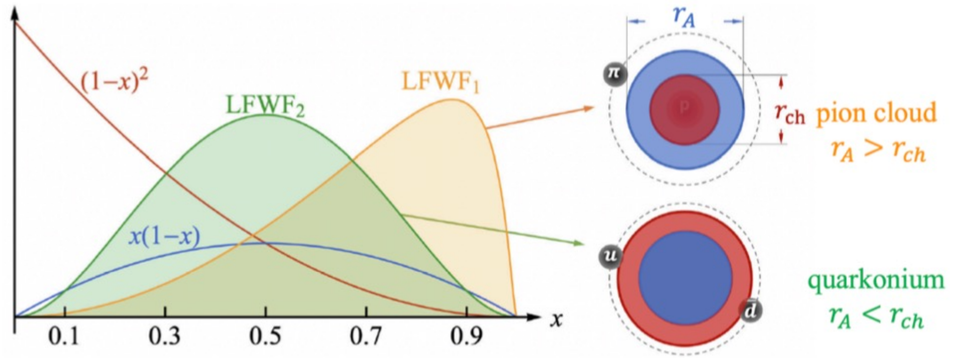
$$\mathcal{A}(r_\perp) = \int [\mathrm{d}x_i \mathrm{d}^2 r_{i\perp}]_n |\tilde{\psi}_n(\{x_i, \vec{r}_{i\perp}\})|^2 \sum_j x_j \delta^{(2)}(r_\perp - r_{j\perp}) \equiv \langle \sum_j x_j \delta^{(2)}(r_\perp - r_{j\perp}) \rangle$$

light-front wave functions

Matter density $\mathcal{A}(r_\perp)$ mainly samples the valance partons $x_j \sim O(1)$; wee parton $x_j \ll 1$ contributions suppressed




$x_i$: Bjorken variable




$r_{ch}^{\pi} \sim 1.6 r_A^{\pi}$

# Microscopic interpolation of EMT

[Cao:2023ohj, Cao:2024fto]

momentum $$t^{++} = \langle \sum_j e^{-i\vec{q}_\perp \cdot \vec{r}_{j\perp}} x_j \rangle$$

shear $$t^{12} = \frac{1}{2} \langle \sum_j e^{-i\vec{q}_\perp \cdot \vec{r}_{j\perp}} \frac{i \overleftrightarrow{\nabla}_j^1 \overleftrightarrow{\nabla}_j^2 - q^1 q^2}{x_j} \rangle$$

energy $$t^{+-} = 2 \langle \underbrace{\sum_j e^{i\vec{q}_\perp \cdot \vec{r}_{j\perp}} \frac{-\frac{1}{4} \overleftrightarrow{\nabla}_j^2 + m_j^2 - \frac{1}{4} q_\perp^2}{x_j}}_{\text{kinetic part}} + \underbrace{V e^{i\vec{r}_{N\perp} \cdot \vec{q}_\perp}}_{\text{potential part}} \rangle$$

$$\int d^3 x T^{+\mu}(x) = P^\mu$$

$$P^- = \frac{P_\perp^2 + M^2}{P^+}$$

light-front coordinates

$$x^\pm = x^0 \pm x^3$$

$$\vec{x}^\perp = (x^1, x^2)$$

The quantum average is defined as

$$\langle \hat{O} \rangle = \int [\mathrm{d}x_i \mathrm{d}^2 \boldsymbol{r}_{i\perp}]_n \psi_n^*(\{x_i, \boldsymbol{r}_{i\perp}\}) \hat{O} \psi_n(\{x_i, \boldsymbol{r}_{i\perp}\})$$

International Journal of Modern Physics A | Vol. 33, No. 26, 1830025 (2018)
| Reviews

**Forces inside hadrons: Pressure, surface tension, mechanical radius, and all that**

Maxim V. Polyakov and Peter Schweitzer

$\hat{T}_{++}$ of the EMT. Being related to the stress tensor $\hat{T}_{ij}$ the form factor $D(t)$ naturally "mixes" good and bad light-front components and is described in terms of transitions between different Fock state components in overlap representation. As a quantity intrinsically nondiagonal in a Fock space, it is difficult to study the $D$-term in approaches based on light-front wave functions. This is due to the rela-

# Application to charmonium

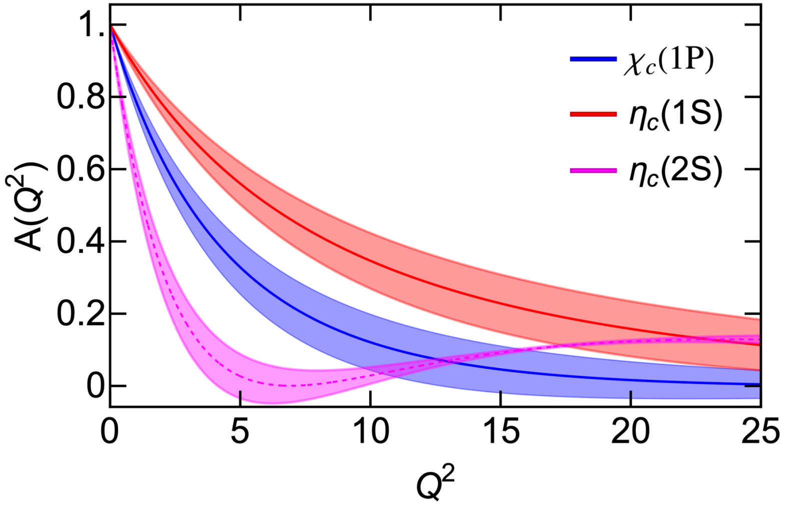
Effective Hamiltonian:

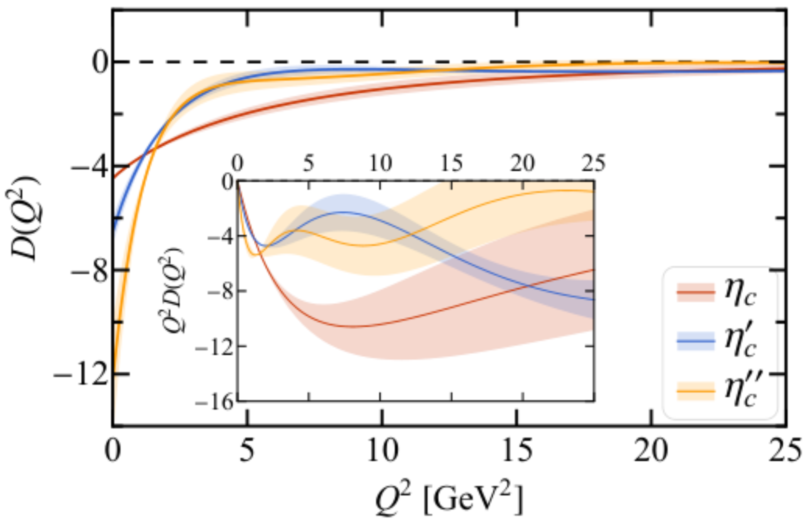
[Li:2017mlw, Xu:2024hfx, Hu:2024edc]

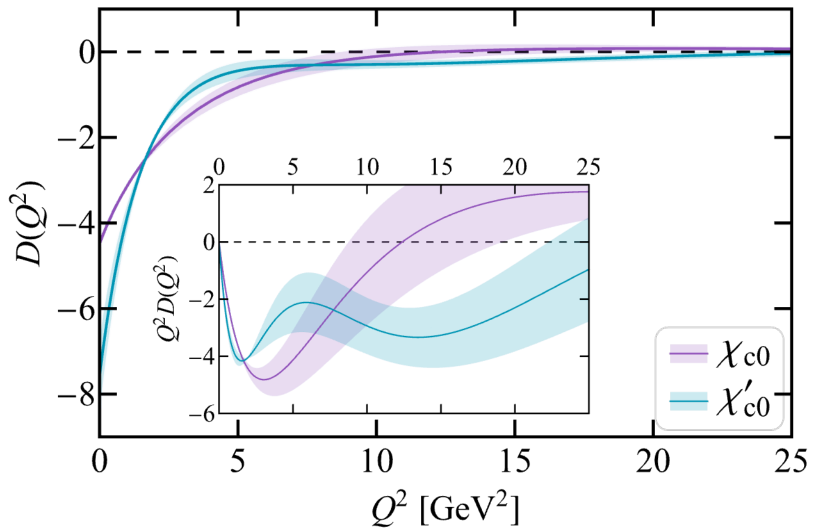
kinetic energy, confinement, one gluon exchange

$$H_{\text{eff}} = \frac{\vec{k}_\perp^2 + m_q^2}{x} + \frac{\vec{k}_\perp^2 + m_{\bar{q}}^2}{1-x} + \kappa^4 x(1-x)\vec{r}_\perp^2 - \frac{\kappa^4}{(m_q+m_{\bar{q}})^2}\partial_x(x(1-x)\partial_x) - \frac{C_F 4\pi\alpha_s(Q^2)}{Q^2}\bar{u}_{s'}(k')\gamma_\mu u_s(k)\bar{v}(\bar{k})\gamma^\mu v_{\bar{s}'}(\bar{k}')$$

Gravitational form factors of $\eta_c$ and $\chi_{c0}$



# Summary

- We obtain a non-perturbative light-front wave function representation to evaluate the gravitational form factors
- The light-front wave function representation provides a microscopic interpolation for energy-momentum tensors
- We apply the light-front wave function representation to charmonium solved from an effective Hamiltonian

Based on:
Cao, Li, Vary, PRD 108, 056026 (2023)
Xu, Cao, Hu, Li, Zhao, Vary, PRD 109, 114024 (2024)
Cao, Li, Vary, PRD 110, 076025 (2024)
Hu, Cao, Xu, Li, Zhao, Vary, PRD 111 (2025) 7, 074031

*Thank you!*

# Backup Slides

# Gravitational form factor D: the last global unknown

The structure of hadrons can be probed by the other fundamental forces

[Polyakov:2018zvc]

| | | | | | |
|---|---|---|---|---|---|
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| | | | | $J$ | $= \frac{1}{2}$ |
| | | | | $\boldsymbol{D}$ | $=$ **?** |

Conservation laws constrain gravitational form factors except $D$

[Cotogno:2019xcl, Lorce:2019sbq]

$$A(0) = 1, \quad J(0) = \frac{1}{2}, \quad \lim_{Q^2 \to 0} Q^2 D(Q^2) = 0$$

# Mechanical properties of hadrons

$D(q^2)$ is related to the pressure and shear forces inside hadrons

[Polyakov:2018zvc]

$$T^{ij}(\boldsymbol{r}) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right)s(r) + \delta^{ij}p(r)$$

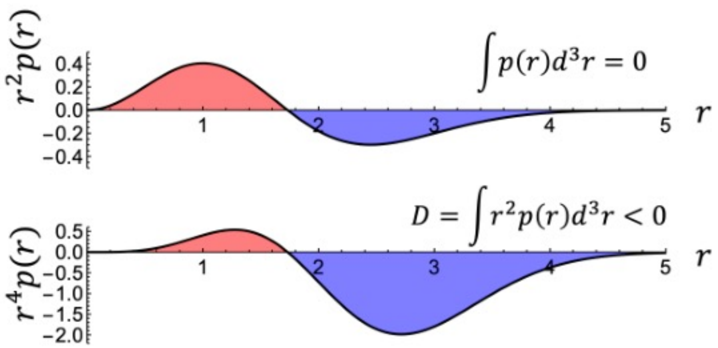
$$\Downarrow$$

$$p(r) = \frac{1}{6M}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}}{\mathrm{d}r}\tilde{D}(r), \qquad s(r) = -\frac{1}{4M}r\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\tilde{D}(r)$$

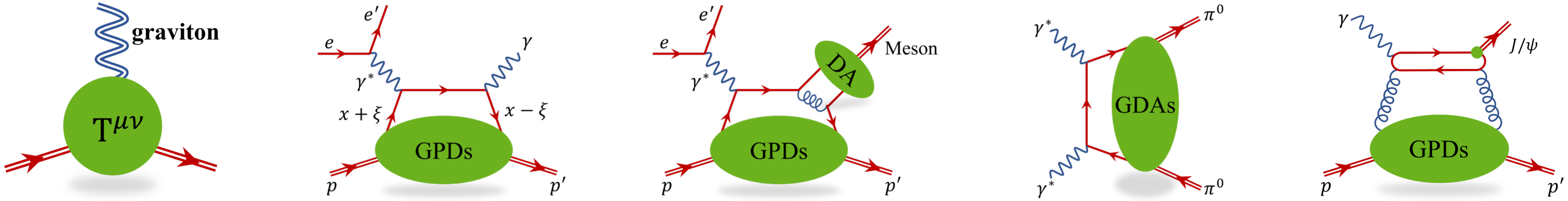
Hadron stability conditions:

[Perevalova:2016dln]

- Force equilibrium (von Laue condition): $\int d^3 r p(r) = 0$
- Stability conjecture: $D(0) = \int \mathrm{d}^3 r r^2 p(r) < 0$



# How to access gravitational form factors




- Deeply virtual Compton scattering
- Deeply virtual meson production
- Two-photon pair production
- $J/\psi$ threshold photoproduction

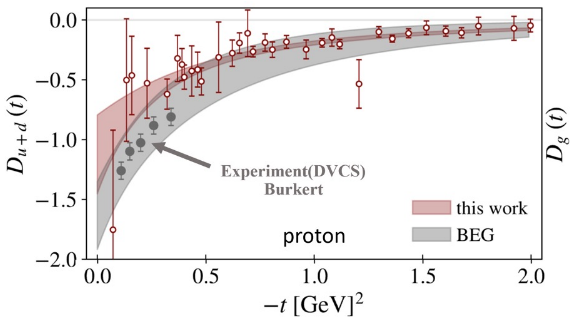
[Kumano:2017lhr, Duran:2022xag, Burkert:2023wzr]

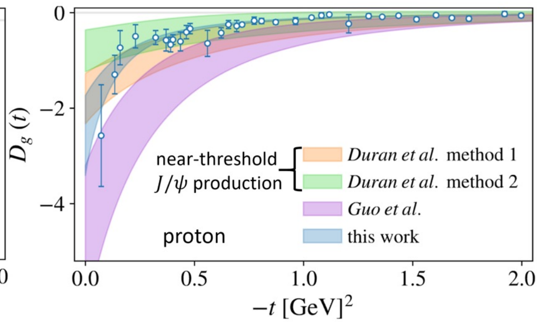
Ji's sum rule:

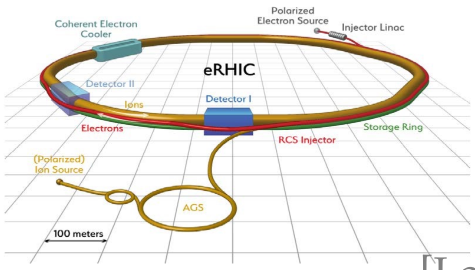
$$\int_{-1}^{1} dx x H^{q,g}(x,\xi,t) = A^{q,g}(t) + \xi^2 D^{q,g}(t), \quad \int_{-1}^{1} dx x E^{q,g}(x,\xi,t) = B^{q,g}(t) - \xi^2 D^{q,g}(t)$$

[Ji:1996nm]

Here, $H^{q,g}$ and $E^{q,g}$ are generalized parton distributions

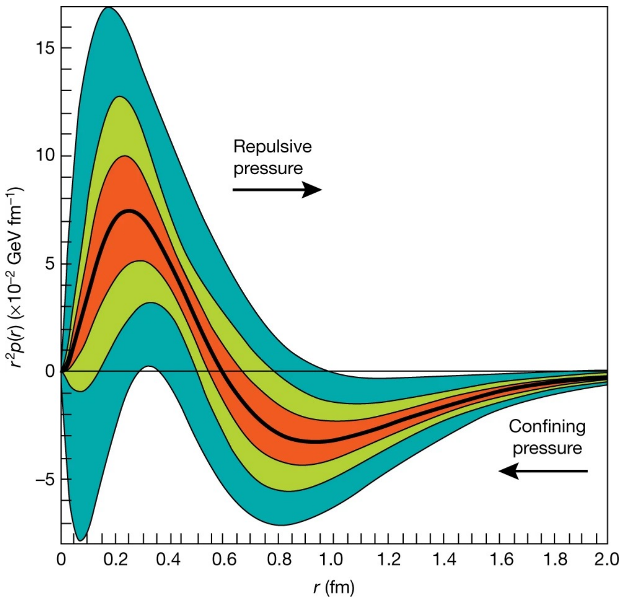


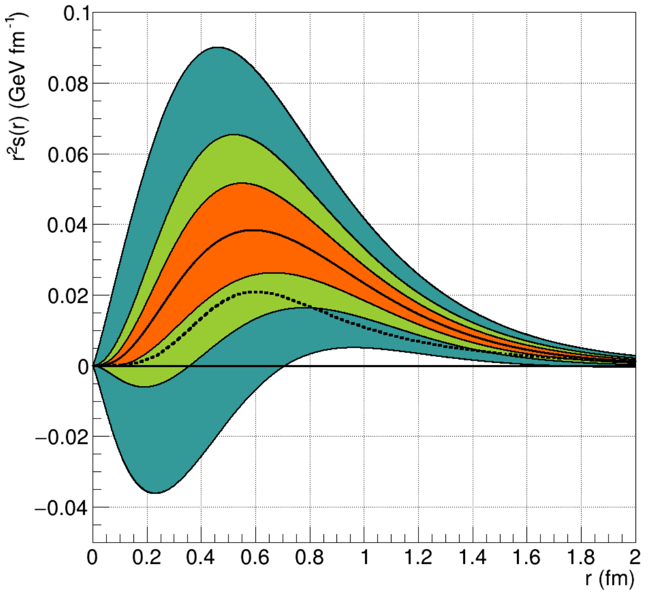




[Lattice'23: Hackett:2023nkr]

# The first measurement of the pressure and shear



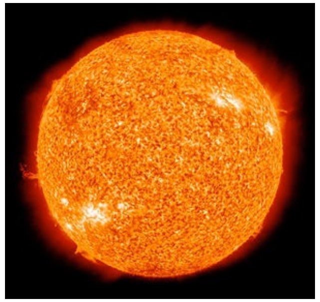

[Burkert:2018bqq]
[Burkert:2021ith]

The proton contains the highest pressure in nature

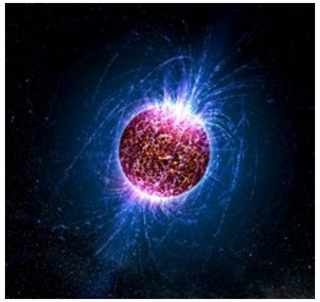
Adapted from Kumano: LC2024

Earth atmosphere
$10^5$Pa

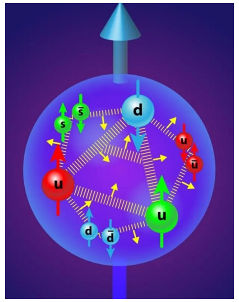
Center of earth
$10^{11}$Pa



Center of sun
$10^{16}$Pa



Neutron star
$10^{34}$Pa

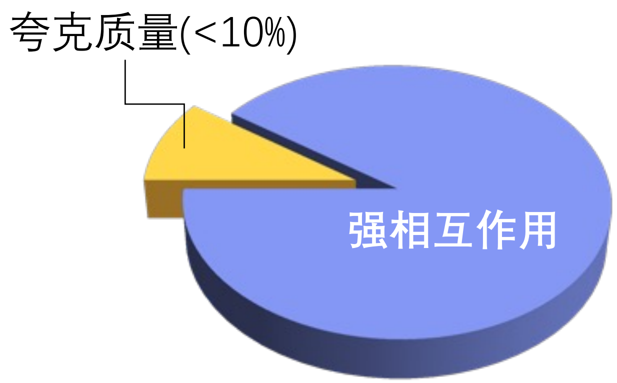


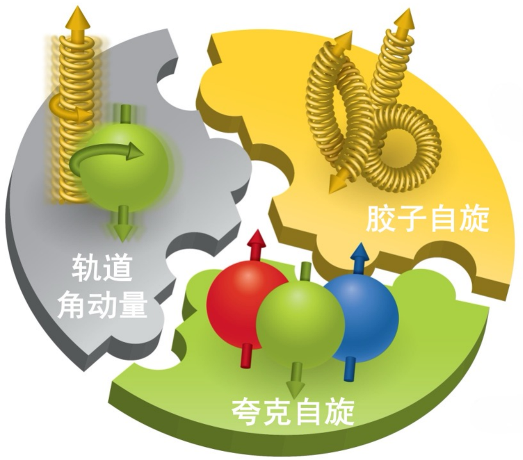

Proton
$10^{35}$Pa

# Big problems related with energy-momentum tensor

- Origin of hadron mass
- Spin decomposition
- Mechanical properties of hadrons

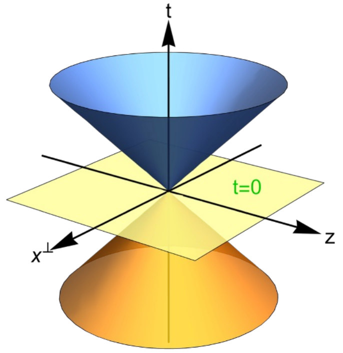
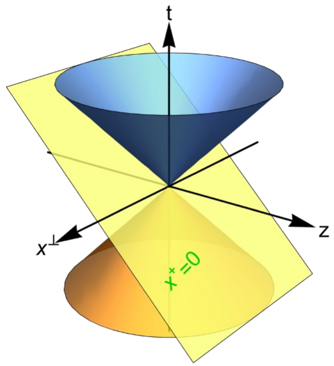
[Burkert:2023wzr]




# Light-front quantization

[Dirac:1949cp]

| | equal time quantization | light-front quantization |
|---|---|---|
| |  |  |
| ■ Time: | $t \equiv x^0$ | $t \equiv x^+ = x^0 + x^3$ |
| ■ Hamiltonian: | $H \equiv P^0$ | $H \equiv P^- = P^0 - P^3$ |
| ■ Dispersion relation: | $p^0 = \sqrt{\vec{p}^2 + m^2}$ | $p^- = (\vec{p}_\perp^2 + m^2)/p^+$ |

light-front coordinates

$$x^\pm = x^0 \pm x^3$$

$$\vec{x}^\perp = (x^1, x^2)$$

Light-front quantization is a Hamiltonian method of the quantum field theory

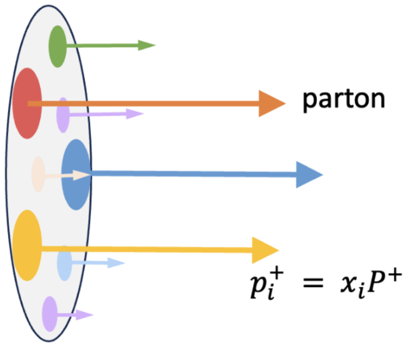
$$(P^+ \hat{P}^- - \vec{P}_\perp^2) \left|\psi_H\right\rangle = M_h^2 \left|\psi_h\right\rangle$$

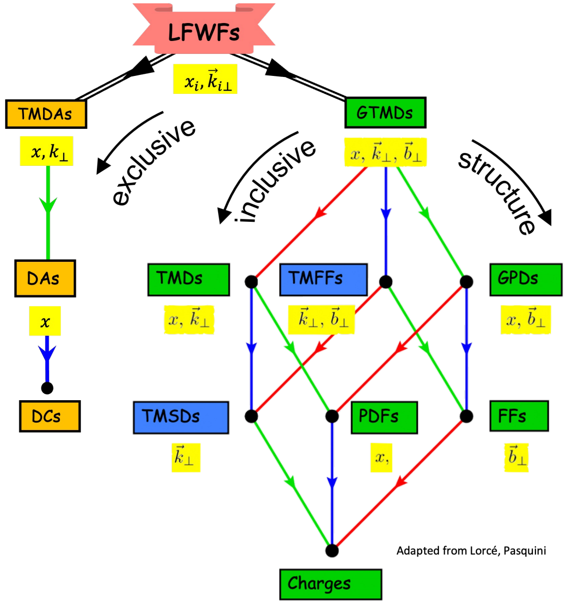
[Review: Brodsky:1997de]

# Light-front wave functions

$$|\psi_h(P,j,\lambda)\rangle = \sum_{n=1}^{\infty}\int [\mathrm{d}x_i \mathrm{d}^2 k_{i\perp}]_n \psi_{h/n}(\{\vec{k}_{i\perp}, x_i, \lambda_i\}_n)\big|\{\vec{p}_{i\perp}, p_i^+, \lambda_i\}_n\rangle$$

- Light-front physics measures hadron structures in high-energy scattering experiments
- Light-front wave functions (LFWFs) provide full information about the hadron structure
- LFWF representation offers simple physical interpretations




[Lorce:2011kd]

# Light-front wave function representation for $D(q^2)$

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https://doi.org/10.1142/S0217751X18300259 | **Cited by:** 241 **(Source: Crossref)**

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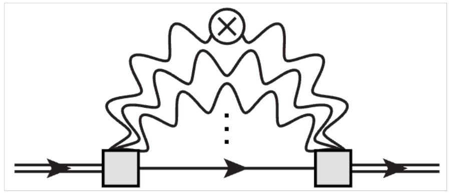
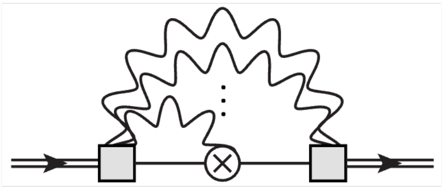


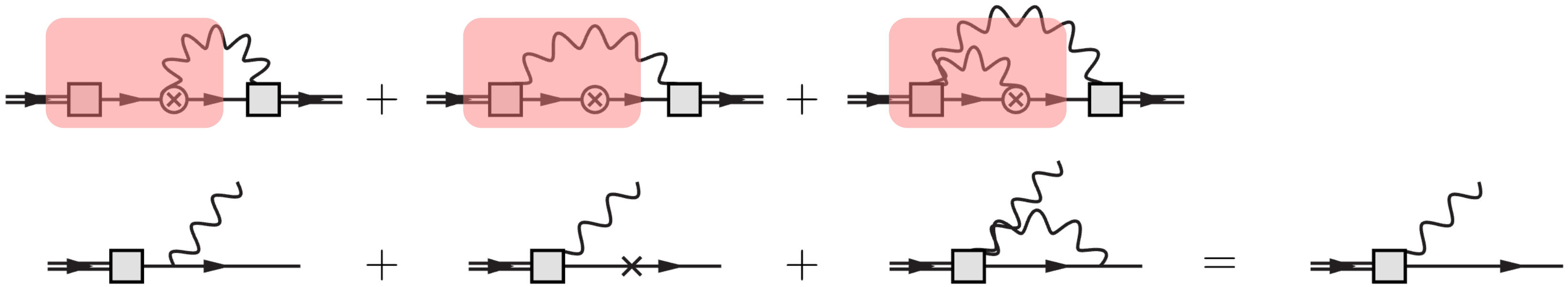
diagonal



non-diagonal

- We start from the scalar Yukawa theory to seek inspiration
- $D(q^2)$ contains the overlap between different Fock state components. However, the non-diagonal diagrams add up to a diagonal diagram

[Cao:2023ohj]



# Scalar Yukawa model

$$\mathcal{L} = \partial_\mu N^\dagger \partial^\mu N - m^2 N^\dagger N + \frac{1}{2}\partial_\mu \pi \partial^\mu \pi - \frac{1}{2}\mu^2\pi^2 + g_0 N^\dagger N \pi + \delta m^2 N^\dagger N$$

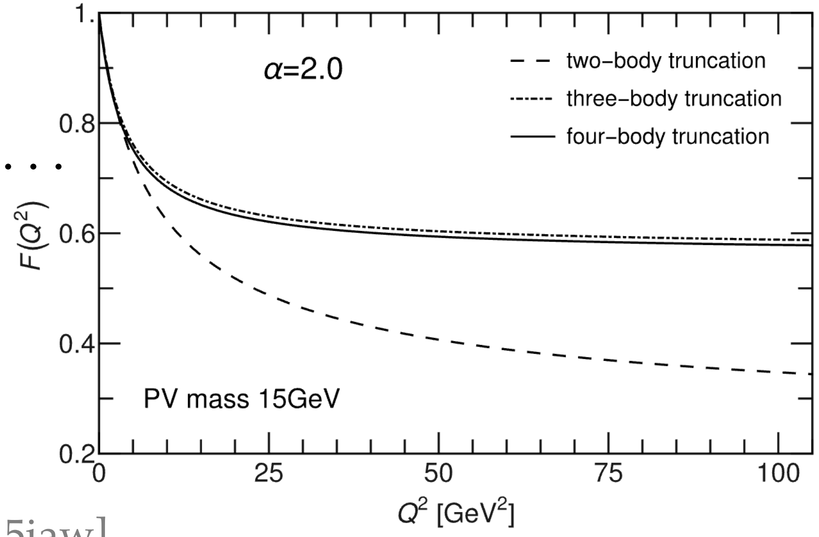
$$\Downarrow$$

$$\hat{T}^{\mu\nu} = \partial^{\{\mu} N^\dagger \partial^{\nu\}} N - g^{\mu\nu}\left[\partial_\sigma N^\dagger \partial^\sigma N - (m^2 - \delta m^2) N^\dagger N\right] - g^{\mu\nu} g_0 N^\dagger N \pi + \partial^\mu \pi \partial^\nu \pi - \frac{1}{2} g^{\mu\nu}\left(\partial^\rho \pi \partial_\rho \pi - \mu_0^2 \pi^2\right)$$

where $m = 0.94\text{GeV}$, $\mu = 0.14\text{GeV}$. $g_0$ and $\delta m^2$ are bare parameters.

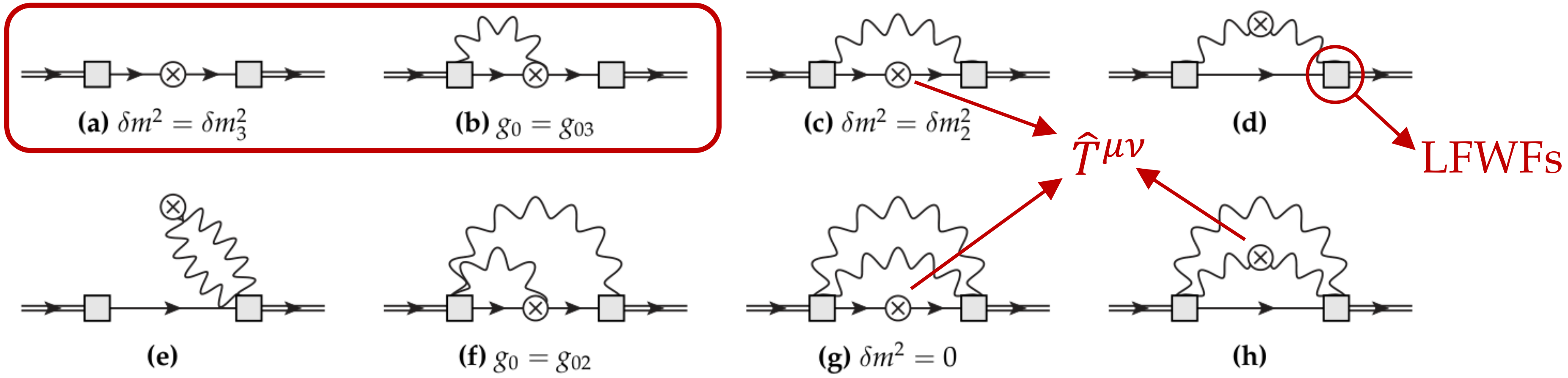
- $N$: mock nucleon, $\pi$: mock pion [Gross:2001ha]
- Quenched approximation: to avoid vacuum instability
- Fock sector expansion: $|p\rangle = |N\rangle + |N\pi\rangle + |N\pi\pi\rangle + |N\pi\pi\pi\rangle \cdots$
- Solved up to $|N\pi\pi\pi\rangle$ sector at non-perturbative couplings
- Fock sector dependent renormalization [Karmanov:2008br]
- Fock sector expansion converged up to $|N\pi\pi\rangle$ sector

$\alpha = g^2/(16\pi m^2), \quad Q^2 = -q^2$




[Li:2015iaw]

# Stress-energy tensor renormalization




- Light-front wave functions (LFWFs) & sector dependent counterterms from [Li, Karmanov & Vary:2015iaw,2016yzu]
- Light-front graphical rules extended to non-perturbative regime using LFWFs [Carbonell:1998rj]
- All divergences cancel out with sector dependent counterterms, e.g. $(a) + (b)$:

$$t_a^{\alpha\beta} = Z[2P^\alpha P^\beta + (\frac{1}{2}q^2 - \delta m_3^2)g^{\alpha\beta} - \frac{1}{2}q^\alpha q^\beta]$$

$$t_b^{\alpha\beta} = -\sqrt{Z}g^{\alpha\beta}\int \frac{dx}{2x(1-x)}\int \frac{d^2k_\perp}{(2\pi)^3} g_{03}\psi_2(x,k_\perp) = g^{\alpha\beta}Z\delta m_3^2$$

$$t^{\alpha\beta} = \langle p'|\hat{T}^{\alpha\beta}(0)|p\rangle$$

# Covariant decomposition on the light-front

[Cao:2024rul]

The hadronic matrix element for spin-0 particles:

$$\langle p'|\,\hat{T}_i^{\alpha\beta}(0)\,|p\rangle = 2P^\alpha P^\beta A_i(q^2) + \frac{1}{2}(q^\alpha q^\beta - q^2 g^{\alpha\beta})D_i(q^2) + 2M^2 g^{\alpha\beta}\bar{c}_i(q^2) + \frac{M^4\omega^\alpha\omega^\beta}{(\omega\cdot P)^2}S_{1i}(q^2) + (V^\alpha V^\beta + q^\alpha q^\beta)S_{2i}(q^2)$$

where $P = (p + p')/2$, $q = p' - p$, $V^\alpha = \epsilon^{\alpha\beta\rho\sigma}P_\beta q_\rho \omega_\sigma/(\omega\cdot P)$. $\omega^\mu = (\omega^+, \omega^-, \boldsymbol{\omega}_\perp) = (0,2,0)$ is a null vector indicating the light-front direction.

- $S_{1,2}(q^2)$ are two spurious gravitational form factors (GFFs) which usually contain uncanceled divergences
- The spurious GFFs appear due to the violation of the full Lorentz symmetry

# Components to extract gravitational form factors

In Drell-Yan-Breit frame ($q^+ = 0, \vec{P}_\perp = 0$):

$$
\begin{aligned}
t_i^{++} &= 2(P^+)^2 A_i(q_\perp^2), \\
t_i^{+-} &= 2(M^2 + \frac{1}{4}q_\perp^2)A_i(q_\perp^2) + q_\perp^2 D_i(q_\perp^2) + 4M^2\bar{c}_i(q_\perp^2), \\
t_i^{12} &= \frac{1}{2}q_\perp^1 q_\perp^2 D_i(q_\perp^2), \\
t_i^{11} + t_i^{22} &= -\frac{1}{2}q_\perp^2 D_i(q_\perp^2) - 4M^2\bar{c}_i(q_\perp^2) + 2q_\perp^2 S_{2i}(q_\perp^2), \\
t_i^{--} &= 2\Big(\frac{M^2 + \frac{1}{4}q_\perp^2}{P^+}\Big)^2 A_i(q_\perp^2) + \frac{4M^4}{(P^+)^2} S_{1i}(q_\perp^2)
\end{aligned}
$$

- $\hat{T}_i^{++}$, $\hat{T}_i^{12}$ and $\hat{T}_i^{+-}$ are three "good currents" which are free of spurious form factors
- Gravitational form factors derived from these currents are consistent with the covariant field theory in the perturbative limit

# Light-front wave function representation

[Cao:2023ohj, Cao:2024fto]

$$t^{12} = \frac{1}{2}\Big\langle \sum_j e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{r}_{j\perp}} \frac{i\overleftrightarrow{\nabla}^1_j\overleftrightarrow{\nabla}^2_j - q^1q^2}{x_j}\Big\rangle$$

$$\int d^3x T^{+\mu}(x) = P^\mu$$

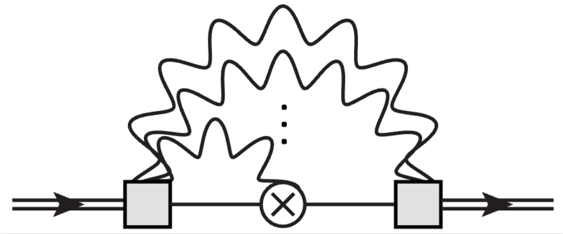
$$t^{+-} = 2\Big\langle \underbrace{\sum_j e^{i\boldsymbol{q}_\perp\cdot\boldsymbol{r}_{j\perp}} \frac{-\frac{1}{4}\overleftrightarrow{\nabla}^2_{j\perp} + m_j^2 - \frac{1}{4}q_\perp^2}{x_j}}_{\text{kinetic part}} + \underbrace{V e^{i\boldsymbol{r}_{N\perp}\cdot\boldsymbol{q}_\perp}}_{\text{potential part}}\Big\rangle$$

$$P^- = \frac{P_\perp^2 + M^2}{P^+}$$

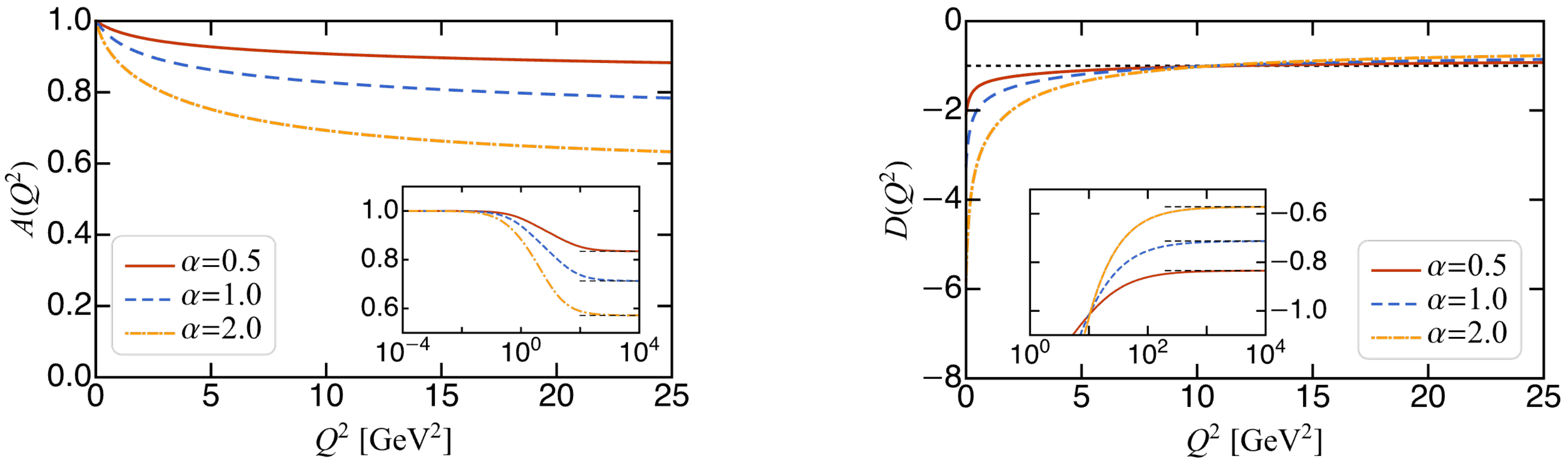
where $V = M^2 - \sum_j \frac{-\nabla^2_{j\perp}+m_j^2}{x_j}$ in the scalar Yukawa model. The quantum average is defined as

$$\langle\hat{O}\rangle = \int [\mathrm{d}x_i\mathrm{d}^2\boldsymbol{r}_{i\perp}]_n \psi_n^*(\{x_i,\boldsymbol{r}_{i\perp}\})\hat{O}\psi_n(\{x_i,\boldsymbol{r}_{i\perp}\})$$

- Modify $V$ in phenomenological models
- $e^{i\boldsymbol{r}_{N\perp}\cdot\boldsymbol{q}_\perp} \xrightarrow{\text{F.T.}} \delta^{(2)}(\boldsymbol{r}_\perp - \boldsymbol{r}_{N\perp})$ indicates the location of interaction



# Strongly-coupled scalar nucleon




[Cao:2023ohj]

$$\alpha = \frac{g^2}{16\pi m^2}$$

- For small coupling, $D(Q^2)$ is close to $-1$, the free scalar particle's result
- In the forward limit ($Q^2 = 0$),

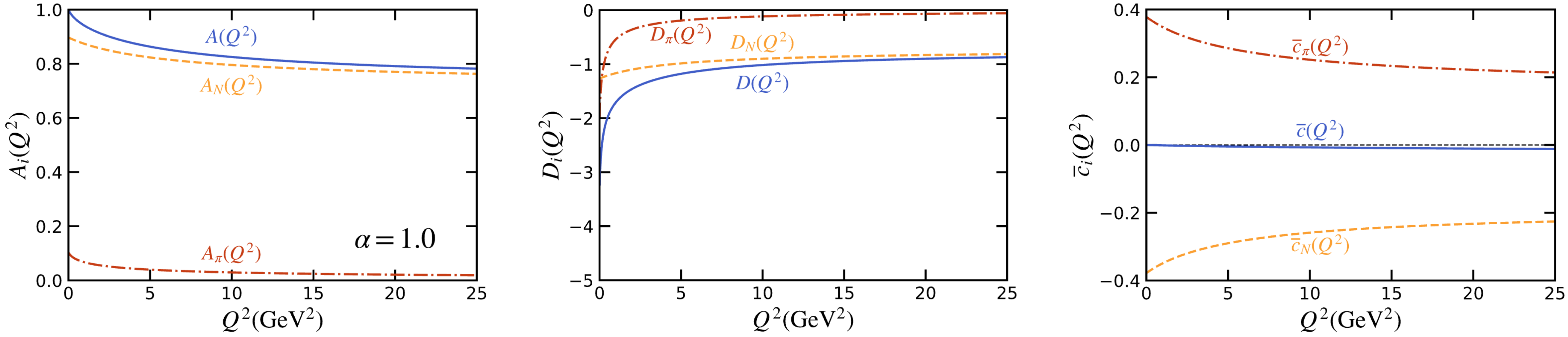
$$\lim_{Q^2=0} A(Q^2) = 1, \qquad \lim_{Q^2\to 0} Q^2 D(Q^2) = 0$$

- For large $Q^2$,

$$\lim_{Q^2\to\infty} A(Q^2) = Z, \qquad \lim_{Q^2\to\infty} D(Q^2) = -Z$$

revealing a pointlike core, consistent with the physical picture of the model

# Dissecting the strongly-coupled scalar nucleon

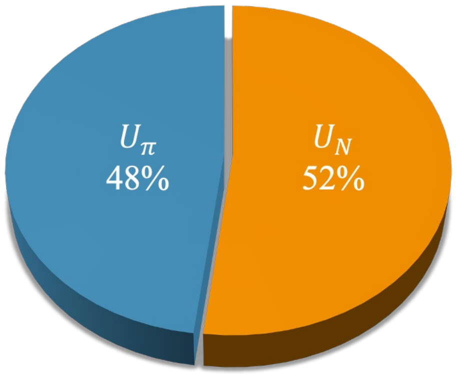


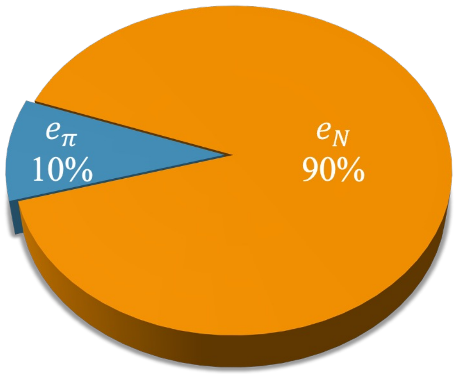

- A nonvanishing but small $\bar{c}(q^2)$ because of Fock space truncation $\sum_i \bar{c}_i(q^2) \neq 0$ [Cao:2024fto]
- Mass decomposition: [Lorce:2017xzd]

$$e_i = \int d^2 r_\perp \mathcal{E}(r_\perp) = A_i(0)$$

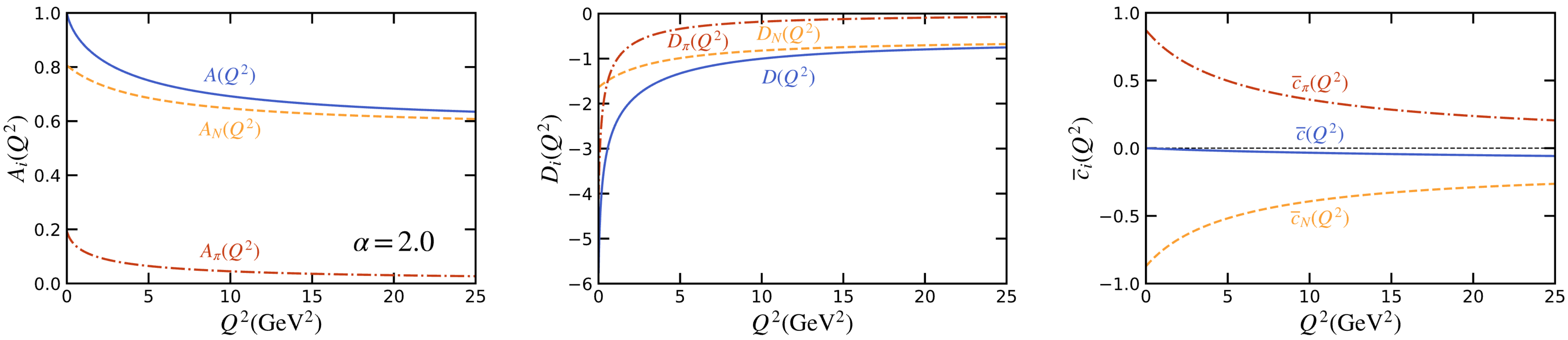
$$\lambda_i = \int d^2 r_\perp \Lambda_i(r_\perp) = \bar{c}_i(0)$$

$$U_i = e_i + \lambda_i$$



# Dissecting the strongly-coupled scalar nucleon

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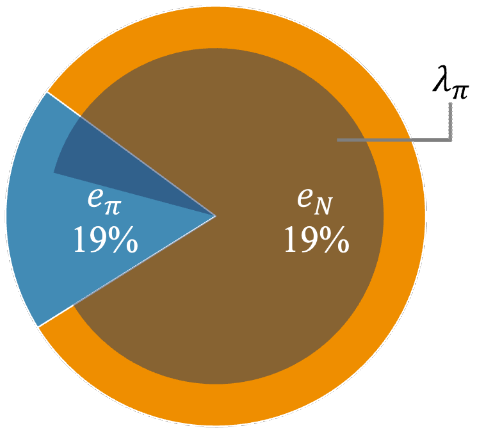


# Charmonium: hydrogen atom of QCD

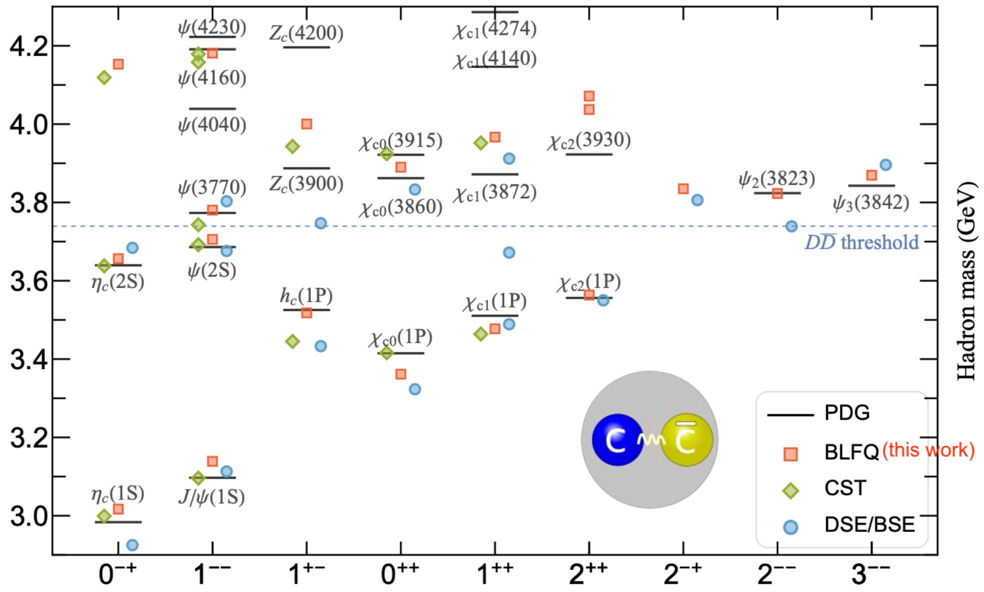
Effective Hamiltonian in the $q\bar{q}$ Fock sector [Li:2015zda, Li:2017mlw, Li:2021ejv]

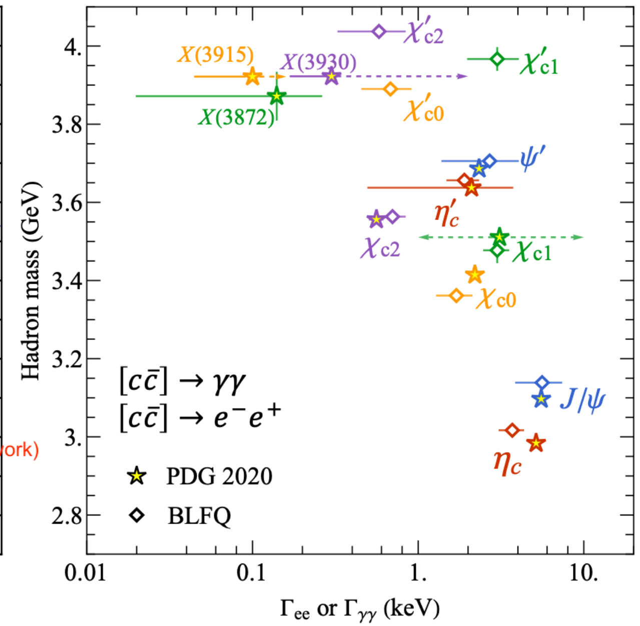
kinetic energy, confinement

$$H_{\text{eff}} = \frac{\vec{k}_\perp^2 + m_q^2}{x} + \frac{\vec{k}_\perp^2 + m_{\bar{q}}^2}{1-x} + \kappa^4 x(1-x)\vec{r}_\perp^2 - \frac{\kappa^4}{(m_q+m_{\bar{q}})^2}\partial_x(x(1-x)\partial_x)$$
$$- \frac{C_F 4\pi\alpha_s(Q^2)}{Q^2}\bar{u}_{s'}(k')\gamma_\mu u_s(k)\bar{v}(\bar{k})\gamma^\mu v_{\bar{s}'}(\bar{k}')$$

one gluon exchange

Two parameters $(m_q, \kappa)$ are fixed by fitting the charmonium mass spectrum



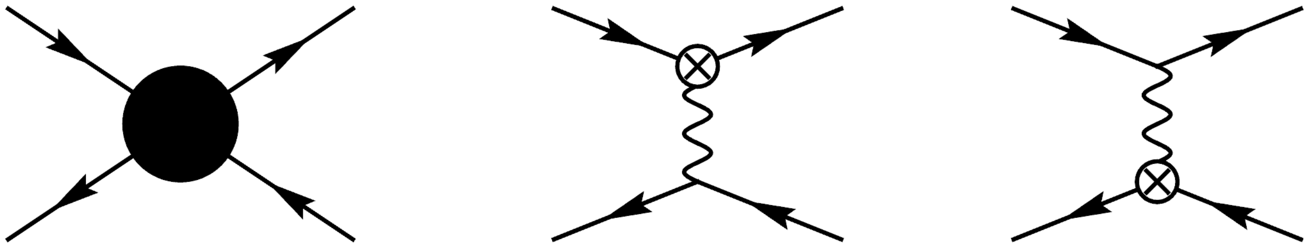



# Impulse ansatz

- From the effective Hamiltonian, we can't give the exact stress-energy operator directly
- We adopt impulse ansatz for the interaction term in $T^{+-}$

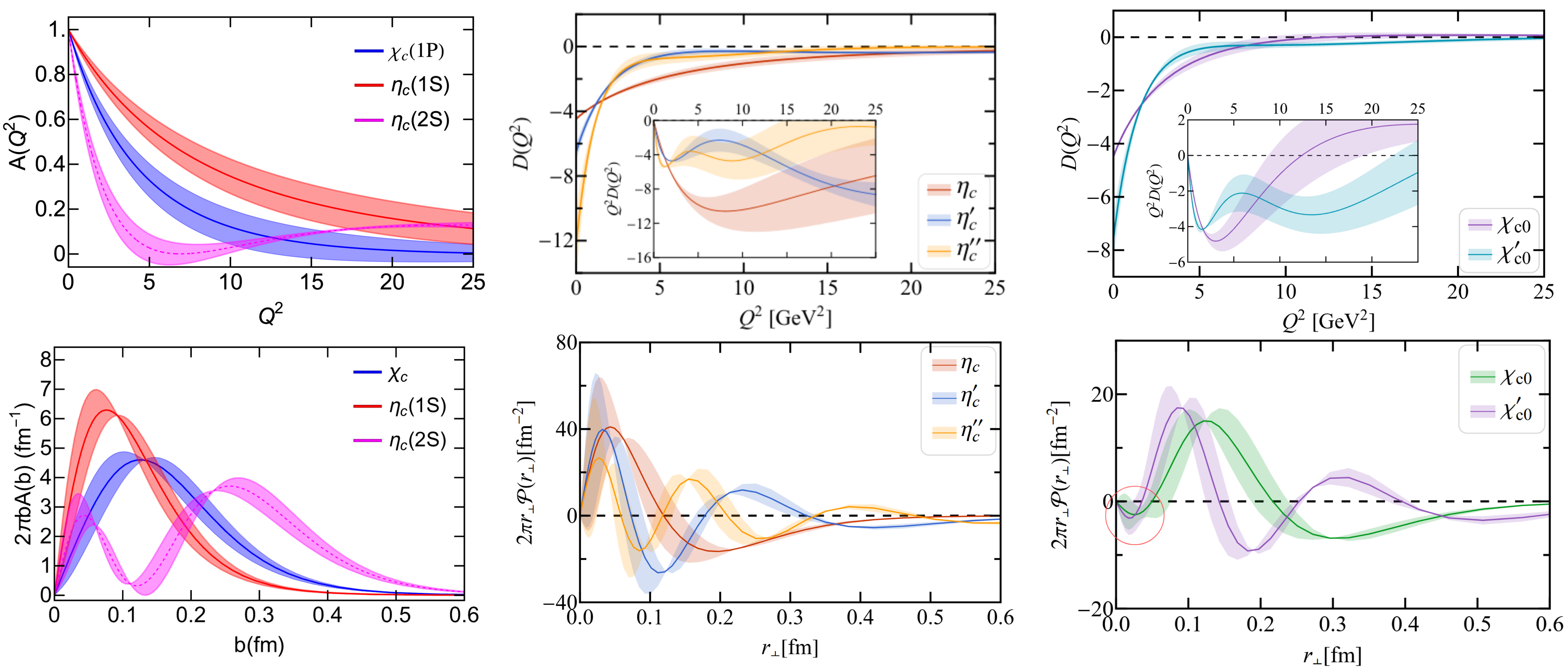
$$t_{\text{int}}^{+-} = \frac{1}{2}\sum_{s,\bar{s}}\int \frac{\mathrm{d}x}{4\pi x(1-x)}\int \mathrm{d}^2 r_\perp \psi_{s\bar{s}}^*(x,\vec{r}_\perp)\left[e^{i\vec{q}_\perp\cdot\vec{r}_{1\perp}} + e^{i\vec{q}_\perp\cdot\vec{r}_{2\perp}}\right]v(x,\vec{r}_\perp,-i\nabla_\perp)\psi_{s\bar{s}}(x,\vec{r}_\perp)$$

where $v(x,\vec{r}_\perp,-i\nabla_\perp) = M^2 - \frac{-\nabla_\perp^2+m_q^2}{x} - \frac{-\nabla_\perp^2+m_{\bar{q}}^2}{1-x}$



# Charmonium gravitational form factors

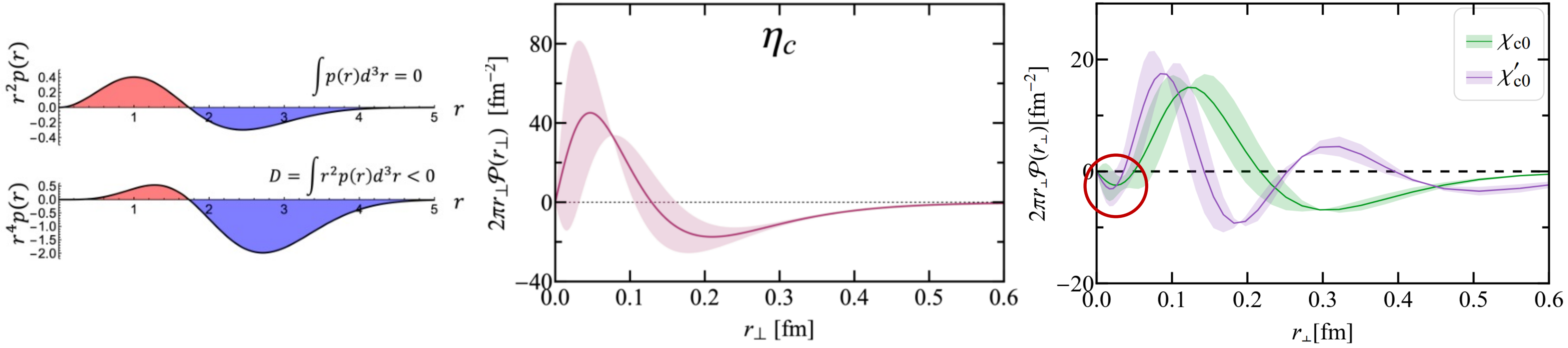
[Xu:2024hfx, Hu:2024edc]



# Mechanical stability

$$D = \int \mathrm{d}^3 r r^2 p(r) < 0$$

- A mechanically stable system should have a repulsive core and an attractive edge
- $\eta_c$ has a repulsive core but $\chi_{c0}$ has an attractive core



# Energy and momentum densities

2D transverse densities on the light-front:

[Xu:2024hfx, Freese:2021czn]

$$\mathcal{T}^{\alpha\beta}(\vec{r}_\perp; P) = \frac{1}{2P^+} \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp \cdot \vec{r}_\perp} \langle P + \frac{1}{2} q | \hat{T}^{\alpha\beta}(0) | P - \frac{1}{2} q \rangle$$

Momentum ($\mu = +, 1, 2$) and energy ($\mu = -$) densities:

$$\mathcal{P}^\mu(r_\perp) \equiv \mathcal{T}^{+\mu}(r_\perp; P) = P^\mu \mathcal{A}(r_\perp),$$

$$\mathcal{P}^-(r_\perp) \equiv \mathcal{T}^{+-}(r_\perp; P) = \frac{P_\perp^2 \mathcal{A}(r_\perp) + \mathcal{M}^2(r_\perp)}{P^+}$$

$$\int d^3x T^{+\mu}(x) = P^\mu$$

Where (for spin-0 particles):

$$\mathcal{A}(r_\perp) = \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp \cdot \vec{r}_\perp} A(q_\perp^2),$$

$$\mathcal{M}^2(r_\perp) = \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp \cdot \vec{r}_\perp} \left[ (M^2 + \frac{1}{4} q_\perp^2) A(q_\perp^2) + \frac{1}{2} q_\perp^2 D(q_\perp^2) \right]$$

$$P^- = \frac{P_\perp^2 + M^2}{P^+}$$

- $\mathcal{A}(r_\perp)$ can be interpreted as the momentum density
- $\mathcal{M}^2(r_\perp)$ can be interpreted as the invariant mass squared density

# Hadron as a relativistic medium

[Li:2024vgv]

- The quantum expectation value of the stress-energy tensor:

$$\langle\Psi|\hat{T}^{\alpha\beta}(x)|\Psi\rangle = \langle\mathcal{E}\mathcal{U}^\alpha\mathcal{U}^\beta - \mathcal{P}\Delta^{\alpha\beta} + \Pi^{\alpha\beta} - g^{\alpha\beta}\Lambda\rangle_\Psi$$

where $\mathcal{U}^\alpha$ is hadronic 4-velocity ($\mathcal{U}^\alpha\mathcal{U}_\alpha = 1$), $\Delta^{\alpha\beta} = g^{\alpha\beta} - \mathcal{U}^\alpha\mathcal{U}^\beta$

- Physical densities:

$$\text{Energy density: } \mathcal{E}(x) = M\int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{iq\cdot x}\left\{A(q^2) - \frac{q^2}{4M^2}[A(q^2)+D(q^2)]\right\}$$

$$\text{Pressure: } \mathcal{P}(x) = \frac{1}{6M}\int\frac{d^3q}{(2\pi)^3}e^{iq\cdot x}q^2D(q^2)$$

$$\text{Shear tensor: } \Pi^{\alpha\beta}(x) = \frac{1}{4M}\int\frac{d^3q}{(2\pi)^3}e^{iq\cdot x}(q^\alpha q^\beta - \frac{q^2}{3}\Delta^{\alpha\beta})D(q^2)$$

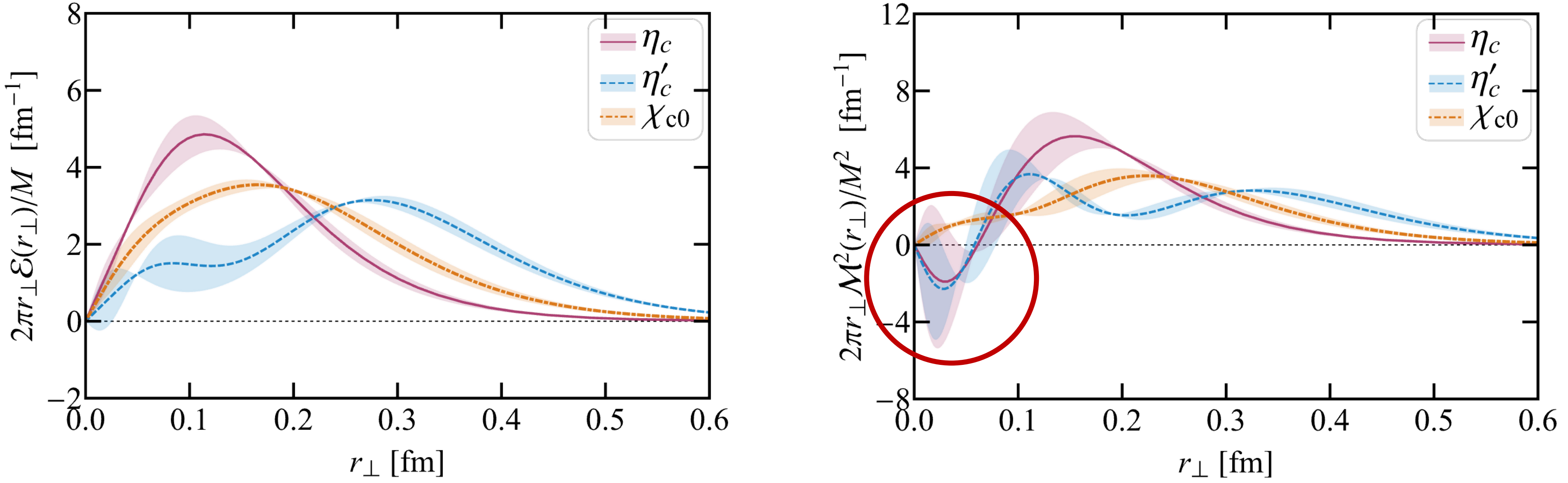
$$\text{Cosmological constant: } \Lambda = -M\int\frac{d^3q}{(2\pi)^3}e^{iq\cdot x}\bar{c}(q^2)$$

# Energy density and invariant mass squared density

$$\mathcal{E}(r_\perp) = M \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{r}_\perp} \left\{ \left(1 + \frac{q_\perp^2}{4M^2}\right) A(q_\perp^2) + \frac{q_\perp^2}{4M^2} D(q_\perp^2) \right\},$$

$$\mathcal{M}^2(r_\perp) = M^2 \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{r}_\perp} \left\{ \left(1 + \frac{q_\perp^2}{4M^2}\right) A(q_\perp^2) + \frac{q_\perp^2}{2M^2} D(q_\perp^2) \right\}$$

- Energy density is positive
- Invariant mass squared density becomes negative at small $r_\perp$: tachyonic core?

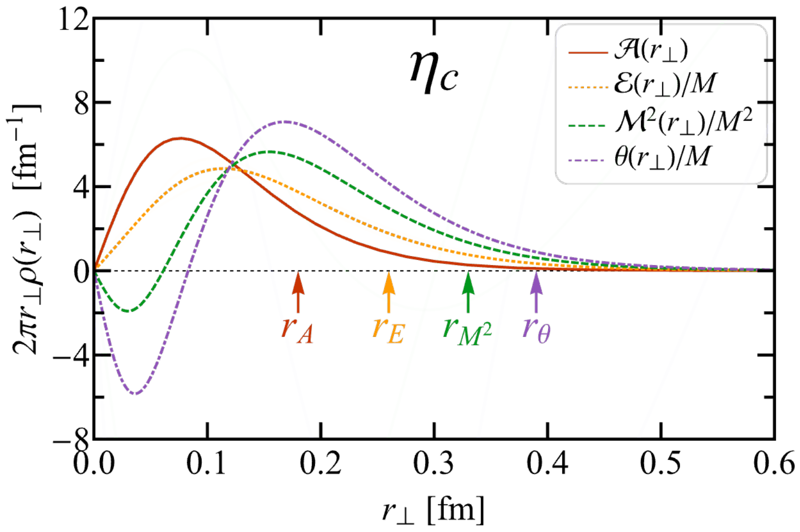


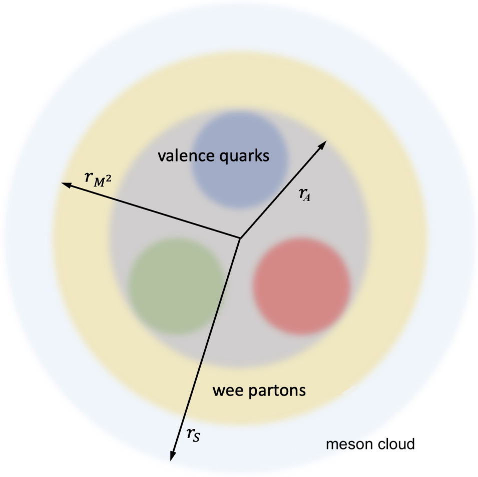
# Physical densities

- Momentum density $\mathcal{A}(r_\perp)$, energy density $\mathcal{E}(r_\perp)$, invariant mass squared density $\mathcal{M}^2(r_\perp)$ and the trace scalar density $\theta(r_\perp) = \mathcal{T}^\alpha_\alpha(r_\perp) = \mathcal{E}(r_\perp) - 3\mathcal{P}(r_\perp)$
- The negative $D$ suggests a chain of inequalities about different radii

$$r_A < r_E < r_{M^2} < r_\theta$$

$$r_A^2 = -6A'(0), \quad r_E^2 = r_A^2 - \frac{3}{2}\lambda_C^2(1+D), \quad r_{M^2}^2 = r_A^2 - \frac{3}{2}\lambda_C^2(1+2D), \quad r_\theta^2 = r_A^2 - \frac{3}{2}\lambda_C^2(1+3D)$$




$$\lambda_C = \frac{1}{M}$$

$$p_i^- = \frac{p_{i\perp}^2 + m_i^2}{x_i p^+}$$

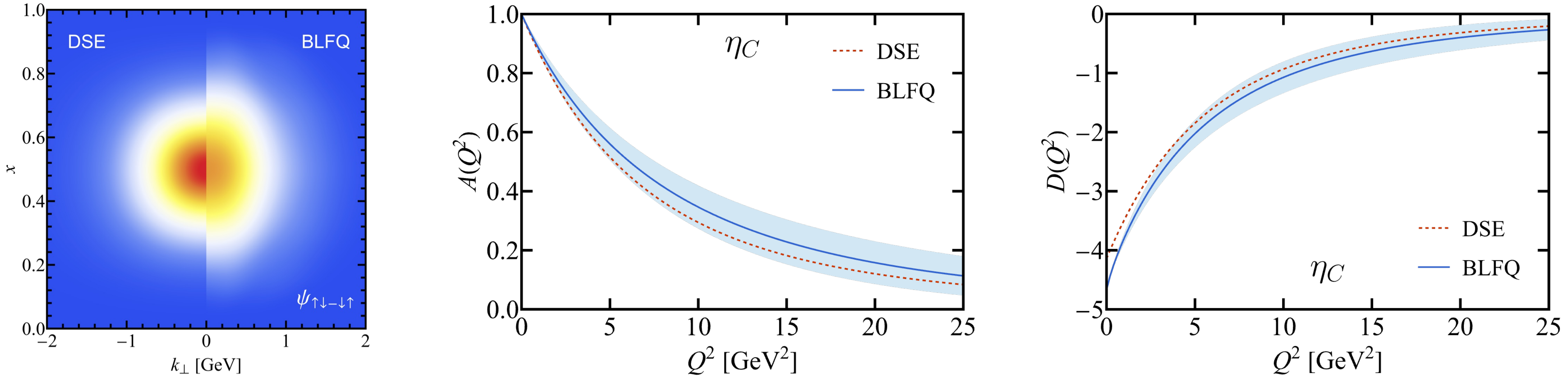
# Charmonium from Dyson-Schwinger equation

- LFWFs can be projected from Bethe-Salpeter amplitude [Shi:2021nvg]

$$\varphi_i(x,\vec{k}_\perp) \sim \int \mathrm{d}k^- \,\mathrm{d}k^+ \delta(xP^+ - k^+)\mathrm{Tr}[\Gamma_i \chi(k,P)]$$

- Distinct wave functions give consistent predictions

[Cao:2025bit]



# Comparison between BLFQ and DSE