

Evolution of Dihadron fragmentation function with nonperturbative behavior

[arXiv:2610.20448v1](https://arxiv.org/abs/2610.20448v1)

Kazuki Makino

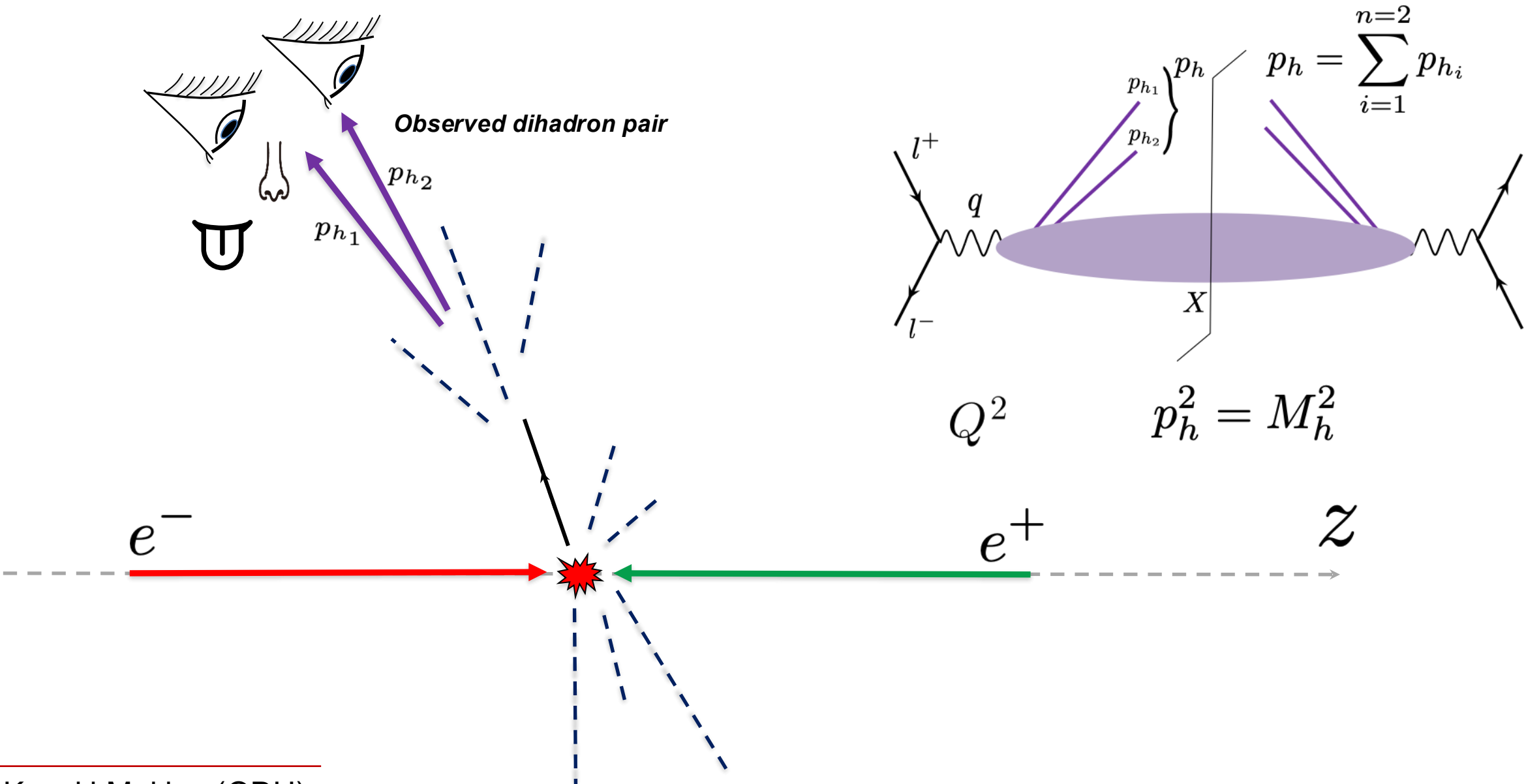
Old Dominion University

In collaboration with Ted Rogers



Oct 22, 2026

Cross section with two hadrons

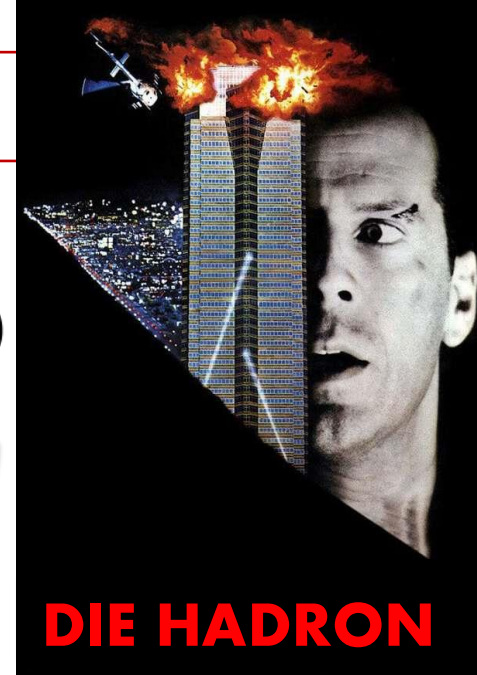
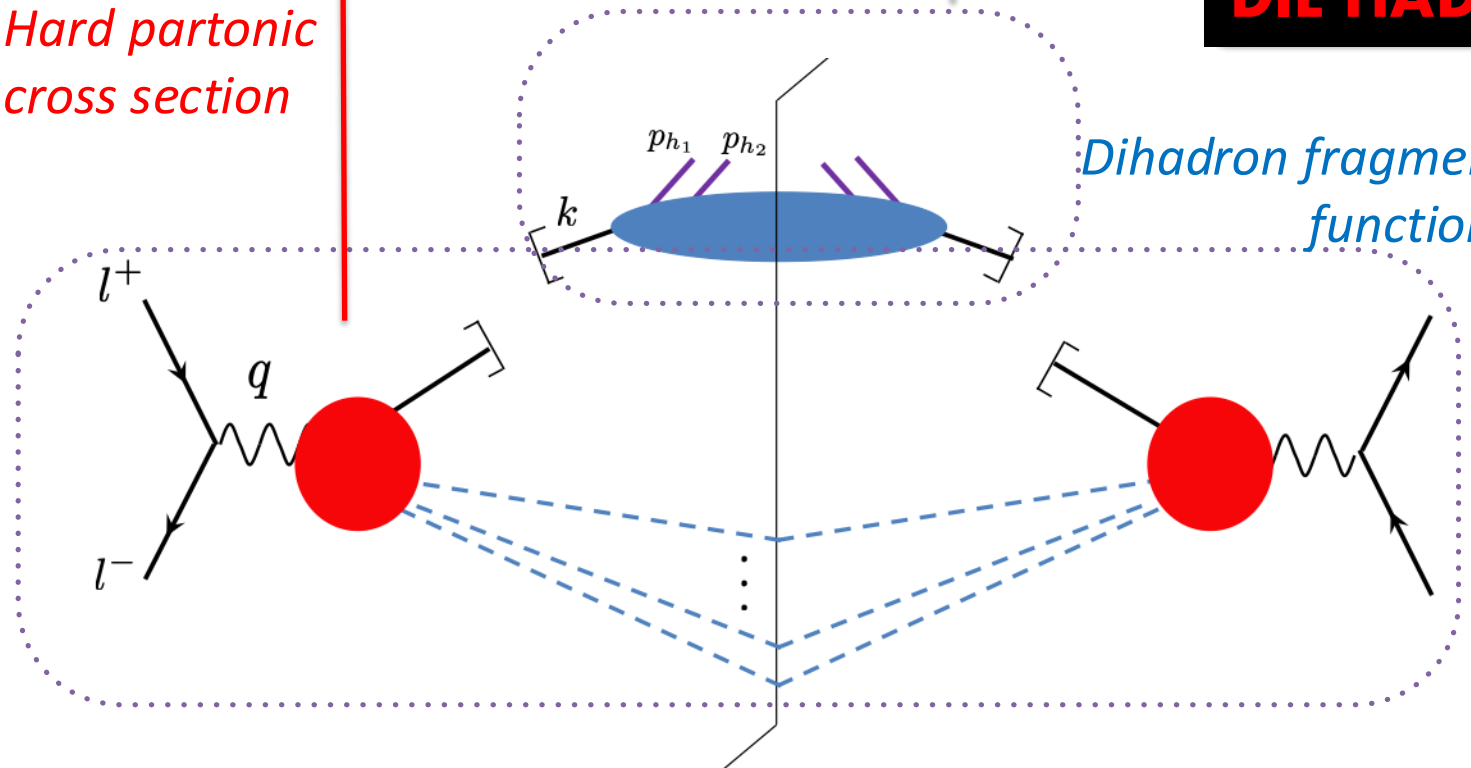


Di-hadron Fragmentation functions

$$\frac{d\sigma}{dY} = \sum_j \int_z^1 \frac{d\xi}{\xi^2} \left(2E_{\hat{k}} (2\pi)^3 \frac{d\hat{\sigma}}{d^3\hat{\mathbf{k}}} \Big|^{l^+ l^- \rightarrow j+X} \right) d_{h_1 h_2 / j}(\xi, p_{h_1}, p_{h_2})$$

Hard partonic cross section

Dihadron fragmentation function



Di-hadron Fragmentation functions

$$\frac{d\sigma}{dY} = \sum_j \int_z^1 \frac{d\xi}{\xi^2} \left(2E_{\hat{k}} (2\pi)^3 \frac{d\hat{\sigma}}{d^3\hat{k}} \Big|^{l^+ l^- \rightarrow j+X} \right) d_{h_1 h_2 / j}(\xi, p_{h_1}, p_{h_2})$$

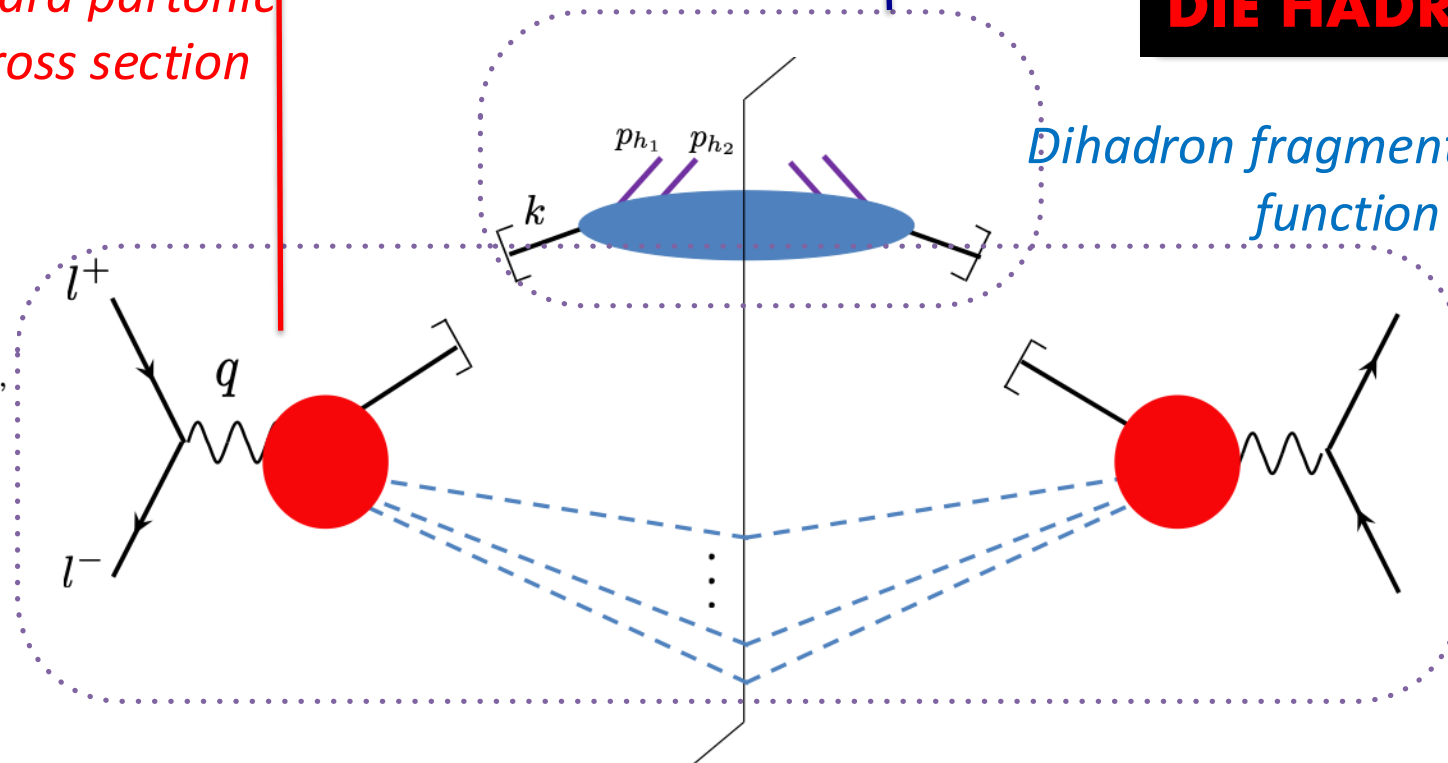
Same collinear factorization as FFs

$$dY \equiv \frac{d^3\mathbf{p}_{h_1}}{2E_{p_{h_1}} (2\pi)^3} \frac{d^3\mathbf{p}_{h_2}}{2E_{p_{h_2}} (2\pi)^3}$$

For subtleties with kinematics
See

Hard partonic
cross section

Dihadron fragmentation
function



QCD factorization with multihadron fragmentation functions

Comment on “QCD factorization with multihadron fragmentation functions”

Reply to “Comment on ‘QCD factorization with multihadron fragmentation functions’ ”



Reply to comment...



Phys.Rev.D 111 (2025) 5, 056001
Phys.Rev.D 113 (2026) 3, 038901
Phys.Rev.D 113 (2026) 3, 038902

Dihadron Fragmentation functions (DiFFs)

$$\frac{d\sigma}{dY} = \sum_j \int_z^1 \frac{d\xi}{\xi^2} \left(2E_{\hat{k}} (2\pi)^3 \frac{d\hat{\sigma}}{d^3\hat{\mathbf{k}}} \Big|^{l^+ l^- \rightarrow j+X} \right) d_{h_1 h_2 / j}(\xi, p_{h_1}, p_{h_2})$$

Same collinear factorization as FFs

For subtleties with kinematics



Reply to comment...



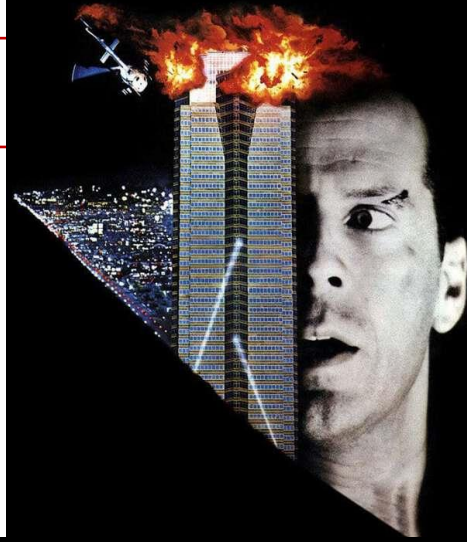
Reply to "Comment on 'QCD factorization with multihadron fragmentation functions' "

Comment on "QCD factorization with multihadron fragmentation functions"

QCD factorization with multihadron fragmentation functions

Phys.Rev.D 111 (2025) 5, 056001
Phys.Rev.D 113 (2026) 3, 038901
Phys.Rev.D 113 (2026) 3, 038902

Kazuki Makino (ODU)



MULTI HADRON

The following analysis applies
equally to multi-hadron
fragmentation functions

DiFFs regions (All collinear DiFFs)

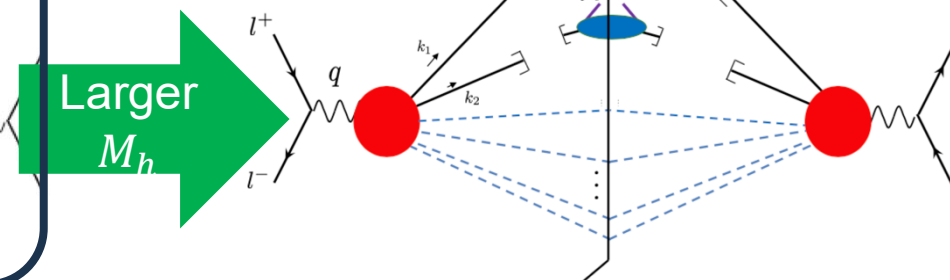
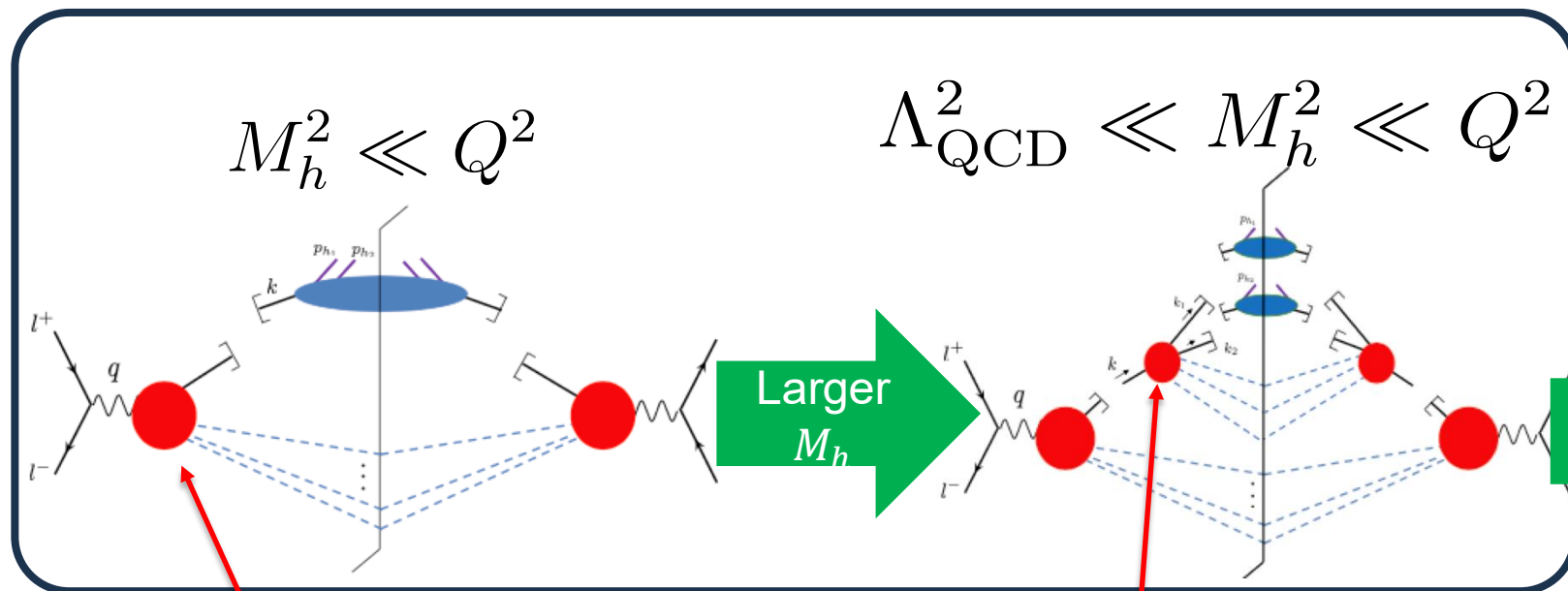
DiFFs

No DiFFs

$$M_h^2 \ll Q^2$$

$$\Lambda_{\text{QCD}}^2 \ll M_h^2 \ll Q^2$$

$$M_h^2 \rightarrow Q^2$$



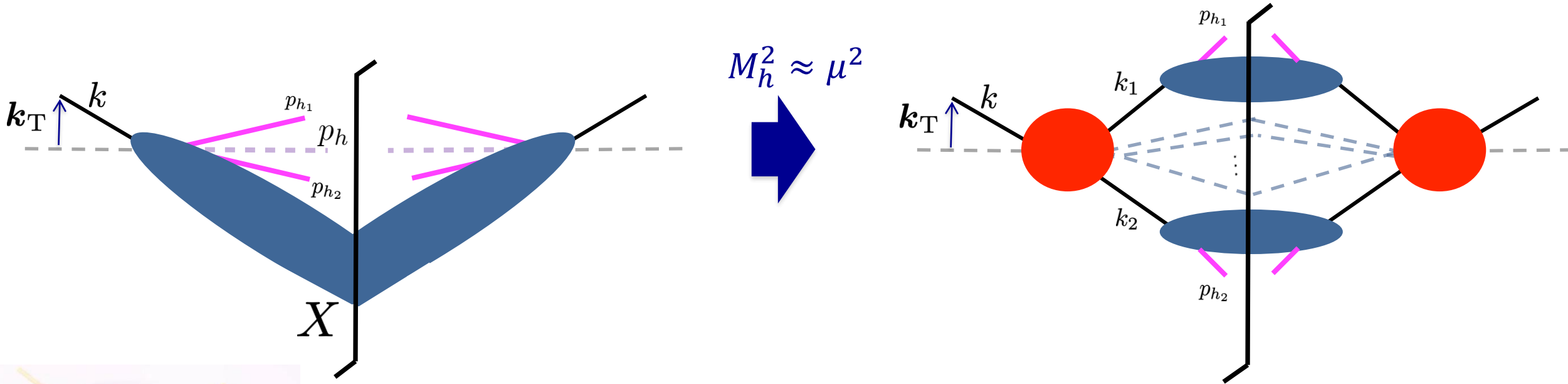
$$\sigma \sim \hat{\sigma} \otimes d_{\text{DiFF}}$$

$$\sigma = \hat{\sigma} \otimes \boxed{\hat{\sigma}' \otimes (d_1 \otimes d_2)} \\ d_{\text{DiFF}}$$

$$\sigma \sim \hat{\sigma} \otimes d_1 \otimes d_2$$

DiFFs at large M_h^2

At large M_h^2 , DiFFs is *perturbatively* describable with single FFs



$$= \int \frac{d\xi_a}{\xi_a^2} \int \frac{d\xi_b}{\xi_b^2} \frac{1}{M_h^2} C_{j/i'j'}^{(n)}(\xi_a, \xi_b, \xi, M_h^2; \mu) d_{h1/j'}(\xi_a; \mu) d_{h2/i'}(\xi_b; \mu)$$

Refactorization

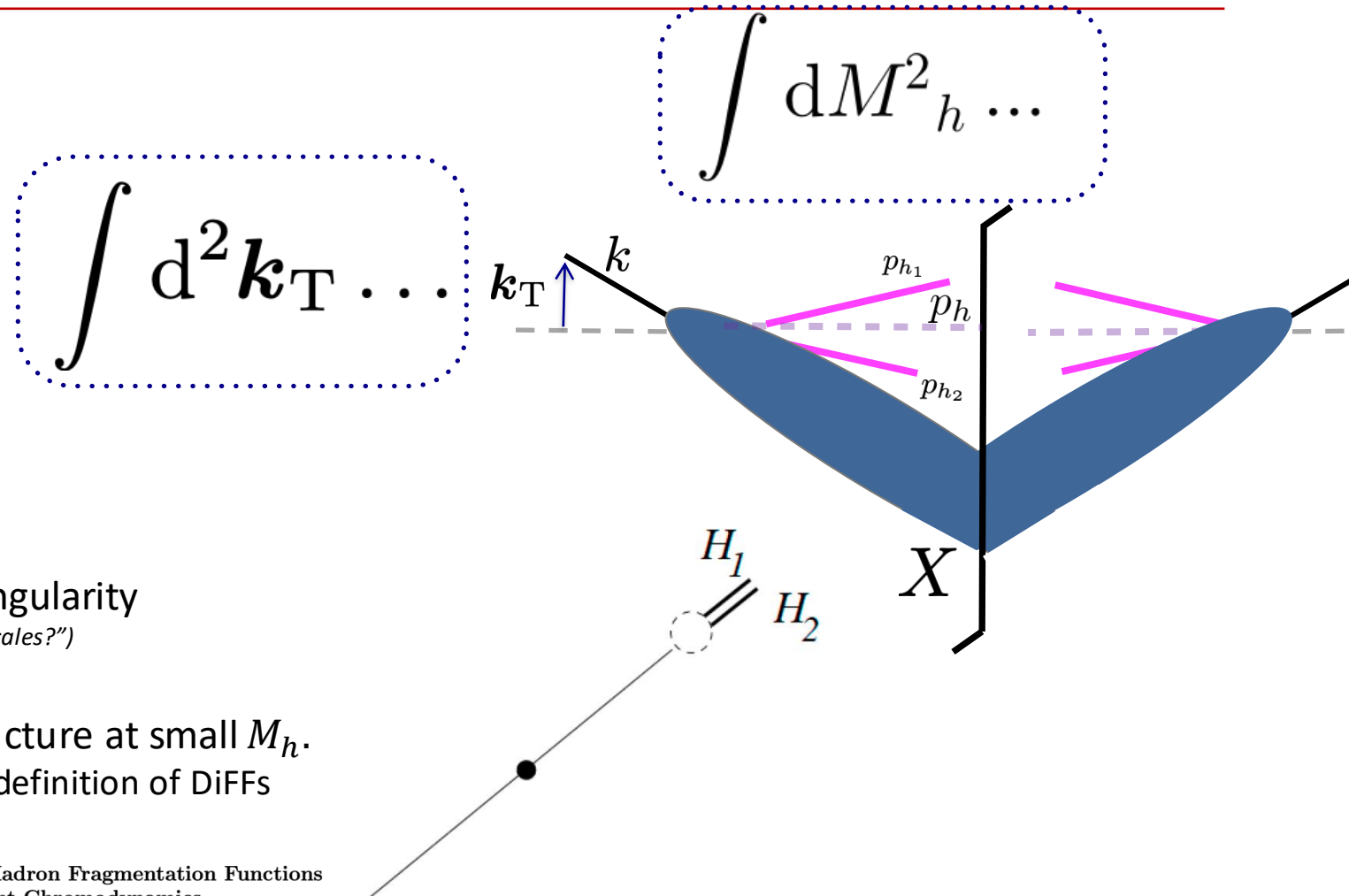


Two scales M_h & μ

- Goal is to obtain nonperturbative structure at small M_h
- Need to separate two scale dependences M_h & μ

- DiFFs has been defined to absorb collinear singularity
(“What are the consequence of independent factorization and renormalization scales?”)
Rogers, Whitehill arxiv.2608.01489

- But the goal is to obtain nonperturbative structure at small M_h .
- Careful treatment is especially needed for fully NP definition of DiFFs



Quantum Computing Hadron Fragmentation Functions in Light-Front Chromodynamics

Juan José Gálvez-Viruet, Felipe J. Llanes-Estrada, and Nicolás Martínez de Arellano
Depto. Física Teórica & IPARCOS, Univ. Complutense de Madrid, 28040 Madrid, Spain

María Gómez-Rocha
Depto. de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional,
Universidad de Granada, 18071 Granada, Spain

T. J. Hobbs
High Energy Physics Division, Argonne National Laboratory, Lemont, IL 60439, USA
(Dated: October 22, 2025)

$\int d^2 \mathbf{k}_T \dots$ Divergence in DiFFs

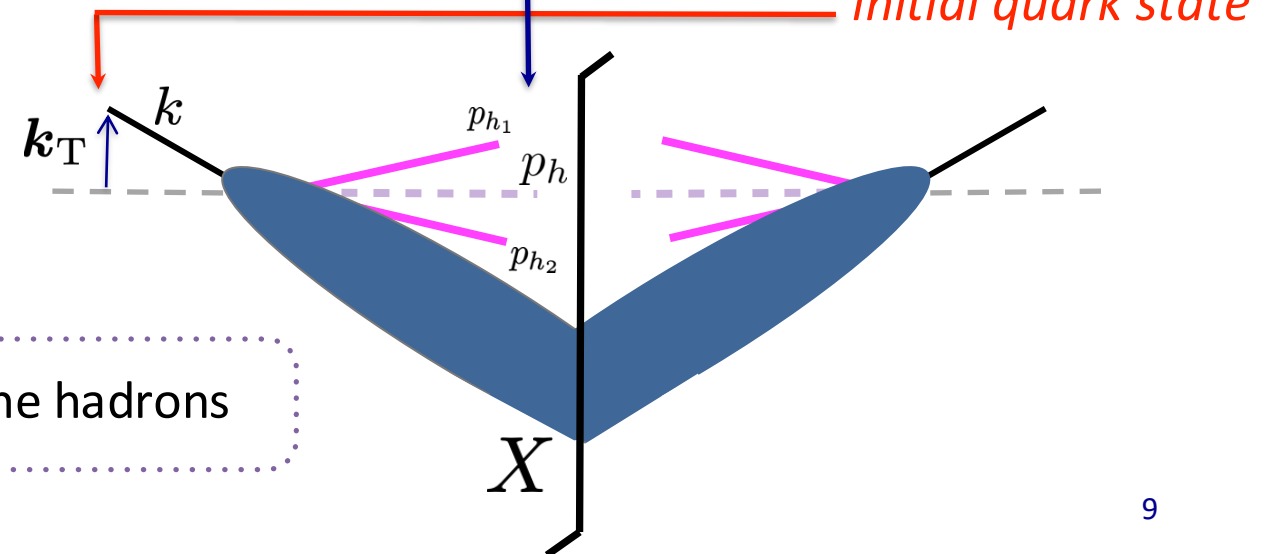
Before renormalization

$$d_{h_1 h_2 / j, (0)}(\xi, \underline{p_{h_1}, p_{h_2}})$$

$$= \frac{1}{N_c} \text{Tr}_C \text{Tr}_D \frac{\xi^{1-2\epsilon}}{4} \sum_X \int \frac{dx^-}{2\pi} e^{ix^- p_{h,H}^+ / \xi} \underbrace{\langle 0 | \gamma^+ \psi_{0,j}(x/2) | p_{h_1}, p_{h_2}, X \rangle}_{\text{final hadronic state}} \underbrace{\langle p_{h_1}, p_{h_2}, X | \bar{\psi}_{0,j}(-x/2) | 0 \rangle}_{\text{initial quark state}}$$

$$\int d^2 \mathbf{k}_T \dots$$

Ultraviolet Divergent



UV div associated to quark fields is independent of the hadrons

Renormalization for DiFFs

After renormalization

$$d_{h_1 h_2 / j}(\xi, \underline{p_{h_1}, p_{h_2}}; \mu) = \sum_{j'} \int_{\xi}^1 \frac{d\xi'}{\xi'} L_{j'j}(\xi') d_{(0), h_1 h_2 / j'}(\xi / \xi', \underline{p_{h_1}, p_{h_2}}) = [L_{j'j} \otimes d_{(0), h_1 h_2 / j'}](\xi, \underline{p_{h_1}, p_{h_2}}; \mu)$$



RG invariance

$$\frac{d}{d \ln \mu^2} d_{(0), h_1 h_2 / j'}(\xi, \underline{p_{h_1}, p_{h_2}}) = 0$$



DGLAP equation

$$\frac{d}{d \ln \mu^2} d_{h_1 h_2 / j}(\xi, \underline{p_{h_1}, p_{h_2}}; \mu) = [P_{j'j} \otimes d_{h_1 h_2 / j'}](\xi, \underline{p_{h_1}, p_{h_2}}; \mu)$$

UV div associated to quark fields is independent of the hadrons

Divergence in DiFFs

UV divergence in M_h^2 must be regulated by cutoff
→ complicates probability interpretations

$$d_{h_1 h_2 / j}^{\text{norm}}(\xi; \mu) = \int_{4m_\pi^2}^{\infty} dM_h^2 d_{h_1 h_2 / j}(\xi, M_h^2; \mu) \quad \Rightarrow \quad d_{h_1 h_2 / j}^{\text{norm}}(\xi; \underline{M_{\text{max}}^2}; \mu) = \int_{4m_\pi^2}^{M_{\text{max}}^2} dM_h^2 d_{h_1 h_2 / j}(\xi, M_h^2; \mu)$$

UV Div!



M_h^2 - dependence of DiFF (Mellin space)

- M_h^2 and μ dependence are independent:

$$\left[\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}(N, M_h^2; \mu) \right] = \underbrace{L_{j'j}(N)}_{\text{No } M_h^2 \text{ dependence}} \left[\frac{\partial}{\partial \ln M_h^2} d_{(0), h_1 h_2 / j'}(N, M_h^2) \right]$$

$$\frac{d}{d \ln \mu^2} \left[\frac{\partial}{\partial \ln M_h^2} d_{(0), h_1 h_2 / j'}(N, M_h^2) \right] = 0$$

$$\frac{d}{d \ln \mu^2} \left[\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}(N, M_h^2; \mu) \right] = \underbrace{P_{j'j}(N)}_{\text{No } M_h^2 \text{ dependence}} \left[\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j'}(N, M_h^2; \mu) \right]$$

$\rightarrow \frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}(N, M_h^2; \mu)$ & $d_{h_1 h_2 / j}(N, M_h^2; \mu)$ follows DGLAP equation

M_h^2 - dependence of DiFF

- M_h^2 and μ dependence are independent:

$$\left[\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}(N, M_h^2; \mu) \right] = \underbrace{L_{j'j}(N)}_{\text{No } M_h^2 \text{ dependence}} \left[\frac{\partial}{\partial \ln M_h^2} d_{(0), h_1 h_2 / j'}(N, M_h^2) \right]$$

$$\frac{d}{d \ln \mu^2} \left[\frac{\partial}{\partial \ln M_h^2} d_{(0), h_1 h_2 / j'}(N, M_h^2) \right] = 0$$

$$\frac{d}{d \ln \mu^2} \left[\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}(N, M_h^2; \mu) \right] = P_{j'j}(N) \left[\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j'}(N, M_h^2; \mu) \right]$$

- Equation in M_h^2

➡

$$\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}(N, M_h^2; \mu) = K_{j'/j}(N, M_h^2) d_{h_1 h_2 / j'}(N, M_h^2; \mu)$$

Separate the different scale dependences

$$\frac{d}{d \ln \mu^2} d_{h_1 h_2 / j}(\xi, M_h^2; \mu) = \sum_{j'} \int_{\xi}^1 \frac{d\xi'}{\xi'} d_{h_1 h_2 / j'}(\xi', M_h^2; \mu) P_{j'j}(\xi/\xi', \mu)$$

No M_h^2 dependence

UV divergence associated to quark fields

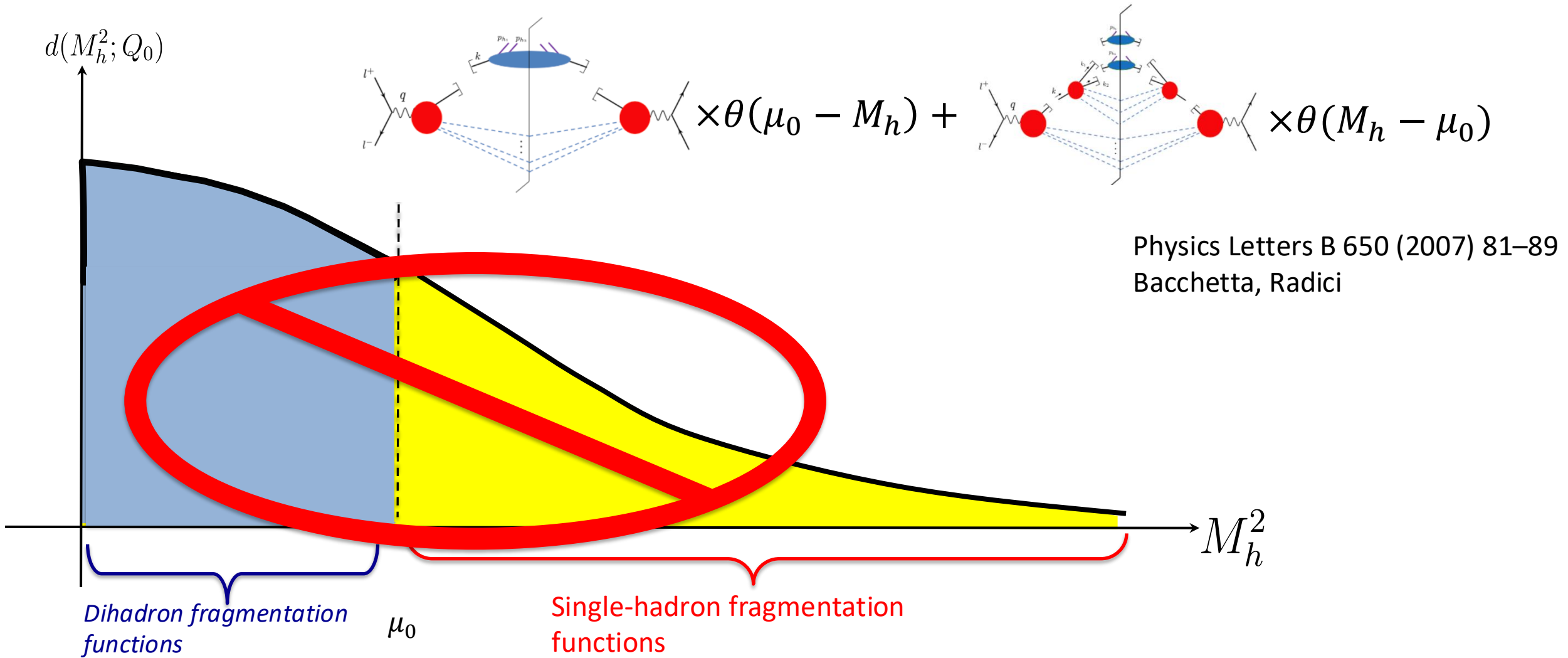
$$\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}(\xi, M_h^2; \mu) = \sum_{j'} \int_{\xi}^1 \frac{d\xi'}{\xi'} d_{h_1 h_2 / j'}(\xi'; M_h^2; \mu) K_{j'/j}(\xi/\xi', M_h^2)$$

No μ^2 dependence

*Depends on hadron identity,
(can be nonperturbative)*

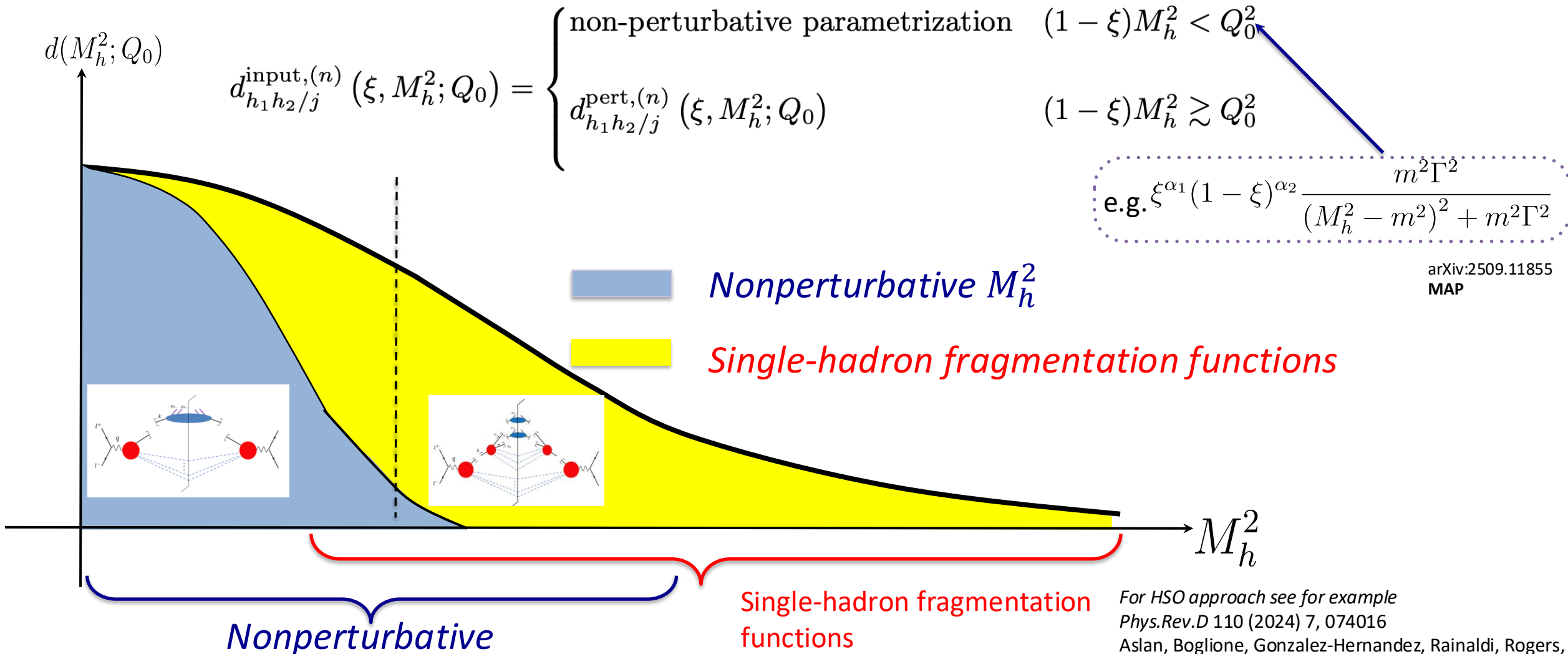
Pheno strategy

Sharp cutoff



Pheno strategy

1. Start by describing an input scale DiFF for $\mu = Q_0$



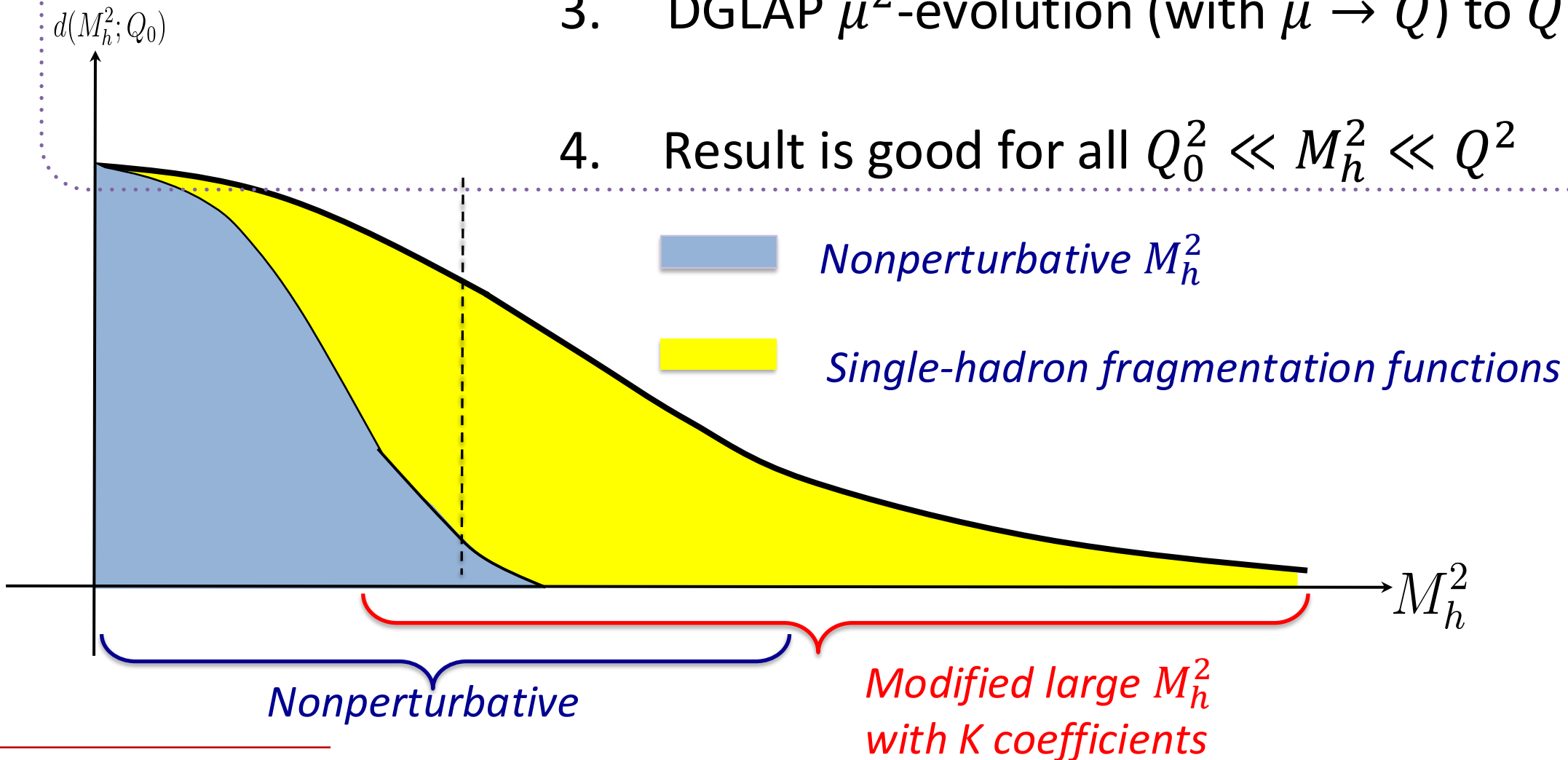
For HSO approach see for example
 Phys.Rev.D 110 (2024) 7, 074016
 Aslan, Boglione, Gonzalez-Hernandez, Rainaldi, Rogers,
 Simonelli

Pheno strategy

2. Use M_h^2 - evolution to transition to a good description for all $M_h^2 \gg Q_0^2$

3. DGLAP μ^2 -evolution (with $\mu \rightarrow Q$) to $Q \gg Q_0$

4. Result is good for all $Q_0^2 \ll M_h^2 \ll Q^2$



Perturbative tail at $M_h^2 \gg Q_0^2$

1. Calculate K from d^{pert} : Perturbatively calculable DiFFs⁽ⁿ⁾

Calculate



$$\frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}^{\text{pert}, (n)}(\xi, M_h^2; \mu) \Big|_{\underline{\mu \rightarrow M_h}} = \left(\sum_i \int_{\xi}^1 \frac{d\xi'}{\xi'} d_{h_1 h_2 / i}^{\text{pert}, (n)}(\xi'; M_h^2; \mu) K_{i/j}^{\text{pert}, (n)}(\xi/\xi', M_h^2) \right) \Big|_{\underline{\mu \rightarrow M_h}}$$

$\mu \rightarrow M_h$

Perturbative tail at $M_h^2 \gg Q_0^2$

1. Calculate K from d^{pert} : Perturbatively calculable DiFFs⁽ⁿ⁾

$$\mu \rightarrow M_h$$

$$\left. \frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}^{pert, (n)}(\xi, M_h^2; \mu) \right|_{\mu \rightarrow M_h} = \left(\sum_i \int_{\xi}^1 \frac{d\xi'}{\xi'} d_{h_1 h_2 / i}^{pert, (n)}(\xi'; M_h^2; \mu) K_{i/j}^{pert, (n)}(\xi/\xi', M_h^2) \right) \Big|_{\mu \rightarrow M_h}$$

Calculate



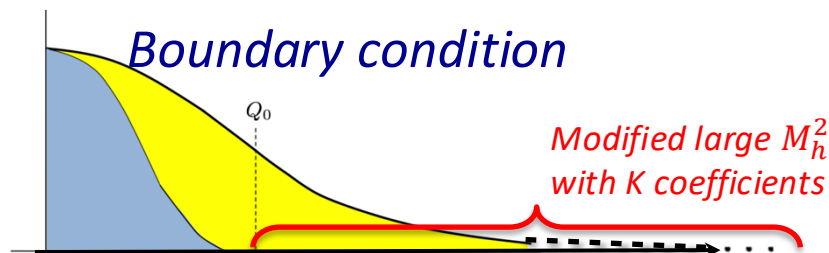
2. Use K to solve for d^{evol} : M_h^2 -evolved DiFFs⁽ⁿ⁾

Solve

$$\mu \rightarrow Q_0$$

$$\left. \frac{\partial}{\partial \ln M_h^2} d_{h_1 h_2 / j}^{evol, (n)}(\xi, M_h^2; \mu) \right|_{\mu \rightarrow Q_0} = \left(\sum_i \int_{\xi}^1 \frac{d\xi'}{\xi'} d_{h_1 h_2 / i}^{evol, (n)}(\xi'; M_h^2; \mu) K_{i/j}^{pert, (n)}(\xi/\xi', M_h^2) \right) \Big|_{\mu \rightarrow Q_0}$$

$$d_{h_1 h_2 / i}^{evol, (n)}(\xi, M_h^2 = Q_0^2; Q_0) = d_{h_1 h_2 / i}^{pert, (n)}(\xi, M_h^2; M_h)$$



Merging $M_h^2 \lesssim Q_0^2$ with $M_h^2 \gg Q_0^2$

Matched asymptotes

$$d(M_h^2; Q_0)^{d_{h_1 h_2/j}}(\xi, M_h^2; Q_0) = d_{h_1 h_2/j}^{\text{input},(n)}(\xi, M_h^2; Q_0) + \underbrace{\left(d_{h_1 h_2/j}^{\text{evol},(n)}(\xi, M_h^2; Q_0) - d_{h_1 h_2/j}^{\text{pert},(n)}(\xi, M_h^2; Q_0) \right)}_{\text{Overall error}} \Xi(M_h^2, \xi)$$

Overall error

$$\sim \alpha_s(Q_0)^{n+1} \ln \frac{M_h^2}{Q_0^2}$$

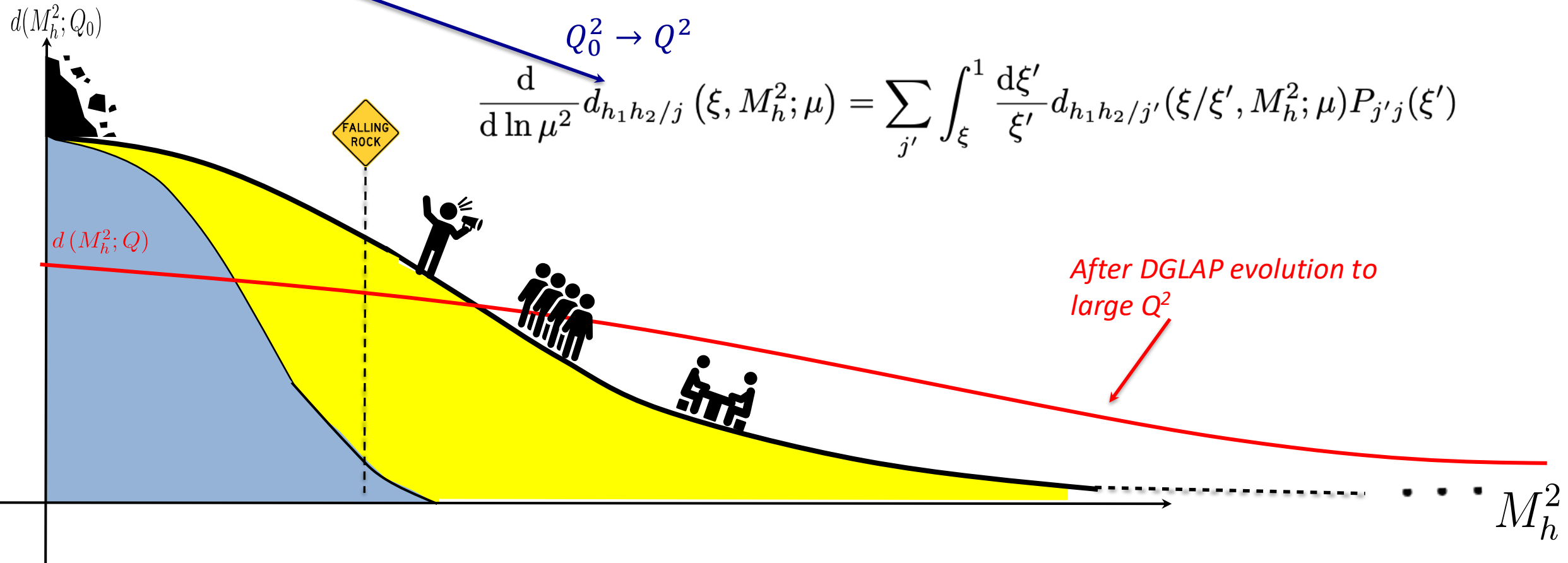
Modified $M_h^2 \gg Q_0^2$

Modified large M_h^2
with K coefficients

Nonperturbative

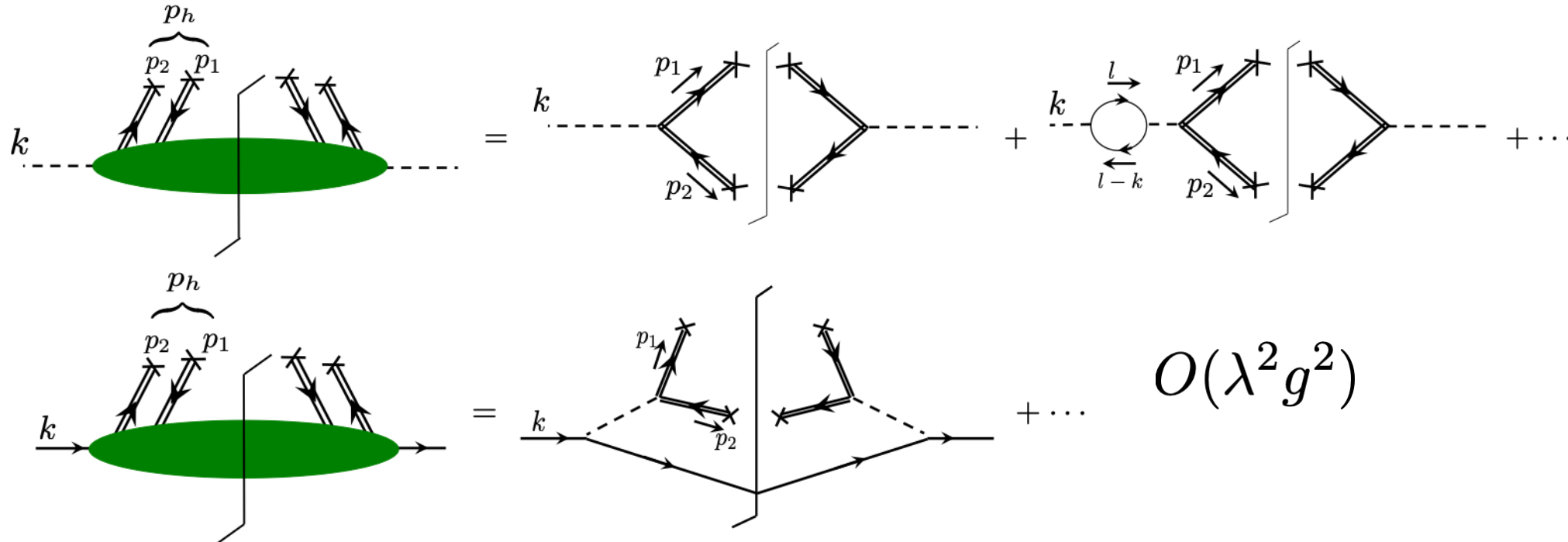
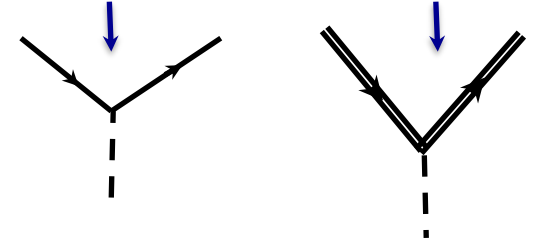
DGLAP evolution

$$d_{h_1 h_2 / j}(\xi, M_h^2; Q_0) = d_{h_1 h_2 / j}^{\text{input},(n)}(\xi, M_h^2; Q_0) + \left(d_{h_1 h_2 / j}^{\text{evol},(n)}(\xi, M_h^2; Q_0) - d_{h_1 h_2 / j}^{\text{pert},(n)}(\xi, M_h^2; Q_0) \right) \Xi(M_h^2, \xi)$$



Example in Yukawa theory

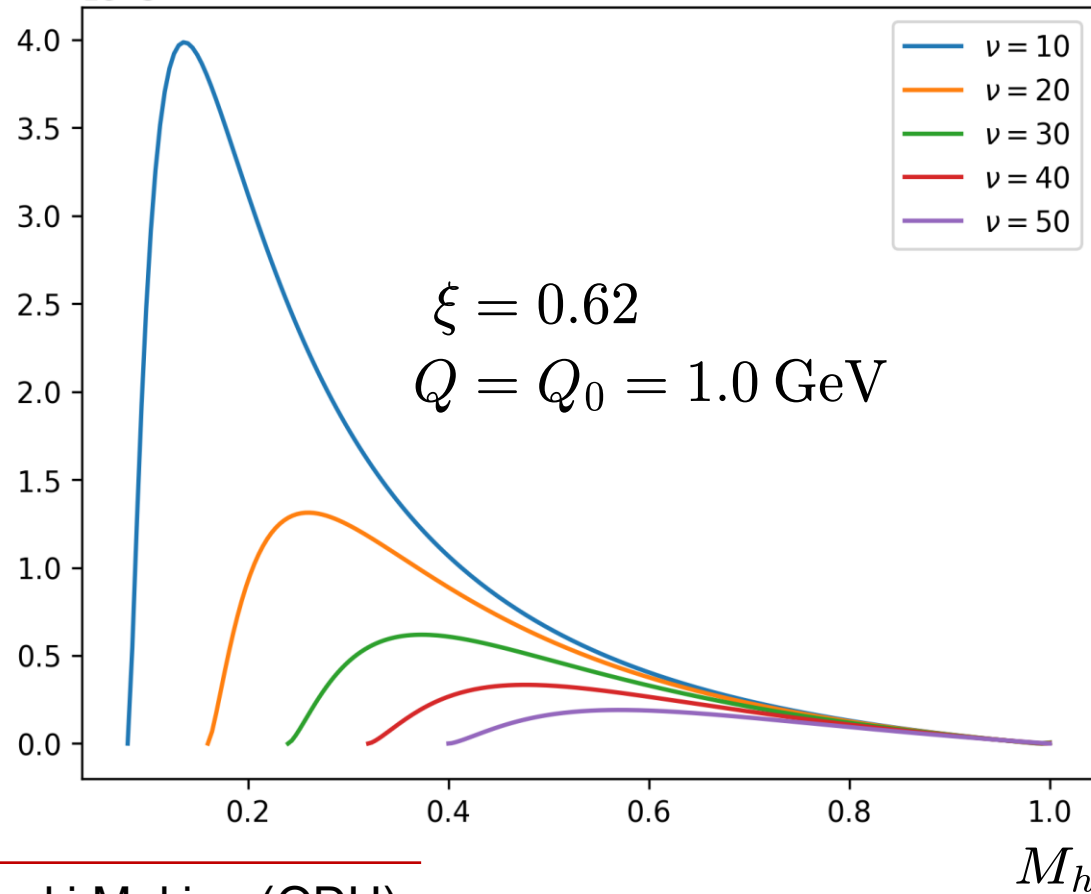
$$\mathcal{L} = i\bar{\psi}_q \not{\partial} \psi_q + i\bar{\Psi}_h \not{\partial} \Psi_h + \frac{1}{2}(\partial\phi)^2 - m_q \bar{\psi}_q \psi_q - m_h \bar{\Psi}_h \Psi_h - \frac{m_s^2}{2}\phi^2 - \lambda \bar{\psi}_q \psi_q \phi - g \bar{\Psi}_h \Psi_h \phi$$



Input scale Fermion DiFFs

Initial scale dihadron fragmentation function

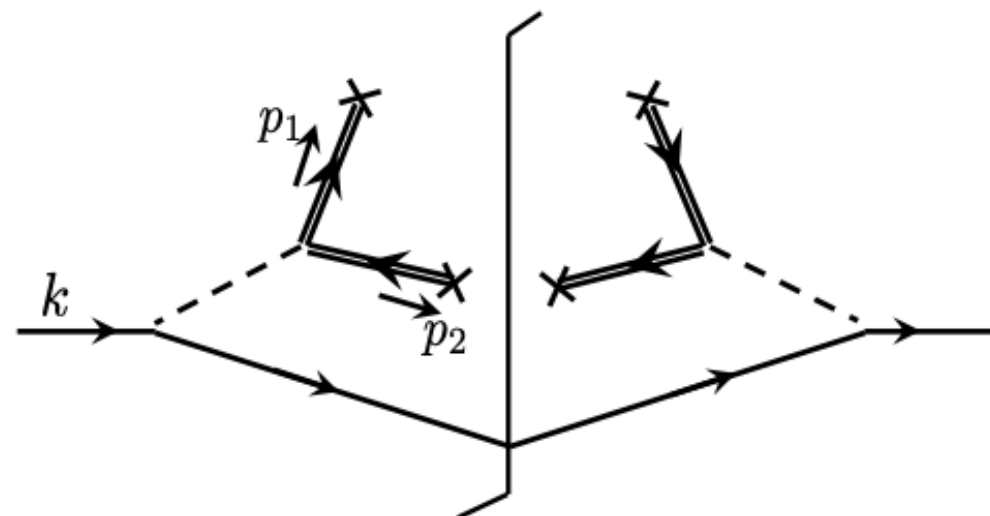
$$M_h |d_{h\bar{h}/f}|$$



Intrinsic mass scales $m_q(Q_0) = m_s(Q_0) = \nu * .0015 \text{ GeV}$

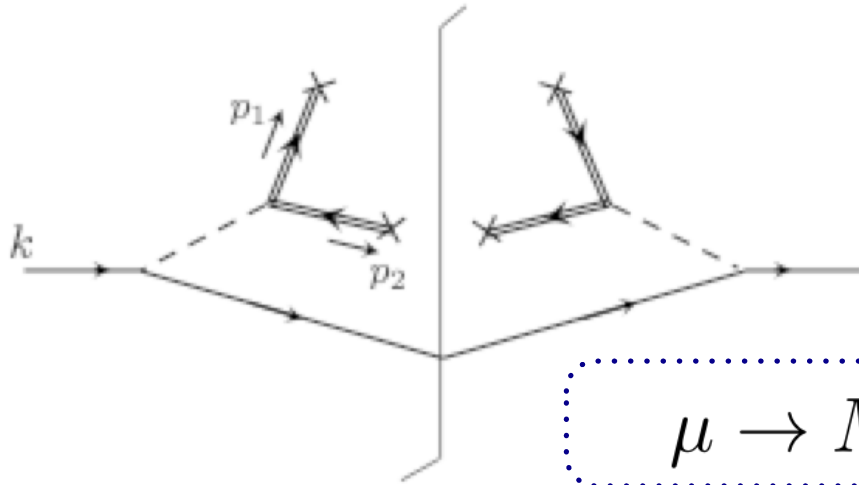
$\nu = 10, 20, 30, 40, 50$

$m_h(Q_0) = \nu * .004 \text{ GeV}$

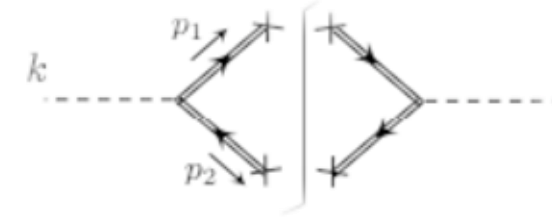


(DiFFs) in Yukawa theory

Evolution in M_h



$$\frac{\partial}{\partial \ln M_h^2} d_{q \rightarrow h \bar{h}}^{\text{pert}}(\xi, M_h^2; Q_0) = d_{s \rightarrow h \bar{h}}^{\text{pert}}(\xi, M_h^2; Q_0) \otimes K_{q \rightarrow s}^{\text{pert}}(\xi, M_h^2) + \dots$$



$$K_{q \rightarrow s}^{\text{pert}, (0,1)}(\xi, M_h^2) = -a_\lambda(M_h) \xi$$

$$K_{q \rightarrow q}^{\text{pert}, (0,0)}(\xi, M_h^2) = -\delta(1 - \xi)$$

DGLAP evolved DiFF

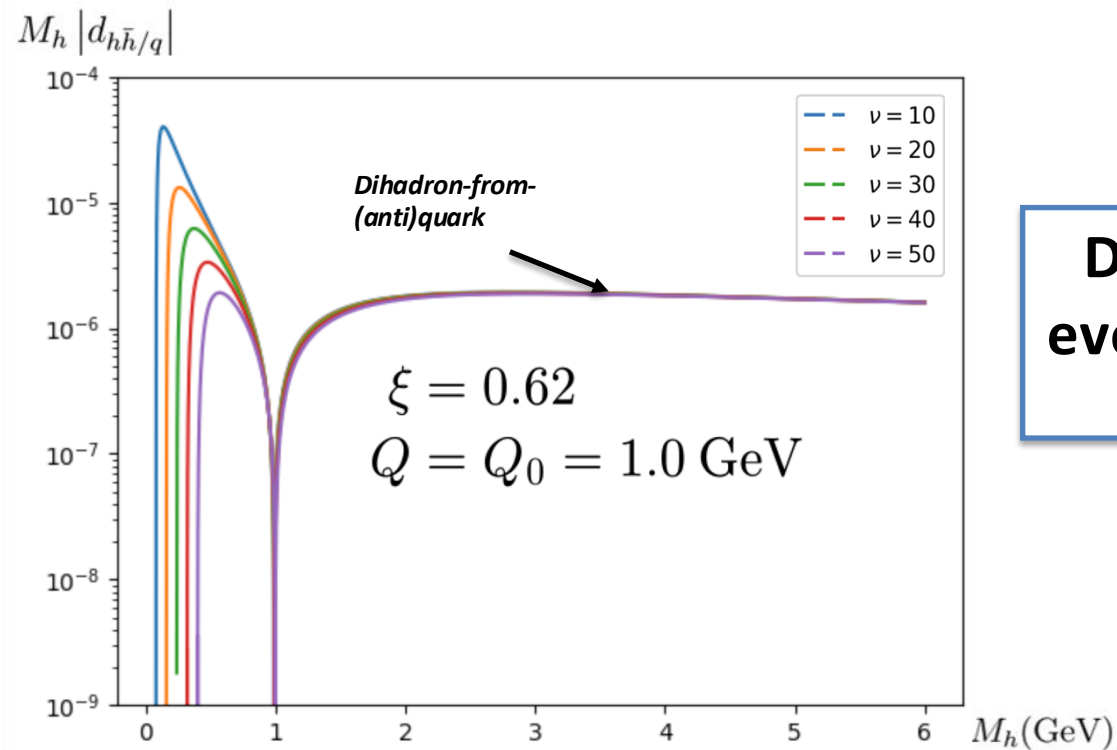
Initial scale dihadron fragmentation function

(y ; log scale)

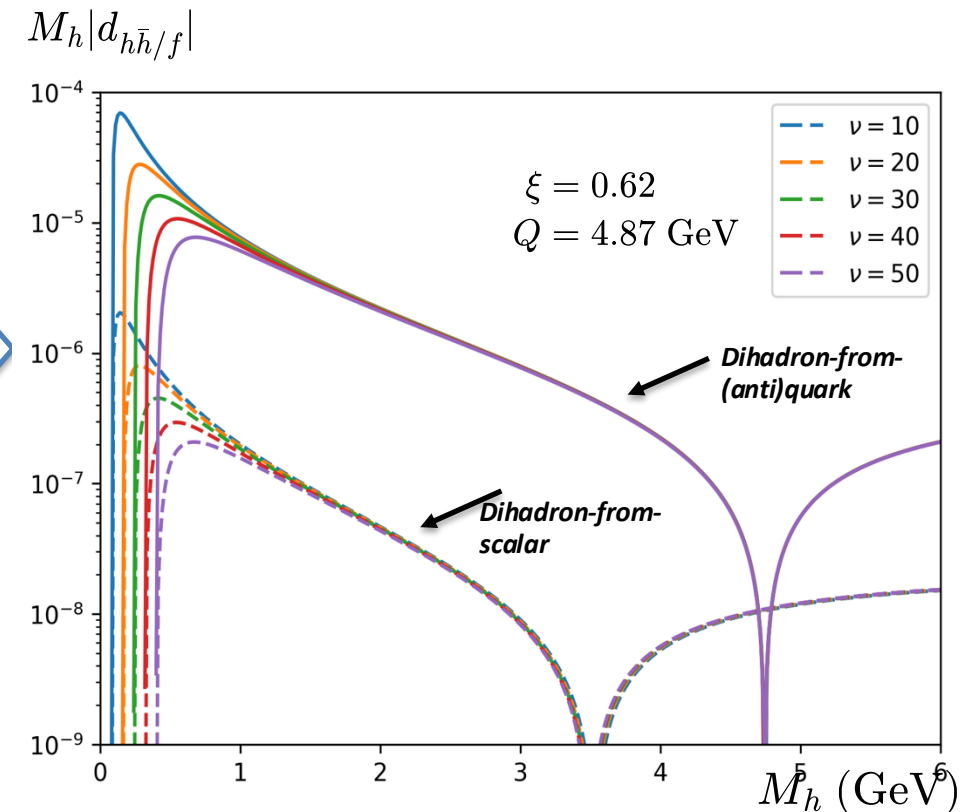
$$\mu \rightarrow Q_0$$

Modified $M_h^2 \gg Q_0^2$

- K kernel
- P kernel



**DGLAP
evolution**



Probability density interpretation

- UV divergence complicates probability density interpretation

$$d_{h_1 h_2 / j}^{\text{norm}}(\xi; \mu) = \int_{4m_\pi^2}^{\infty} dM_h^2 d_{h_1 h_2 / j}(\xi, M_h^2; \mu) \quad \xrightarrow{\text{UV Div!}} \quad d_{h_1 h_2 / j}^{\text{norm}}(\xi; \underline{M_{\text{max}}^2}; \mu) = \int_{4m_\pi^2}^{\underline{M_{\text{max}}^2}} dM_h^2 d_{h_1 h_2 / j}(\xi, M_h^2; \mu)$$

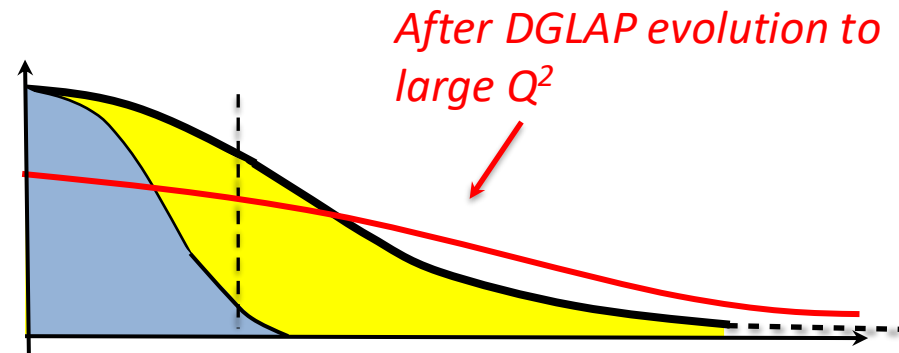
$$\langle M_h \rangle(\mu) = \frac{1}{d_{h_1 h_2 / j}^{\text{norm}}(\xi; \mu)} \int_{4m_\pi^2}^{\infty} dM_h^2 M_h d_{h_1 h_2 / j}(\xi, M_h^2; \mu) \quad \xrightarrow{\text{linear UV Div!}} \quad \langle M_h \rangle(\underline{M_{\text{max}}^2}; \mu) = \frac{1}{d_{h_1 h_2 / j}^{\text{norm}}(\xi; \underline{M_{\text{max}}^2}; \mu)} \int_{4m_\pi^2}^{\underline{M_{\text{max}}^2}} dM_h^2 M_h d_{h_1 h_2 / j}(\xi, M_h^2; \mu)$$

Probability density interpretation

- Probability density depends on the choice of M_{max} cut-off

$$d_{h_1 h_2 / j}^{\text{norm}}(\xi; \underline{M_{max}^2}; \mu) = \int_{4m_\pi^2}^{\underline{M_{max}^2}} dM_h^2 d_{h_1 h_2 / j}(\xi, M_h^2; \mu)$$

- Constant M_{max} vs M_{max} that grow as Q



Probability density interpretation (Constant $M_{\max}$)

$$M_{\max}^2 = \text{constant}$$

$$\mu = Q$$

$$d_{h_1 h_2 / j}^{\text{norm}}(\xi; \underline{M_{\max}^2}; \mu) = \int_{4m_\pi^2}^{\underline{M_{\max}^2}} dM_h^2 d_{h_1 h_2 / j}(\xi, M_h^2; \mu)$$

DGLAP evolution of normalized DiFFs

*Ceccopieri, Radici, and Bacchetta
Physics Letters B 650 (2007) 81–89
de Florian, Vanni
Phys.Lett.B 578 (2004) 139-149*

$$\frac{d}{d \ln Q^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi; M_{\max}^2; \mu) = \frac{d \ln \mu^2}{d \ln Q^2} \frac{\partial}{\partial \ln \mu^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi; M_{\max}^2; \mu) + M_{\max}^2 \frac{d \ln M_{\max}^2}{d \ln Q^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi, M_{\max}^2; \mu)$$

$$\frac{d}{d \ln Q^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi; M_{\max}^2 = 4 \text{ GeV}^2; Q) = \frac{\partial}{\partial \ln Q^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi; M_{\max}^2 = 4 \text{ GeV}^2; Q) \Big|_{\text{fixed } M_{\max}}$$

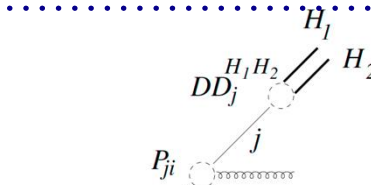
Probability density interpretation ($M_{max} = M_{max}(Q)$)

$$M_{max}^2 = Q^2$$

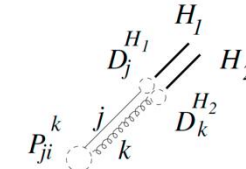
$$\mu = Q$$

$$d_{h_1 h_2 / j}^{\text{norm}}(\xi; \underline{M_{max}^2}; \mu) = \int_{4m_\pi^2}^{\underline{M_{max}^2}} dM_h^2 d_{h_1 h_2 / j}(\xi, M_h^2; \mu)$$

$$\frac{d}{d \ln Q^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi; M_{max}^2; \mu) = \frac{d \ln \mu^2}{d \ln Q^2} \frac{\partial}{\partial \ln \mu^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi; M_{max}^2; \mu) + M_{max}^2 \frac{d \ln M_{max}^2}{d \ln Q^2} d_{h_1 h_2 / j}(\xi, M_{max}^2; \mu)$$



+



$$\frac{d}{d \ln Q^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi; Q^2; Q) = \underbrace{\frac{\partial}{\partial \ln Q^2} d_{h_1 h_2 / j}^{\text{norm}}(\xi; M_{max}^2; Q)}_{\text{DGLAP evolution}} \Big|_{M_{max}=Q} + \underbrace{Q^2 d_{h_1 h_2 / j}(\xi, Q^2; Q)}_{\frac{1}{Q^2} \mathcal{H} \otimes d_1 \otimes d_2 \text{ Inhomogeneous correction}}$$

Fig. from

Probability density interpretation

- The mean depends on the choice of M_{max} cut-off

$$\langle M_h \rangle(\underline{M_{max}^2}; \mu) = \frac{1}{d_{h_1 h_2 / j}^{\text{norm}}(\xi; \underline{M_{max}^2}; \mu)} \int_{4m_\pi^2}^{\underline{M_{max}^2}} dM_h^2 M_h d_{h_1 h_2 / j}(\xi, M_h^2; \mu)$$

- The choice of $M_{max} = M_{max}(Q)$ vs $M_{max} = 2\text{GeV}$

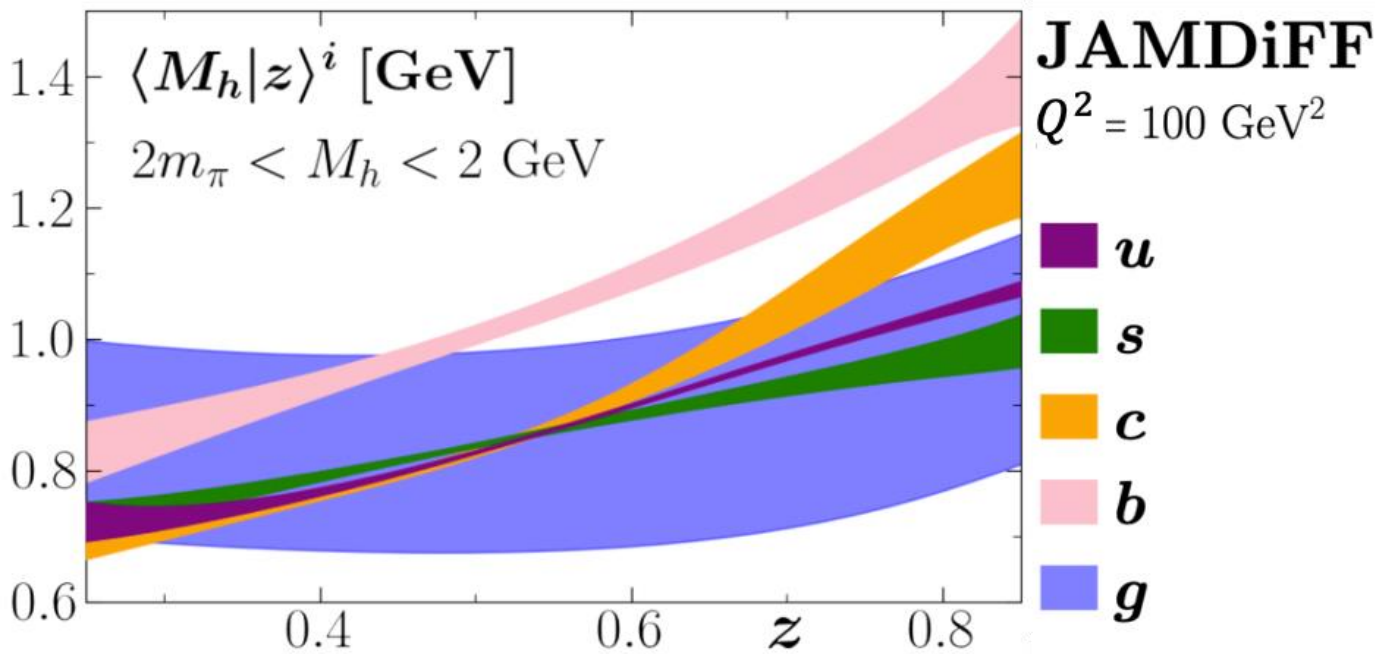
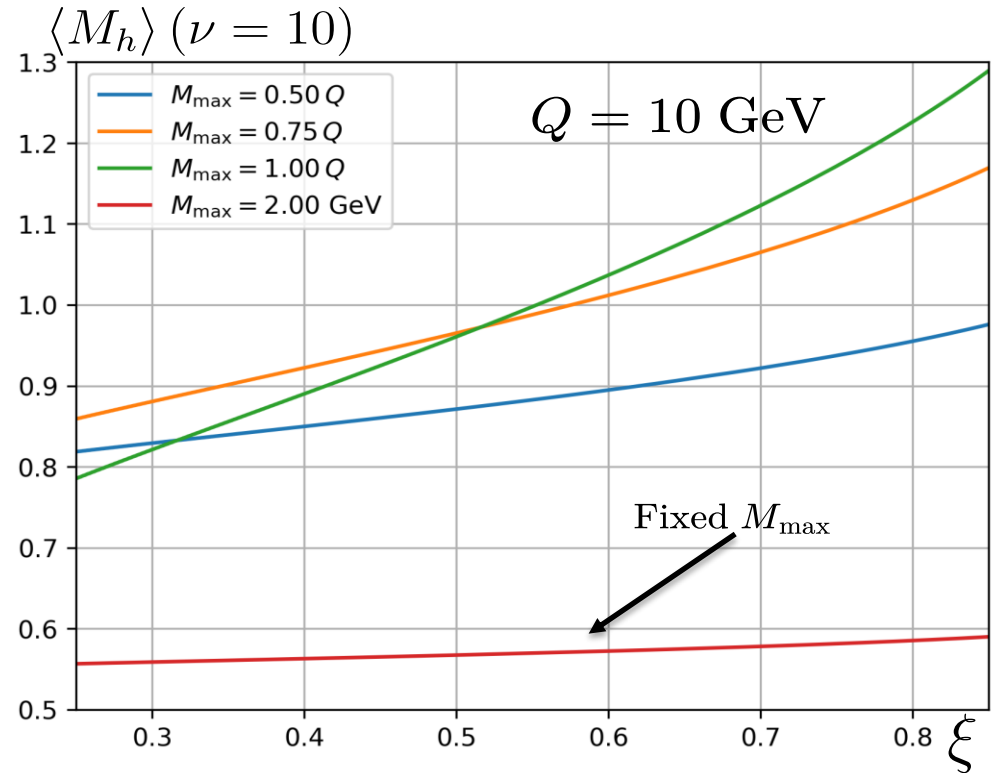


Fig.13 in arXiv:2308.14857

JAM

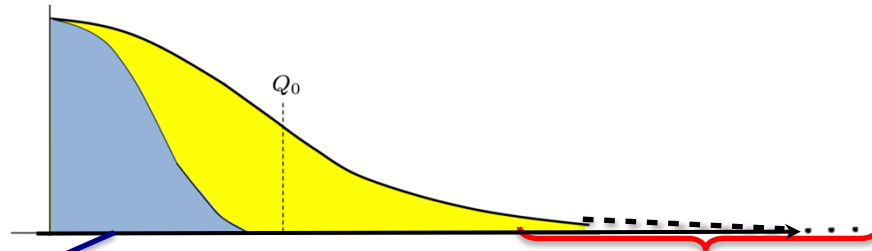
The first data-driven extractions of flavored DiFFs!



The End

Summary

Thank you



Separate dependence on Q^2 and M_h^2

Hadron Structure Oriented approach

Modified $M_h^2 \gg Q_0^2$

Phys.Rev.D 106 (2022) 3, 034002

Phys.Rev.D 107 (2023) 9, 094029

Phys.Rev.D 110 (2024) 7, 074016

arXiv:2610.20448v1

Next steps: merge with phenomenology

DiFFs in different variables

$$d_{h_1 h_2 / j} (\xi, M_h^2, \zeta, \phi_R) = d_{h_1 h_2 / j} (\xi, p_{h_1}, p_{h_2})$$

Basic DiFF

$$d_{h_1 h_2 / j}^{\text{norm}} (\xi) = \int_{4m_\pi^2}^{\infty} \frac{dM_h^2}{(4\pi)^3} \int_{-\zeta_{\text{max}}}^{\zeta_{\text{max}}} d\zeta \int_0^{2\pi} d\phi_R d_{h_1 h_2 / j} (\xi, M_h^2, \zeta, \phi_R)$$

Normalized with respect to all variables except ξ

$$\underbrace{d_{h_1 h_2 / j} (\xi, M_h^2)}_{\text{“Extended” DiFF}} = \frac{1}{(4\pi)^3} \int_{-\zeta_{\text{max}}}^{\zeta_{\text{max}}} d\zeta \int_0^{2\pi} d\phi_R d_{h_1 h_2 / j} (\xi, M_h^2, \zeta, \phi_R)$$

“Extended” DiFF

Ceccopieri, Radici, Bacchetta, Phys.Lett.B 650 (2007) 81-89

Probability density interpretation

- The choice of $M_{max} = M_{max}(Q)$ vs $M_{max} = 2\text{GeV}$

