

Pion–Nucleon Scattering in BChPT combined with $1/N_c$ Expansion

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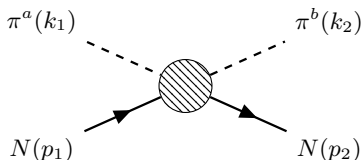
Work with José L. Goity

Phys. Rev. D 112, 114048

Outline

- 1 Motivation & Introduction
- 2 BChPT $\times 1/N_c$: combined chiral and $1/N_c$ expansions
- 3 $\pi N \rightarrow \pi N$ at NNLO: one-loop amplitude and renormalization
- 4 Fits to partial wave amplitudes
- 5 Summary

Pion–Nucleon Scattering



- Simplest meson–baryon scattering process
- A clean laboratory for low-energy QCD
- Low-energy dynamics governed by spontaneously broken chiral symmetry (pions: Goldstone bosons)
- Sufficient availability of data to test the theory

The $1/N_c$ Expansion of QCD

- QCD has a hidden expansion parameter $1/N_c$
 - ▶ N_c : number of colors, gauge group $SU(N_c)$
 - ▶ real world $N_c = 3$
 - ▶ Large- N_c limit: N_c is a large odd number
- Usually applied at the quark–gluon level, but it can also be applied **at the hadronic level (EFT)**:
 - ▶ spin-flavor $SU(4)$ symmetry for baryons
 - ▶ mesons: $M_\pi = \mathcal{O}(N_c^0)$, $F_\pi = \mathcal{O}(\sqrt{N_c})$
 - ▶ baryons (N_c quarks): $m_B = \mathcal{O}(N_c)$
 - ▶ $g_{\pi NN} = \mathcal{O}(N_c^{3/2})$, πN amplitudes must be $\mathcal{O}(N_c^0)$
 - ▶ N and Δ belong to one spin-flavor multiplet

Motivation to Combine $1/N_c$ with BChPT

- Ordinary BChPT (only $S = 1/2$ baryons) has poor convergence:
 - ▶ light Δ : $m_\Delta - m_N \simeq 293$ MeV
 - ▶ large πN coupling: large counterterms needed
- Including the Δ improves convergence: $S = \frac{1}{2}$ and $S = \frac{3}{2}$ loop contributions partially cancel
 - ▶ in the $1/N_c$ expansion N and Δ are in one spin-flavor multiplet
 $\Rightarrow \Delta$ **is naturally included** in the combined approach
- $m_B = \mathcal{O}(N_c) \Rightarrow 1/m_B$ corrections are $1/N_c$ -suppressed
 $\Rightarrow$ Heavy-Baryon ChPT treatment is natural

BChPT Combined with $1/N_c$

- Symmetries: chiral $SU_L(2) \times SU_R(2)$ and spin-flavor $SU(4)$
- $\mathcal{L} \sim \mathbf{B}^\dagger T_\chi T_{SF} \mathbf{B}$: chiral order from T_χ , $1/N_c$ order from T_{SF}

$$\mathbf{B} = \begin{pmatrix} B_{S=1/2} \\ B_{S=3/2} \\ \vdots \\ B_{S=N_c/2} \end{pmatrix}$$

ξ expansion: BChPT $\times 1/N_c$: $\mathcal{O}(p) = \mathcal{O}(1/N_c) = \mathcal{O}(\xi)$

$$\mathcal{L} = B^\dagger \left(iD_0 + \not{g}_A u^{ia} \hat{G}^{ia} - \frac{C_{HF}}{N_c} \hat{S}^2 + \frac{c_1}{2\Lambda} \hat{\chi}_+ \right) B \\ + \frac{F_\pi^2}{4} \left\langle D_\mu U (D^\mu U)^\dagger \right\rangle + \frac{F_\pi^2}{4} \left\langle \chi U^\dagger + U \chi^\dagger \right\rangle$$

$$D_\mu = \partial_\mu + \frac{1}{2} (u^\dagger \partial_\mu u + u \partial_\mu u^\dagger)$$

$$u^i = i (u^\dagger \partial^i u - u \partial^i u^\dagger)$$

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u$$

$$\tilde{\chi}_\pm = \chi_\pm^a I^a$$

$$c_1 = \text{quark mass contribution term}$$

$$u^{ia} = \frac{1}{2} \langle u^i \tau^a \rangle$$

$$\chi = 2B_0 M_\pi$$

$$C_{HF} = m_\Delta - m_N$$

$$\chi_\pm^a = \langle \chi_\pm I^a \rangle$$

$$F_\pi = 92 \text{ MeV}$$

$$u^2 = U = \exp(i\tau^a \pi^a / F_\pi)$$

$$G^{ia} : \text{spin-flavor operator}$$

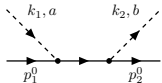
$$\not{g}_A : \text{axial coupling constant}$$

$$\hat{\chi}_+ = \tilde{\chi}_+ + N_c \chi_+^0$$

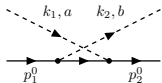
$$\Lambda = 770 \text{ MeV}$$

Large N_c Consistency Conditions

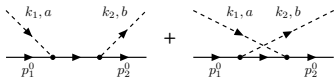
- Since $G^{ia} = \mathcal{O}(N_c)$ and $F = \mathcal{O}(\sqrt{N_c})$



$$= -i \frac{\hat{g}_A^2}{F^2} k_1^i k_2^j \frac{G^{jb} G^{ia}}{\omega} = \mathcal{O}(N_c)$$



$$= -i \frac{\hat{g}_A^2}{F^2} k_1^i k_2^j \frac{G^{ia} G^{jb}}{-\omega} = \mathcal{O}(N_c)$$

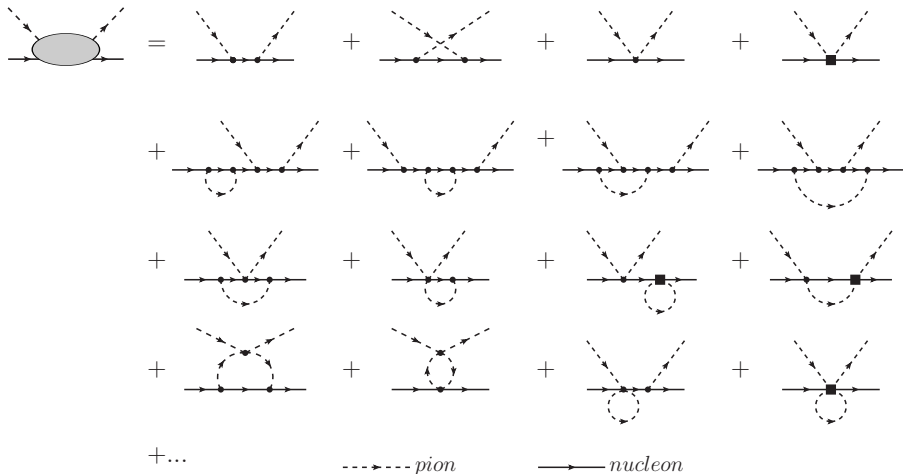


$$= -i \frac{\hat{g}_A^2}{F^2} k_1^i k_2^j \frac{[G^{jb}, G^{ia}]}{\omega} = \mathcal{O}(N_c^{-1})$$

$$[G^{ia}, G^{jb}] = \frac{i}{2} \delta^{ij} \epsilon^{abc} I^c + \frac{i}{2} \delta^{ab} \epsilon^{ijk} S^k = \mathcal{O}(N_c^0)$$

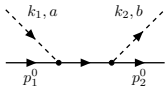
- This kind of large N_c cancellations are very common

Scattering Amplitude Calculation

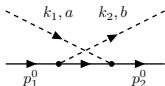


- 82 (49 non-zero; including cross diagrams) diagrams were calculated

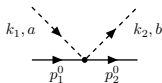
Scattering Amplitude at LO



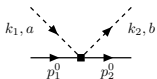
$$= iT_{D1}^{ba} = -i \frac{\tilde{g}_A^2}{F^2} k_1^i k_2^j \frac{G^{jb} \mathcal{P}_n G^{ia}}{p_1^0 + k_1^0 - \delta m_n}$$



$$= iT_{D2}^{ba} = -i \frac{\tilde{g}_A^2}{F^2} k_1^i k_2^j \frac{G^{ia} \mathcal{P}_n G^{jb}}{p_1^0 - k_2^0 - \delta m_n}$$



$$= iT_{D3}^{ba} = \frac{(k_1^0 + k_2^0)}{2F^2} \epsilon^{abc} I^c$$



$$= iT_{D4}^{ba} = -i \frac{2c_1 N_c M_\pi^2}{\Lambda F^2} \delta^{ab}$$

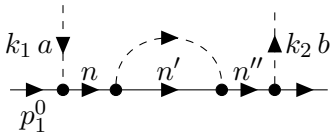
Loop Diagrams and UV Divergences

$$\begin{array}{c} \text{p} \\ \text{---} \bullet \text{---} \bullet \text{---} \\ \text{m}_n \end{array} \quad \begin{array}{c} \text{---} \text{---} \text{---} \end{array} = \frac{\hat{g}_A^2}{F_\pi^2} \hat{G}^{ia} P_n \hat{G}^{ja} \int \frac{d^4 k}{(2\pi)^4} \frac{k^i k^j}{(k^2 - M_\pi^2 + i\epsilon)(p^0 - k^0 - \delta m_n + i\epsilon)}$$

- Dimensional regularization ($d = 4 - 2\epsilon$, $\lambda_\epsilon = \frac{1}{\epsilon} - \gamma + \log 4\pi$) isolates the UV divergences

$$\begin{aligned}
 &= \frac{i\hat{g}_A^2}{144\pi^2 F_\pi^2} \hat{G}^{ia} P_n \hat{G}^{ia} \left(\underbrace{-3(3M_\pi^2 - 2(p^0 - \delta m_n)^2)(p^0 - \delta m_n)\lambda_\epsilon}_{\text{Not Finite (UV divergence)}} \right. \\
 &+ \underbrace{(p^0 - \delta m_n)(-21M_\pi^2 + 16(p^0 - \delta m_n)^2)}_{\text{Finite}} \\
 &+ \underbrace{3(p^0 - \delta m_n)(3M_\pi^2 - 2(p^0 - \delta m_n)^2) \log(M_\pi^2/\mu^2)}_{\text{Finite}} \left. + \underbrace{12(M_\pi^2 - (p^0 - \delta m_n)^2)^{3/2} \left(\frac{\pi}{2} + \arctan \left((p^0 - \delta m_n)/\sqrt{M_\pi^2 - (p^0 - \delta m_n)^2} \right) \right)}_{\text{Finite}} \right)
 \end{aligned}$$

Example Diagram: No-pole UV Divergence



$$\begin{aligned}
 &= \frac{g_A^4 k_1^i k_2^j}{48 F^4 \pi^2} \left(\frac{\delta^{ab} \delta^{ij}}{9216} \left(4(k_1^0 + k_2^0)(3N_c(N_c + 4)((3N_c(N_c + 4) - 16) - 16S^2) + 64(5S^2 + S^4)) \right. \right. \\
 &\quad \left. \left. - \frac{C_{HF}}{N_c} \left(N_c(N_c + 4)(3(183N_c(N_c + 4) - 2128) \right. \right. \right. \\
 &\quad \left. \left. \left. + (45N_c(N_c + 4) - 5984)S^2 - 240S^4) + 64(681S^2 + 214S^4 + 5S^6) \right) \right) \right. \\
 &\quad \left. + \frac{i\delta^{ab} \epsilon^{ijk}}{1536} \left(\frac{C_{HF}}{N_c} (3(N_c(N_c + 4)(8N_c(N_c + 4) - 527) + 624)S^k \right. \right. \\
 &\quad \left. \left. - 245N_c(N_c + 4)S^k S^2 + 536(15S^k S^2 + S^k S^4) \right. \right. \\
 &\quad \left. \left. + 4(k_1^0 + k_2^0)((17N_c(N_c + 4) - 16)S^k - 56S^k S^2) \right) \right) \\
 &\quad + i\delta^{ij} \epsilon^{abc}(\dots) \\
 &\quad + \dots
 \end{aligned}$$

All Diagrams: No-pole UV Divergence

Projection	UV no-pole
$J = 0, I = 0$	$2g_A^2 \alpha \delta^{ab} \frac{C_{HF}}{N_c} (3N_c(4 + N_c) - 20S^2) (4k_1.k_2 - 3M_\pi^2)$ $+ 48\alpha \delta_{ab} \frac{c_1 N_c M_\pi^2}{\Lambda} (8k_1.k_2 - 5M_\pi^2)$
$J = 1, I = 0$	$-ig_A^4 \alpha k_1^i k_2^j (k_1^0 + k_2^0) \delta^{ab} \epsilon^{ijk} S^k$
$J = 0, I = 1$	$-ig_A^4 \alpha (k_1^0 + k_2^0) \vec{k}_1.\vec{k}_2 \epsilon^{abc} I^c +$ $5ig_A^2 \alpha (k_1^0 + k_2^0) (\vec{k}_1 - \vec{k}_2)^2 \epsilon^{abc} I^c$ $+ 4i\alpha (k_1^0 + k_2^0) (10M_\pi^2 + 4k_1.k_2 - 3(k_1^0 + k_2^0)^2) \epsilon^{abc} I^c$
$J = 1, I = 1$	$g_A^4 \alpha k_1^i k_2^j \epsilon^{ijk} \epsilon^{abc} \frac{C_{HF}}{N_c} (8(2 + N_c) G^{kc} - 24S^k I^c)$ $+ 24g_A^2 \alpha k_1^i k_2^j \epsilon^{abc} \epsilon^{ijk} \frac{C_{HF}}{N_c} ((2 + N_c) G^{kc} - 3S^k I^c)$
$J = 2, I = 0$	$16g_A^4 \alpha k_1^i k_2^j \delta^{ab} \frac{C_{HF}}{N_c} (SS) _{J=2}^{ij}$
$J = 0, I = 2$	$16g_A^4 \alpha \vec{k}_1.\vec{k}_2 \frac{C_{HF}}{N_c} (II) _{I=2}^{ab} - 48g_A^2 \alpha k_1.k_2 \frac{C_{HF}}{N_c} (II) _{I=2}^{ab}$
$J = 2, I = 1$	0
$J = 1, I = 2$	0
$J = 2, I = 2$	$-192g_A^4 \alpha k_1^i k_2^j \frac{C_{HF}}{N_c} (GG) _{J=I=2}^{ijab}$

Here, $\alpha = \frac{i}{768\pi^2 F_\pi^4}$

Counter-term Lagrangian

$$\mathcal{L}_{LO} = \mathbf{B}^\dagger \left(iD_0 + \hat{g}_A u^{ia} \hat{G}^{ia} - \frac{C_{HF}}{N_c} \hat{S}^2 + \frac{c_1}{2\Lambda} \hat{\chi}_+ \right) \mathbf{B} + \frac{F_\pi^2}{4} \langle D_\mu U (D^\mu U)^\dagger \rangle + \frac{F_\pi^2}{4} \langle \chi U^\dagger + U \chi^\dagger \rangle$$

$$\mathcal{L}_{CT}(\hat{g}_A^4) = \mathbf{B}^\dagger \left(\frac{f_{10}}{\Lambda^2} \varepsilon^{ijk} S^k u_i^a \overleftrightarrow{D}^0 u_j^a + \frac{f_{01}}{\Lambda^2} \varepsilon^{abc} I^c u_i^a \overleftrightarrow{D}^0 u_i^b + \frac{f_{11}^{(1)}}{\Lambda} \varepsilon^{ijk} \varepsilon^{abc} u_i^a u_j^b G^{kc} + \frac{f_{11}^{(2)}}{\Lambda} \varepsilon^{ij\varepsilon} \varepsilon^{abc} u_i^a u_j^b S^k I^c \right. \\ \left. + \frac{f_{20}}{\Lambda} u_i^a u_j^a \frac{1}{N_c} S^i S^j \Big|_{J=2} + \frac{f_{02}}{\Lambda} u_i^a u_i^b \frac{1}{N_c} I^a I^b \Big|_{I=2} + \frac{f_{22}}{\Lambda} u_i^a u_j^b \frac{1}{N_c} G^{ia} G^{jb} \Big|_{J=I=2} \right) \mathbf{B}$$

$$\mathcal{L}_{CT}(\hat{g}_A^2) = \mathbf{B}^\dagger \left(g_{00}^{(1)} \frac{\langle \chi_+ \rangle}{\Lambda} + g_{00}^{(2)} \frac{\langle \chi_+ \rangle}{\Lambda} \frac{\hat{S}^2}{N_c} + \frac{g_{00}^{(3)}}{\Lambda} u_\mu^a u^{a\mu} + \frac{g_{00}^{(4)}}{\Lambda} u_\mu^a u^{a\mu} \frac{\hat{S}^2}{N_c} + i \frac{g_{01}^{(1)}}{\Lambda^2} \varepsilon^{abc} \chi^a u_0^b I^c \right. \\ \left. + \frac{g_{01}^{(2)}}{\Lambda^2} \varepsilon^{abc} u_\mu^a \overleftrightarrow{D}^0 u^{b\mu} I^c + \frac{g_{02}^{(1)}}{\Lambda} u_\mu^a u^{b\mu} I^a I^b \Big|_{I=2} + \frac{g_{11}^{(1)}}{\Lambda} \varepsilon^{ijk} \varepsilon^{abc} u^{ia} u^{jb} G^{kc} + \frac{g_{11}^{(2)}}{\Lambda} \varepsilon^{ijk} \varepsilon^{abc} u^{ia} u^{jb} \frac{1}{N_c} S^k I^c \right) \mathbf{B}$$

$$\mathcal{L}_{CT}(\hat{g}_A^0) = \mathbf{B}^\dagger \left(h_{00}^{(1)} \frac{N_c}{\Lambda^3} \langle \chi_+^2 \rangle + h_{00}^{(2)} \frac{N_c}{\Lambda^3} \langle \chi_-^2 \rangle + h_{00}^{(3)} \frac{N_c}{\Lambda^3} \langle \chi_+ \rangle u_\mu^a u^{a\mu} + h_{01}^{(1)} \frac{\varepsilon^{abc}}{\Lambda^2} \chi_-^a u_0^b I^c + h_{01}^{(2)} \frac{\varepsilon^{abc}}{\Lambda^2} u_\mu^a \overleftrightarrow{D}^0 u^{\mu b} I^c \right) \mathbf{B}$$

- 21 Low Energy Constants (LEC); When $\pi N \rightarrow \pi N$ only 9 LEC

Pion Nucleon Scattering Partial Wave Data

- SAID database maintained by George Washington University
 - It provides Single Energy Value (SEV) data points with errorbars and WI08 Solution
 - Notation: $L_2 I_2 J$
- $S_{31} \rightarrow$ isospin 3/2 and spin 1/2

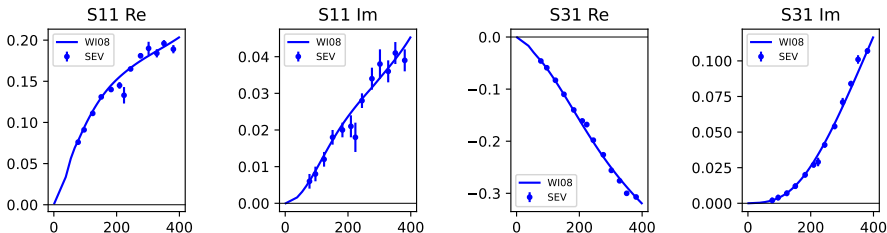
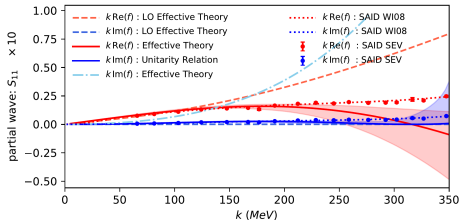


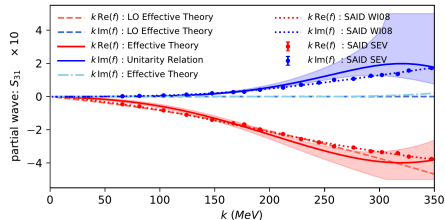
Figure: S partial wave: the horizontal axes represent the pion momentum in the lab frame (in MeV) and the vertical axes represent dimensionless partial waves

Theory vs Experimental Data - S11 and S31

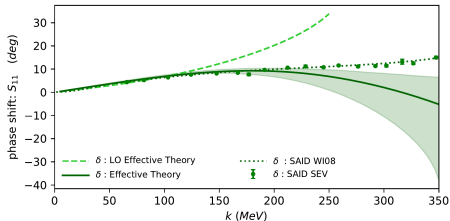
S11 Partial Wave Vs CM Momentum



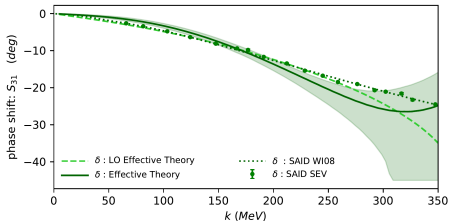
S31 Partial Wave Vs CM Momentum



S11 Phase Shift Vs CM Momentum

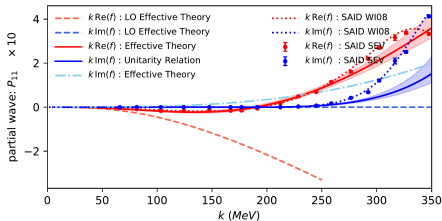


S31 Phase Shift Vs CM Momentum

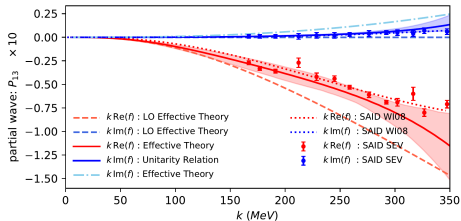


Theory vs Experimental Data - P11 and P13

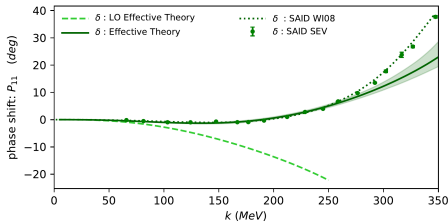
P11 Partial Wave Vs CM Momentum



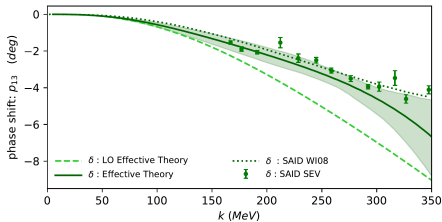
P13 Partial Wave Vs CM Momentum



P11 Phase Shift Vs CM Momentum

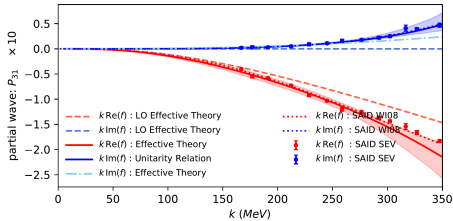


P13 Phase Shift Vs CM Momentum

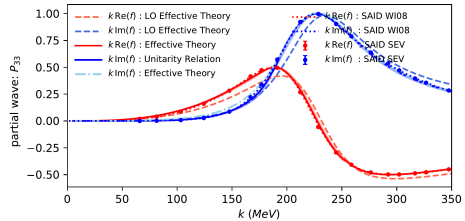


Theory vs Experimental Data - P31 and P33

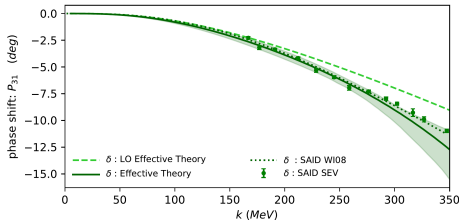
P31 Partial Wave Vs CM Momentum



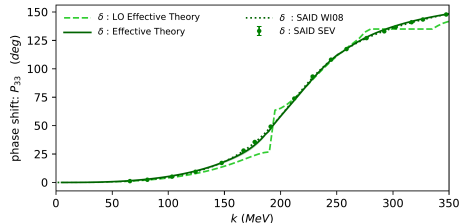
P33 Partial Wave Vs CM Momentum



P31 Phase Shift Vs CM Momentum

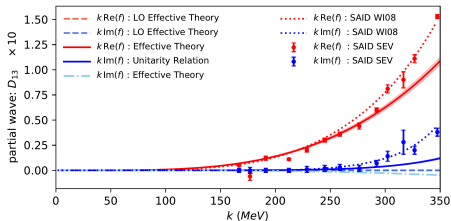


P33 Phase Shift Vs CM Momentum

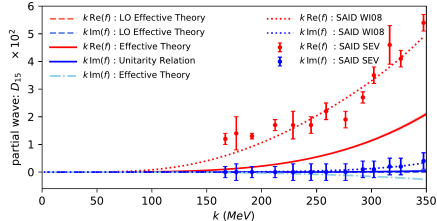


Theory vs Experimental Data - D13 and D15

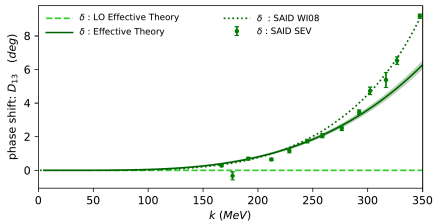
D13 Partial Wave Vs CM Momentum



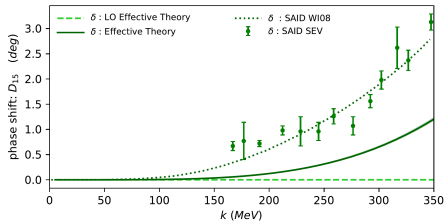
D15 Partial Wave Vs CM Momentum



D13 Phase Shift Vs CM Momentum

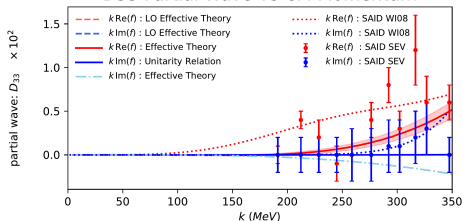


D15 Phase Shift Vs CM Momentum

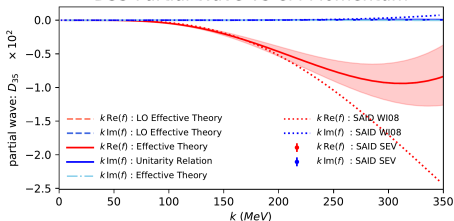


Theory vs Experimental Data - D33 and D35

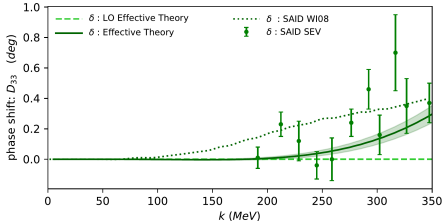
D33 Partial Wave Vs CM Momentum



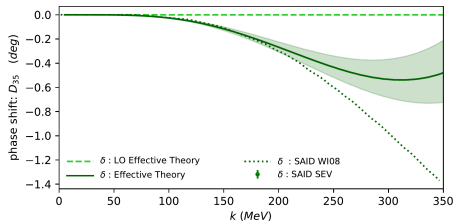
D35 Partial Wave Vs CM Momentum



D33 Phase Shift Vs CM Momentum



D35 Phase Shift Vs CM Momentum



Results from the Fit

- Fit range: S-waves < 250 MeV, others < 350 MeV

	Fit	Reference	LEC	
C_{HF} [MeV]	304.3(0.8)	—	$\bar{\alpha}_{00}^{(3)}$	−4.43(0.27)
Γ_{Δ} [MeV]	119.7(1.4)	117 (114–120)	$\alpha_{00}^{(5)}$	0.35(0.31)
$\check{g}_A$	1.18	—	$\alpha_{01}^{(1)}$	3.90(1.29)
$\bar{c}_1$	−0.90(0.05)	—	$\alpha_{01}^{(2)}$	−1.99(0.81)
g_{ANN}	1.58(0.03)	$\frac{6}{5}g_A = 1.53$	$\alpha_{01}^{(3)}$	0.81(0.20)
$g_{AN\Delta}$	1.81(0.01)	—	$\alpha_{01}^{(4)}$	−0.49(0.27)
			$\alpha_{10}^{(1)}$	2.60(0.57)
a_0^+	$0.006/M_{\pi}$	$0.0078(28)/M_{\pi}$	$\alpha_{11}^{(1)}$	−6.17(0.74)
a_0^-	$0.087/M_{\pi}$	$0.0866(10)/M_{\pi}$	$\alpha_{11}^{(2)}$	4.28(1.96)

Summary and Conclusions

- Applied BChPT $\times 1/N_c$ to πN scattering at NNLO (one loop), renormalized at generic N_c
- $SU(4)$ spin-flavor symmetry: $\mathcal{O}(N_c)$ pieces cancel among diagrams $\Rightarrow$ drastically improved convergence of the low energy expansion
- Fit to SAID partial waves: good agreement up to $k_{\text{CM}} \approx 250$ MeV; P_{33} up to 350 MeV
- Δ extends the range, especially in the P_{33} partial wave
- NNLO significantly improves on LO, with natural-size of corrections
- D-waves appear at one loop: finite and predictive (no counterterms) $\Rightarrow$ improving them requires a higher-order ($N^3\text{LO}$) calculation

Future Work

- Extract the pion–nucleon σ -term, in $\text{BChPT} \times 1/N_c$
- Extend this work to $\pi N \rightarrow \pi \Delta$

- For further details, please see our paper: Phys. Rev. D 112, 114048
<https://doi.org/10.1103/xbrd-8p5x>

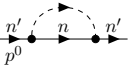
Thank you !



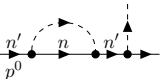
This work is supported by the National Science Foundation,
Grants PHY-1613951, PHY-1913562, and PHY-2412757

BACKUP

Example of extracting UV No-pole-term

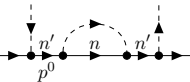


$$= -i\Sigma_{UV} = \frac{ig_A^2}{48\pi^2 F_\pi^2} G^{lc} p_n G^{lc} \left(3M^2 (\delta m_n - p^0) - 2 (\delta m_n - p^0)^3 \right)$$



$$= \left(\frac{g_A}{F_\pi} k^j G^{jb} \right) \left(\frac{i p_n}{\mathbf{p}^0} \right) \frac{ig_A^2}{48\pi^2 F_\pi^2} G^{lc} p_n G^{lc} \left(\overbrace{3M^2 (\delta m_n - \delta m_{n'}) - 2 (\delta m_n - \delta m_{n'})^3}^{\text{mass renormalization}} \right. \\ \left. + \mathbf{p}^0 \left(-3M^2 + 6 (\delta m_n - \delta m_{n'})^2 \right) + (\mathbf{p}^0)^2 \left(-6 (\delta m_n - \delta m_{n'}) \right) + 2(\mathbf{p}^0)^3 \right)$$

Here $p^0 = \delta m_{n'} + \mathbf{p}^0$, where $\mathbf{p}^0$ is the small kinetic energy of the nucleon



$$= -\frac{g_A^3}{48\pi^2 F_\pi^2} k^j G^{jb} p_{n'} G^{lc} p_n G^{lc} \left(\overbrace{\left(-3M^2 + 6 (\delta m_n - \delta m_{n'})^2 \right)}^{\text{vertex renormalization}} \right. \\ \left. + \mathbf{p}^0 \left(-6 (\delta m_n - \delta m_{n'}) \right) + 2(\mathbf{p}^0)^2 \right) \left(\frac{i p_n}{\mathbf{p}^0} \right) \left(-\frac{g_A}{F_\pi} k^i G^{ia} \right)$$

Projecting spin-flavor operators

- Composite operators can be written in terms of basis operators

Example:

$$(G^{ia} S^2 G^{jb})_{I=0, J=0} = \delta^{ij} \delta^{ab} \left(\frac{1}{24} N_c (4 + N_c) + \frac{1}{144} (-40 + 3 N_c (4 + N_c)) S^2 - \frac{1}{18} S^4 \right)$$

$$(G^{ia} S^2 G^{jb})_{I=0, J=1} = i \delta^{ab} \epsilon^{ijk} \left(\frac{1}{24} (2 - N_c (4 + N_c)) S^k + \frac{7}{24} S^k S^2 \right)$$

$$(G^{ia} S^2 G^{jb})_{I=1, J=1} = \epsilon^{ijk} \epsilon^{abc} \left(\frac{3}{8} S^k I^c + \frac{1}{8} S^k I^c S^2 - \frac{1}{8} (2 + N_c) G^{kc} - \frac{1}{16} (2 + N_c) \{S^2, G^{kc}\} \right)$$

$$(G^{ia} S^2 G^{jb})_{I=0, J=2} = \delta^{ab} \left(\frac{1}{12} \{S^i, S^j\} |_{J=2} + \frac{1}{24} S^2 \{S^i, S^j\} |_{J=2} \right)$$

$$(G^{ia} S^2 G^{jb})_{I=1, J=2} = i \epsilon^{abc} \left(\frac{1}{8} (2 + N_c) \{S^i, G^{jc}\} |_{J=2} - \frac{1}{8} I^c \{S^i, S^j\} |_{J=2} \right)$$

$$(G^{ia} S^2 G^{jb})_{I=2, J=2} = -\frac{1}{2} \{G^{ia}, G^{jb}\} |_{I=2, J=2} + \frac{1}{4} \{S^2, \{G^{ia}, G^{jb}\} |_{I=2, J=2}\}$$

$g^\pm$ and $h^\pm$ from Data

- Experimental data for $g^\pm$ and $h^\pm$ amplitudes can be obtained from:

$$g^\pm(\omega, \theta) = \sum_{l=0}^{\infty} \frac{8\pi\sqrt{s}}{E + m_N} \left((l+1)f_{l+}^\pm + lf_{l-}^\pm \right) P_l(\cos \theta)$$

$$h^\pm(\omega, \theta) = \sum_{l=0}^{\infty} \frac{8\pi\sqrt{s}}{(E + m_N)k^2 \sin^2 \theta} \left(\left((l+1)f_{l+}^\pm P_{l+1}(\cos \theta) + lf_{l-}^\pm P_{l-1}(\cos \theta) \right) \right. \\ \left. - \left((l+1)f_{l+}^\pm + lf_{l-}^\pm \right) \cos \theta P_l(\cos \theta) \right)$$

- Here, $f_{l\pm}^\pm$: SAID pw data

Partial Waves and Phase Shifts

- Partial waves (independent of θ : scattering angle) are given by:

$$f_{l\pm}^{\pm}(\omega) = \frac{\sqrt{\omega^2 - M^2 + m_N^2} + m_N}{16\pi(\sqrt{\omega^2 + m_N^2 - M_\pi^2} + \omega)} \int_0^\pi \sin \theta d\theta \left(g^{\pm}(\omega, \theta) P_l(\cos \theta) \right. \\ \left. + (w^2 - M_\pi^2) h^{\pm}(\omega, \theta) \left[P_{l\pm 1}(\cos \theta) - \cos \theta P_l(\cos \theta) \right] \right)$$

- S-channel partial waves, $f_{l\pm}^I$:

$$\begin{pmatrix} f_{l\pm}^{1/2} \\ f_{l\pm}^{3/2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} f_{l\pm}^+ \\ f_{l\pm}^- \end{pmatrix}$$

- Phase shifts:

$$\delta_{l\pm}^I = \arctan \left(\frac{2\text{Im}(k f_{l\pm}^I) + \eta_{l\pm}^I - 1}{2\text{Re}(k f_{l\pm}^I)} \right)$$

- Notation: $L_{2I\,2J}$; here, $J = l\pm$; **S31** $\rightarrow$ isospin 3/2 and spin 1/2

Power Counting

Linked power counting ξ is defined as $O(1/N_c) = O(p) = O(\xi)$ [19]. The ξ -order of a given Feynman diagram, ν_ξ , is simply $\nu_p + \nu_{1/N_c}$, as given by Eq. 2.36, Eq. 2.37, and 2.38:

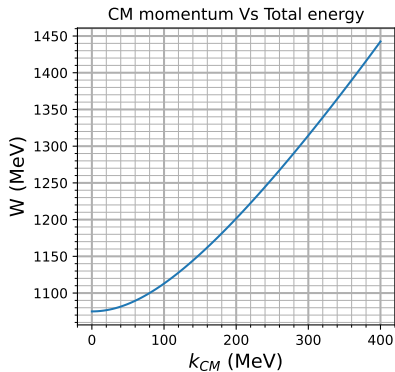
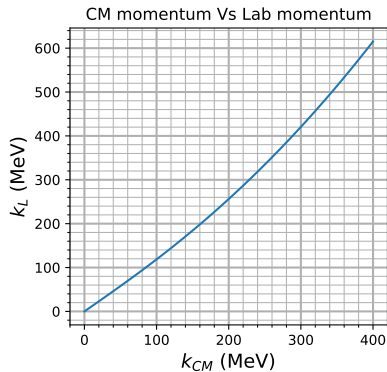
$$\nu_\xi = 1 + 3L + \frac{n_\pi}{2} - n_{\mathcal{L}\pi} + \sum_i n_i (\nu_{Oi} + \nu_{pi} - 1) \quad (2.39)$$

$$\langle S' S'_3 I'_3 | S^i | S S_3 I_3 \rangle = \sqrt{S(S+1)} \delta_{SS'} \delta_{I_3 I'_3} \langle S S_3, 1i | S' S'_3 \rangle \quad (2.22)$$

$$\langle S' S'_3 I'_3 | I^a | S S_3 I_3 \rangle = \sqrt{S(S+1)} \delta_{SS'} \delta_{S_3 S'_3} \langle S I_3, 1a | S' I'_3 \rangle$$

$$\langle S' S'_3 I'_3 | G^{ia} | S S_3 I_3 \rangle = \frac{1}{4} \sqrt{\frac{2S+1}{2S'+1}} \zeta(N_c, S, S') \times \langle S S_3, 1i | S' S'_3 \rangle \langle S I_3, 1a | S' I'_3 \rangle$$

Here $\zeta(N_c, S, S') = \sqrt{(2+N_c)^2 - (S-S')^2(S+S'+1)^2}$. It is important to no-



$g^\pm$ and $h^\pm$ Amplitudes

- General pion nucleon scattering amplitude, $T_{\pi N}^{ba}$ can be decomposed into $g^\pm$ and $h^\pm$ amplitudes

$$T_{\pi N}^{ba} = \left(\frac{E + m_N}{2m_N} \right) \left(\delta^{ba} \left[g^+(\omega, t) + i\sigma \cdot (\vec{k}_2 \times \vec{k}_1) h^+(\omega, t) \right] + i\epsilon^{bac} \tau^c \left[g^-(\omega, t) + i\sigma \cdot (\vec{k}_2 \times \vec{k}_1) h^-(\omega, t) \right] \right)$$

- g^+ : isoscalar non-spin-flip: $I = J = 0$
- g^- : isovector non-spin-flip: $I = 1, J = 0$
- h^+ : isoscalar spin-flip: $I = 0, J = 1$
- h^- : isovector spin-flip: $I = J = 1$

$g^\pm$ and $h^\pm$ from LO

$$g^+ = \frac{g_A^2}{144F^2}(N_c - 1)(5 + N_c)(2M^2 - t - 2\omega^2) \left(\frac{1}{\omega + \delta m_N - \delta m_\Delta} + \frac{1}{\delta m_N - \delta m_\Delta - \omega} \right) - \frac{2c_1 M_\pi^2 N_c}{F^2 \Lambda}$$

$$g^- = \frac{\omega}{2F^2} + \frac{g_A^2}{288F^2}(2 + N_c)^2(2M^2 - t - 2\omega^2) \left(\frac{2}{\omega} \right) + \frac{g_A^2}{288F^2}(N_c - 1)(5 + N_c)(2M^2 - t - 2\omega^2) \left(\frac{-1}{\omega + \delta m_N - \delta m_\Delta} + \frac{1}{\delta m_N - \delta m_\Delta - \omega} \right)$$

$$h^+ = \frac{g_A^2}{144F^2}(2 + N_c)^2 \left(\frac{-2}{\omega} \right) + \frac{g_A^2}{144F^2}(N_c - 1)(5 + N_c) \left(\frac{1}{\omega + \delta m_N - \delta m_\Delta} - \frac{1}{\delta m_N - \delta m_\Delta - \omega} \right)$$

$$h^- = -\frac{g_A^2}{288F^2}(N_c - 1)(5 + N_c) \left(\frac{1}{\omega + \delta m_N - \delta m_\Delta} + \frac{1}{\delta m_N - \delta m_\Delta - \omega} \right)$$

Renormalization

$$\begin{aligned}\mathcal{L}_{CT} &= \mathbf{B}^\dagger \left(\frac{f_{10}}{\Lambda^2} \varepsilon^{ijk} S^k u_i^a \overleftrightarrow{D}^0 u_j^a \right) \mathbf{B} \\ &= \mathbf{B}^\dagger \pi^a \pi^b \frac{f_{10}}{\Lambda^2 F_\pi^2} k_1^i k_2^j (k_1^0 + k_2^0) \delta^{ab} \varepsilon^{ijk} S^k \mathbf{B}\end{aligned}$$

$$\text{counter-term} = \left(\underbrace{f_{10}^{(0)}(\mu)}_{\text{Low Energy Constant (LEC)}} + \underbrace{\frac{\tilde{g}_A^4 \Lambda^2}{96 F_\pi^2}}_{\beta_{f_{10}}^{(0)}} \frac{\lambda_\epsilon}{(4\pi)^2} \right) \frac{1}{\Lambda^2 F_\pi^2} k_1^i k_2^j (k_1^0 + k_2^0) \delta^{ab} \varepsilon^{ijk} S^k$$

Low Energy Constant (LEC)

$\beta_{f_{10}}^{(0)}$

Renormalization

$$\begin{aligned}\mathcal{L}_{CT} &= \mathbf{B}^\dagger \left(\frac{f_{10}}{\Lambda^2} \varepsilon^{ijk} S^k u_i^a \overleftrightarrow{D}^0 u_j^a \right) \mathbf{B} \\ &= \mathbf{B}^\dagger \pi^a \pi^b \frac{f_{10}}{\Lambda^2 F_\pi^2} k_1^i k_2^j (k_1^0 + k_2^0) \delta^{ab} \varepsilon^{ijk} S^k \mathbf{B}\end{aligned}$$

$$\text{counter-term} = \left(\underbrace{f_{10}^{(0)}(\mu)}_{\text{Low Energy Constant (LEC)}} + \underbrace{\frac{\dot{g}_A^4 \Lambda^2}{96 F_\pi^2}}_{\beta_{f_{10}}^{(0)}} \frac{\lambda_\epsilon}{(4\pi)^2} \right) \frac{1}{\Lambda^2 F_\pi^2} k_1^i k_2^j (k_1^0 + k_2^0) \delta^{ab} \varepsilon^{ijk} S^k$$

Low Energy Constant (LEC)

$\beta_{f_{10}}^{(0)}$

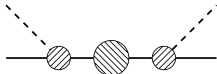
UV No-pole from all 45 diagrams

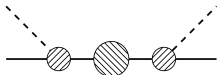
Projection	UV no-pole
$J = 0, I = 0$	$2g_A^2 \alpha \delta^{ab} \frac{C_{HF}}{N_c} (3N_c(4 + N_c) - 20S^2) (4k_1 \cdot k_2 - 3M^2)$ $+ 48\alpha \delta_{ab} \frac{c_1 N_c M^2}{\Lambda} (8k_1 \cdot k_2 - 5M^2)$
$J = 1, I = 0$	$-ig_A^4 \alpha k_1^i k_2^j (k_1^0 + k_2^0) \delta^{ab} \varepsilon^{ijk} S^k$

Finite NNLO

$$\text{NNLO} = \text{LO} + \text{Loops}$$

$$= \text{LO}_{no-pole} + \text{LO}_{pole} + \text{Loos}_{no-pole} + \text{Loops}_{pole}$$

$$= \text{LO}_{no-pole} + \text{Loops}_{no-pole} +$$


$$= \text{LO}_{no-pole} + \text{Loops}_{no-pole} +$$


$$+ \text{CounterTerms}$$

NNLO Theory vs Experimental Data - $Re(g^+)$

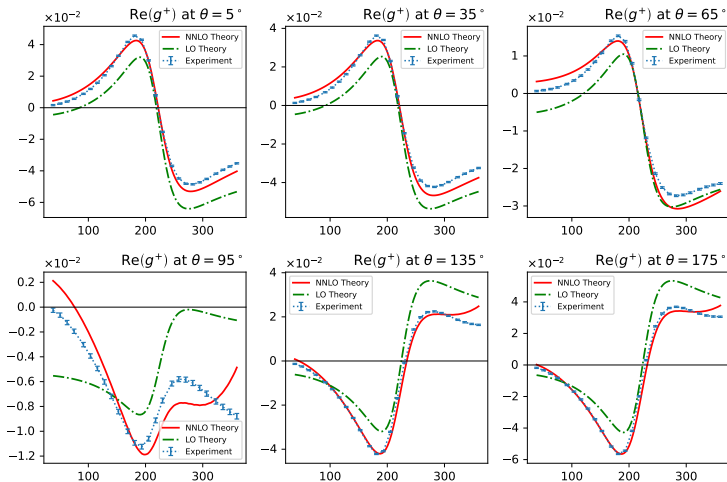


Figure: $Re(g^+)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Re(g^+)$ in units of $1/\text{MeV}$ for different CM angles.

NNLO Theory vs Experimental Data - $Im(g^+)$

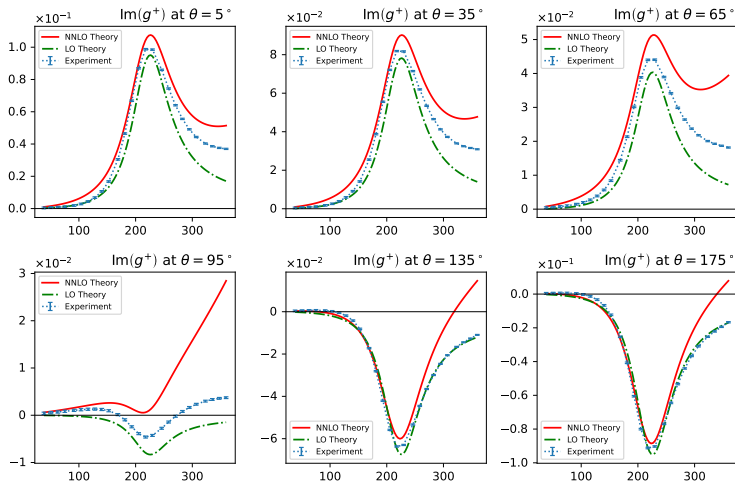


Figure: $Im(g^+)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Im(g^+)$ in units of $1/\text{MeV}$ for different CM angles.

NNLO Theory vs Experimental Data - $Im(g^+)$

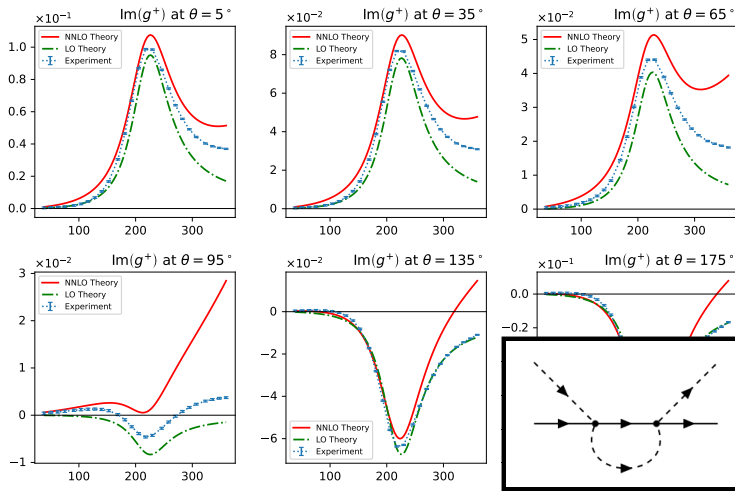


Figure: $Im(g^+)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Im(g^+)$ in units of $1/\text{MeV}$ for different CM angles.

NNLO Theory vs Experimental Data - $Re(h^+)$

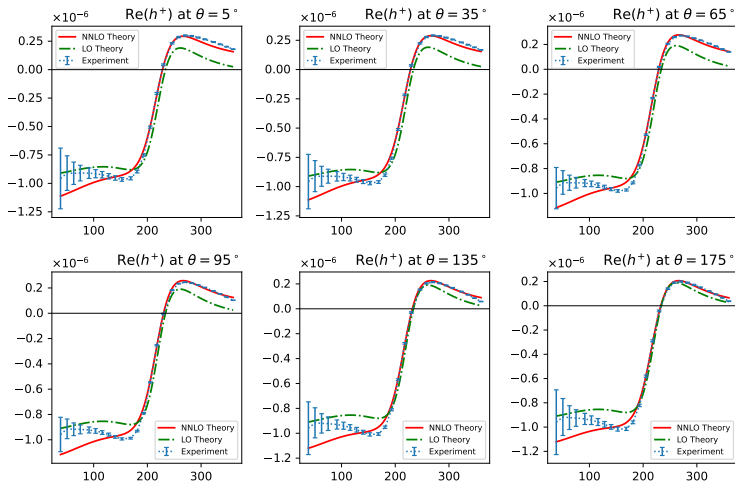


Figure: $Re(h^+)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Re(h^+)$ in units of $(1/\text{MeV})^3$ for different CM angles.

NNLO Theory vs Experimental Data - $Im(h^+)$

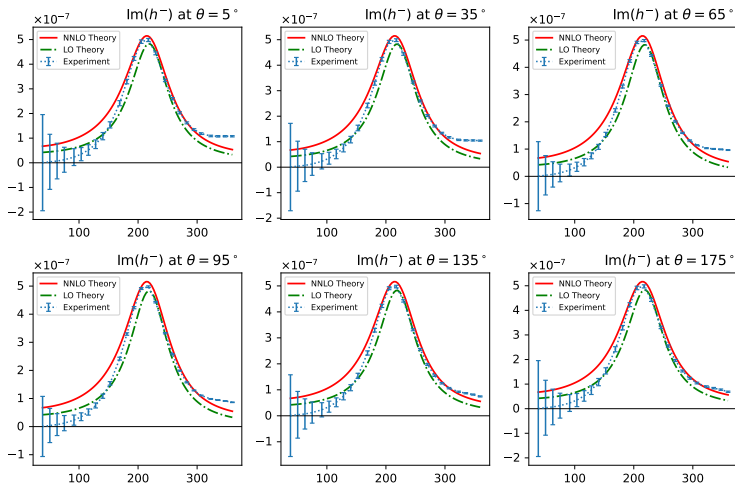


Figure: $Im(h^+)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Im(h^+)$ in units of $(1/\text{MeV})^3$ for different CM angles.

NNLO Theory vs Experimental Data - $Re(g^-)$

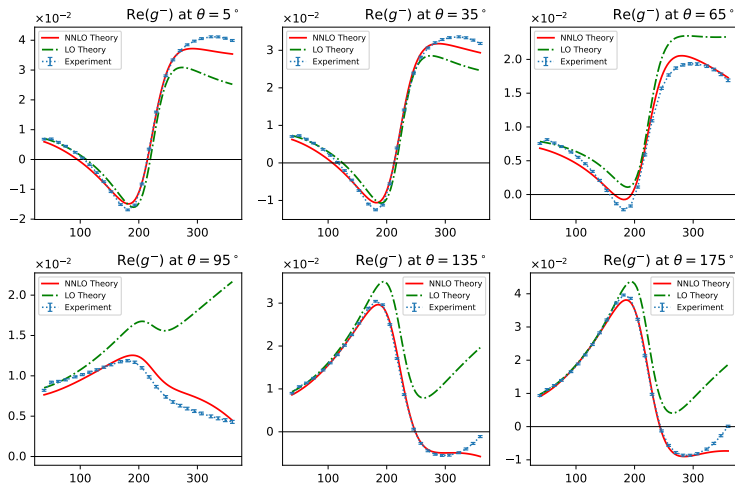


Figure: $Re(g^+)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Re(g^+)$ in units of $1/\text{MeV}$ for different CM angles.

NNLO Theory vs Experimental Data - $Im(g^-)$

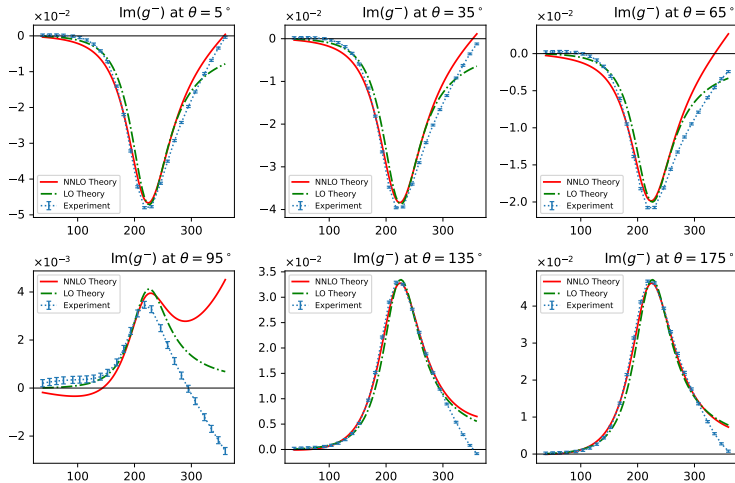


Figure: $Im(g^-)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Im(g^-)$ in units of $1/\text{MeV}$ for different CM angles.

NNLO Theory vs Experimental Data - $Re(h^-)$

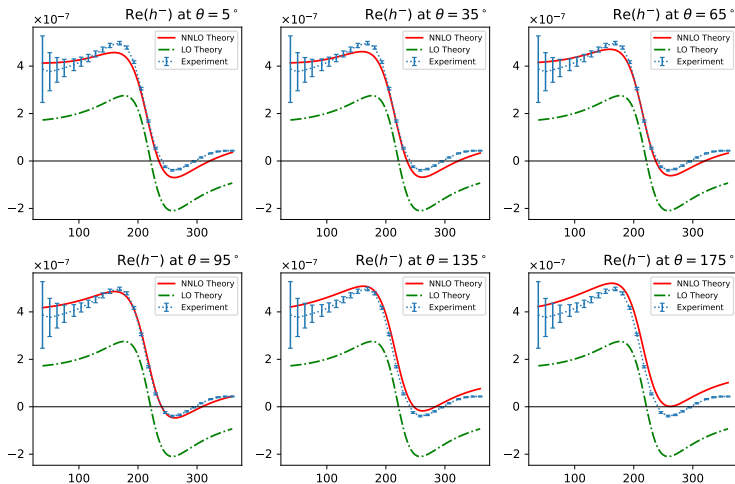


Figure: $Re(h^-)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Re(h^-)$ in units of $(1/\text{MeV})^3$ for different CM angles.

NNLO Theory vs Experimental Data - $Im(h^-)$

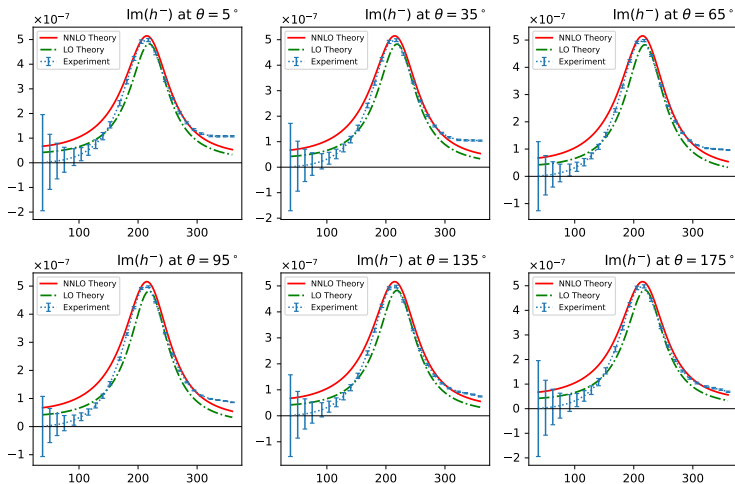


Figure: $Im(h^-)$ from NNLO: The horizontal axes show the CM pion momentum in MeV, and the vertical axes show $Im(h^-)$ in units of $(1/\text{MeV})^3$ for different CM angles.

Low Energy Fit

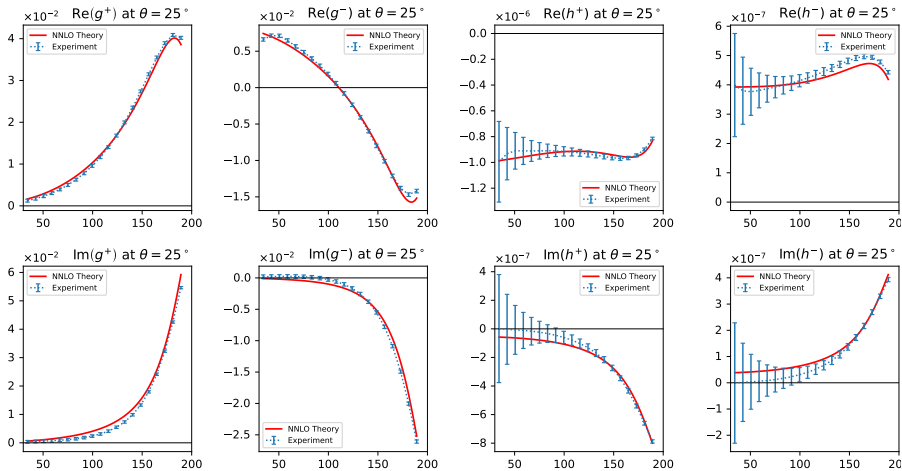


Figure: Horizontal axes: $k_{CM} = 30 - 200$ MeV , vertical axes: $g^\pm$ (1/MeV) or $h^\pm$ (1/MeV)³

$$\chi^2/\text{dof} = 45; g_A^\circ = 0.96; g_{ANN} = 1.36; g_{AND} = 1.48; c_1 = 0.29;$$

$$C_{HF} = 295 \text{ MeV}; \Gamma_\Delta = 79.9 \text{ MeV}$$

Uncertainties Associated with Fits

• Fits with bootstrapped uncertainty

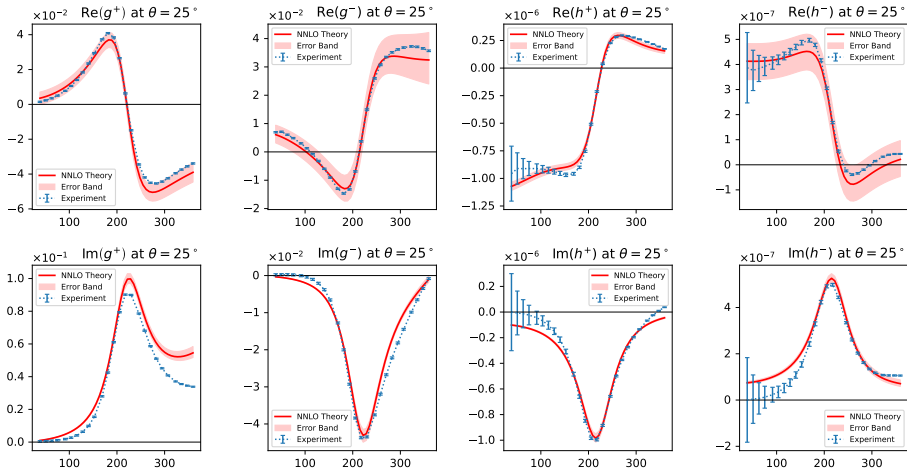


Figure: Horizontal axes: k_{CM} in MeV , vertical axes: $g^\pm$ ($1/\text{MeV}$) or $h^\pm$ ($1/\text{MeV}$)³