

# Hadronic Matrix Elements of the QCD EMT

*—Lattice validation of hadron mass and trace-anomaly sum rules*

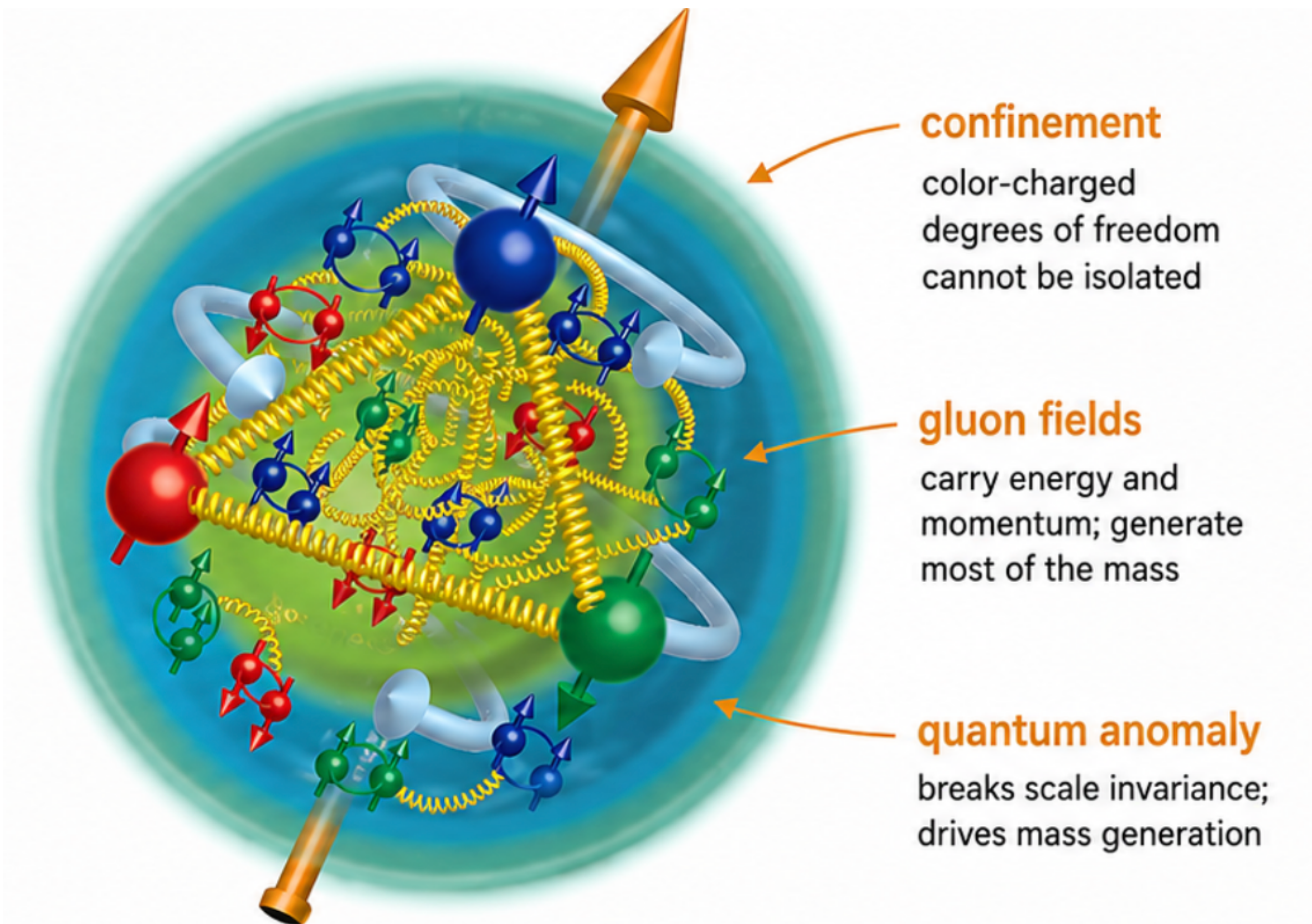
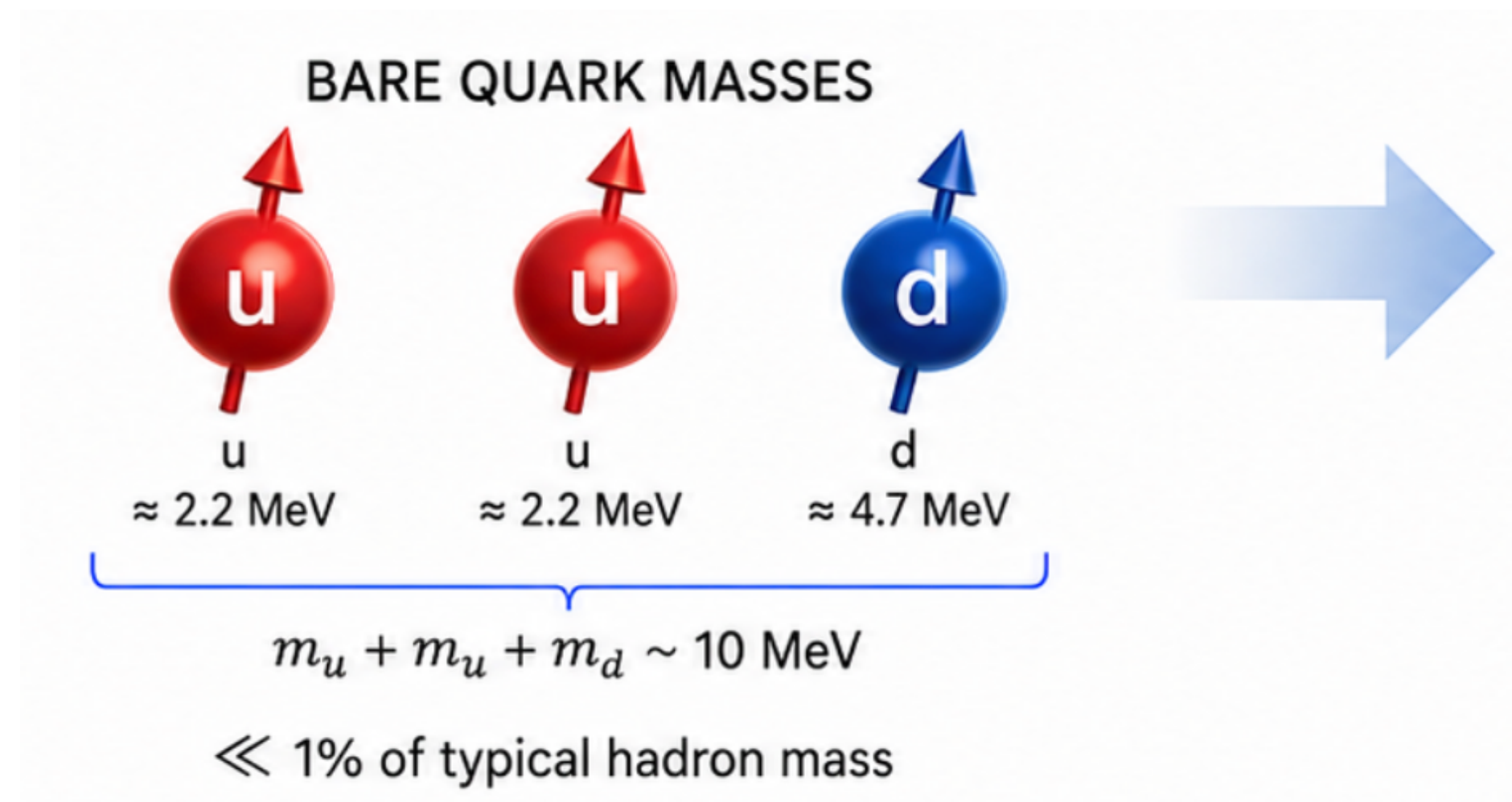
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Based on arXiv: [2601.13070](https://arxiv.org/abs/2601.13070)

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# Hadron mass is a QCD question

Most visible mass is generated by **QCD dynamics**, **not** by bare quark masses.



**How to understand the hadron mass in terms of quark and gluon fields?**

# The QCD EMT: from quark-gluon fields to hadron mass

To understand hadron mass in terms of **quarks** and **gluons**, we need the operator that measures **energy** and **momentum** in QCD.

$$T^{\mu\nu} = -F_a^{\mu\alpha}F_{a,\alpha}^{\nu} + \frac{1}{4}g^{\mu\nu}F_a^{\alpha\beta}F_{a,\alpha\beta} + \frac{i}{4}\sum_f \bar{\psi}_f \left( \gamma^\mu \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\mu \right) \psi_f$$

Noether current of  
spacetime translations

$$\partial_\mu T_{\text{QCD}}^{\mu\nu} = 0$$

$$P^\mu = \int d^3x T^{0\mu}$$

$$H_{\text{QCD}} = \int d^3x T^{00}$$

Hadron matrix elements

$$\langle H | T_{q,g}^{\mu\nu} | H \rangle$$

$\langle p' | T^{\mu\nu} | p \rangle \longrightarrow$  *gravitational form factors*

$\langle p | T^{00} | p \rangle \longrightarrow$  *energy / mass density*

$\langle p | T^\mu_\mu | p \rangle \longrightarrow$  *scale breaking / trace anomaly*

# How do we weigh a hadron?

$$M_H = \frac{1}{2M_H} \langle H | T^{00} | H \rangle$$

Rest-frame Hamiltonian definition of mass

**Total  $T^{\mu\nu}$  is conserved; q/g components are scale and scheme dependent:**

$$T^{\mu\nu} = T_{q,R}^{\mu\nu}(\mu) + T_{g,R}^{\mu\nu}(\mu)$$

## ● Two-terms

$$M = \langle T_{q,R}^{00} \rangle + \langle T_{g,R}^{00} \rangle$$

*Quark and gluon  
energy contributions*

## ● Three-terms

$$M = \sigma_m + (\langle T_{q,R}^{00} \rangle - \sigma_m) + \langle T_{g,R}^{00} \rangle$$
$$\sigma_m \equiv \frac{1}{2M_H} \langle \sum_f m_f \bar{\psi}_f(x) \psi_f(x) \rangle$$

*Explicit quark-mass  
contribution isolated*

## ● Ji' four-term decomposition

$$T^{00} = \bar{T}_{q,R}^{00} + \bar{T}_{g,R}^{00} + \frac{1}{4} T_{\mu}^{\mu}$$
$$M = M_q + M_g + M_a + \sigma_m$$

*Traceless and trace  
parts separated*

# How does QCD acquire a scale?

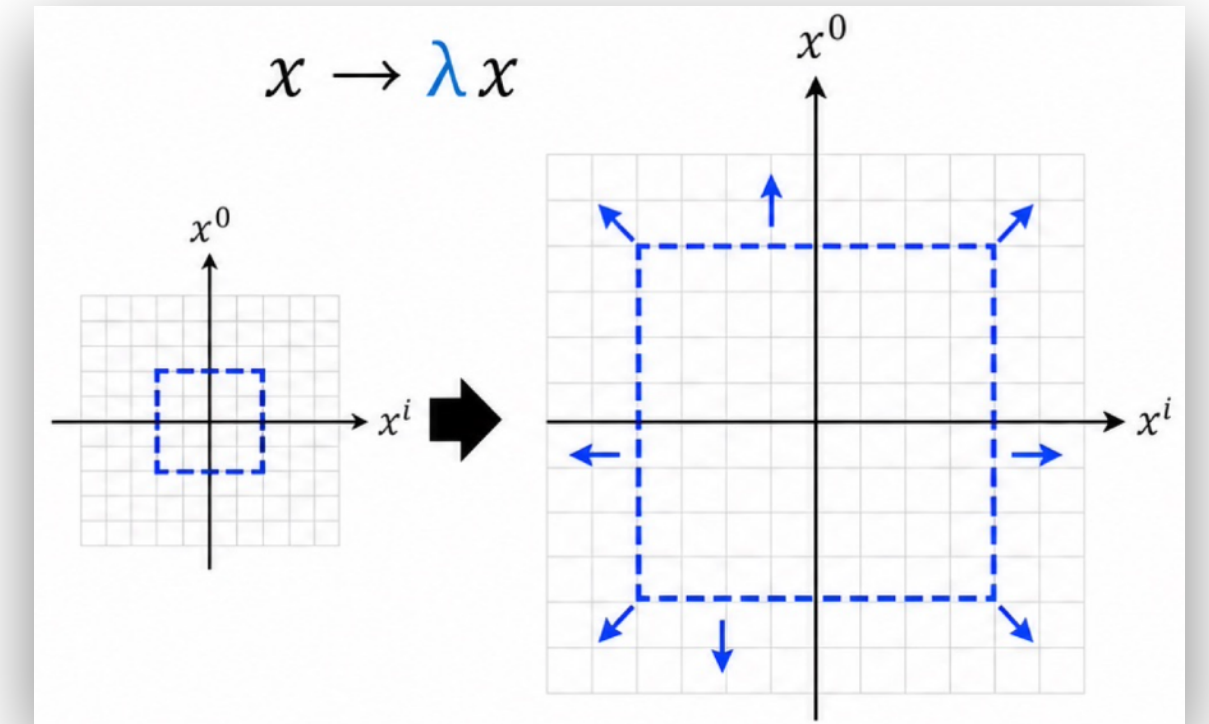
## ● Massless classical QCD: scale invariant

$$D^\mu = x_\nu T^{\mu\nu}, \quad \partial_\mu D^\mu = T^\mu_\mu$$

$$T^\mu_\mu = m_q \bar{\psi}\psi \xrightarrow{m_q=0} 0$$

No intrinsic dimensional parameter in the classical massless theory

dilatation symmetry



## ● Quantum and explicit scale breaking

$$T^\mu_\mu = \frac{\beta(g)}{2g} (F^2)_R + (1 + \gamma_m) (m_q \bar{\psi}\psi)$$

Anomalous (quantum) scale breaking, survives at  $m_q = 0$

Explicit scale breaking



## ● Hadron mass scale

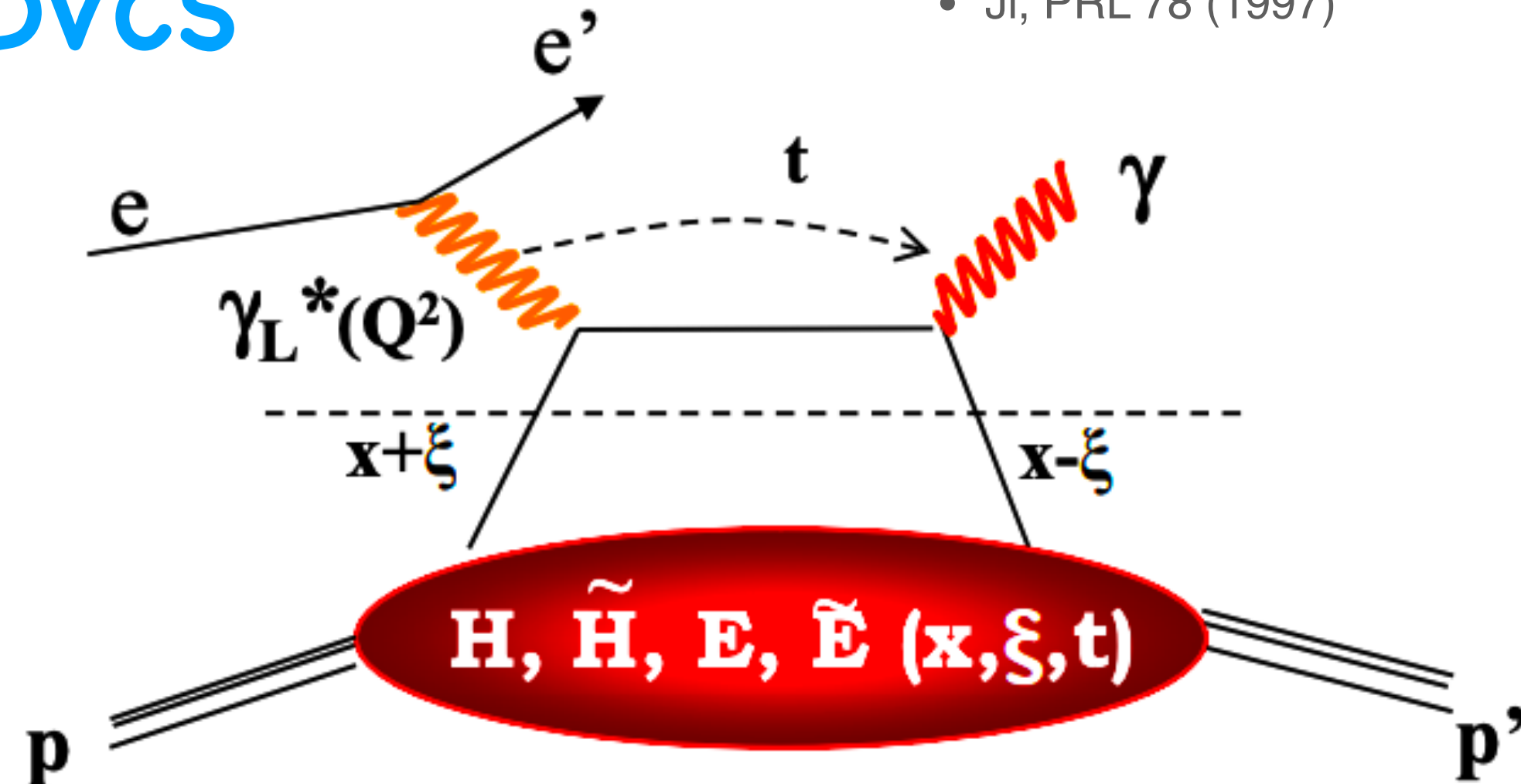
$$\langle P | T^\mu_\mu | P \rangle = 2P^\mu P_\mu = 2M^2$$

$$M = \frac{1}{2M} \left( \langle (T_{q,R})^\mu_\mu \rangle + \langle (T_{g,R})^\mu_\mu \rangle \right)$$

# Experimental access: GFFs from exclusive processes

DVCS

• Ji, PRL 78 (1997)



Leading-twist GPD moments:

$$\int_{-1}^1 dx x H^q(x, \xi, t) = A^q(t) + \xi^2 C^q(t)$$

$$\int_{-1}^1 dx x E^q(x, \xi, t) = B^q(t) - \xi^2 C^q(t)$$

$$\langle p', s' | T_{q,g}^{\mu\nu} | p, s \rangle = \bar{u}(p', s') \left[ A_{q,g}(t) \gamma^{(\mu} P^{\nu)} + B_{q,g}(t) \frac{P^{(\mu} i \sigma^{\nu)\rho} \Delta_\rho}{2M} \right. \\ \left. + C_{q,g}(t) \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{4M} + \bar{C}_{q,g}(t) M g^{\mu\nu} \right] u(p, s).$$

$$P = \frac{p' + p}{2}, \quad \Delta = p' - p, \quad t = \Delta^2. \quad \text{not fixed by twist-2 GPD moments}$$

Exclusive processes constrain the symmetric-traceless EMT sector

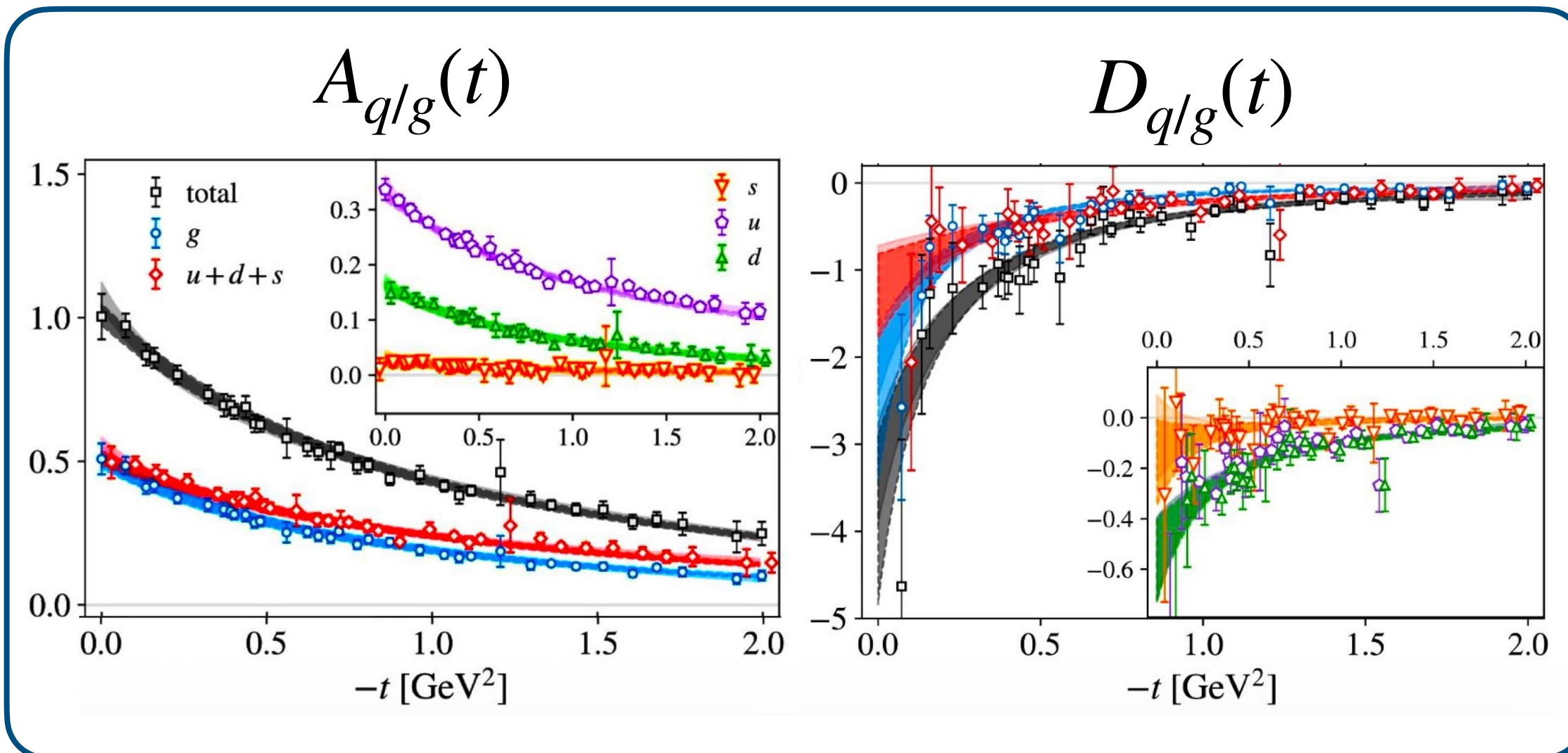
# Lattice QCD access: local and nonlocal routes

- X. Ji, PRL 110 (2013); SCPMA57 (2014);
- X. Ji, Y. Zhao, et al, RMP 93 (2021).
- A. Radyushkin, PRD 100 (2019)
- A. Radyushkin, Int.J.Mod.Phys.A 2020

## Local twist-2 operators

$$O_q^{\{\mu\nu\}} = \bar{\psi} \gamma^{\{\mu} i \overleftrightarrow{D}^{\nu\}} \psi - \text{traces}$$

$$\langle p' | T_{q/g}^{\mu\nu} | p \rangle \longrightarrow A(t), B(t), D(t)$$

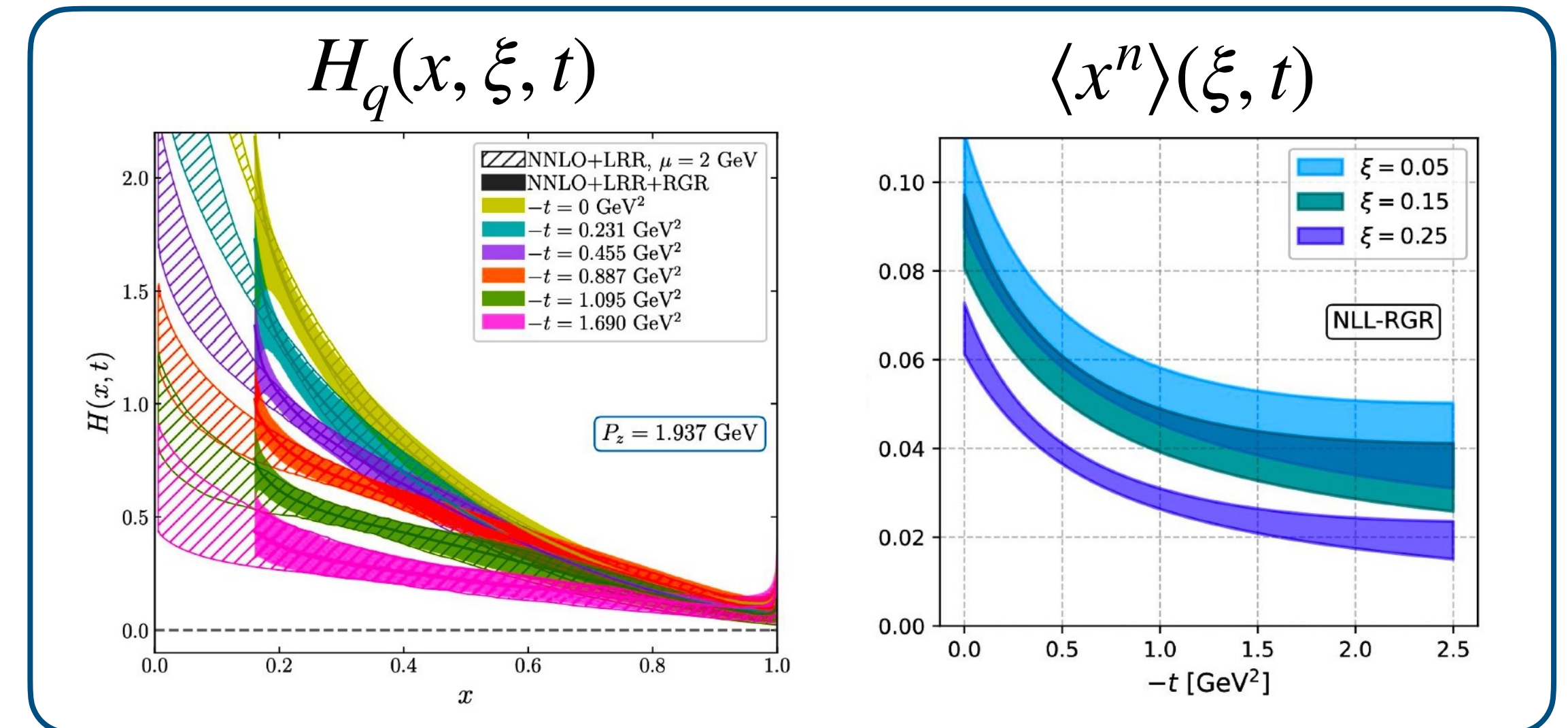


• MIT, PRL 132 (2024) 25, 251904

## Nonlocal operators

$$\tilde{h}_F(z, P, \Delta) = \langle p' | \bar{\psi}(0) \Gamma \mathcal{W}(0, z) \psi(z) | p \rangle$$

Euclidean spatial correlator  $\longrightarrow$  LaMET / SDF matching  $\longrightarrow$  ***x-dependent GPDs / GPD moments***



- H.-T. Ding, XG, Qi Shi et. al., JHEP 02 (2025) 056
- XG, F. Yao et. al., PRD 113 (2026) 1, 014505

# What remains missing? The scalar trace

$$T^{\mu\nu} = \underbrace{\left( T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T^\alpha_\alpha \right)}_{\bar{T}^{\mu\nu} : \text{symmetric traceless, twist-2}} \underbrace{\frac{1}{4} g^{\mu\nu} T^\alpha_\alpha}_{\text{scalar trace, twist-4}}$$

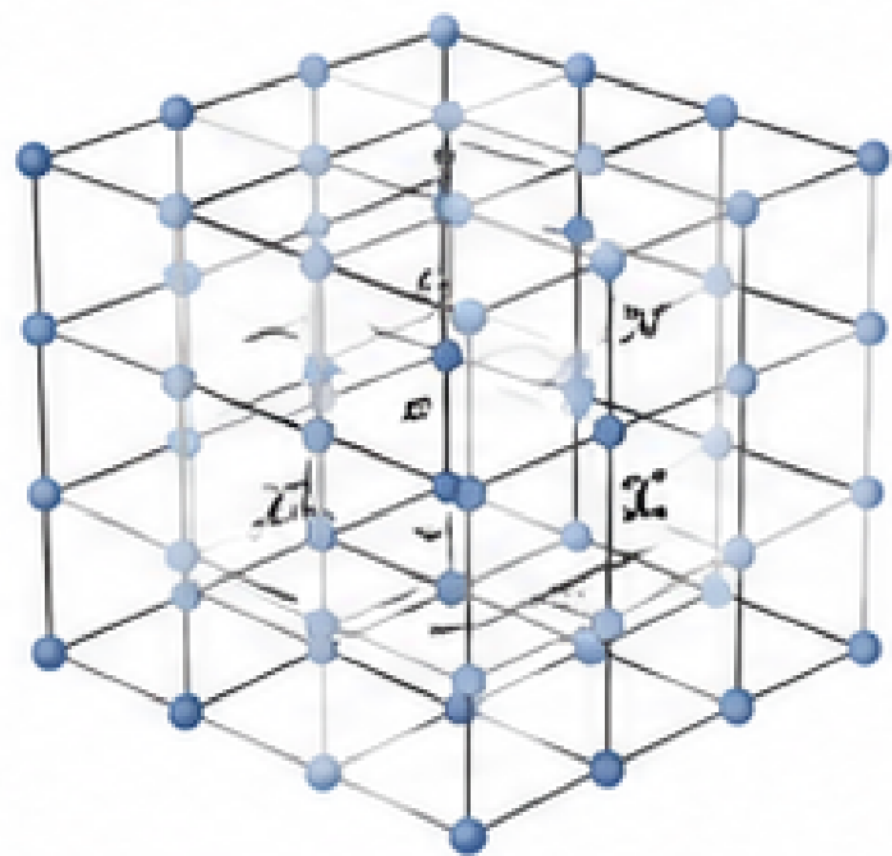
- GPDs from **twist expansion** gives access to the traceless EMT: the **trace sector is absent**; trace-related  $\bar{C}_{q/g}$  form factors are not constrained.
- **No independent test** of mass/trace-anomaly sum rules.

# Full EMT matrix elements from lattice QCD: why hard?

$$T^{\mu\nu} = O_{1,\mu\nu} - \frac{1}{4}O_{2,\mu\nu} + \frac{1}{4}O_{3,\mu\nu} = \underbrace{-F_a^{\mu\alpha}F_{a,\alpha}^{\nu}}_{T_g^{\mu\nu}} + \frac{1}{4}g^{\mu\nu}F_a^{\alpha\beta}F_{a,\alpha\beta} + \underbrace{\sum_f i\bar{\psi}_f\gamma^{\{\mu}D^{\nu\}}\psi_f}_{T_q^{\mu\nu}}$$

Bare lattice EMT

$$T_{\mu\nu}^{\text{bare}}(x)$$



- **Reduced symmetry:**  $O(4) \rightarrow H(4)$   
 ,only traceless components in  $H(4)$   
 irreps are safe.

- **Power-divergent mixing (trace):**

$$\langle \bar{\psi} \gamma_\mu \overleftrightarrow{D}^\mu \psi \rangle^R \sim \lim_{a \rightarrow 0} \left[ \langle \bar{\psi} \gamma_\mu \overleftrightarrow{D}^\mu \psi \rangle + \frac{1}{a} \langle \bar{\psi} \Gamma \psi \rangle + \dots \right]$$

**No  $a \rightarrow 0$  continuum limit!**

- **No direct access to full EMT.**

Renormalized full EMT  
including trace

$$T_{\mu\nu}^R(x)$$



# Gradient Flow and Finite-Resolution Operators

Gauge-covariant diffusion in four dimensions:

$$\partial_{t_f} B_\mu(t_f, x) = D_\nu F_{\nu\mu}(t_f, x), \quad B_\mu(0, x) = A_\mu(x)$$

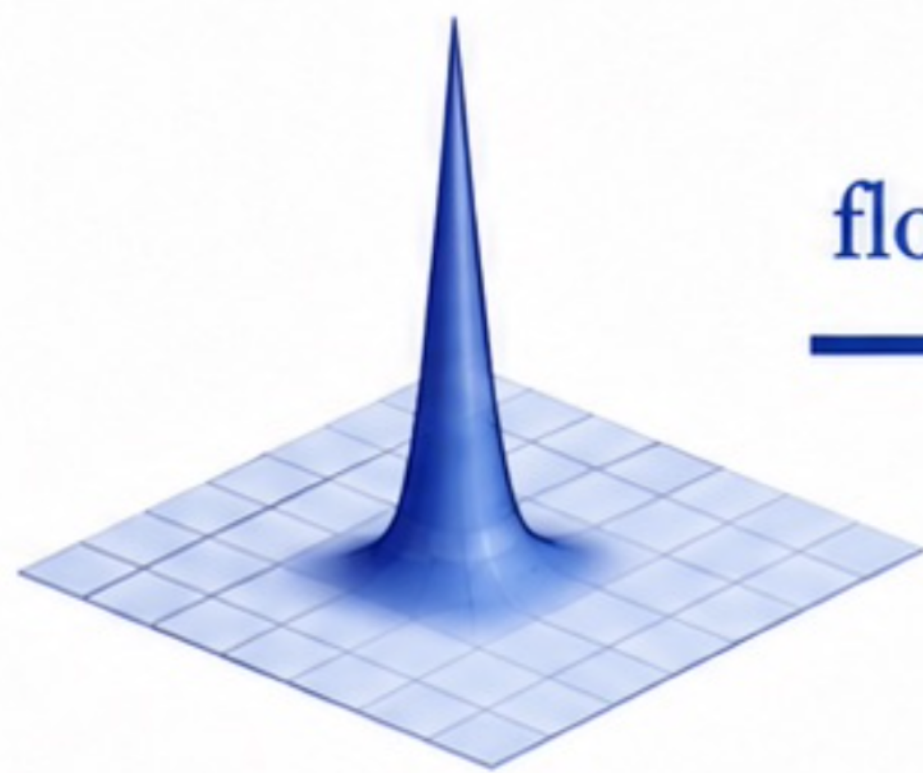
- M. Luscher, Commun.Math.Phys. 293 (2010)
- M. Luscher, JHEP 08 (2010) 071
- M. Luscher, P. Weisz, JHEP 02 (2011) 051

Local QCD field

Flowed field/operator

$$A_\mu(x)$$

$$t_f = 0$$

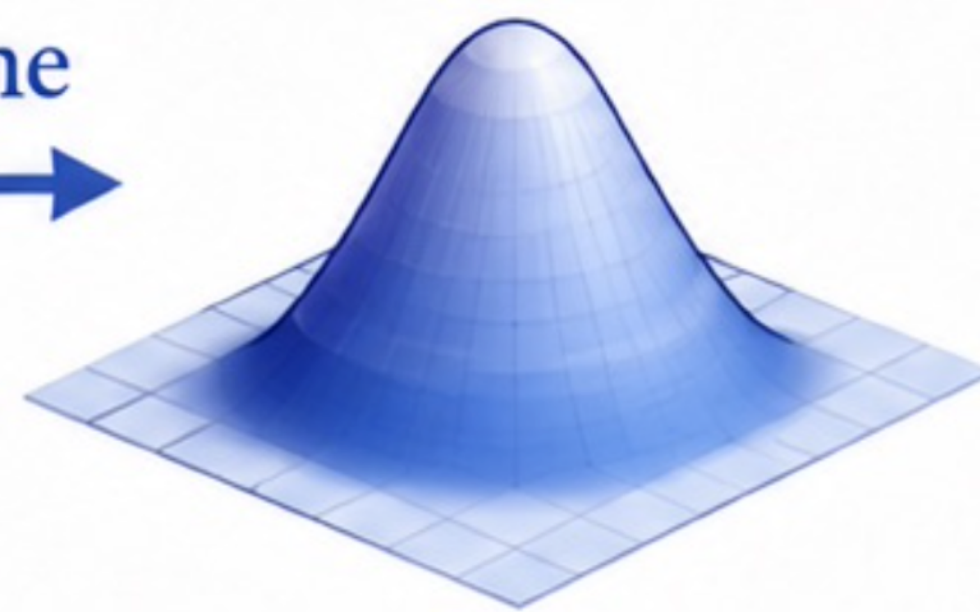


Local operator  
pointlike

flow time  
→

$$B_\mu(t_f, x)$$

$$t_f > 0$$



Smeared over  
 $r_{\text{flow}} \simeq \sqrt{8t_f}$

$$r_{\text{flow}} \simeq \sqrt{8t_f}, \quad \mu_t \sim \frac{1}{\sqrt{8t_f}}$$

- High-momentum modes are exponentially suppressed
- $t_f$  has dimension of **[length<sup>2</sup>]**.
- The operator acquires a physical resolution scale  $\mu_t \sim 1/\sqrt{8t_f}$ .

**From UV-sensitive pointlike operators to UV-finite smeared probes**

# UV-finite flowed operators and the continuum limit

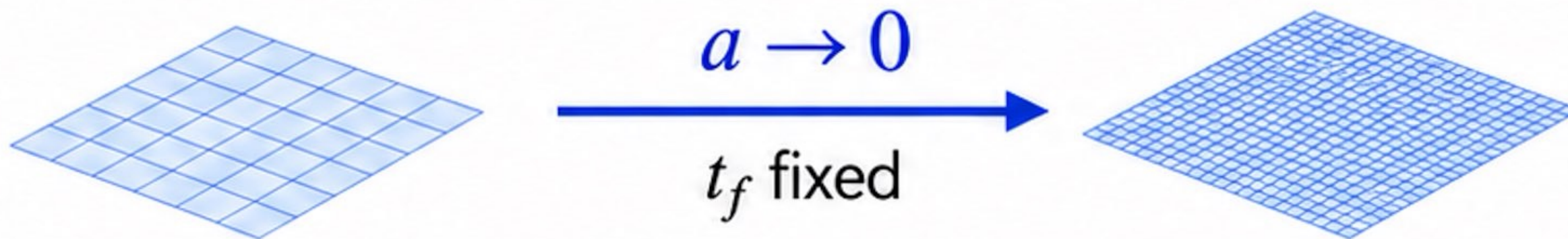
## UV-finite flowed operator

$$\tilde{\mathcal{O}}_i(t_f, x) = \mathcal{O}_i[B_\mu(t_f, x), \dot{\chi}(t_f, x), \ddot{\bar{\chi}}(t_f, x)]$$

finite composite operator at  $t_f > 0$

## Continuum limit at fixed physical flow time

$$\tilde{\mathcal{O}}_i(t_f, a) = \tilde{\mathcal{O}}_i(t_f, 0) + \mathcal{O}(a^2/t_f, a^2)$$



1

$$t_f > 0$$

**UV-finite** flowed basis



2

**regulator independent**

lattice, dimensional regularization, ...



3

**safe continuum limit**

$a \rightarrow 0$  at fixed  $t_f$

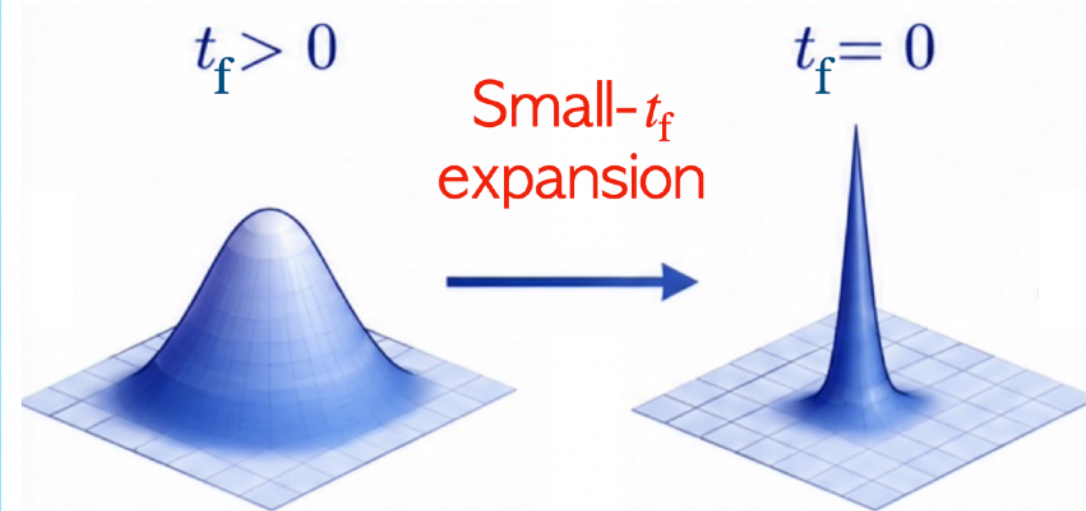
**UV-finite flowed operators allow a controlled continuum limit at**

**fixed  $t_f > 0$**

# From Flowed Operators to the Local QCD EMT

$$t_f > 0$$

- UV-finite flowed operator basis
- finite resolution
- not yet the local EMT



$$t_f = 0 \text{ boundary}$$

local 4D QCD

$$[T^\mu_\mu]_R = \frac{\beta(g)}{2g} (F^2)_R + (1 + \gamma_m)(m\bar{\psi}\psi)_R$$

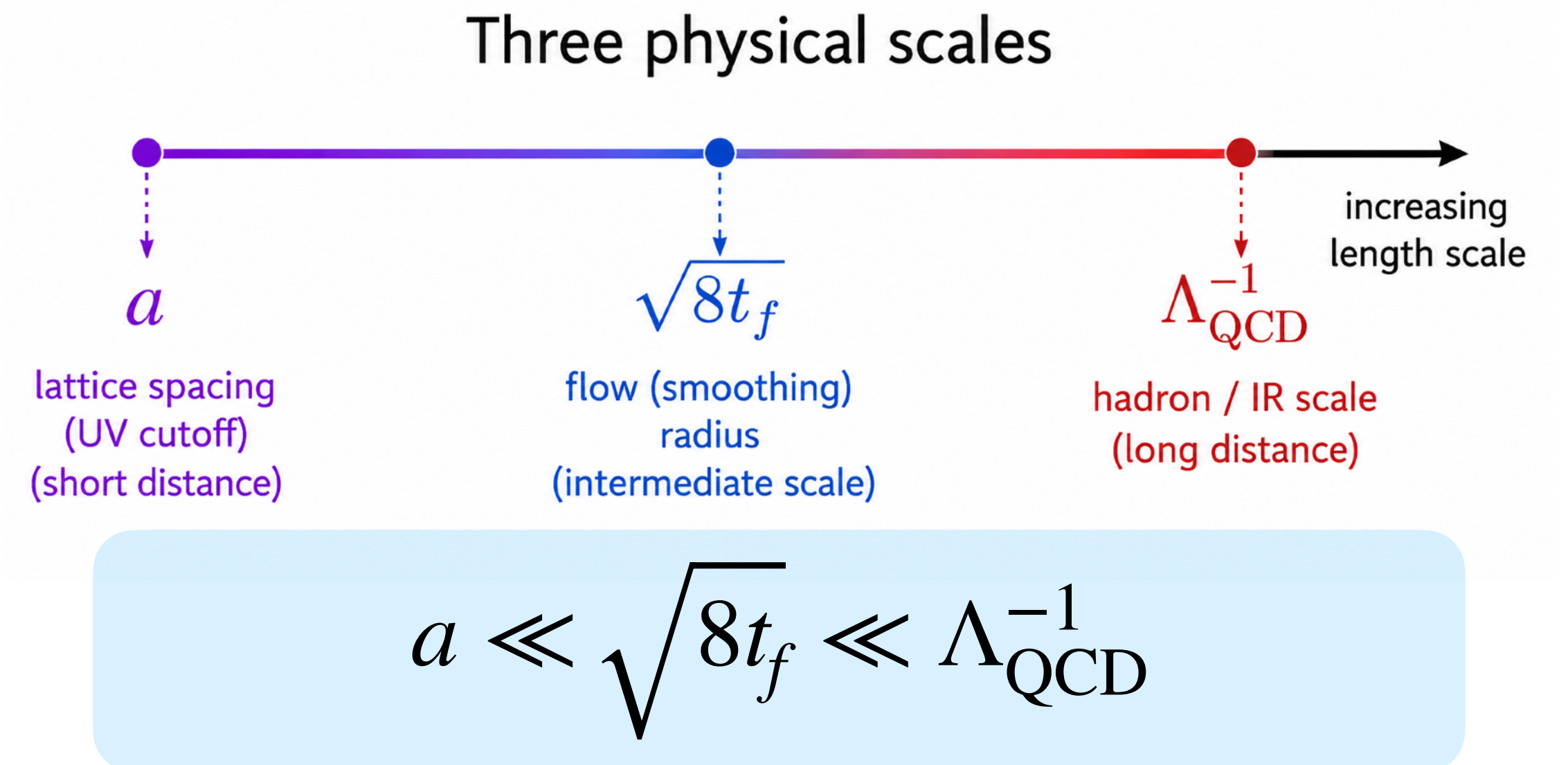
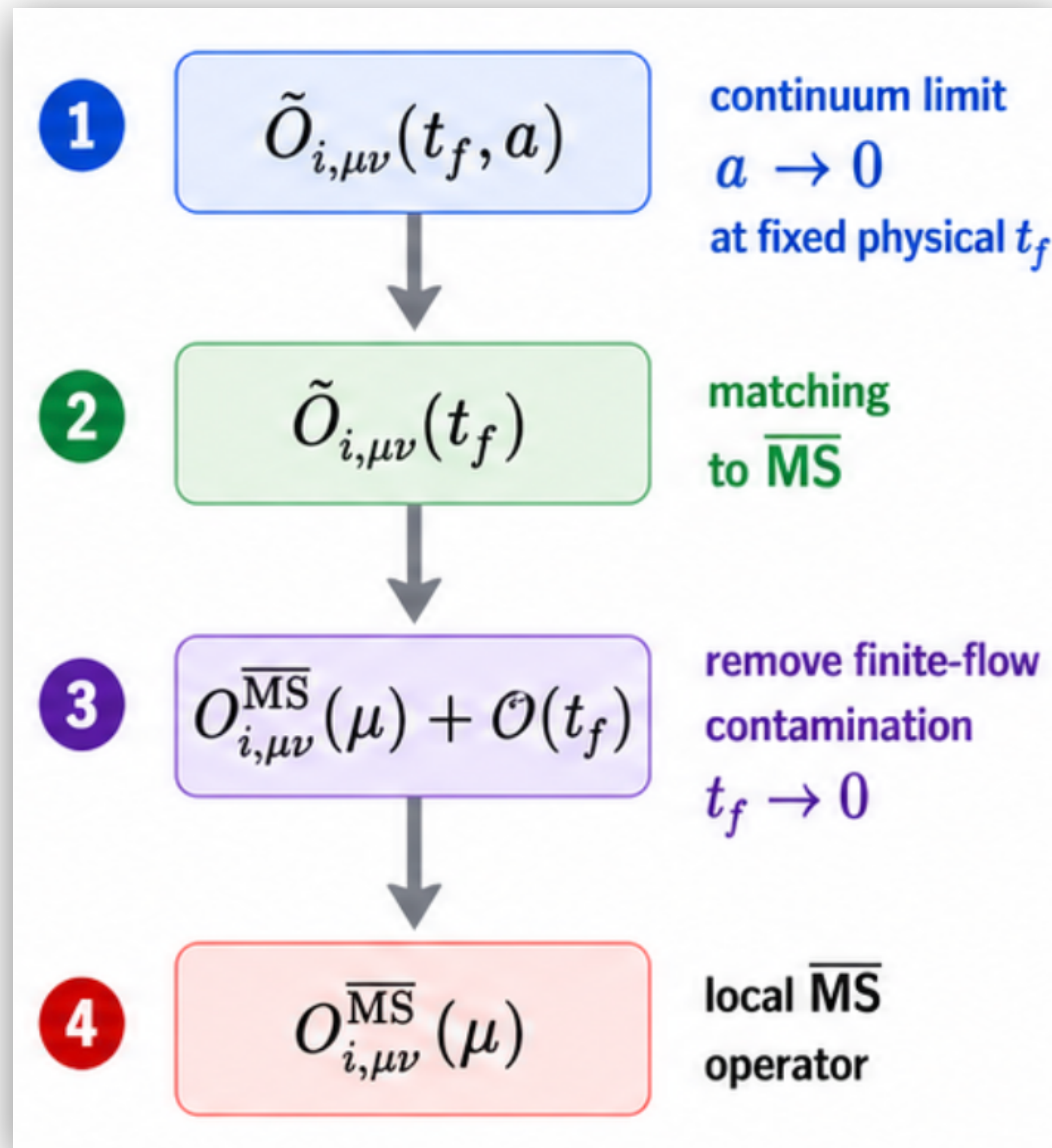
trace anomaly lives here

$$\mathcal{O}_i(t_f, x) = \sum_j C_{ij}(t_f, \mu) \mathcal{O}_j^{\overline{\text{MS}}}(\mu, x) + \mathcal{O}(t_f)$$

- **Matching** relates the flowed basis to the local renormalized EMT
- The local EMT is recovered in the  $t_f \rightarrow 0$  limit

**Small-flow-time matching and  $t_f \rightarrow 0$  recovers the local EMT**

# A controlled path to the full local EMT



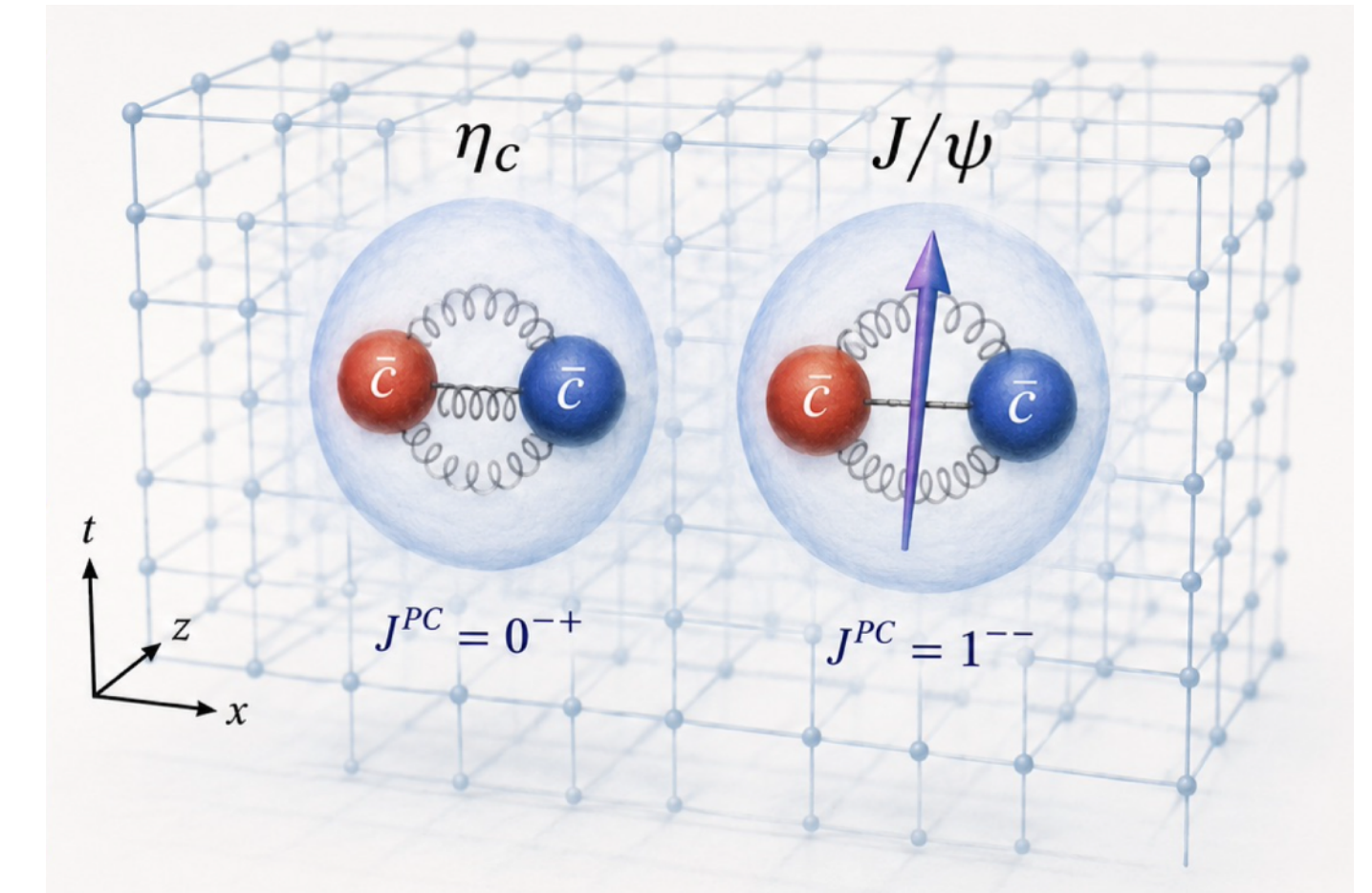
- $t_f$  too small: **cutoff** artifacts dominate ( $\sim a^2/t_f, a^2, \dots$ ).
- $t_f$  too large: **oversmearing**, loss of short-distance information.

**All quark and gluon EMT components, including the trace, are derived in one common  $\overline{\text{MS}}$  scheme and scale**

# Lattice calculation: charmonium states as a demonstration

- $\eta_c$  and  $J/\psi$  states: **three ensembles** with physical  $\eta_c$  and  $J/\psi$  meson masses; Light quark contribution ignored.

| $a$ [fm] | $N_s$ | $N_t$ | $m_s^{\text{sea}}/m_l^{\text{sea}}$ | $M_\pi^{\text{sea}}$ [MeV] | $c_{\text{sw}}$ | $am_c^{\text{val}}$ |
|----------|-------|-------|-------------------------------------|----------------------------|-----------------|---------------------|
| 0.06     | 48    | 64    | 20                                  | 160                        | 1.0336          | 0.306               |
| 0.05     | 64    |       |                                     |                            | 1.030934        | 0.236               |
| 0.04     | 64    |       |                                     |                            | 1.02868         | 0.167               |



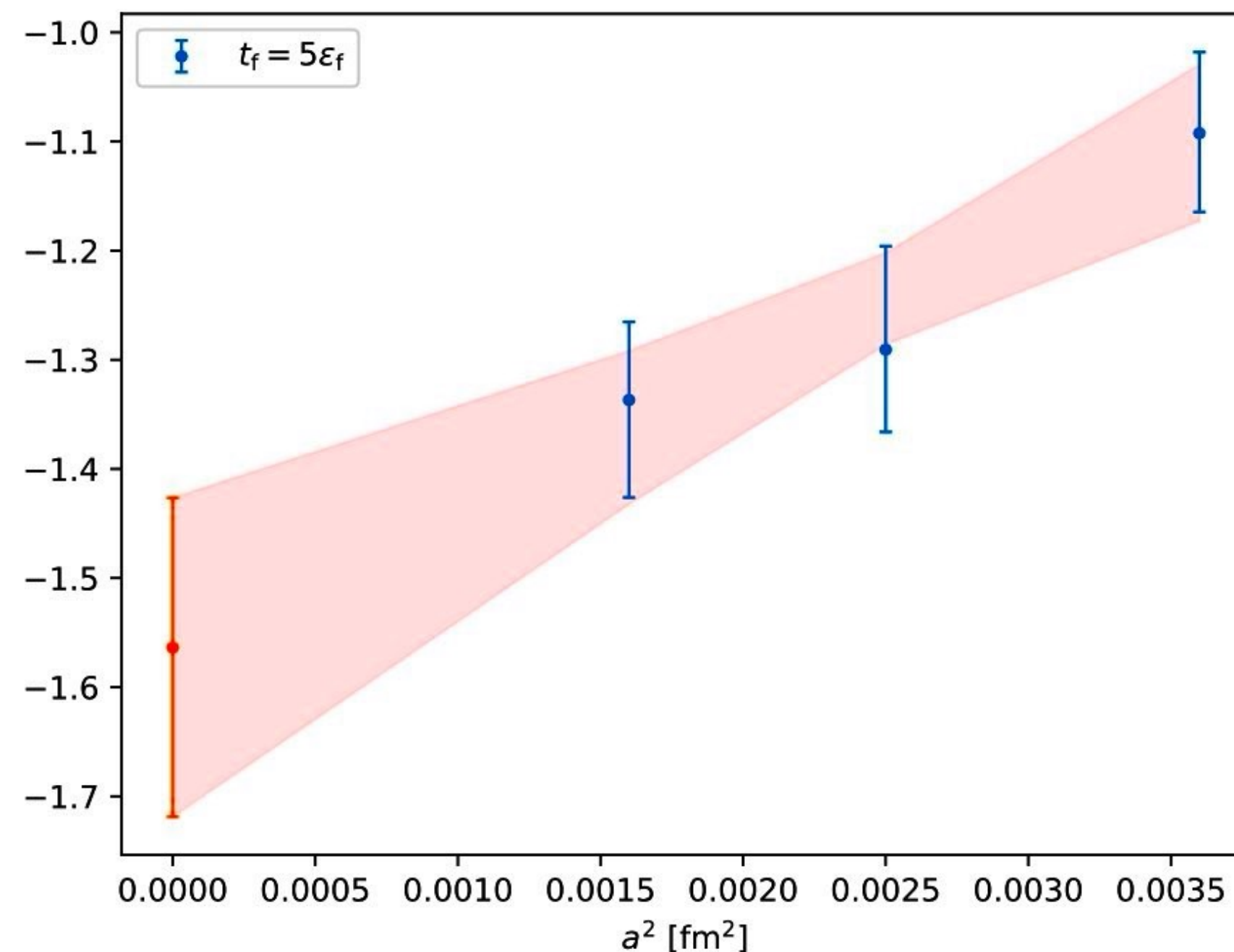
- Start with the rest frame:  **$P = 0$** :

$$\frac{1}{2M} \langle P = 0 | T_{q/g}^{\mu\nu} | P = 0 \rangle = M \begin{pmatrix} A_{q/g}(0) + \bar{C}_{q/g}(0) & 0 & 0 & 0 \\ 0 & -\bar{C}_{q/g}(0) & 0 & 0 \\ 0 & 0 & -\bar{C}_{q/g}(0) & 0 \\ 0 & 0 & 0 & -\bar{C}_{q/g}(0) \end{pmatrix}$$

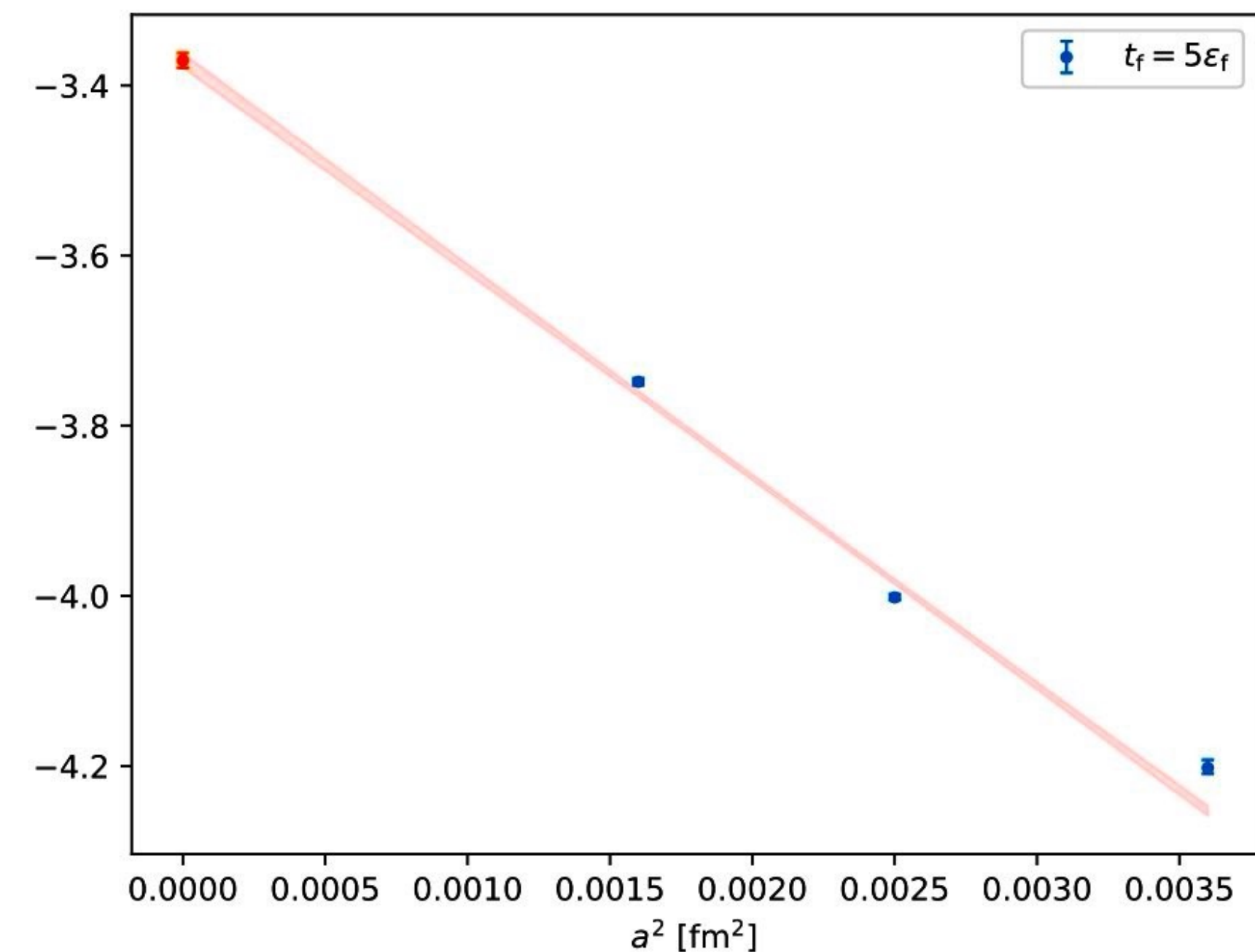
# Continuum extrapolation at fixed flow time

$$\langle \eta_c | \tilde{O}_{i,\mu\nu}(t_f, a) | \eta_c \rangle = \langle \eta_c | \tilde{O}_{i,\mu\nu}(t_f) | \eta_c \rangle + a^2 X_{i,\mu\nu}(t_f)$$

gluon EMT  $\langle \eta_c | \tilde{O}_{1,\mu\nu}(t_f) | \eta_c \rangle$



quark EMT  $\langle \eta_c | \tilde{O}_{3,\mu\nu}(t_f) | \eta_c \rangle$

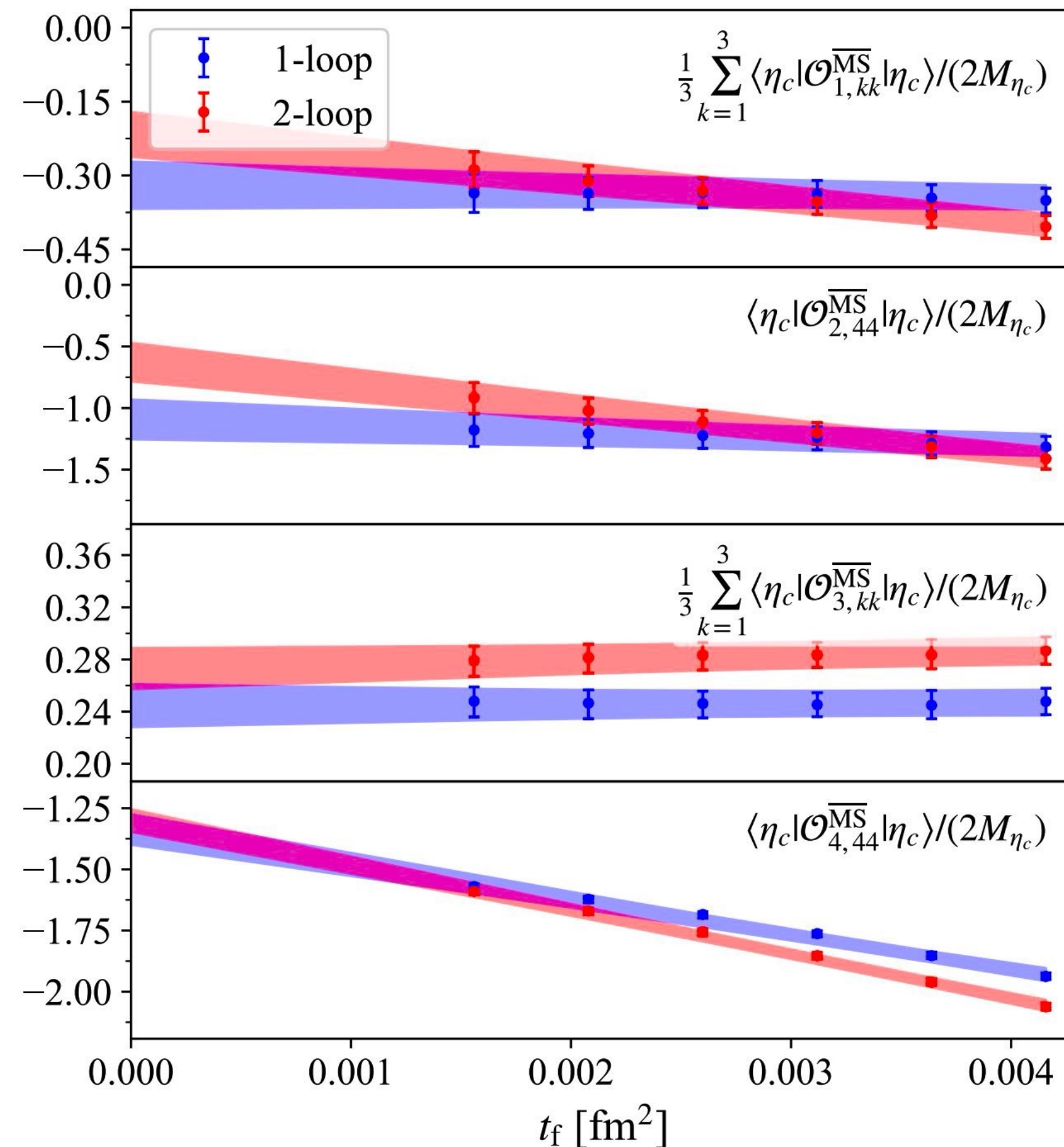


**At fixed physical  $t_f$ , the continuum extrapolation is well controlled due to UV finiteness**

# Matching to $\overline{\text{MS}}$ and extrapolation to zero flow time

$$O_{i,\mu\nu}^{\overline{\text{MS}}}(\mu) = M_{ij}^{\overline{\text{MS}} \leftarrow \text{Flow}}(\mu, t_f) \tilde{O}_{j,\mu\nu}(t_f) + \mathcal{O}(t_f \Lambda^2)$$

- H. Suzuki, PTEP 2013 (2013) 083B03
- H. Makino, H. Suzuki, PTEP 2014 (2014) 063B02
- R. Harlander, Y. Kluth, F. Lange, Eur.Phys.J.C 78 (2018) 11, 944



- One- and two-loop matching are close.
- **Small-flow-time** window is under control: linear in  $t_f$ .
- The final result is the **local  $\overline{\text{MS}}$**  EMT matrix element.

# GFFs from EMT matrix elements (rest frame $P=0$ )

- $\eta_c$  (spin-0) in forward limit:

$$\langle P | T_i^{\mu\nu} | P \rangle = 2P^\mu P^\nu A_i(0) + 2M^2 g^{\mu\nu} \bar{C}_i(0)$$

- $J/\psi$  (spin-1): denote the polarization by  $\gamma_\lambda$

$$\begin{aligned} \langle P, \gamma_\lambda | T_i^{\mu\nu} | P, \gamma_\lambda \rangle &= 2P^\mu P^\nu A_{0,i}(0) \\ &+ \left( 2\delta^{\mu\lambda} \delta^{\nu\lambda} + \frac{1}{2} g^{\mu\nu} \right) M^2 \bar{f}_i(0) \\ &- g^{\mu\nu} M^2 \bar{C}_{0,i}(0). \end{aligned}$$

|          | $\eta_c$  |              | $J/\psi$  |              |              |
|----------|-----------|--------------|-----------|--------------|--------------|
|          | $A(0)$    | $\bar{C}(0)$ | $A(0)$    | $\bar{C}(0)$ | $\bar{f}(0)$ |
| gluon(g) | 0.084(45) | 0.058(18)    | 0.077(35) | -0.146(29)   | -0.012(32)   |
| charm(c) | 0.918(17) | -0.068(4)    | 0.926(34) | 0.176(87)    | 0.023(5)     |
| total    | 1.000(46) | -0.010(18)   | 0.997(43) | 0.030(30)    | 0.010(32)    |

- $A_{q/g}$  probes the **momentum fraction**  $\langle x \rangle_{q/g}$ .
- $\bar{C}_{q/g}$  probes the **trace-related** channel.
- The quark and gluon sectors combine into the **conserved total EMT**:  
 $A_g(0) + A_q(0) \approx 1, \bar{C}_g(0) + \bar{C}_q(0) \approx 0$

# The sigma term from the QCD equation of motion

- $\sigma$  term: quantifies the **explicit quark-mass contribution** to the hadron mass and is **scale and scheme invariant**.

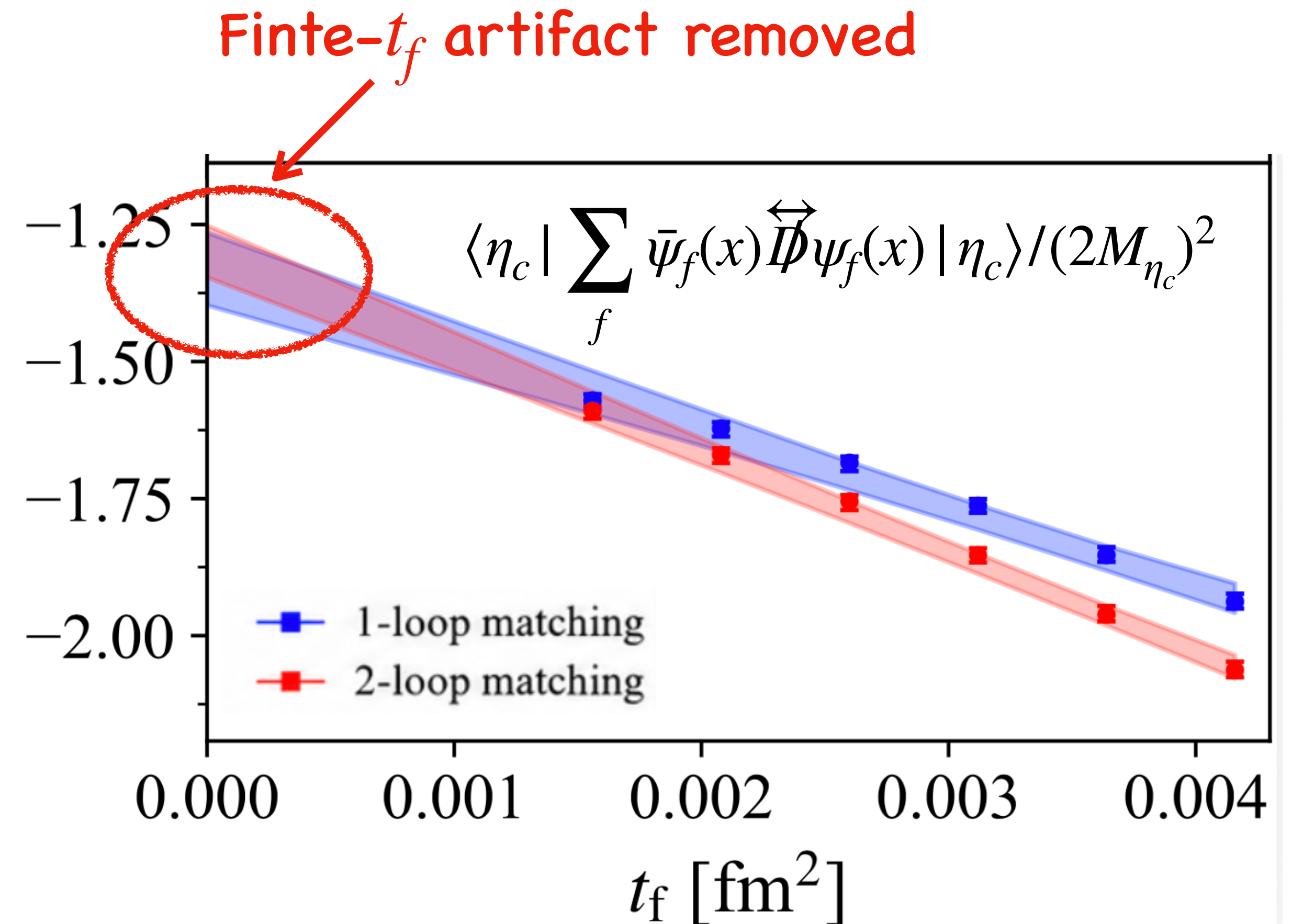
$$\sigma \equiv \frac{1}{2M} \langle P | \sum_f m_f \bar{\psi}_f(x) \psi_f(x) | P \rangle$$

- For physical-state matrix elements, the **QCD EOM** gives,

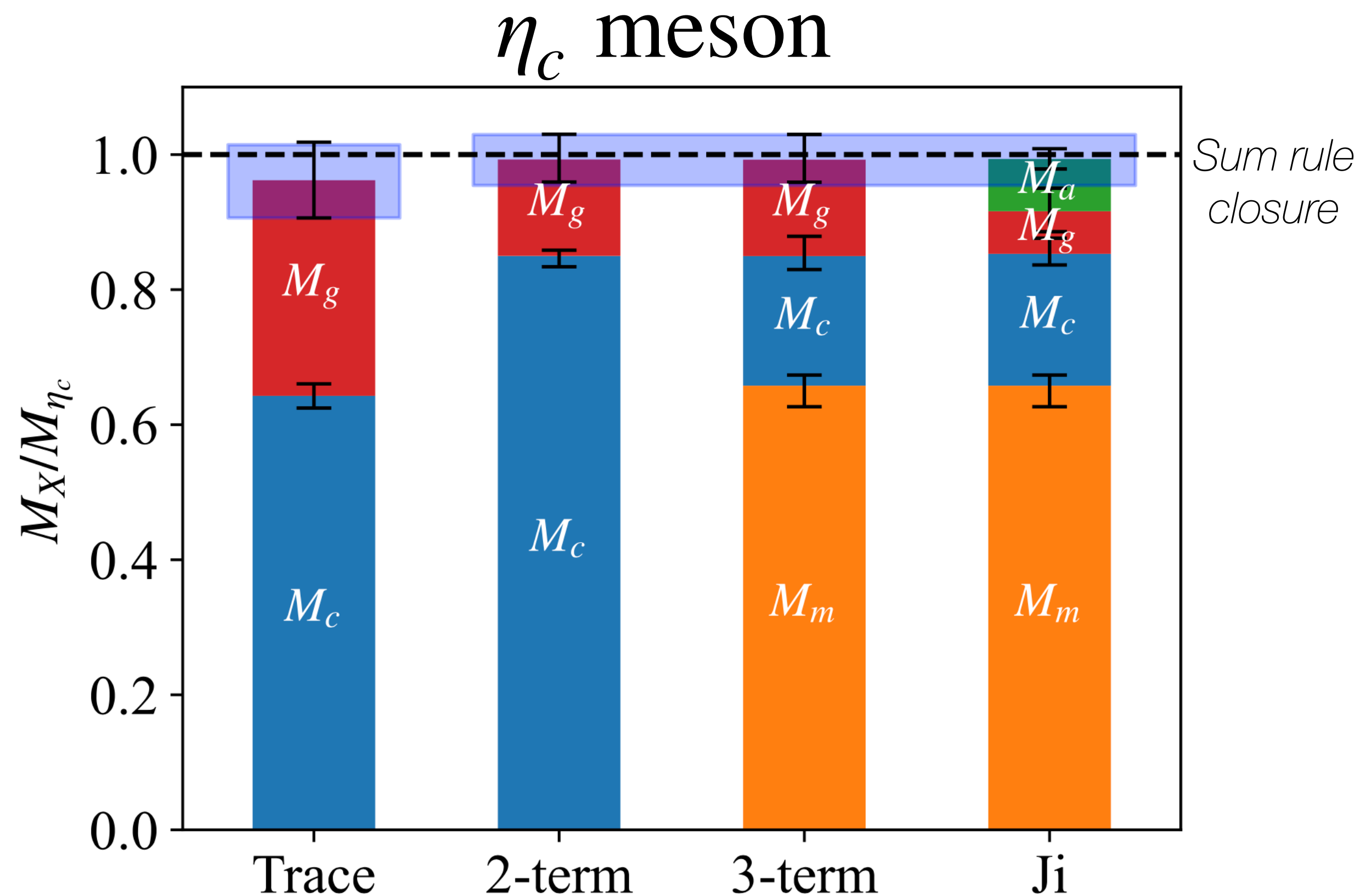
$$\sum_f \bar{\psi}_f(x) \overleftrightarrow{D} \psi_f(x) = -2 m_f \bar{\psi}_f(x) \psi_f(x)$$

$$\sigma = -\frac{1}{4M} \langle P | \sum_f \bar{\psi}_f(x) \overleftrightarrow{D} \psi_f(x) | P \rangle$$

Trace operator: power divergent  
mixing now avoided through GF



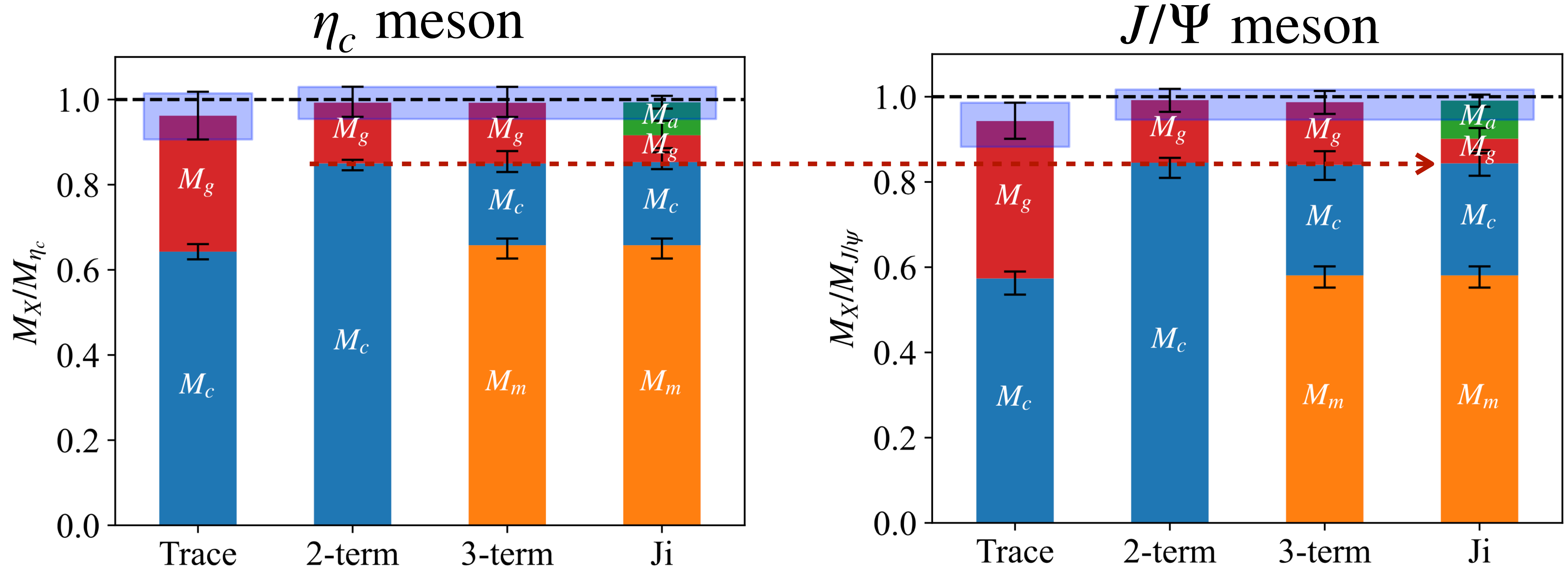
# Quark-gluon decomposition of mass and trace anomaly



All ingredients are determined in a common  $\overline{\text{MS}}$  framework.

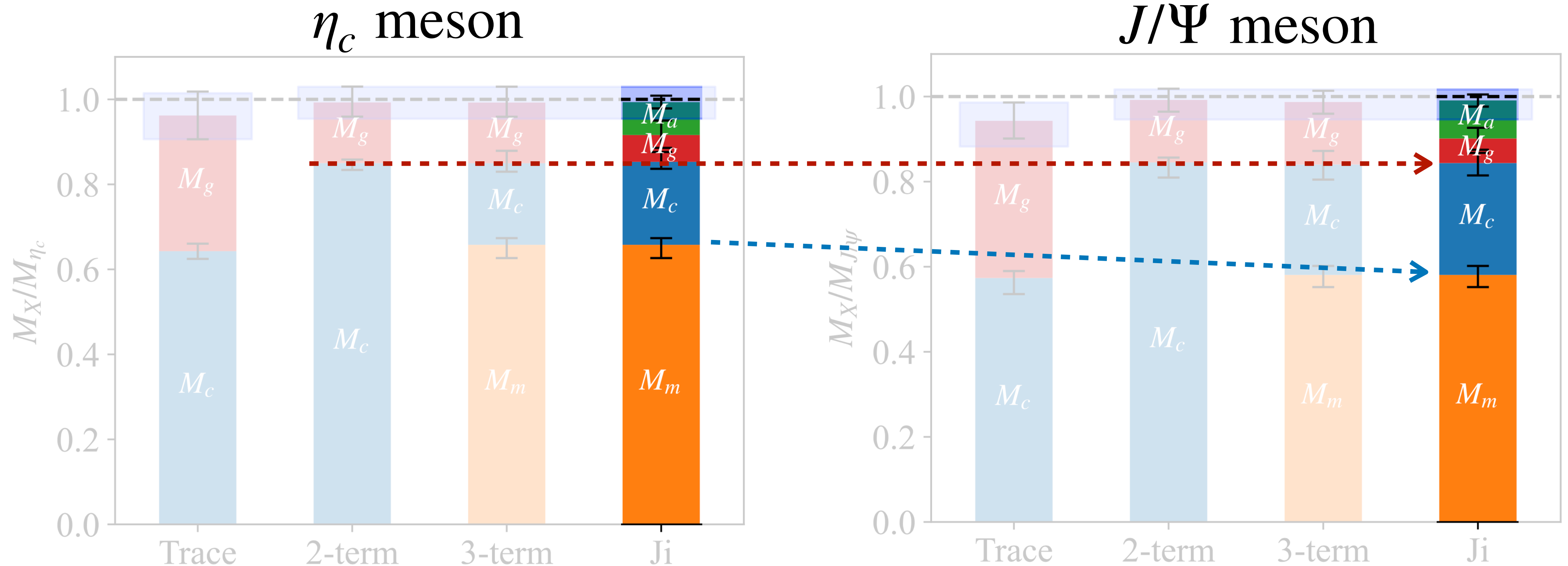
- Trace, two-term, three-term, and Ji decompositions **all close**.
- **First simultaneous lattice validation** in a common scheme.
- The **gluon** sector remains substantial even in heavy quarkonium,

# Quark-gluon decomposition of mass and trace anomaly



- **Gluon** contributions are **remarkably similar** between  $\eta_c$  and  $J/\psi$ .
- Changing heavy-quark spin primarily **reorganizes** the charm **charm energy/sigma contribution**.

# Quark-gluon decomposition of mass and trace anomaly



- **Gluon** contributions are **remarkably similar** between  $\eta_c$  and  $J/\psi$ .
- Changing heavy-quark spin primarily **reorganizes** the charm **charm energy/sigma contribution**.

# Summary

- **Full EMT beyond the twist-2 sector.** Connecting hadron mass, gravitational form factors, and scale-symmetry breaking requires both the traceless and scalar-trace sectors.
- **A controlled lattice determination via gradient flow.** UV-finite operators, continuum extrapolation, and matching to  $\overline{\text{MS}}$  enable renormalized quark and gluon matrix elements, including the scalar trace.
- **First simultaneous validation of mass and trace-anomaly sum rules.**  
Multiple sum rules are consistently satisfied for  $\eta_c$  and  $J/\psi$ , revealing sizable gluonic contributions and state dependence dominated by the charm sector.

**A general first-principles framework for resolving the QCD operator structure of hadron mass, with applications to other hadrons and nuclei.**

**Thanks for your attention!**