

Stress in Two-Body Systems

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 Jefferson Lab

 CENTER for
NUCLEAR FEMTOGRAPHY

🍁 Stresses in quantum systems quantified by $D(\Delta^2)$ and $\bar{c}(\Delta^2)$ form factors

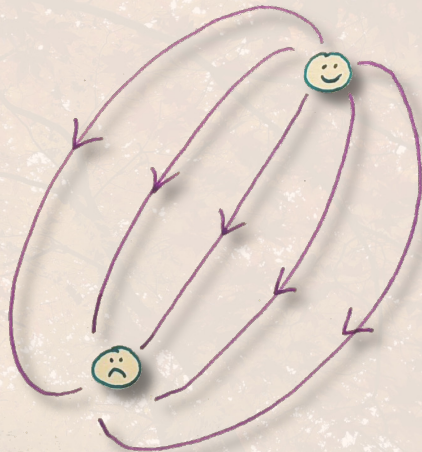
- 🍂 $\bar{c}(\Delta^2)$ quantifies forces on open subsystems
- 🍂 $D(\Delta^2)$ quantifies balancing stresses

🍁 I consider systems that:

- 🍂 Are non-relativistic
- 🍂 Consist of two bodies
- 🍂 Are bound by abelian force fields
- 🍂 Have no angular momentum

🍁 My main finding:

- 🍂 $D(0) < 0$ if force carrier is spin-even
- 🍂 $D(0) > 0$ if force carrier is spin-odd
- 🍂 The sign has nothing to do with mechanical stability



A photograph of a brick building with autumn trees in the foreground and a clear blue sky. The text "The Energy-Momentum Tensor" is overlaid in a stylized font.

The Energy-Momentum Tensor

- ✳ The **energy-momentum tensor** (EMT) describes **density** and **flow** of energy & momentum.
- ✳ Also known as the stress-energy tensor.

Energy density

Momentum densities

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

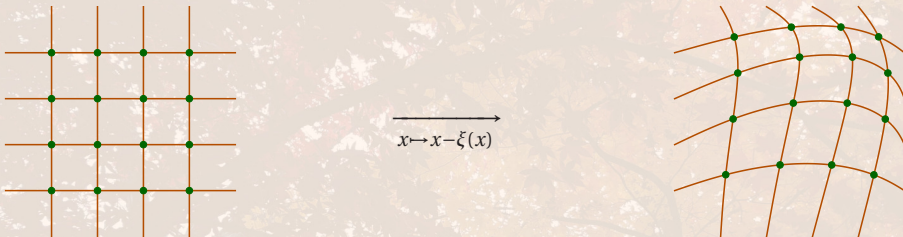
Energy fluxes

Stress tensor

The diagram illustrates the components of the energy-momentum tensor $T^{\mu\nu}(x)$. The tensor is represented as a 4x4 matrix. The components are color-coded and labeled with arrows:

- Energy density:** $T^{00}(x)$ (blue box, blue arrow from the label).
- Momentum densities:** $T^{01}(x)$, $T^{02}(x)$, and $T^{03}(x)$ (green boxes, green arrow from the label).
- Energy fluxes:** $T^{10}(x)$, $T^{20}(x)$, and $T^{30}(x)$ (red boxes, red arrow from the label).
- Stress tensor:** $T^{11}(x)$, $T^{12}(x)$, $T^{13}(x)$, $T^{21}(x)$, $T^{22}(x)$, $T^{23}(x)$, $T^{31}(x)$, $T^{32}(x)$, and $T^{33}(x)$ (orange boxes, orange arrow from the label).

- Conserved current from *local* spacetime translations (**Noether's second theorem**):



➤ **Noether's theorems:** symmetries imply conservation laws

➤ *Local* translation: move spacetime differently everywhere

- The EMT is the associated conserved quantity

$$\delta_{\xi} S_{\text{QCD}} = - \int d^4x \underbrace{T_{\text{QCD}}^{\mu\nu}(x)}_{\text{EOM are satisfied}} \partial_{\mu} \xi_{\nu}(x) \xrightarrow{\text{EOM are satisfied}} 0$$

$$\sum_q \left\{ \frac{i}{2} \bar{q}(x) \gamma^{\mu} \overleftrightarrow{D}^{\nu} q(x) \right\} - 2 \text{Tr}[G^{\mu\lambda} G^{\nu}_{\lambda}] + \frac{1}{2} g^{\mu\nu} \text{Tr}[G^{\lambda\sigma} G_{\lambda\sigma}]$$

Felsager, *Geometry, Particles, and Fields*

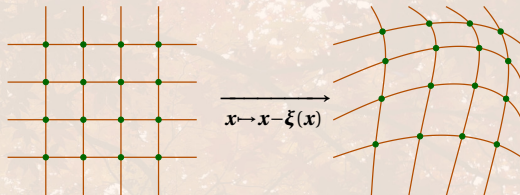
Montesinos & Flores, *Rev Mex Fis* 52 (2006) 29

AF, *Phys Rev D* 113 (2026) 016011

🍁 3×3 sub-matrix of the energy-momentum tensor.

$$\begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

↑
Stress tensor



🍁 Work needed to deform a system:

$$W[x \mapsto x - \xi(x)] = - \int d^3x \overbrace{T_{\text{QCD}}^{ij}(x)}^{\text{stress tensor}} \left(\overbrace{\frac{1}{2}(\partial_i \xi_j(x) + \partial_j \xi_i(x))}^{\text{strain}} + \overbrace{\frac{1}{2}(\partial_i \xi_j(x) - \partial_j \xi_i(x))}^{\text{torsion}} \right)$$

🍁 **Stress:** internal forces resisting *spatial* deformation

🍁 Deformations are a virtual pressure gauge

- Continuity equation (integral form):

$$\frac{d}{dt} \underbrace{\left[\int_V d^3x T^{0j}(\mathbf{x}, t) \right]}_{\text{Momentum in region}} = - \underbrace{\oint_{\partial V} dS \hat{n}_i T^{ij}(\mathbf{x}, t)}_{\text{Flux out of region}}$$

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

- Stresses are **momentum flux densities**

Include **particle fluxes & forces**



Particle flux



Force

- Kinetic pressure is pressure**

- It's the microscopic origin of gas & fluid pressure
- Photon flux is the origin of radiation pressure

- Stress tensor parametrized using **EMT form factors (EMT-FFs)**

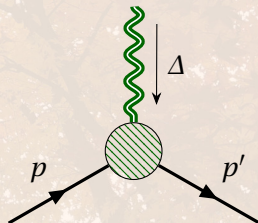
$$\langle \mathbf{p}' | \hat{T}_a^{ij}(0) | \mathbf{p} \rangle = \frac{P^i P^j}{M} A_a(\Delta^2) + \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M} D_a(\Delta^2) - M \delta^{ij} \bar{c}_a(\Delta^2)$$

- Non-relativistic ket normalization
- Also known as **gravitational form factors**
- EMT is source of gravitation, but we're not doing gravitational physics

- From Galilei covariance (non-relativistic)
- EMT-FFs encode composite system structure

$$\mathbf{P} = \frac{1}{2}(\mathbf{p} + \mathbf{p}')$$

$$\Delta = \mathbf{p}' - \mathbf{p}$$



🍁 Energy-momentum tensor parametrized using **EMT form factors**

$$\langle \mathbf{p}' | \hat{T}_a^{ij}(0) | \mathbf{p} \rangle = \overbrace{\frac{P^i P^j}{M} A_a(\Delta^2)}^{\text{barycentric motion \& dispersion}} + \underbrace{\frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M} D_a(\Delta^2) - M \delta^{ij} \bar{c}_a(\Delta^2)}_{\text{internal pressure \& forces}}$$

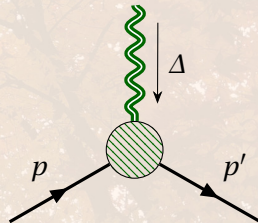
🍁 Form from Galilei covariance (non-relativistic)

🍁 **EMT-FFs encode composite system structure**

- 👉 But in a momentum space form
- 👉 Spatial picture via Fourier transforms

$$\mathbf{P} = \frac{1}{2}(\mathbf{p} + \mathbf{p}')$$

$$\Delta = \mathbf{p}' - \mathbf{p}$$



Kobzarev & Okun, *Sov Phys JETP* 16 (1963) 1343

H. Pagels, *Phys Rev* 144 (1966) 1250

X. Ji, *PRL* 78 (1997) 610

A photograph of a brick building with autumn trees in the foreground and a clear blue sky. The text "Particle Stresses" is overlaid in a white, cursive font with a red shadow effect.

Particle Stresses

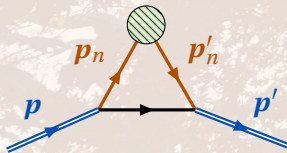
Pauli stress tensor:

$$\hat{T}_n^{ij}(\mathbf{x}) = \frac{1}{4m_n} \left[\hat{p}_n^i, [\hat{p}_n^j, \delta^{(3)}(\mathbf{x} - \hat{\mathbf{q}}_n)]_+ \right]_+$$

- Weyl-quantized form of operator
- Classically just particle motion
- Ensures pointlike particle has $D(\Delta^2) = 0$

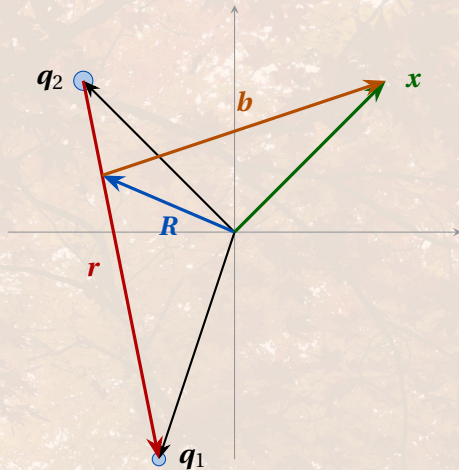
McCoy's formula gives:

$$\langle \mathbf{p}' | \hat{T}_n^{ij}(\mathbf{x}) | \mathbf{p} \rangle = \frac{p_n^i p_n^j}{m_n} \langle \mathbf{p}' | \delta^{(3)}(\mathbf{x} - \hat{\mathbf{q}}_n) | \mathbf{p} \rangle$$



$$\mathbf{P} = \frac{1}{2}(\mathbf{p} + \mathbf{p}')$$

$$\mathbf{P}_n = \frac{1}{2}(\mathbf{p}_n + \mathbf{p}'_n)$$



W. Pauli, *Quantentheorie* (1933) Chapter 2

N. McCoy, *Trans Amer Math Soc* 31 (1929) 793

🍁 $|\mathbf{p}\rangle$ in $\langle \mathbf{p}' | \hat{T}_a^{ij}(0) | \mathbf{p} \rangle$ implicitly contains internal state:

$$|\mathbf{p}\rangle \equiv |\mathbf{p}; n, l, m_l\rangle$$

🍂 $\mathbf{p}$ is the *barycentric momentum*.

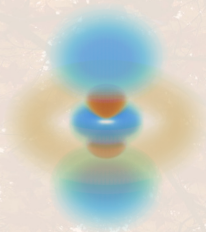
🍁 Every $|n, l, m_l\rangle$ state has different form factors!

🍂 In effect are different particles—analogy to proton, N^* , etc.

🍁 I will focus on an $l = 0$ ground state:

$$\langle \mathbf{r} | n = 1, l = 0, m_l = 0 \rangle = \psi(\mathbf{r}) = \frac{u(r)}{\sqrt{4\pi} r}$$

🍂 Form factor calculations involve expectation values with this wave function.



🍁 Direct matrix element gives:

$$\langle \mathbf{p}' | \hat{T}_n^{ij}(0) | \mathbf{p} \rangle = \frac{1}{M} \int d^3r \psi^*(\mathbf{r}) \left(\overbrace{\chi_n P^i P^j}^{\text{Barycentric motion \& dispersion (related to } A_n(\Delta^2))} - \underbrace{\frac{i\eta_n}{2} (P^i \overleftrightarrow{\nabla}^j + \overleftrightarrow{\nabla}^i P^j)}_{\text{Orbital fluxes (zero for spin-zero)}} - \overbrace{\frac{\overleftrightarrow{\nabla}^i \overleftrightarrow{\nabla}^j}{4\chi_n}}^{\text{internal stresses (related to } D(\Delta^2) \text{ and } \bar{c}(\Delta^2))} \right) \psi(\mathbf{r}) e^{i\eta_n \bar{\chi}_n \Delta \cdot \mathbf{r}}$$



Image: dynamic pressure from wind

AF, 2609.33151

🍁 Several auxiliary quantities:

$$\chi_n = \frac{m_n}{M}$$

Mass fraction

$$\bar{\chi}_n = 1 - \chi_n$$

Complement

$$\mathbf{r} = \mathbf{q}_1 - \mathbf{q}_2$$

$$\equiv \eta_1 \mathbf{q}_1 + \eta_2 \mathbf{q}_2$$

👉 Actually make formulas more compact...

🍁 Particle form factor results:

$$A_n(\Delta^2) = \chi_n \int_0^\infty dr u^2(r) j_0(\bar{\chi}_n \Delta r)$$

$$D_n(\Delta^2) = \frac{2}{\chi_n \Delta^2} \int_0^\infty dr \left(\frac{2u^2(r)}{r^2} - \frac{u(r)u'(r)}{r} + u(r)u''(r) - (u'(r))^2 \right) j_2(\bar{\chi}_n \Delta r)$$

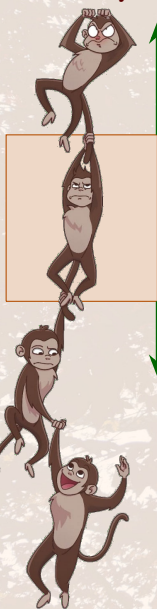
$$\bar{c}_n(\Delta^2) = \frac{1}{2\chi_n \bar{\chi}_n M^2 \Delta} \int_0^\infty dr \left(u'(r)u''(r) - u(r)u'''(r) \right) j_1(\bar{\chi}_n \Delta r)$$

🍁 $\bar{c}_n(\Delta^2)$ can be rewritten:

$$\bar{c}_n(\Delta^2) = \frac{1}{M\Delta} \int_0^\infty dr u^2(r) (-V'(r)) j_1(\bar{\chi}_n \Delta r)$$

🍁 **Can be inverted to find the force law using $\bar{c}(\Delta^2)$!**

$$-V'(r) = \frac{1}{\bar{\chi}_n^3 |\psi(r)|^2} \frac{d}{db} \left[\int_0^\infty d\Delta \Delta^2 \bar{c}_n(\Delta^2) j_0(\Delta b) \right] \Big|_{b=\bar{\chi}_n r}$$



$$F_{\text{up}} = +3Mg$$

🍁 **Cauchy momentum equation** for an *open, static* system:

$$f_{\text{net}}^j(\mathbf{x}, t) = f_{\text{external}}^j(\mathbf{x}, t) - \nabla_i T^{ij}(\mathbf{x}, t) = 0$$

- 👉 Balance between **external forces** & **internal forces**
- 👉 Can infer external forces from stress tensor!

$$F_g = -Mg$$

🍁 Works for expectation values in quantum systems!

$$\langle f_{\text{external}}^j(\mathbf{x}, t) \rangle = \nabla_i \langle T^{ij}(\mathbf{x}, t) \rangle$$

$$F_{\text{down}} = -2Mg$$

🍁 With probability density, can invert to get force law

- 👉 Precise statement for two-body systems on previous slide

Schrödinger, [Ann. Phys. 387 \(1927\) 265](#) (original quantum stress paper)

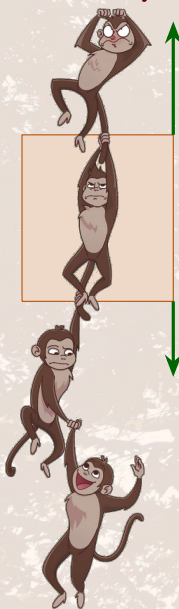
Nielsen & Martin, [PRB 32 \(1985\) 3780](#) (highly influential quantum stress paper)

Polyakov & Son, [JHEP 09 \(2018\) 156](#) (application of Cauchy equation to hadrons)

AF, [Phys Rev D 111 \(2025\) 034047](#) (inversion of Cauchy equation to get force law)

Ji, Miller & Yang, [Research 9 \(2026\) 1369](#) (application of inversion idea to proton)

Monkey image by Angie Freese ([Instagram](#))



$$F_{\text{up}} = +3Mg$$

Internal stress tensor:

$$t_n^{ij}(\mathbf{b}) = \int \frac{d^3\Delta}{(2\pi)^3} \left(\frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M} D_n(\Delta^2) - M \delta^{ij} \bar{c}_n(\Delta^2) \right) e^{-i\Delta \cdot \mathbf{b}}$$

$$F_g = -Mg$$

$\bar{c}(\Delta^2)$ quantifies force imbalance

Hence provides access to net external force

$$F_{\text{down}} = -2Mg$$

$D(\Delta^2)$ quantifies balancing forces from all sides

- Provides ambient pressures
- Similar to deep sea pressure, stress in bridge, etc.
- All monkeys have the same $\bar{c}$ —differ only in D
- This is physical and can be felt**

A photograph of a brick building with autumn trees in the foreground and a clear blue sky. The text "Static Field Stresses" is overlaid in a white, cursive font with a red shadow effect.

Static Field Stresses

Assume **static abelian** fields

- Static: time-independent
- Abelian: no self-interactions

Field depends on:

- $\mathbf{x}$: c-number
- $\hat{q}_1$: operator
- $\hat{q}_2$: operator

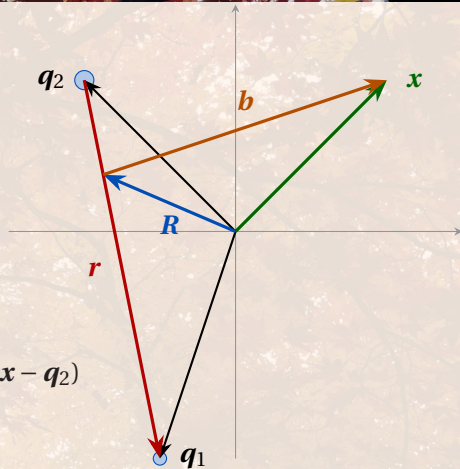
Field becomes c-number in $\mathbf{q}$ -representation

$$\langle \mathbf{p}' | \hat{T}_\phi^{ij}(\mathbf{x}) | \mathbf{p} \rangle = \int d^3 r |\psi(\mathbf{r})|^2 \int d^3 R e^{-i\Delta \cdot \mathbf{R}} T_\phi^{ij}(\mathbf{x} - \mathbf{q}_1, \mathbf{x} - \mathbf{q}_2)$$

End result:

$$\langle \mathbf{p}' | \hat{T}_\phi^{ij}(0) | \mathbf{p} \rangle = \underbrace{\int d^3 r |\psi(\mathbf{r})|^2 \int d^3 b e^{i\Delta \cdot \mathbf{b}} T_\phi^{ij}(0; \mathbf{b} - \eta_1 \bar{\chi}_1 \mathbf{r}, \mathbf{b} - \eta_2 \bar{\chi}_2 \mathbf{r})}_{\text{Ensemble average over particle configurations}}$$

Fourier transform of classical field configuration



🍁 Spin-zero EMT (canonical):

$$T_{\text{scalar}}^{\mu\nu} = (\partial^\mu \phi)(\partial^\nu \phi) - \frac{1}{2} g^{\mu\nu} ((\partial\phi)^2 - \mu^2 \phi^2)$$

🍁 Spin-one EMT (Belinfante):

$$T_{\text{vector}}^{\mu\nu} = F^{\mu\rho} F_{\rho}^{\nu} + \mu^2 A^{\mu} A^{\nu} + \frac{1}{4} g^{\mu\nu} F^2 - \frac{1}{2} g^{\mu\nu} \mu^2 A^2$$

🍁 EMT for static Newtonian gravitational field (spin-two):

$$T_{\text{gravity}}^{\mu\nu} = \frac{1}{4\pi G} \left\{ (\partial^\mu \phi)(\partial^\nu \phi) - \frac{1}{2} g^{\mu\nu} (\partial\phi)^2 \right\}$$

Embacher, Grubl & Patek [Phys Rev D 33 \(1986\) 1162](#)

Leader & Lorcé, [Phys Rept 541 \(2014\) 163](#)

AF, [2609.33151](#)

🍁 Spin-zero stress tensor (canonical):

$$T_{\text{scalar}}^{ij} = (\nabla_i \phi)(\nabla_j \phi) - \frac{1}{2} \delta^{ij} ((\nabla \phi)^2 + \mu^2 \phi^2)$$

🍁 Spin-one stress tensor (Belinfante) :

$$T_{\text{vector}}^{ij} = -(E^i E^j + B^i B^j) + \frac{1}{2} \delta^{ij} (E^2 + B^2) + \mu^2 A^i A^j + \frac{1}{2} \delta^{ij} \mu^2 (\phi^2 - A^2)$$

🍁 Stress tensor for static Newtonian gravitational field (spin-two):

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Embacher, Grubl & Patek [Phys Rev D 33 \(1986\) 1162](#)

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AF, [2609.33151](#)

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- 🍁 Spin-one stress tensor (Belinfante) ...electrostatic case, Coulomb gauge:

$$T_{\text{vector}}^{ij} = -(\nabla_i \phi)(\nabla_j \phi) + \frac{1}{2} \delta^{ij} ((\nabla \phi)^2 + \mu^2 \phi^2)$$

- 🍁 Stress tensor for static Newtonian gravitational field (spin-two):

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- Stress tensor for static Newtonian gravitational field (spin-two):

$$T_{\text{gravity}}^{ij} = \frac{1}{4\pi G} \left\{ (\nabla_i \phi)(\nabla_j \phi) - \frac{1}{2} \delta^{ij} (\nabla \phi)^2 \right\}$$

- Unified formula:**

$$T_{\text{static}}^{ij} = (-1)^s \left\{ (\nabla_i \phi)(\nabla_j \phi) - \frac{1}{2} \delta^{ij} ((\nabla \phi)^2 + \mu^2 \phi^2) \right\}$$

☞ ...absorb $\frac{1}{\sqrt{4\pi G}}$ into gravitational field

Embacher, Grubl & Patek [Phys Rev D 33 \(1986\) 1162](#)

Leader & Lorcé, [Phys Rept 541 \(2014\) 163](#)

AF, [2609.33151](#)

- Field configuration (single particle configuration $\mathbf{q}$):

$$\phi(\mathbf{x}; \mathbf{q}_1, \mathbf{q}_2) = \frac{g_1}{4\pi} \frac{e^{-\mu|\mathbf{x}-\mathbf{q}_1|}}{|\mathbf{x}-\mathbf{q}_1|} + \frac{g_2}{4\pi} \frac{e^{-\mu|\mathbf{x}-\mathbf{q}_2|}}{|\mathbf{x}-\mathbf{q}_2|}$$

Need $\text{sign}(g_1) = \text{sign}(g_2)$ for spin-even, $\text{sign}(g_1) = -\text{sign}(g_2)$ for spin-odd

- Stress tensor (single particle configuration $\mathbf{q}$):

$$T_{\phi}^{ij}(\mathbf{x}; \mathbf{q}_1, \mathbf{q}_2) = (-1)^s \left\{ (\nabla_i \phi)(\nabla_j \phi) - \frac{1}{2} \delta^{ij} ((\nabla \phi)^2 + \mu^2 \phi^2) \right\}$$

- Recipe for form factors:

$$\langle \mathbf{p}' | \hat{T}_{\phi}^{ij}(0) | \mathbf{p} \rangle = \underbrace{\int d^3 r |\psi(\mathbf{r})|^2}_{\text{Ensemble average over particle configurations}} \overbrace{\int d^3 b e^{i\Delta \cdot \mathbf{b}} T_{\phi}^{ij}(0; \mathbf{b} - \eta_1 \bar{\chi}_1 \mathbf{r}, \mathbf{b} - \eta_2 \bar{\chi}_2 \mathbf{r})}^{\text{Fourier transform of classical field configuration}}$$

- Easiest to break into self-field stresses ($T_{\bullet}^{ij}$) and interference/cross terms ($T_{\times}^{ij}$) ...

$$(\phi_1(\mathbf{x}) + \phi_2(\mathbf{x}))^2 = \underbrace{\phi_1^2(\mathbf{x}) + \phi_2^2(\mathbf{x})}_{\text{Self-fields}} + 2\phi_1(\mathbf{x})\phi_2(\mathbf{x})$$

🍁 Result is divergent—get finite value through analytic continuation

$$\langle \mathbf{p}' | \hat{T}_{\bullet,n}^{ij}(0) | \mathbf{p} \rangle = \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M} \times \underbrace{\frac{(-1)^{s+1} g_n^2 M}{8\pi \Delta} \left\{ \left(1 - \frac{4\mu^2}{\Delta^2} \right) \text{atan} \left(\frac{\Delta}{2\mu} \right) + \frac{2\mu}{\Delta} \right\} \int_0^\infty dr u^2(r) j_0(\bar{\chi}_n \Delta r)}_{= D_{\bullet,n}(\Delta^2)}$$

🍁 Regularization implicitly introduces Poincaré stresses

🍁 **Sign depends on spin of field**

🍁 No $\bar{c}(\Delta^2)$

🍁 Particle + self-field is closed system

🍁 Stability of particle under self-interaction requires implicit Poincaré stresses

L. Euler, [Novi Commentarii academiae scientiarum Petropolitanae 5, 205 \(1760\)](#) (English translation)

Varma & Schweitzer, [Phys Rev D 102 \(2020\) 014047](#)

Mejia & Schweitzer, [Phys Rev D 113 \(2026\) 054016](#)

AF, [2609.33151](#)

$$(\phi_1(\mathbf{x}) + \phi_2(\mathbf{x}))^2 = \phi_1^2(\mathbf{x}) + \phi_2^2(\mathbf{x}) + \underbrace{2\phi_1(\mathbf{x})\phi_2(\mathbf{x})}_{\text{Cross-term}}$$

🍁 Complicated result for $D_\times(\Delta^2)$:

$$D_\times(\Delta^2) = \frac{(-1)^{s+1} g_1 g_2 M}{4\pi \Delta^2} \int_0^\infty dr u^2(r) \left\{ \frac{(1-\omega)\Delta}{2} \Phi\left(\frac{\Delta r}{2}, \omega, \chi_2 - \chi_1\right) + \left[\frac{6(1+\mu r)}{\Delta r} (j_1(\chi_1 \Delta r) + j_1(\chi_2 \Delta r)) - (j_0(\chi_1 \Delta r) + j_0(\chi_2 \Delta r)) \right] \frac{e^{-\mu r}}{r} \right\}$$

👉 Shorthand notation:

$$\omega = \frac{4\mu^2}{\Delta^2} \quad \Phi(\zeta, \omega, \delta) = \int_{-1}^1 dy \frac{e^{-\zeta \sqrt{1-y^2+\omega}}}{\sqrt{1-y^2+\omega}} j_0((y+\delta)\zeta)$$

👉 $\text{sign}((-1)^{s+1} g_1 g_2) = -1$ is fixed for attractive force

👉 **Sign of $D(\Delta^2)$ will be determined by self-field stresses**

🍁 Simpler expression for $\bar{c}_\times(\Delta^2)$:

$$\bar{c}_\times(\Delta^2) = \frac{(-1)^s g_1 g_2}{4\pi M \Delta} \int_0^\infty dr u^2(r) \frac{(1+\mu r) e^{-\mu r}}{r^2} (j_1(\chi_1 \Delta r) + j_1(\chi_2 \Delta r))$$

🍁 $D(0)$ has attracted great interest—“last global unknown”

- Sign of $D(0)$ same as sign of long-distance stresses (positive=compressive, negative=tensile)
- Commonly conjectured to be negative, but counterexamples exist

🍁 Massive case:

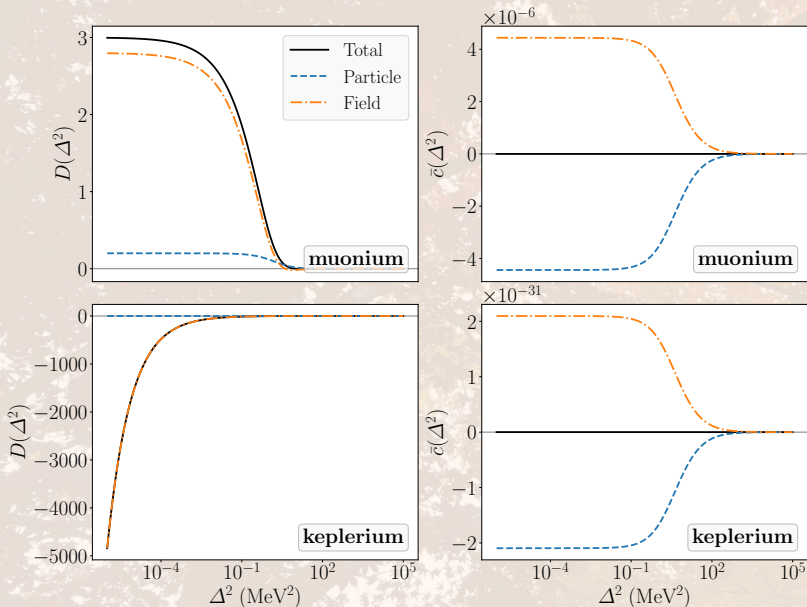
$$D_\phi(0; \mu) = \frac{(-1)^{s+1} M}{4\pi} \left\{ \frac{(g_1 + g_2)^2}{3\mu} + \int_0^\infty dr r u^2(r) e^{-\mu r} \left[\frac{1 - \frac{\mu r}{9}}{5} - \frac{2 e^{\mu r} - 1}{3 \mu r} - \frac{(1 + \mu r)(\chi_2 - \chi_1)^2}{15} \right] \right\}$$

- Dominated by first term at small μ
- $D(0) < 0$ for spin-even
- $D(0) > 0$ for spin-odd

🍁 Massless case:

$$D_\phi(\Delta^2; \mu = 0) = (-1)^{s+1} M \left\{ \frac{(g_1 + g_2)^2}{16\Delta} - \frac{g_1 g_2 (7 + (\chi_2 - \chi_1)^2)}{60\pi} \int_0^\infty dr r u^2(r) \right\} + \mathcal{O}(\Delta)$$

- $D(0) = -\infty$ for spin-even
- $D(0) > 0$ for spin-odd
- $D(0)$ finite iff $g_1 = -g_2$ (requires spin-odd field)



🍁 **Muonium:**
 $\mu^+ \mu^-$ state

🍁 $D(0) = 3 > 0$

🍁 **Keplerium:**
 gravitational onium

🍁 Same Bohr radius
 as muonium

🍁 Same wave function

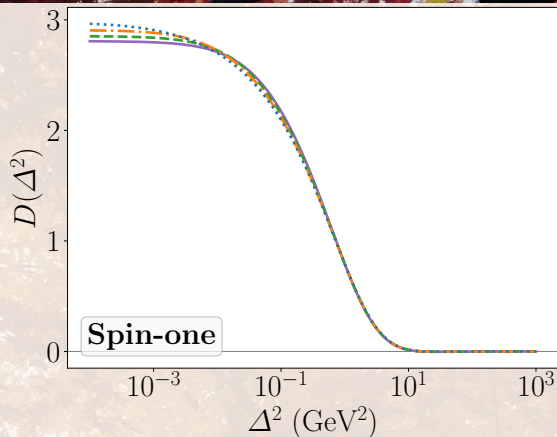
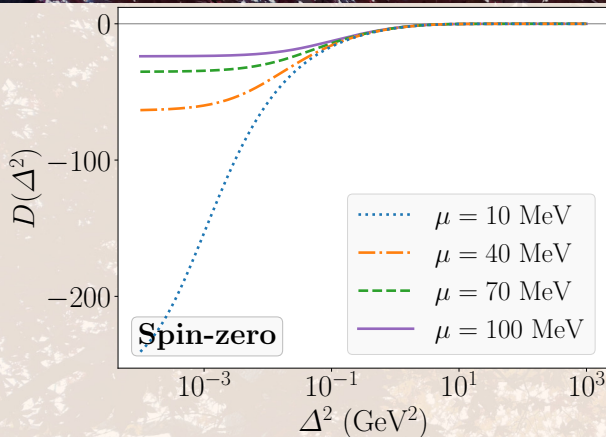
🍁 Kinda absurd mass

$$m_k = \sqrt[3]{\frac{\alpha_{\text{QED}} m_\mu}{G}}$$

$$\approx 486.19 \text{ EeV}$$

$$\approx 866.72 \text{ pg}$$

🍁 $D(0) = -\infty$



🍁 Massive exchange fields $\Rightarrow$ Yukawa potential

🍁 Same potential can give different $D(\Delta^2)$

👉 $D(0) = \text{finite} < 0$ for massive spin-even fields

👉 $D(0) > 0$ for massive spin-odd fields

$$V(r) = -\frac{\alpha e^{-\mu r}}{r}$$

$$m_1 = m_2 = 1 \text{ GeV} \quad \alpha = 1$$

A photograph of a brick building with autumn trees in the foreground and a clear blue sky. The word "Summary" is written in a white, cursive font with a red shadow, centered over the image.

Summary

- ❖ $D(\Delta^2)$ underdetermined by potential
 - 👉 Potential only tells you $\bar{c}(\Delta^2)$
 - 👉 Need microscopic details about how field carries stress
- ❖ Sign of $D(\Delta^2)$ depends on spin of field for abelian forces
 - 👉 Spin-even $\implies D(0) < 0$
 - 👉 Spin-odd $\implies D(0) > 0$
- ❖ QCD breaks this rule—but why?
 - 👉 Of course, QCD is non-abelian
 - 👉 It's not a stability condition
 - 👉 Something to learn here!

Thank you for your time!