

Theoretical Hadron Spectroscopy as a Probe of QCD

Andrew W. Jackura

William & Mary

Department of Physics

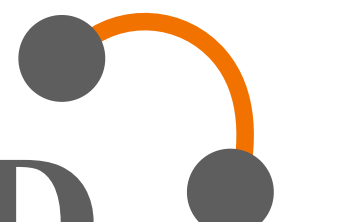
MENU 2026

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Stony Brook, NY



had spec

EXOHAD 
EXOTIC HADRONS TOPICAL COLLABORATION



awjackura@wm.edu



[ajackura.github.io](https://github.com/ajackura)



WILLIAM & MARY

CHARTERED 1693

Hadrons & QCD

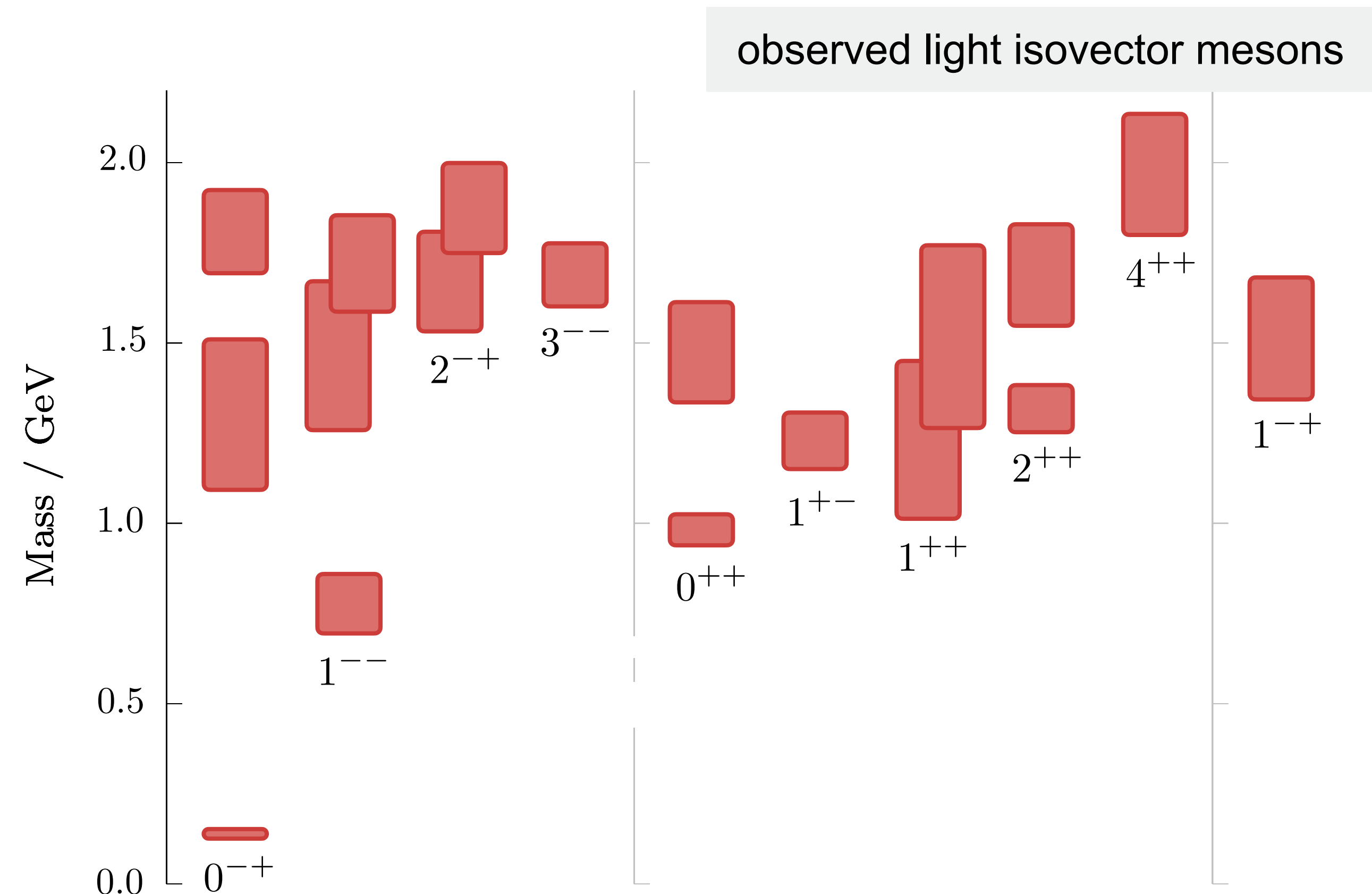
QCD is a deceptively simple looking theory

- We observe **Hadrons**, not quarks and gluons
- Interactions are strong at low energies — perturbative approximations fail for hadrons
- Observed spectrum of hadrons show complexity not easily explained by “*simple*” pictures

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\not{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

Key Questions

How are quarks and gluons confined into hadrons and nuclei?
How do nuclear interactions emerge from QCD dynamics?
What is the nature of new/exotic states of matter?



Hadrons & QCD

Must resort to non-perturbative methods to probe hadron physics

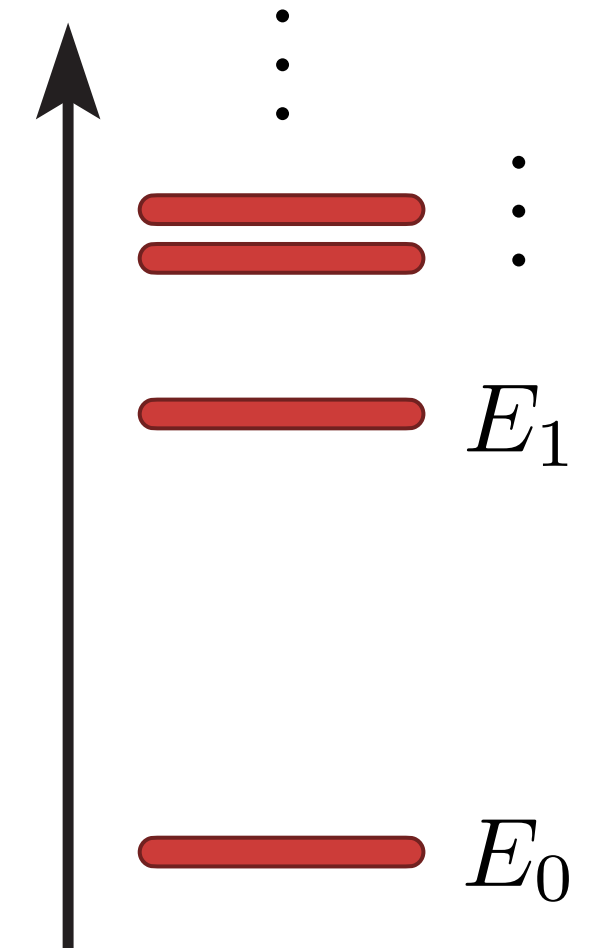
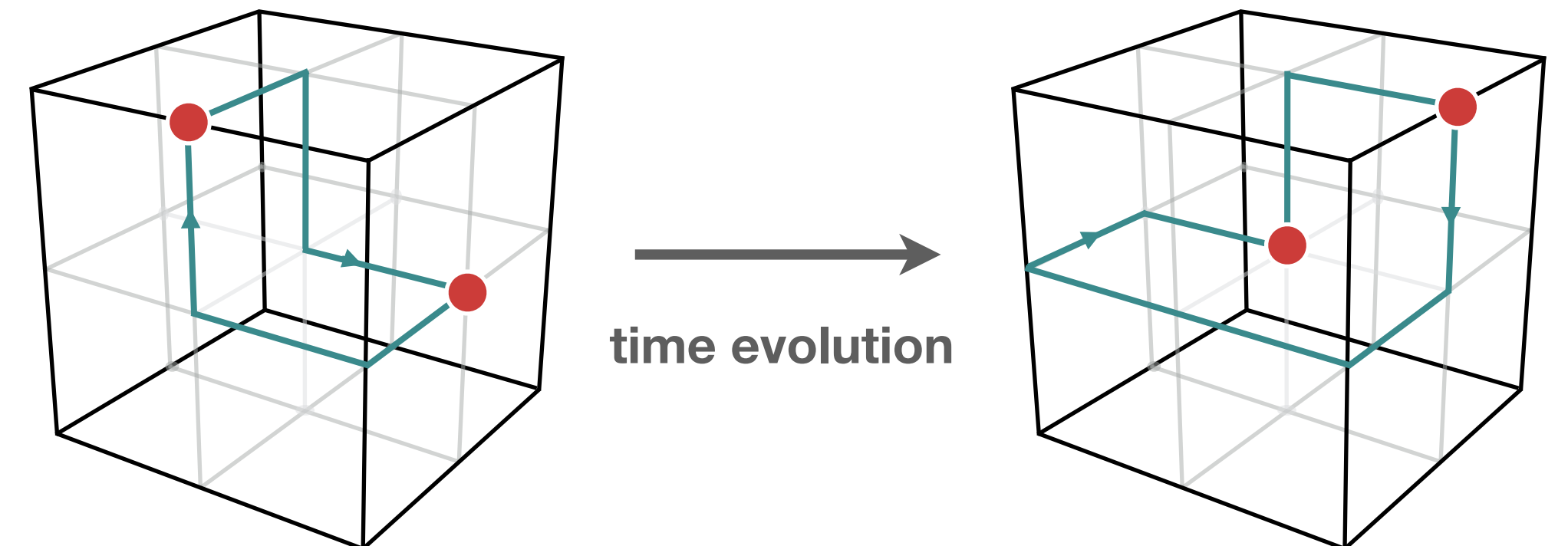
- Lattice QCD offers systematically improvable, non-perturbative approach
- Stochastically evaluate QCD path integral and correlation functions

$$\begin{aligned}
 \mathcal{C}_L^{(E)}(\tau) &= \langle \mathcal{O}(\tau) \mathcal{O}^\dagger(0) \rangle_L \\
 &= \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \int \mathcal{D}U e^{-S_{\text{QCD}}^{(E)}} \mathcal{O}(\tau) \mathcal{O}^\dagger(0) \approx \frac{1}{N_{\text{cfg}}} \sum_{j=1}^{N_{\text{cfg}}} \mathcal{C}_j(\tau) + \sigma_{\mathcal{C}} \\
 &= \sum_{\mathbf{n}} Z_{\mathbf{n}} Z_{\mathbf{n}}^\dagger e^{-E_{\mathbf{n}} \tau}
 \end{aligned}$$

Hadron/Multi-Hadron operators constructed from QCD fields

Lattice Systematics

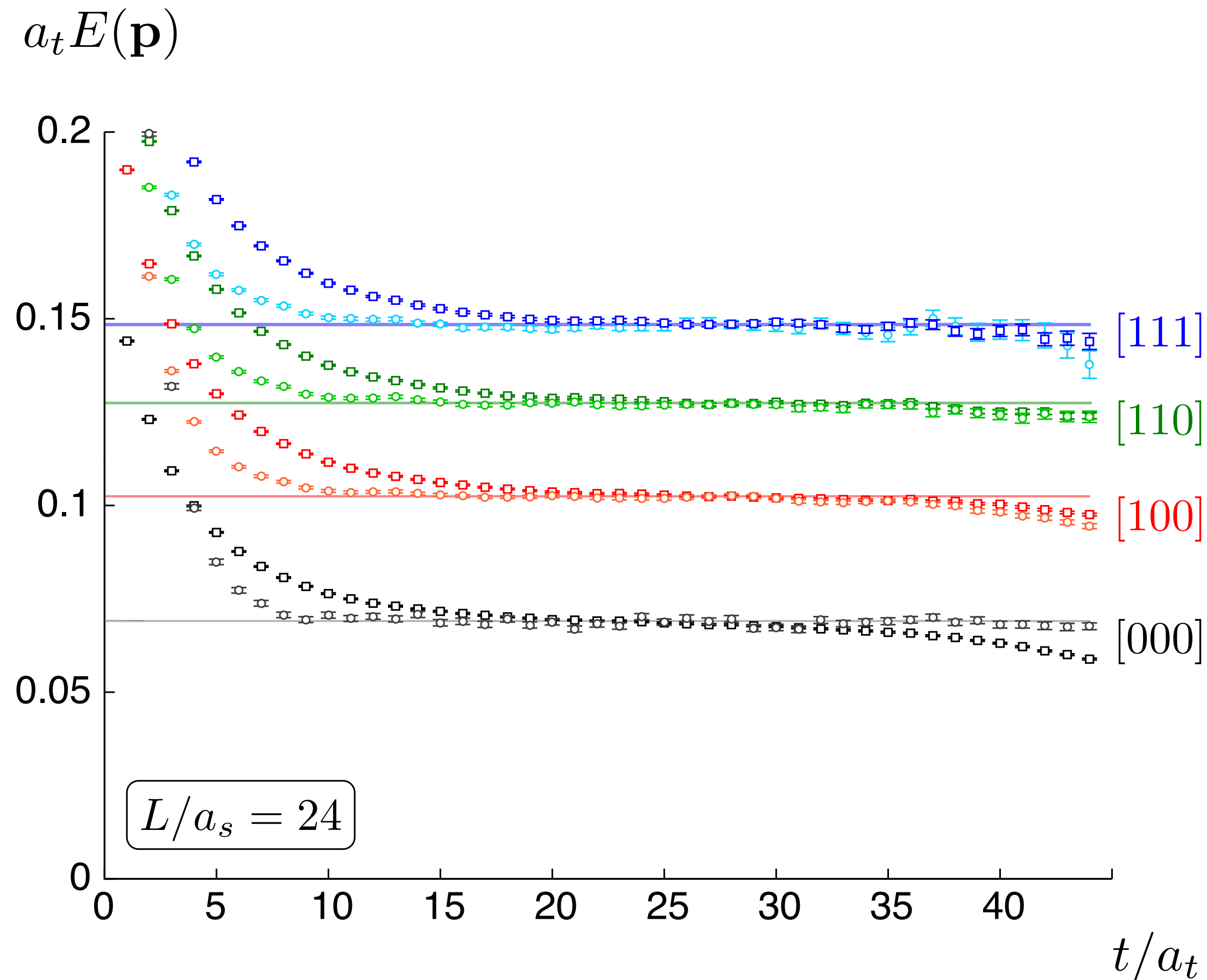
- Discrete spacetime lattice, a
- Finite volume, L
- Euclidean spacetime, $t \rightarrow -i\tau$
- $m_q > m_q^{(\text{phys.})}$



Hadrons & QCD

Must resort to non-perturbative methods to probe hadron physics

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$$E(\mathbf{p}) = \sqrt{m(L)^2 + \mathbf{p}^2}$$

$$\mathbf{p} = \frac{2\pi}{L} \mathbf{n} \quad \mathbf{n} \in \mathbb{Z}^3$$

Can assess finite-volume dependence

$$m_h(L) = m_h + \mathcal{O}(e^{-m_\pi L})$$

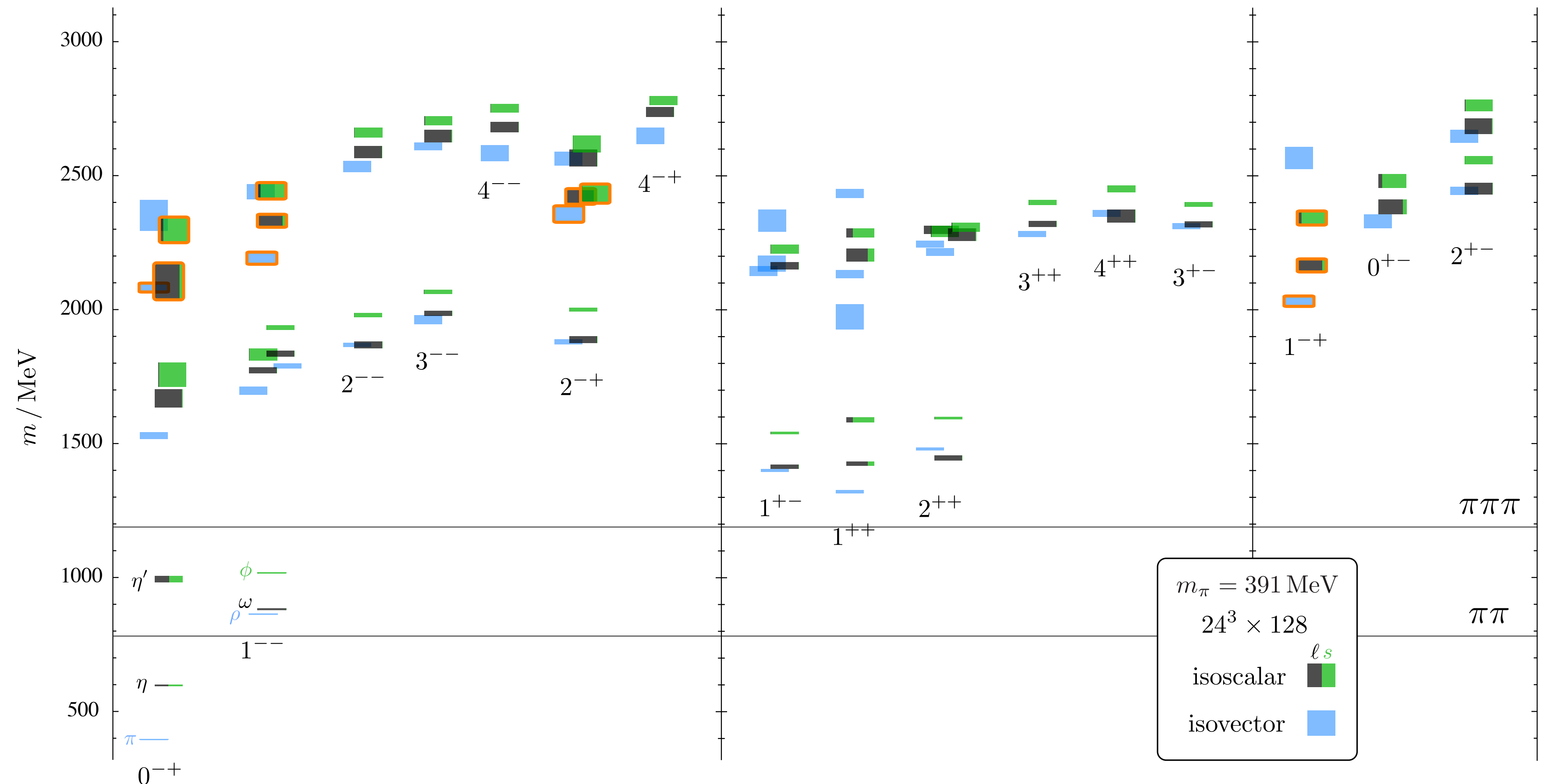
Hadrons & QCD

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Static Spectrum

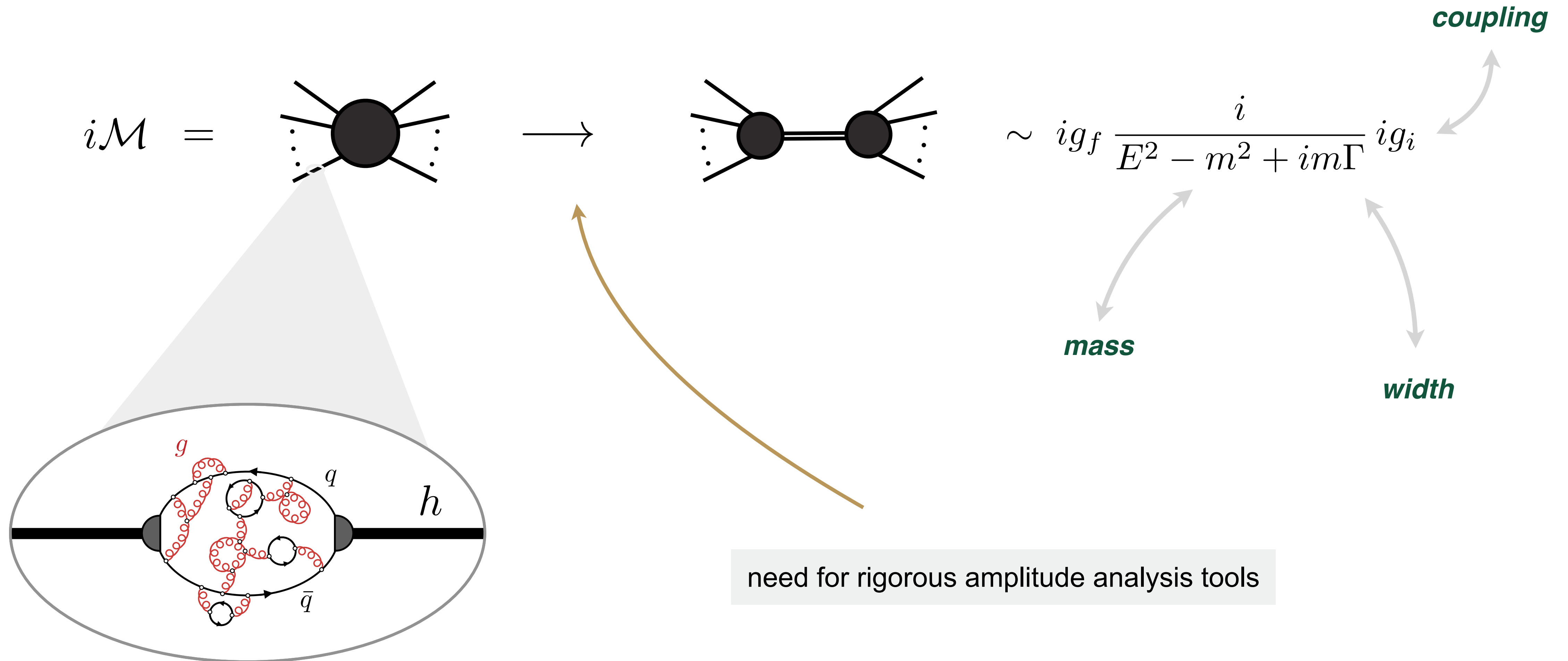
- No decay-coupling information
- An incomplete picture...



Hadron Spectroscopy

Most hadrons in the excited spectrum are unstable **resonances**

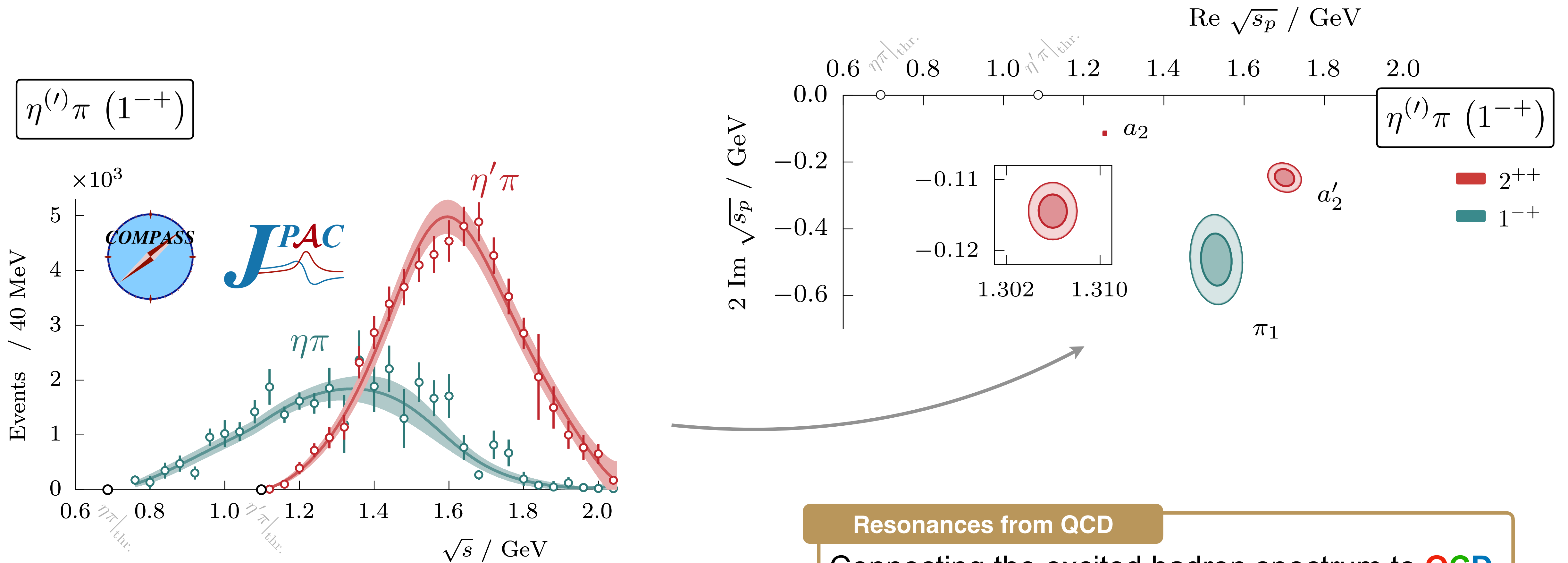
- Observed through reactions and decays
- Defined as poles of corresponding **amplitudes**



Hadron Spectroscopy

Most hadrons in the excited spectrum are unstable **resonances**

- Observed through reactions and decays
- Defined as poles of corresponding **amplitudes**



A. Rodas et al. [JPAC]
Phys. Rev. Lett. **122**, 042002 (2019)

Resonances from QCD

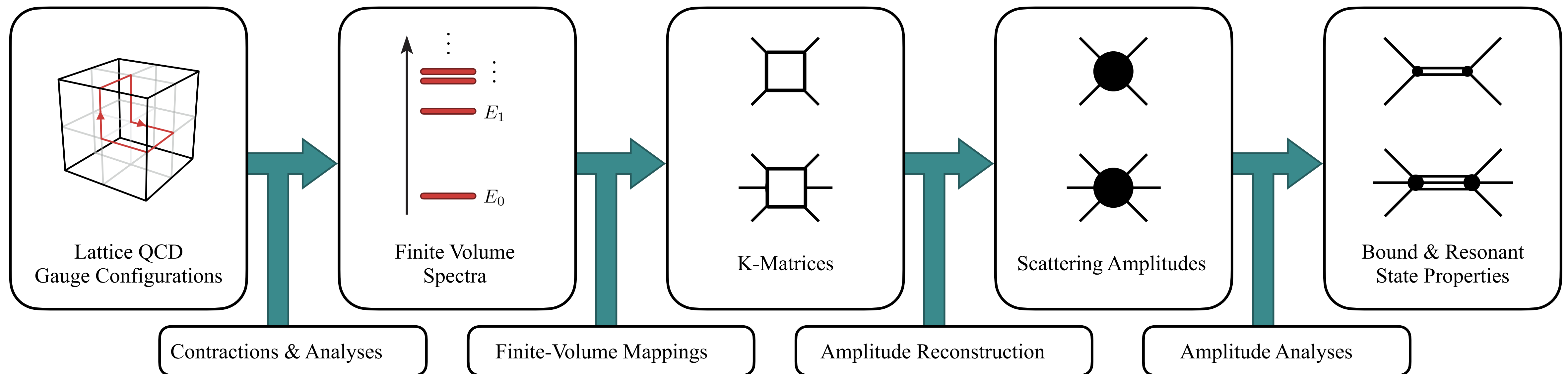
Connecting the excited hadron spectrum to **QCD** requires knowledge of reaction amplitudes

Hadron Spectroscopy from Lattice QCD

Goal: Understand the excited hadron spectrum from **QCD**

- Establish masses, lifetimes, couplings, etc. in a rigorous manner
- Requires obtaining **reaction amplitudes** — key objects which contain *all* information
- Have ability to study hadron physics in manner commensurate with experimental colleagues

Lüscher Methodology



M. Lüscher
Commun.Math.Phys. **105**, 153 (1986)
Nucl.Phys. **B354**, 531 (1991)

Many many others...

Hadron Spectroscopy from Lattice QCD

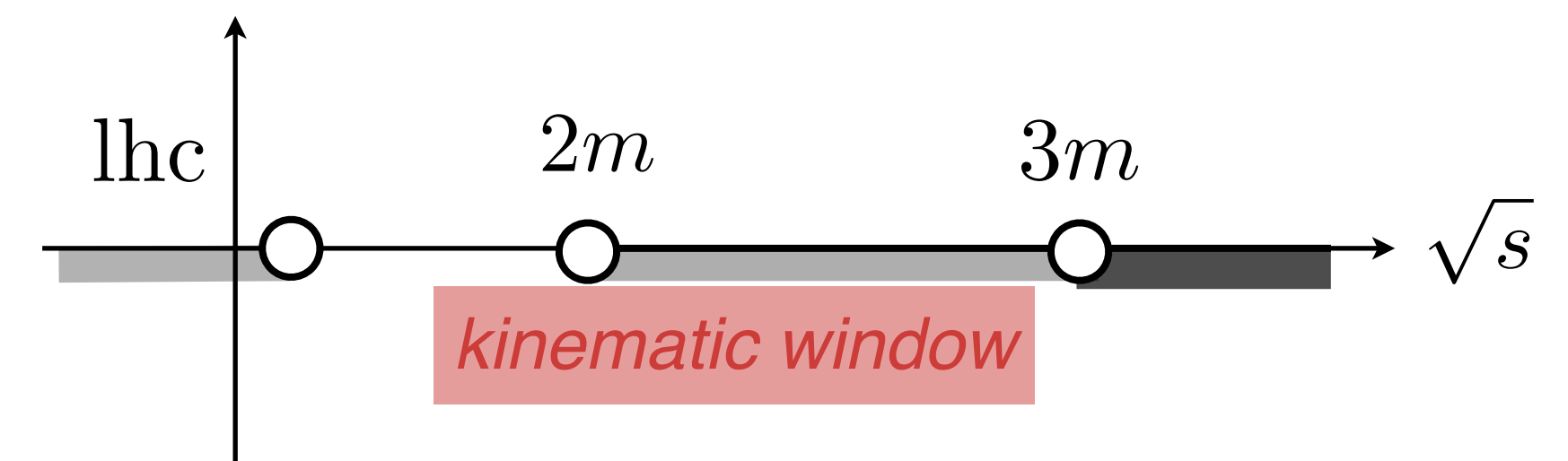
Key Idea: Enforce probability conservation — **unitarity**

- Constrains analytic structure of infinite-volume amplitudes
- Allows for systematic “correction” of analogous **finite-volume** amplitude

Infinite-Volume

$$\begin{aligned} i\mathcal{M}_2 &= \text{[solid black circle]} = \text{[circle with cross]} + \text{[circle with two internal circles]} + \text{[circle with three internal circles]} + \dots \\ &= \text{[square with cross]} + \text{[square with one internal circle and vertical line]} + \text{[square with two internal circles and vertical lines]} + \dots = i\mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2} \end{aligned}$$

Note: In the diagram, the first square is labeled $i\mathcal{K}_2$ in red, and the vertical lines in the subsequent terms are labeled ρ in blue.



Hadron Spectroscopy from Lattice QCD

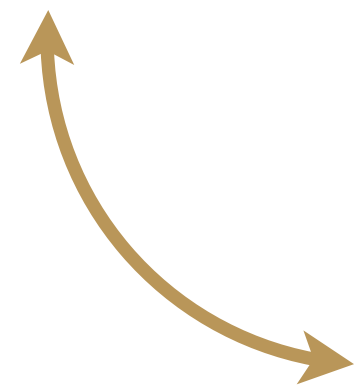
Key Idea: Enforce probability conservation — **unitarity**

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Finite-Volume

$$\mathcal{C}_L^{(M)}(E) = i \sum_{\mathbf{n}} \frac{Z_{\mathbf{n}} Z_{\mathbf{n}}^\dagger}{E - E_{\mathbf{n}}}$$

$$= \text{[Diagram: bubble with } V \text{]} + \text{[Diagram: two bubbles with } V \text{]} + \text{[Diagram: three bubbles with } V \text{]} + \dots$$



$$\text{[Diagram: bubble with } V \text{]} \sim \text{[Diagram: empty bubble]} + F_{2,L}$$

Finite-volume correction (known)

Hadron Spectroscopy from Lattice QCD

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Finite-Volume

$$\begin{aligned} C_L^{(M)}(E) &= i \sum_{\mathbf{n}} \frac{Z_{\mathbf{n}} Z_{\mathbf{n}}^\dagger}{E - E_{\mathbf{n}}} \\ &= \text{[Diagram: circle with two external lines and vertex } V] + \text{[Diagram: two circles with two external lines and vertices } V] + \text{[Diagram: three circles with two external lines and vertices } V] + \dots \\ &= \text{[Diagram: circle with two external lines, vertex } V, \text{ and a vertical red line]} + \text{[Diagram: two circles with two external lines, vertices } V, \text{ and vertical red lines]} + \text{[Diagram: three circles with two external lines, vertices } V, \text{ and vertical red lines]} + \dots \longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_2 \cdot F_{2,L}} \end{aligned}$$

$F_{2,L}$

Quantization Condition

$$\det[1 + \mathcal{K}_2 \cdot F_{2,L}] \Big|_{E=E_{\mathbf{n}}} = 0$$

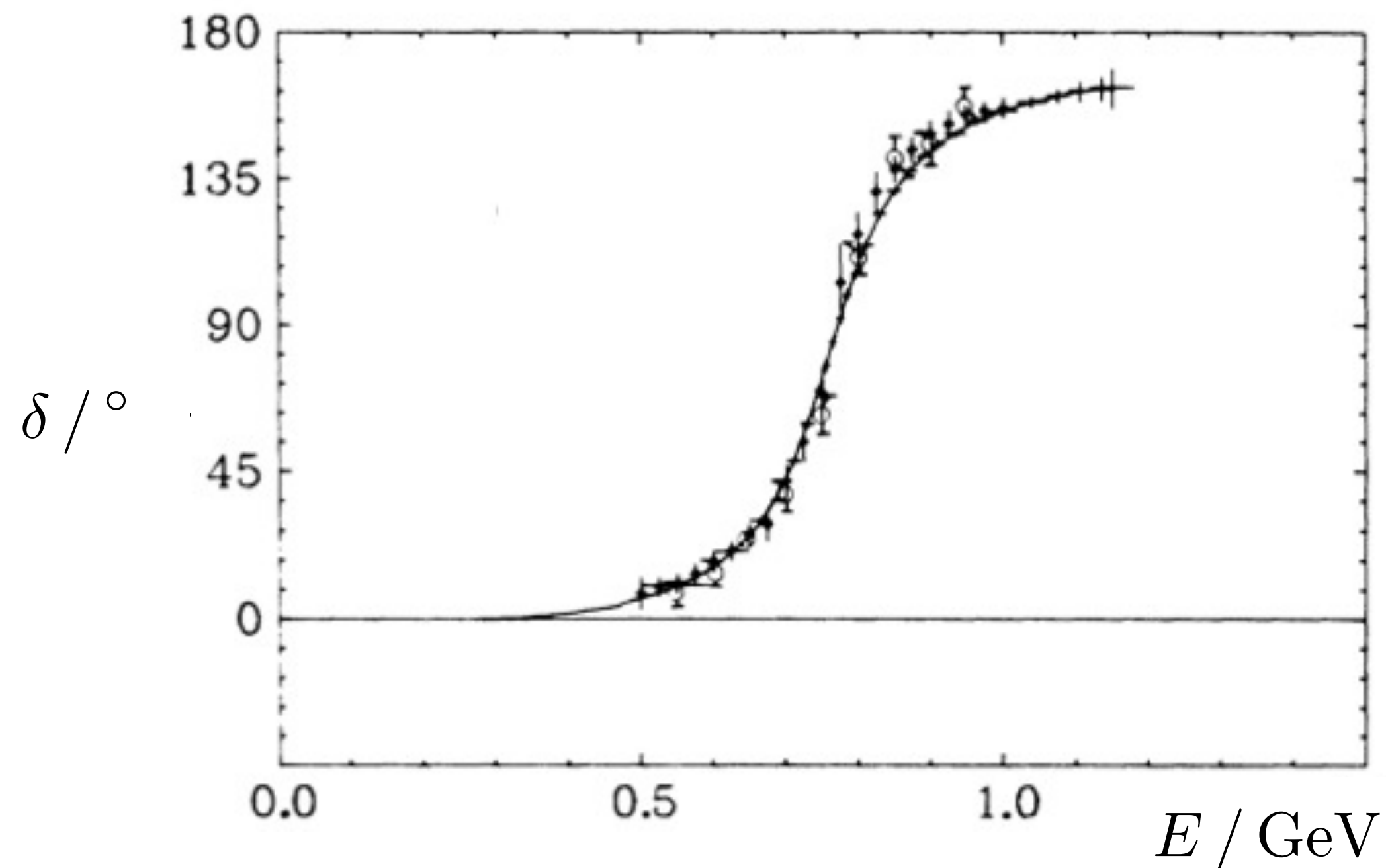
Hadron Spectroscopy from Lattice QCD

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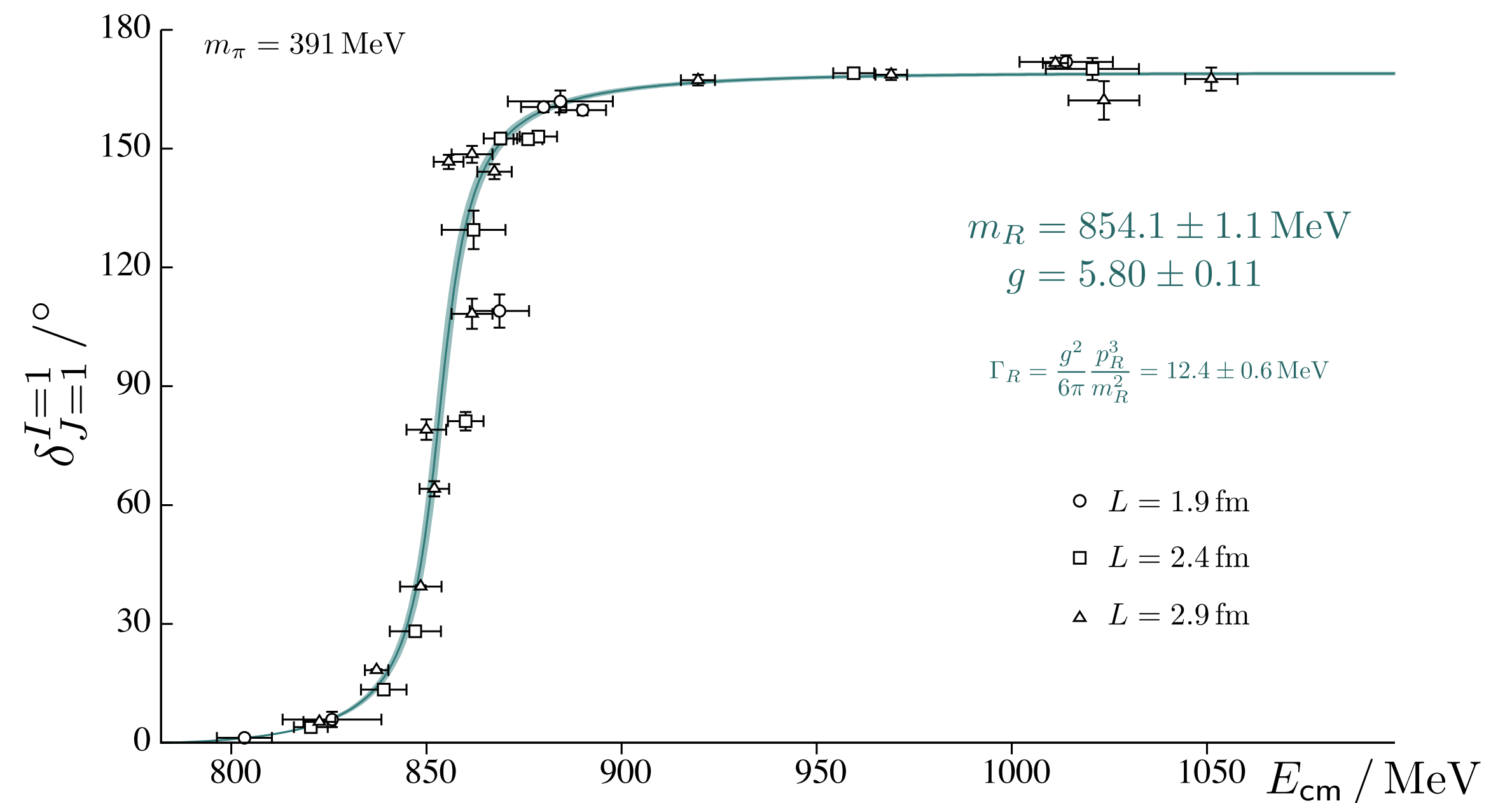
$\pi\pi$ ($I = 1$) Scattering

S.Protopopescu, et al
Phys.Rev.D, **7** (1973) 1279



J.Dudek, R.Edwards, & C.Thomas
Phys.Rev.D, **87** (2013) 034505

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for the Lattice QCD enthusiast

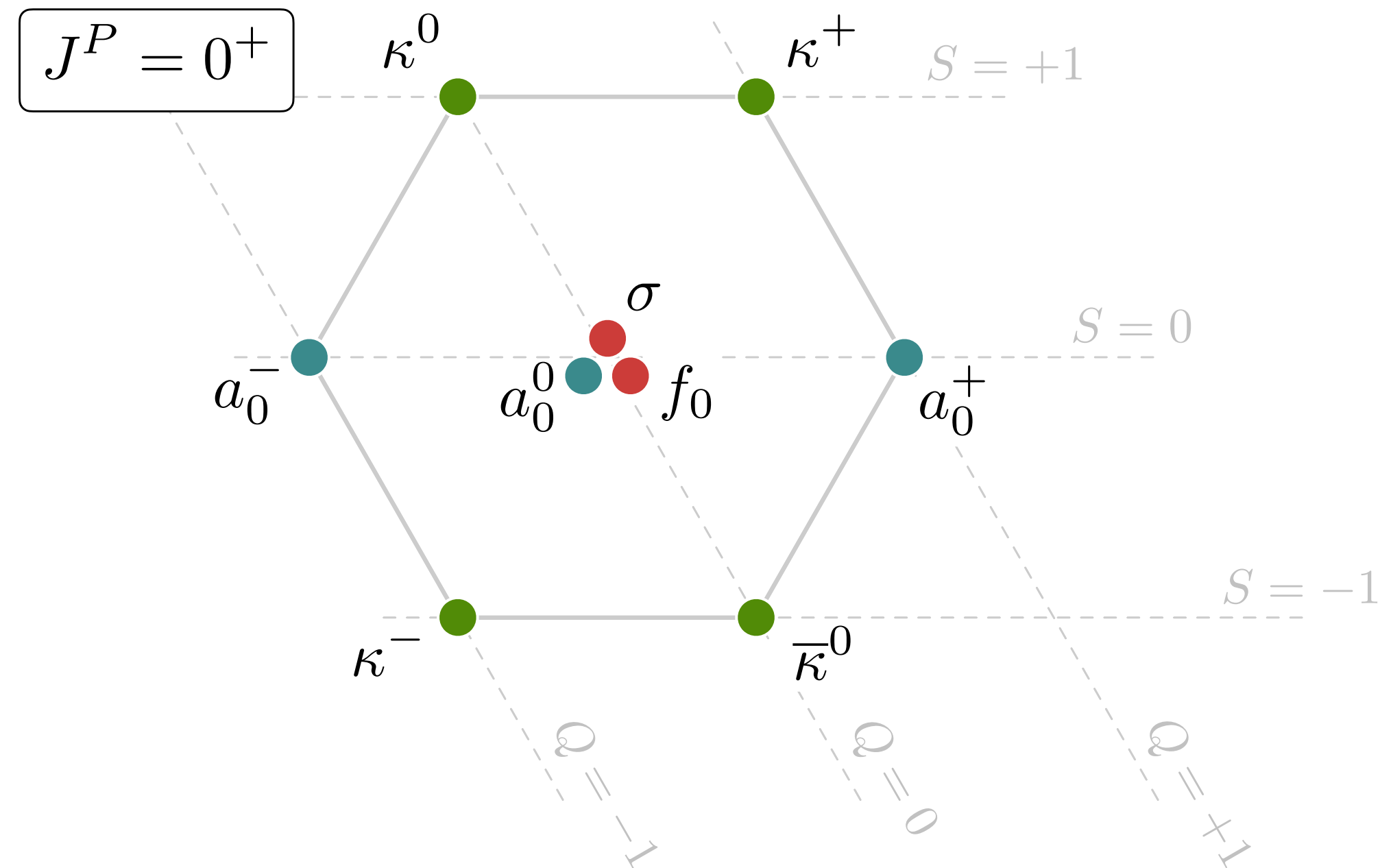
— we use anisotropic lattices with clover fermions and a Symanzik improved gauge action

— Distillation used to smear quark fields

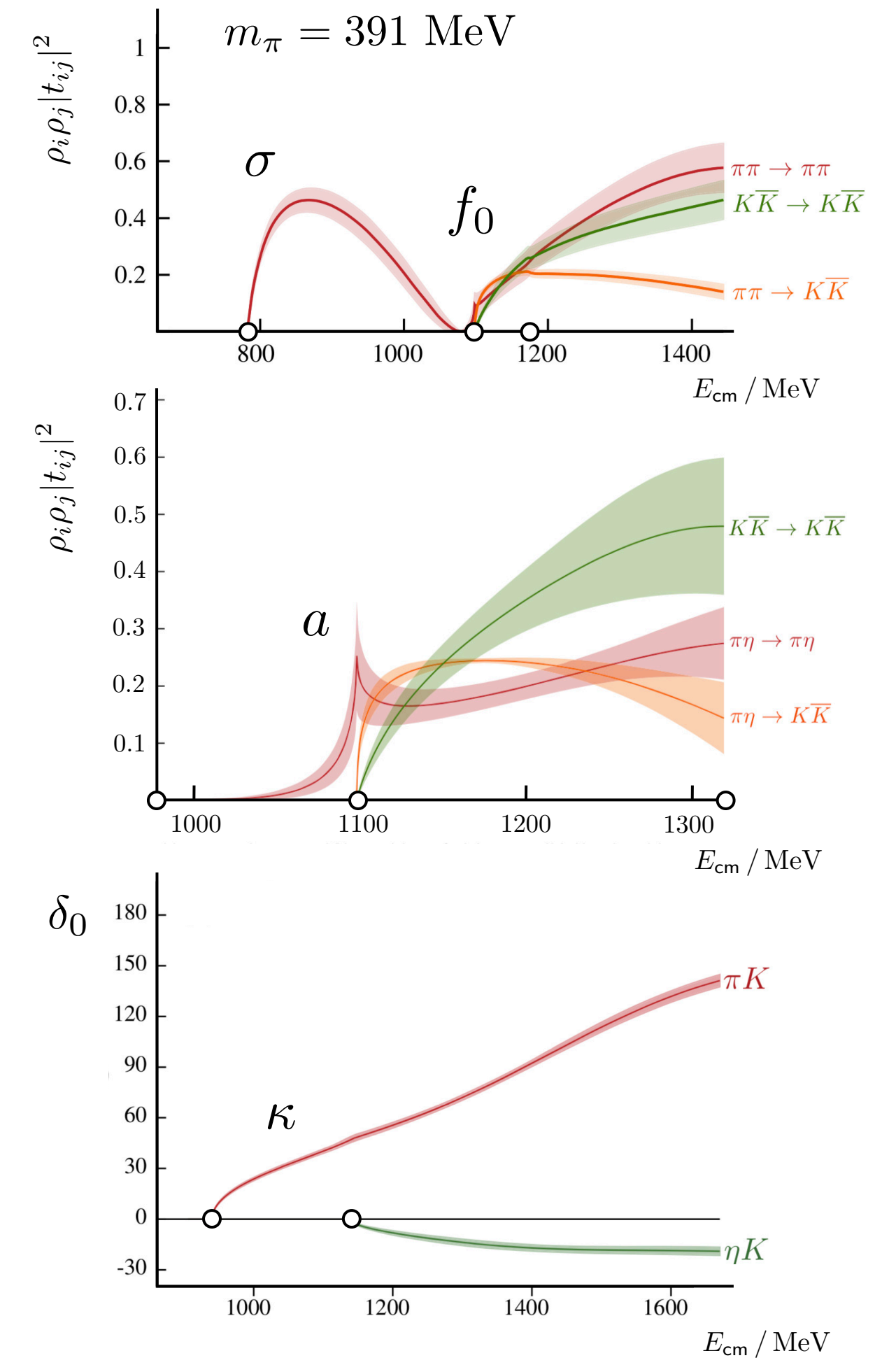
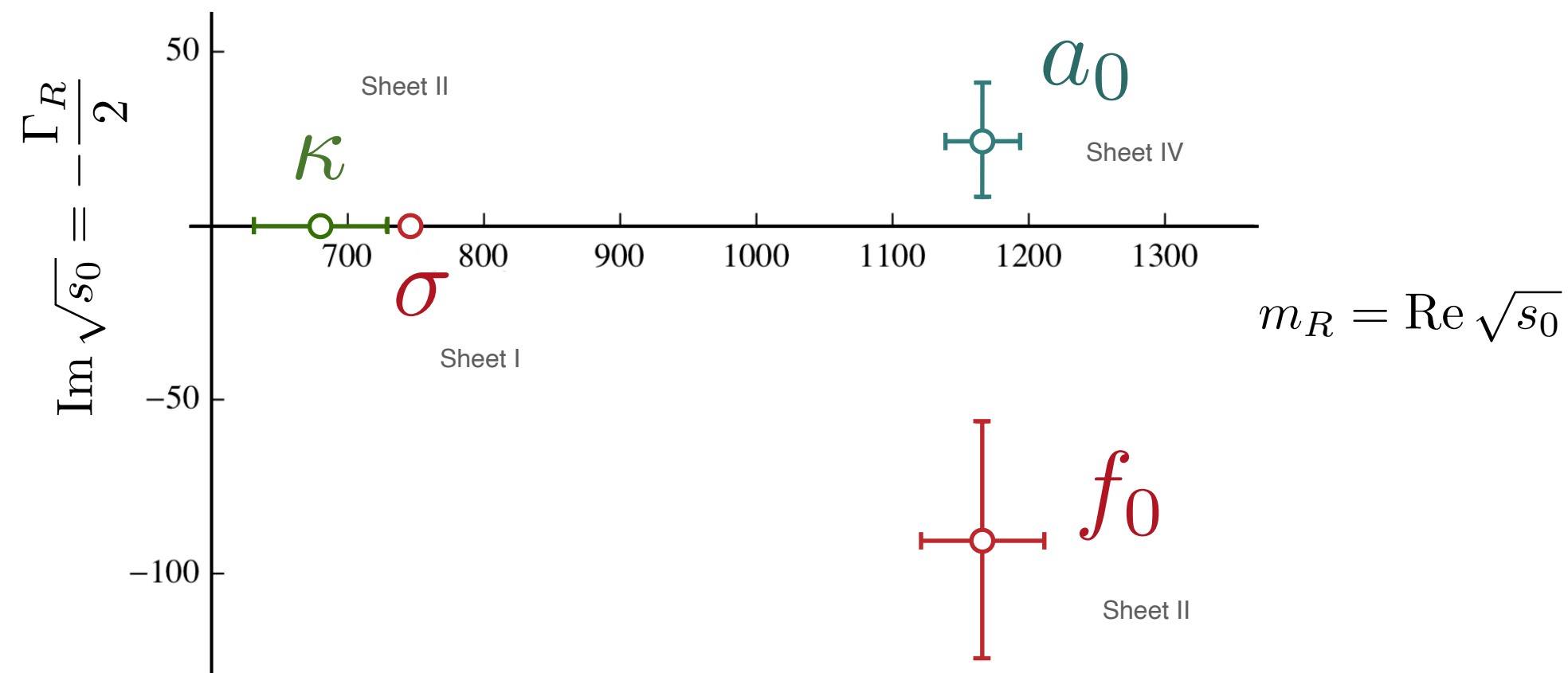
Two-Hadron Systems

Many systems investigated, including coupled channels

light scalars



$m_\pi = 391$ MeV



R.A. Briceño et al. [HadSpec]
Phys. Rev. **D97**, 054513 (2018)

J.J. Dudek et al. [HadSpec]
Phys. Rev. **D93**, 094506 (2016)

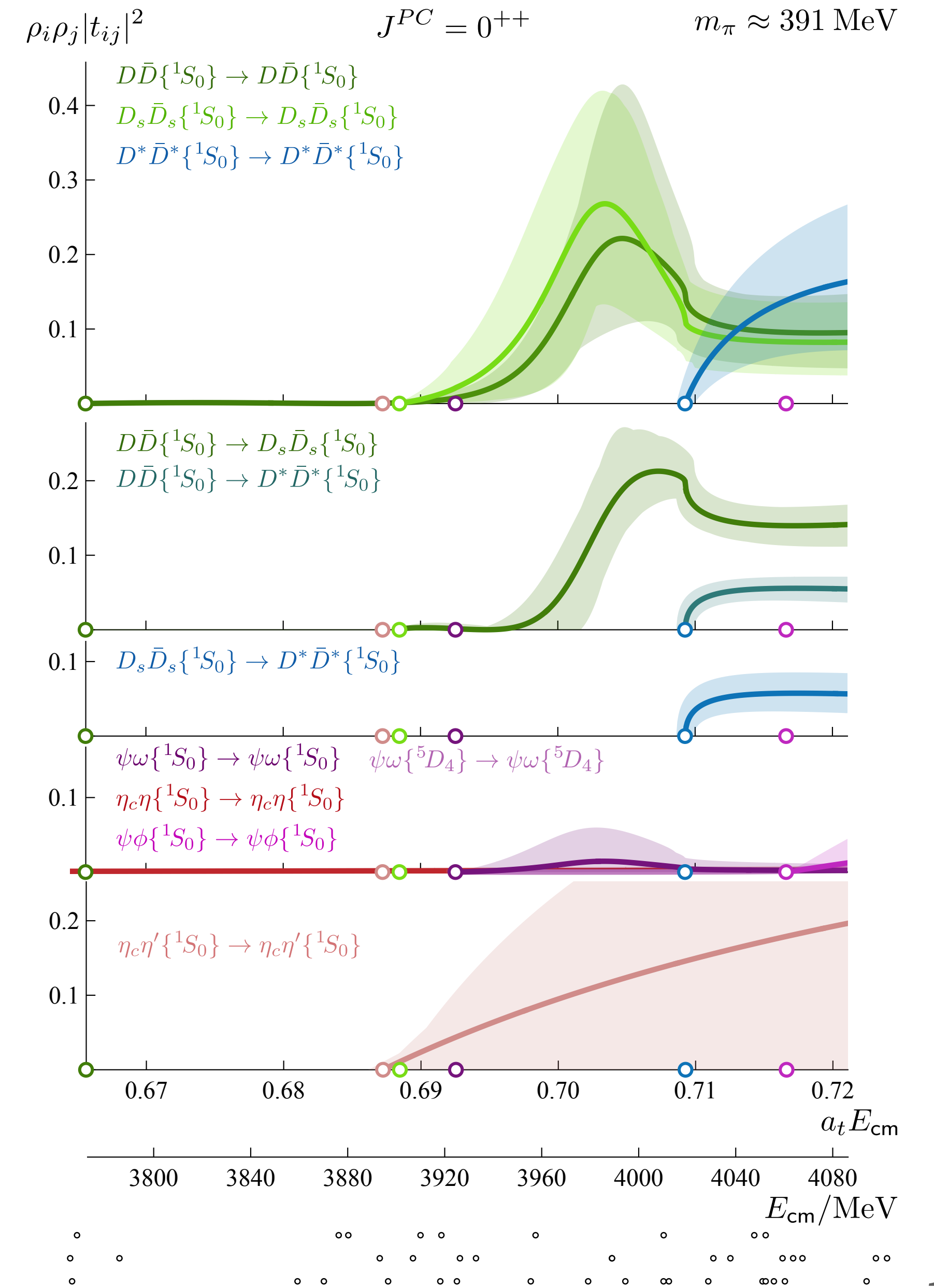
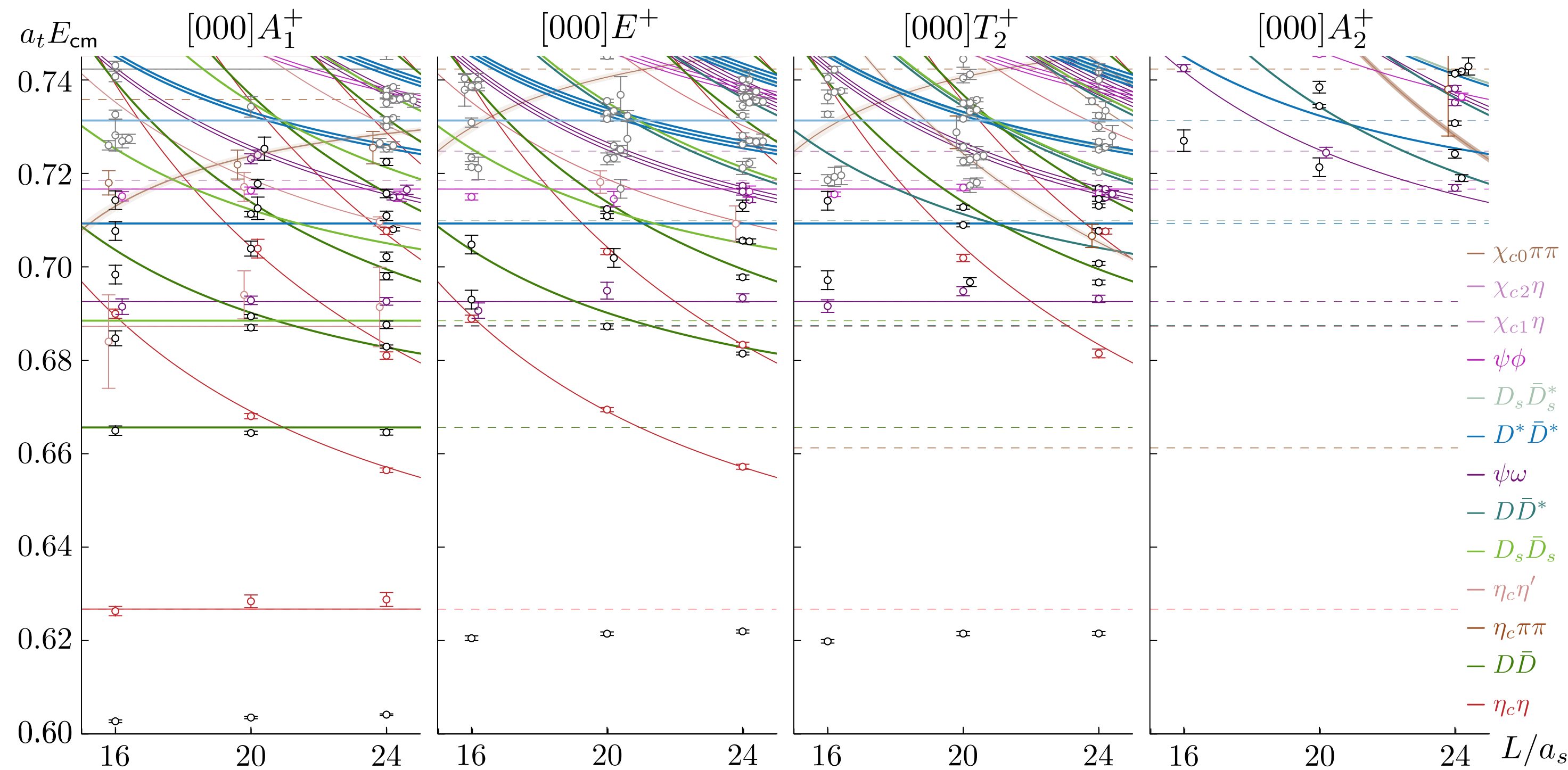
J.J. Dudek et al. [HadSpec]
Phys. Rev. Lett. **113**, 182001 (2014)

had spec

Two-Hadron Systems

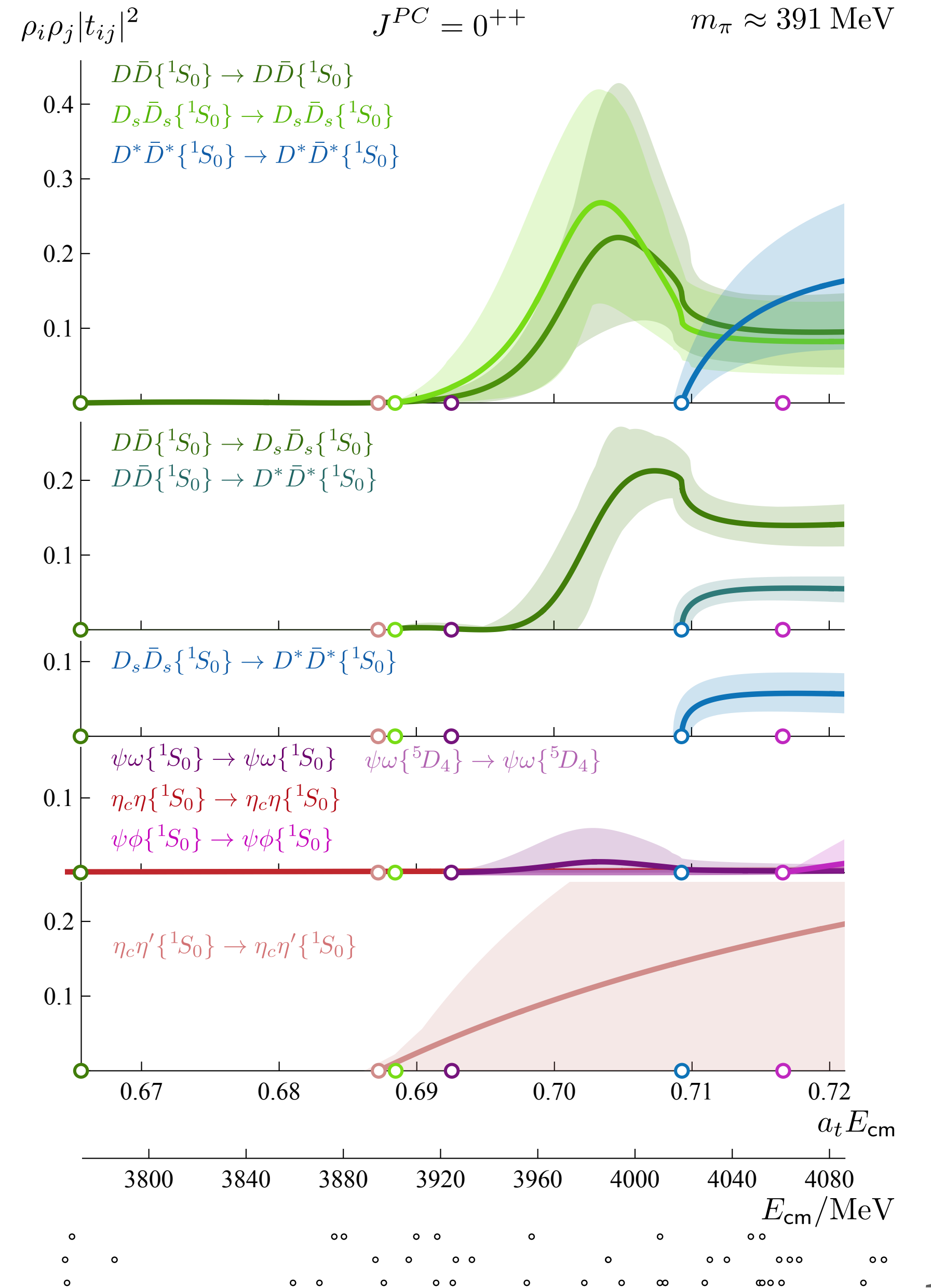
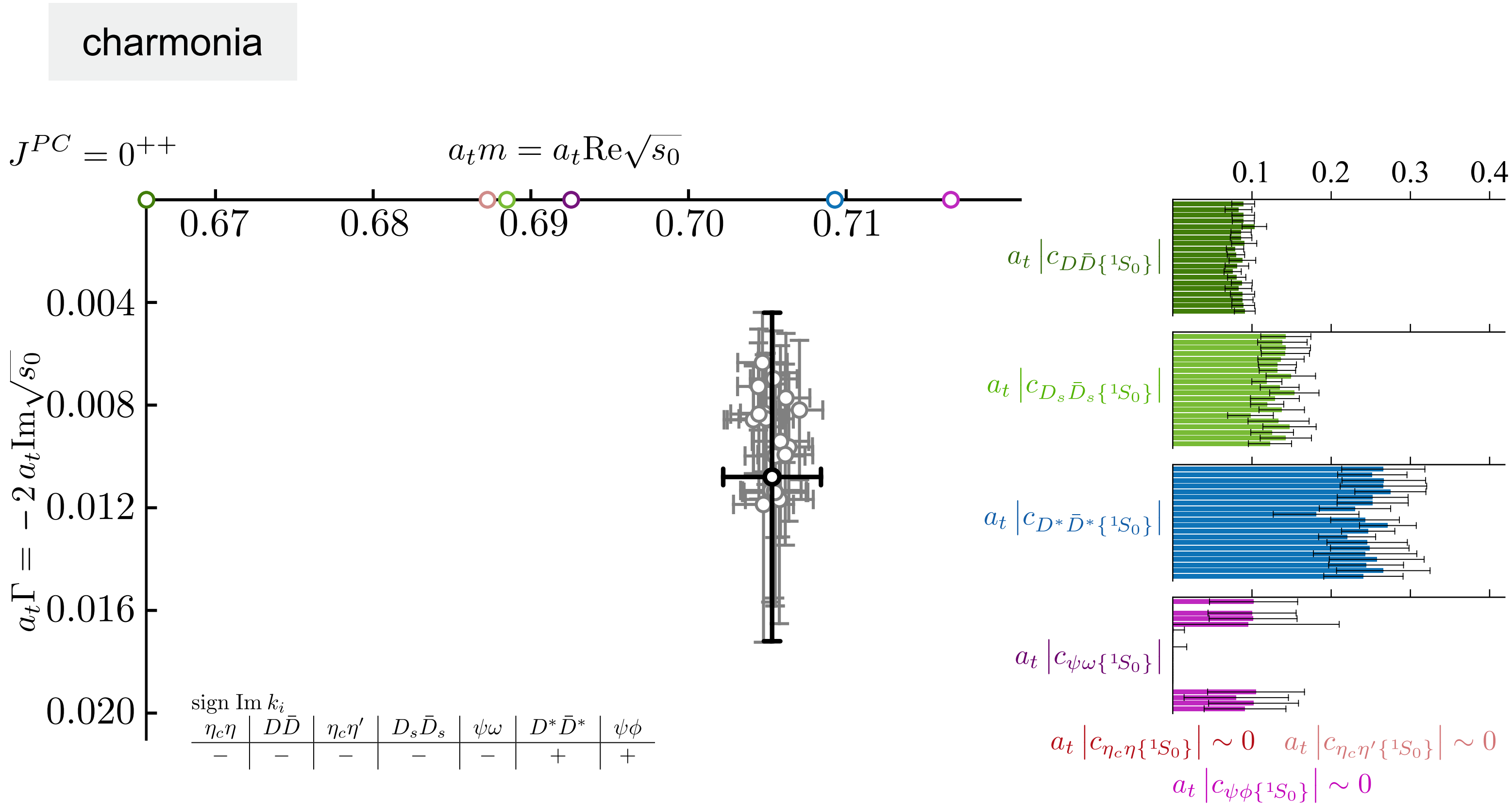
Many systems investigated, including coupled channels

charmonia



Two-Hadron Systems

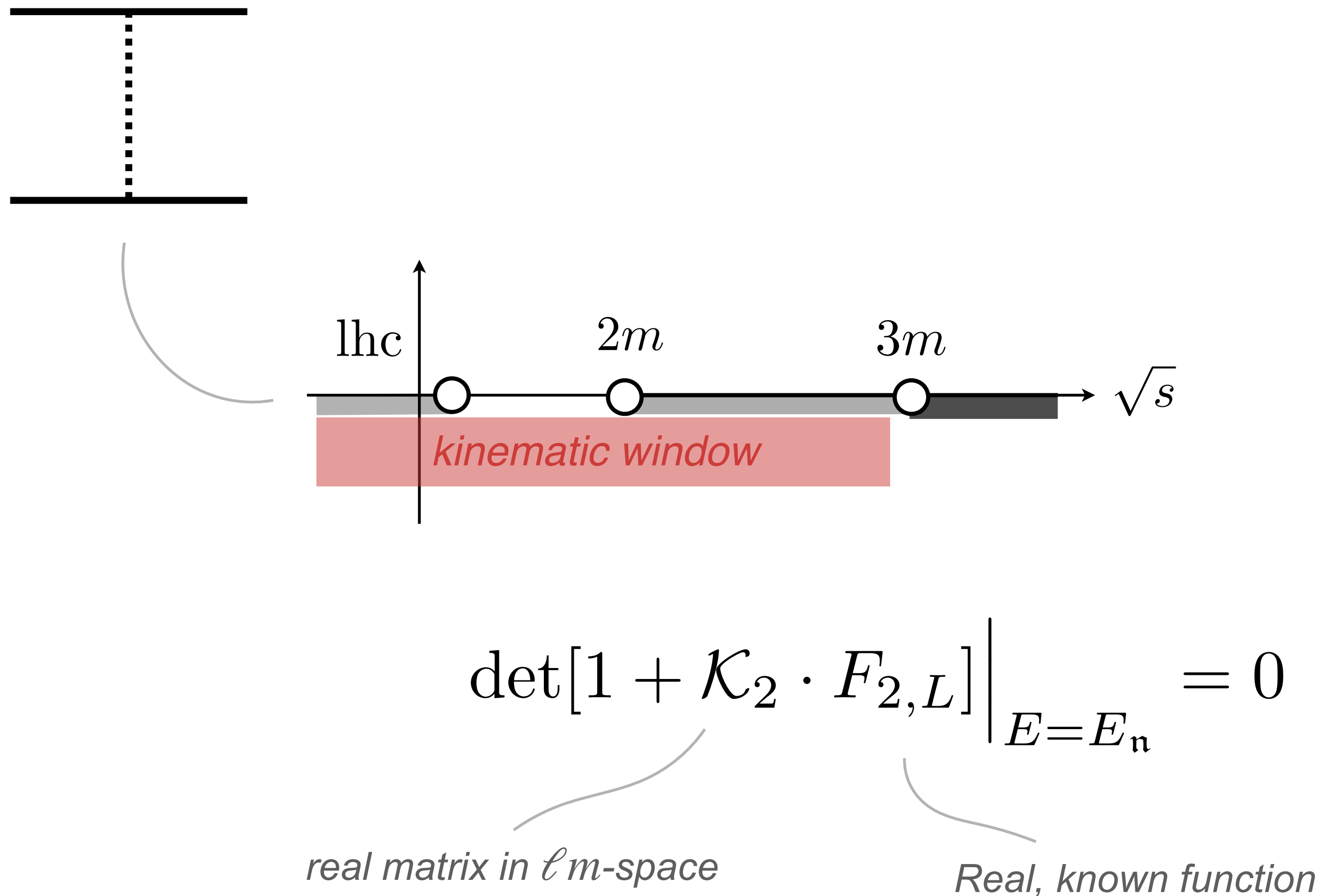
Many systems investigated, including coupled channels



Including Left-Hand Cuts

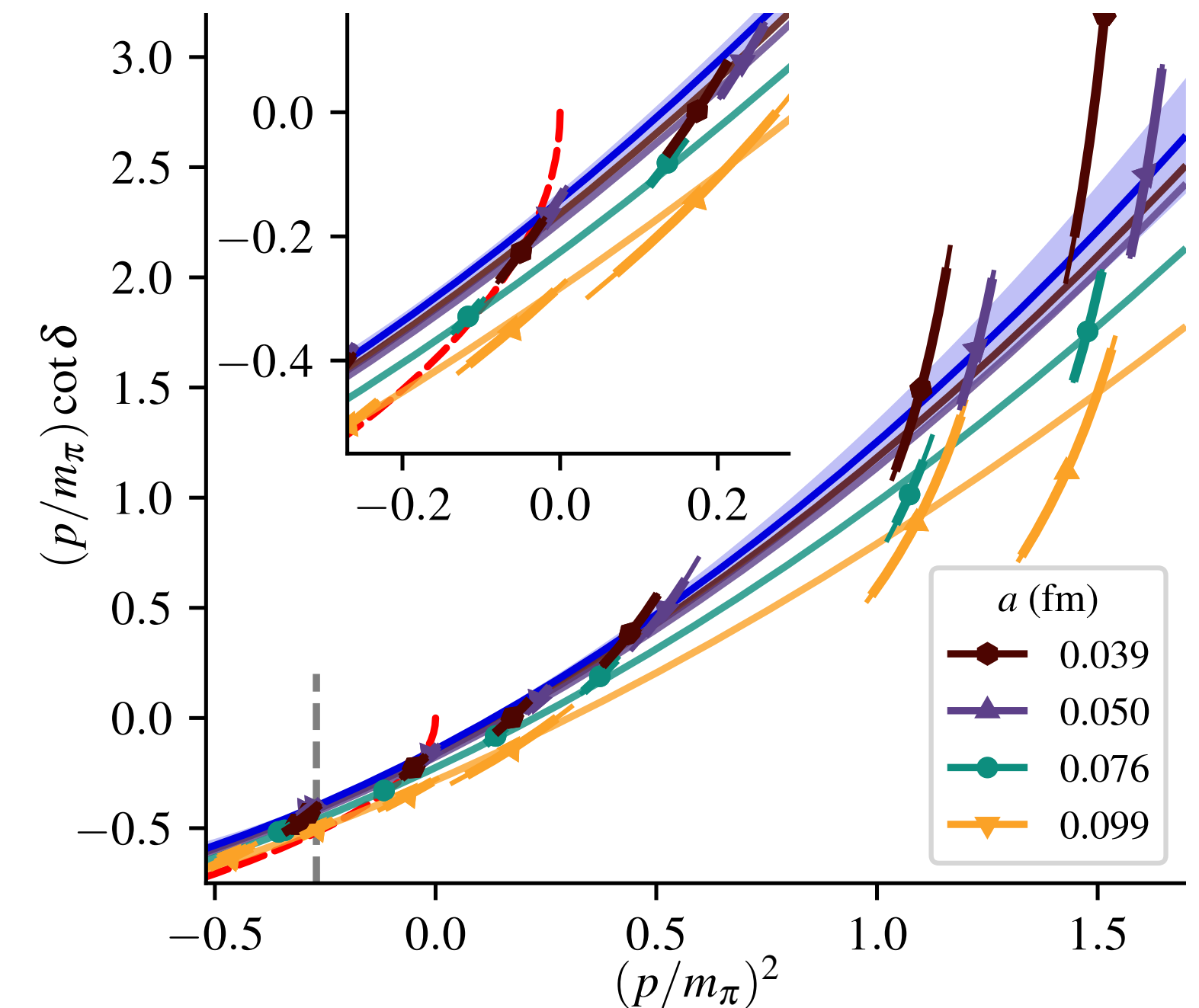
Certain systems may exhibit sensitivity to left-hand cuts

- Thresholds could be such that exchange cuts lie near thresholds
- Finite-volume spectra may lie below left-hand cuts $\implies$ simple Lüscher condition invalid



H-dibaryon at the SU(3)_F-symmetric point $m_\pi \sim 417$ MeV

J. R. Green, A. D. Hanlon, P. M. Junnarkar, H. Wittig,
Phys. Rev. Lett. **127** (24) (2021) 242003.



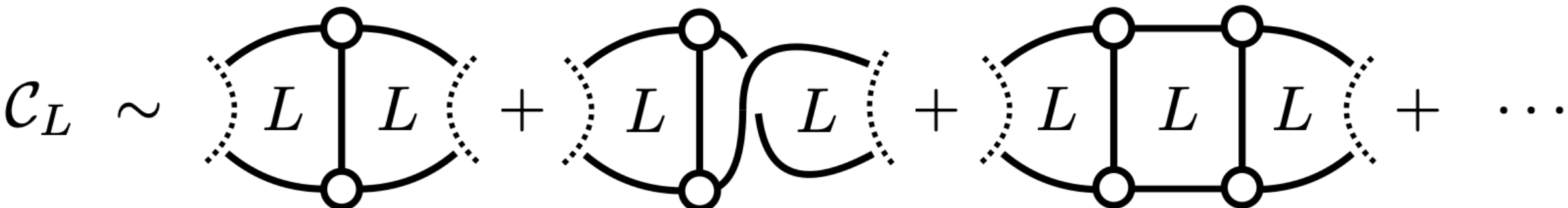
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Generalization — include one-particle exchange explicitly in diagrammatic expansion

$$\det_{aJm_J\ell S} [\mathcal{M}_0^{-1} + F + \mathcal{C}_L] = 0$$


 $\mathcal{C}_L \sim$

$$\mathcal{M}^{J^P} = \mathcal{M}_{\mathcal{E}}^{J^P} + \left(1 + i\mathcal{L}^{J^P}\right) \left[\left(\mathcal{M}_0^{J^P}\right)^{-1} + \mathcal{C}^{J^P} \right]^{-1} \left(1 + i\mathcal{R}^{J^P}\right)$$

A. Baião Raposo and M. T. Hansen,
JHEP **08** (2024) 075

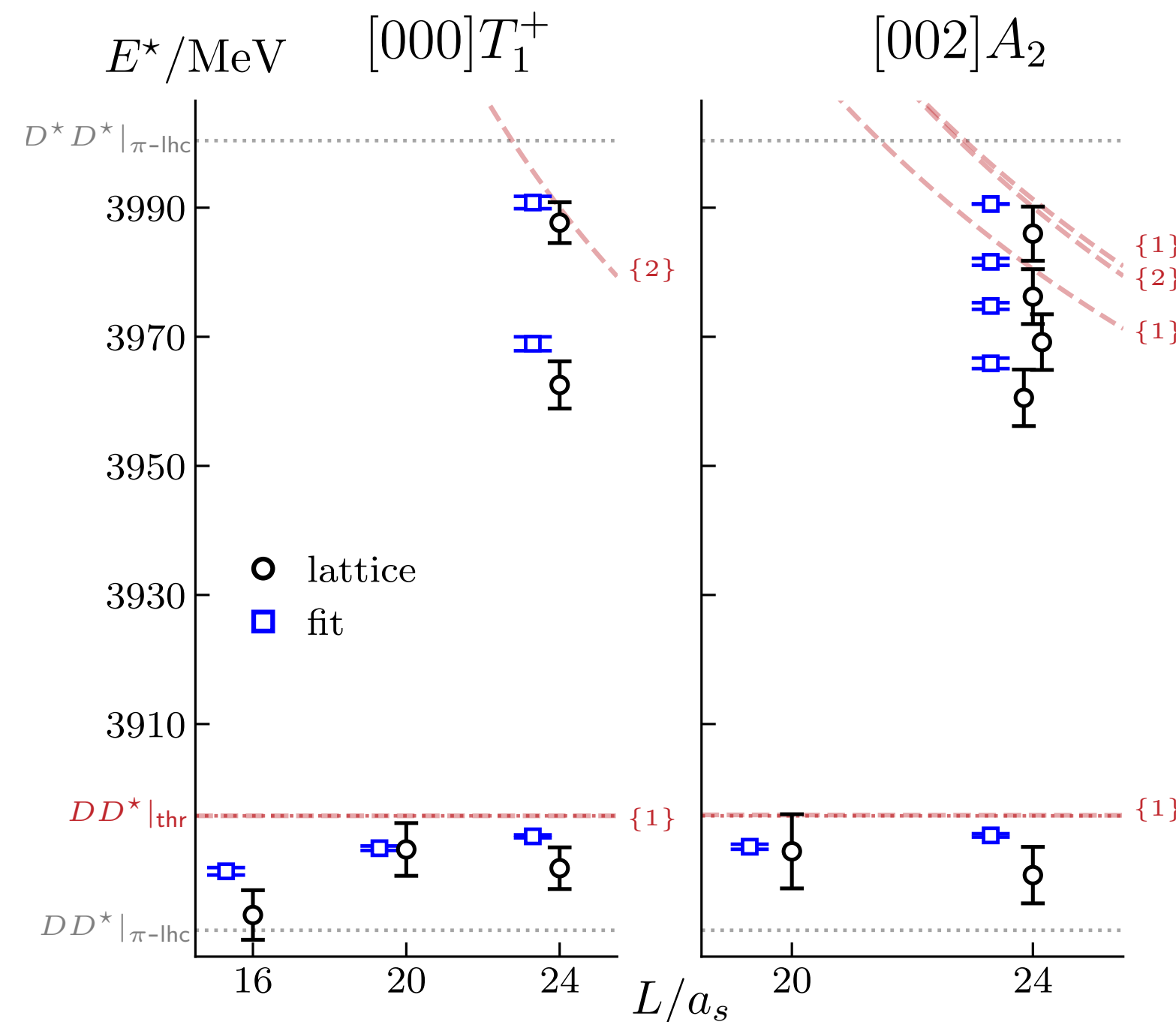
Baião Raposo, R. A. Briceño, M. T. Hansen and AJ,
JHEP **06** (2025) 186

Including Left-Hand Cuts

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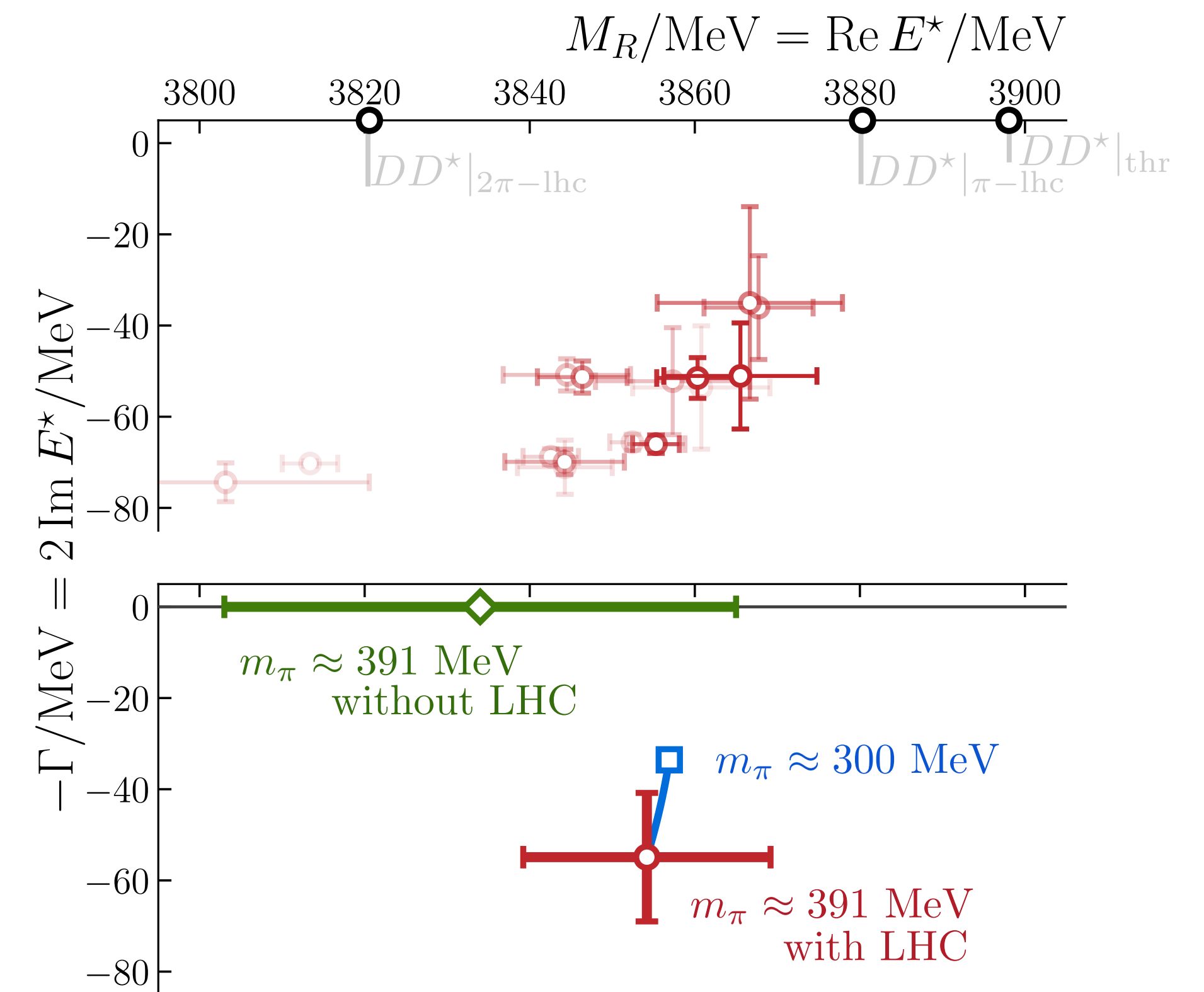
$T_{cc} @ m_\pi \approx 400 \text{ MeV}$



had spec

T. Whyte, D. Wilson, and C. Thomas
Phys. Rev. D **111**, 034511 (2025)

N. Lachini and R. Briceño
arXiv:2607.19009



Including Left-Hand Cuts

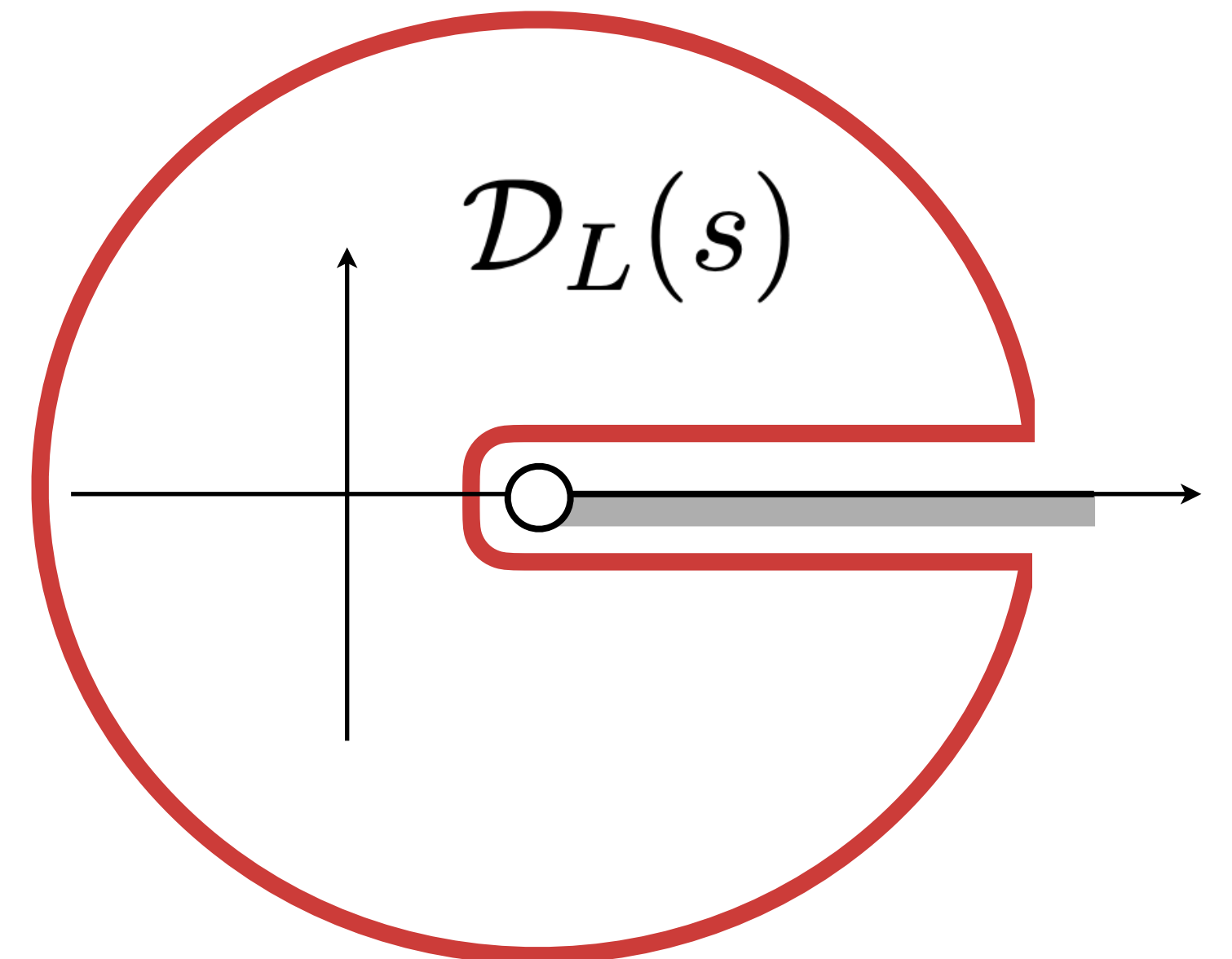
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N/D method provides alternative path to incorporate LHCs

$$\mathcal{M}_L(E) = \mathcal{N}_L(E) [\mathcal{D}_L(E)]^{-1}$$


$$\mathcal{D}_L(s) = 1 + \frac{1}{L^3} \sum_{\mathbf{k}} \frac{1}{s - E^*(\mathbf{k})^2} \mathcal{R}(\mathbf{k}) \mathcal{N}(E^*(\mathbf{k}))$$

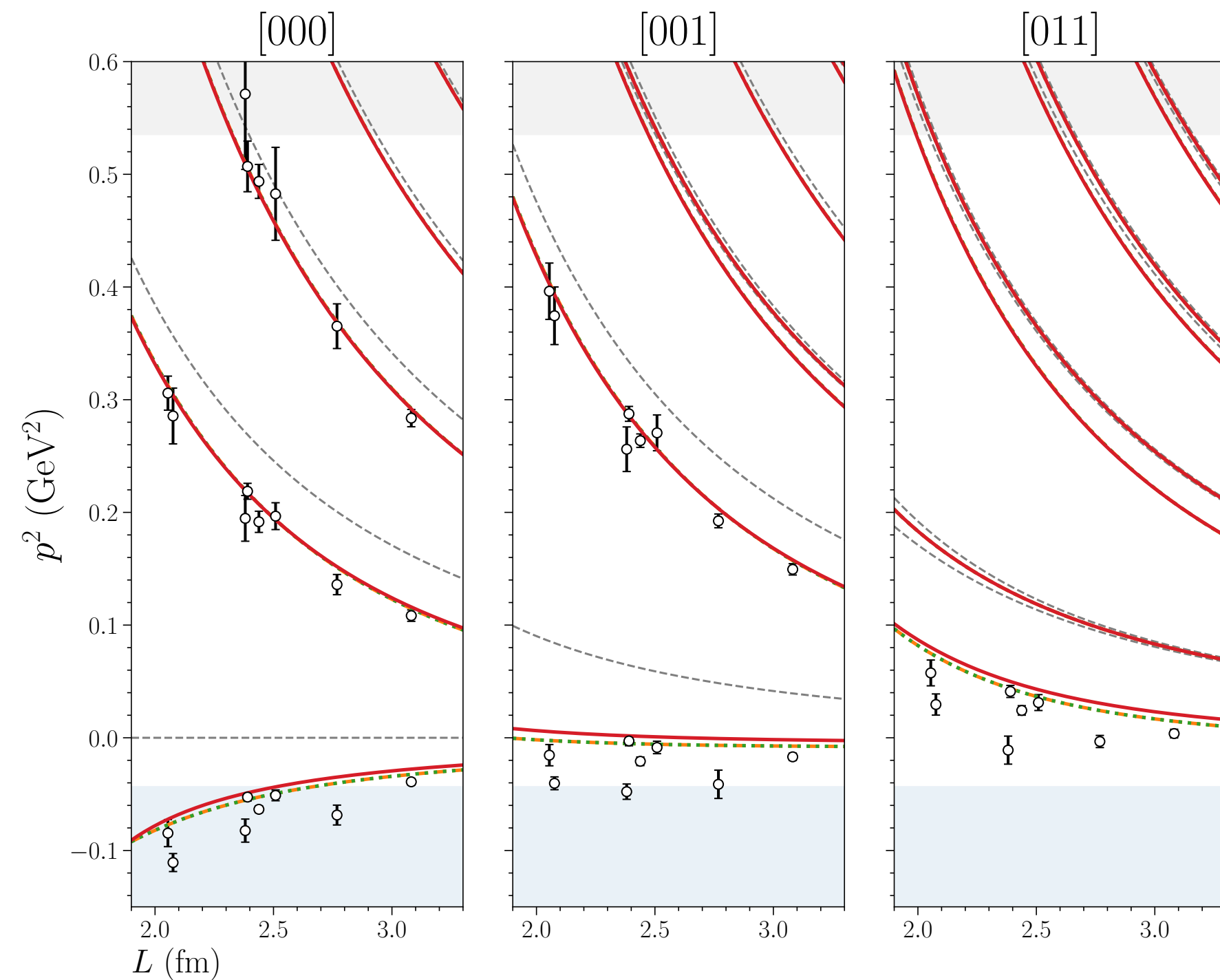


Including Left-Hand Cuts

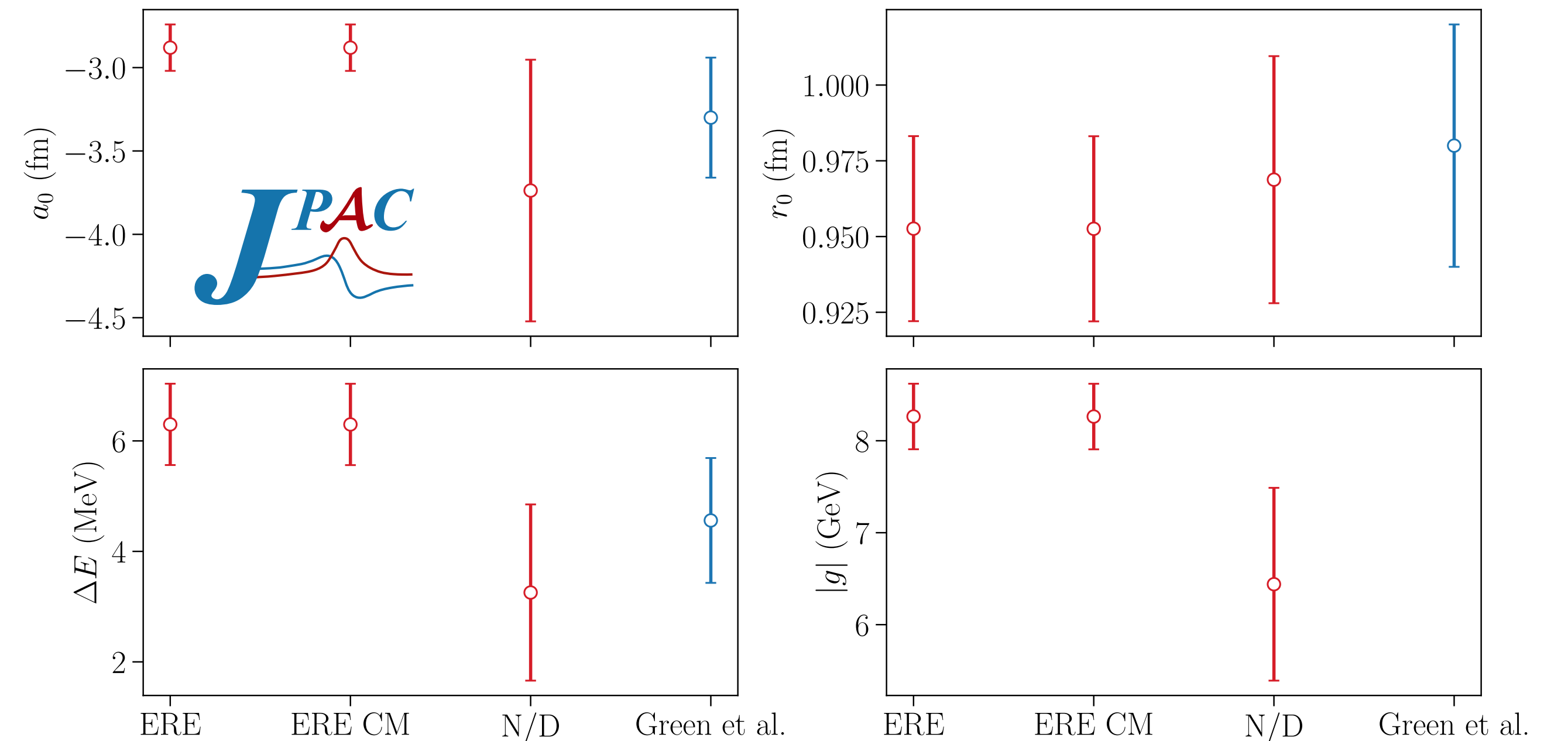
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H-dibaryon at the $SU(3)_F$ -symmetric point



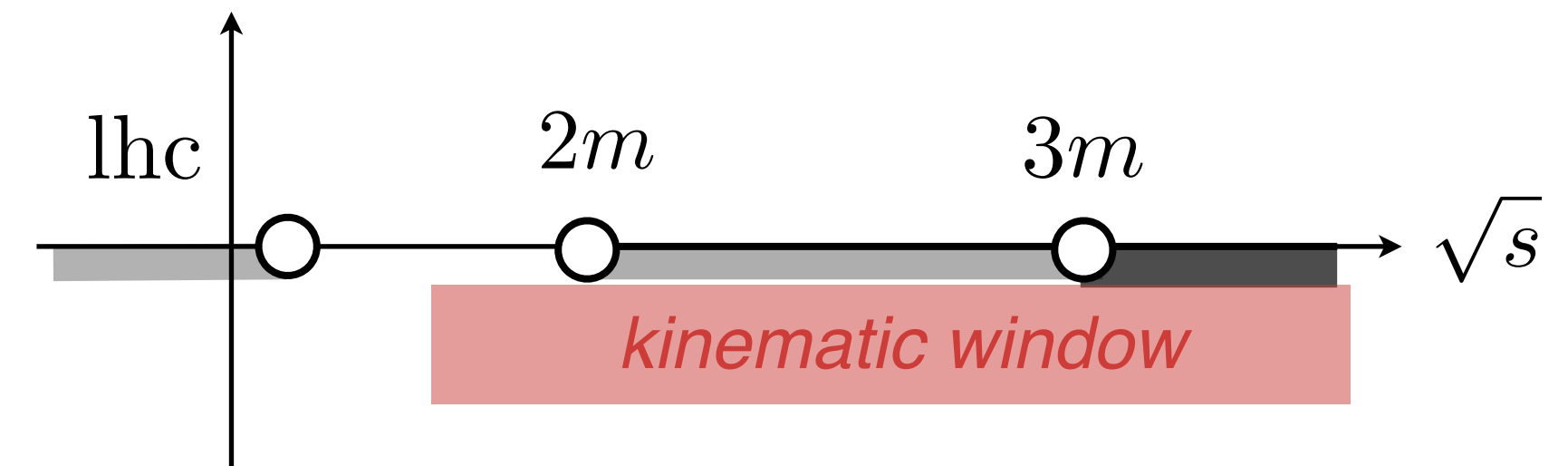
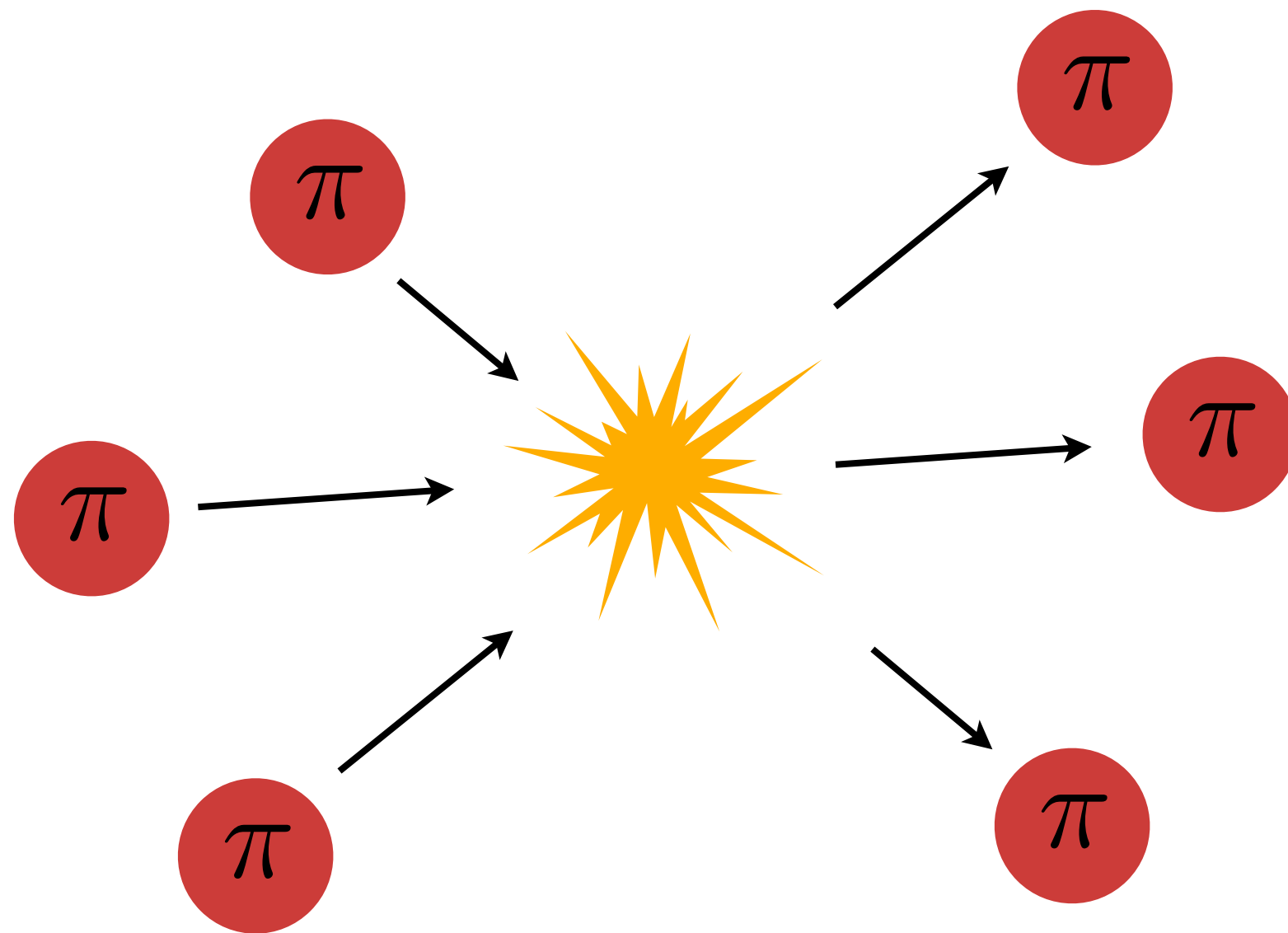
A. Rodas et al [JPAC],
arXiv:2605.22957v1 (2026)



Three-Hadron Systems

Many hadronic resonances couple to three hadrons.

- Advancements in past decade generalizing Lüscher framework to three-body processes
- Many contributions formally and numerically, both in finite- and infinite-volumes
- Will skip many details and progress...



Equivalence proofs

M. Hansen and S. Sharpe
Phys. Rev. D **90**, (2014) 116003

M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

and many more....

AJ et al. [JPAC]
Phys. Rev. D **100**, 034508 (2019)

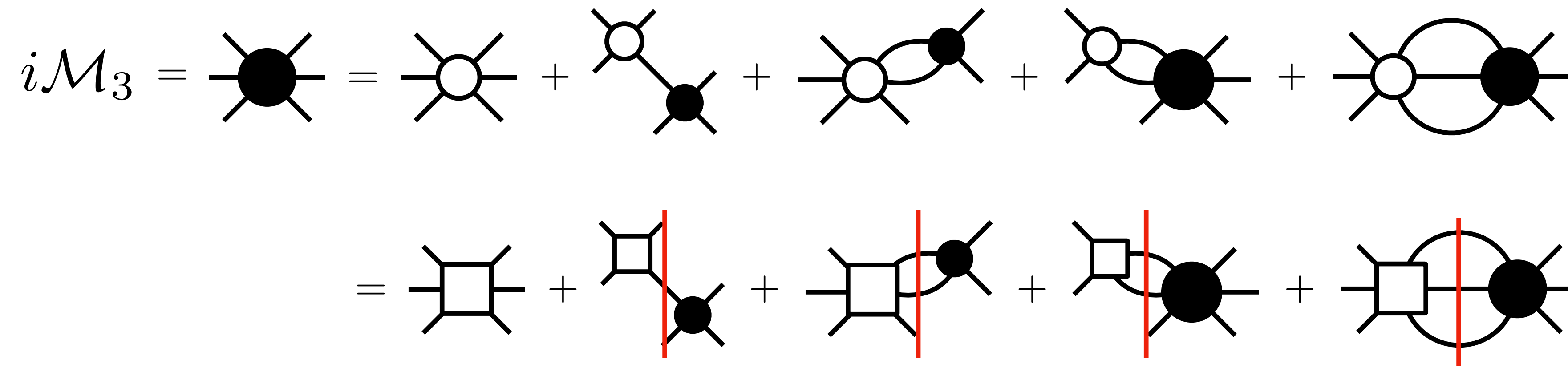
T. Blanton and S. Sharpe
Phys. Rev. D **102**, 054515 (2020)

AJ
Phys. Rev. D **108**, 034505 (2023)

Three-Hadron Systems

Repeat same procedure for three-body case

- S-matrix unitarity enforces K matrix structure
- Have integral equations and more complicated analytic structures

$$i\mathcal{M}_3 = \text{diagram} = \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram}$$


K-Matrix

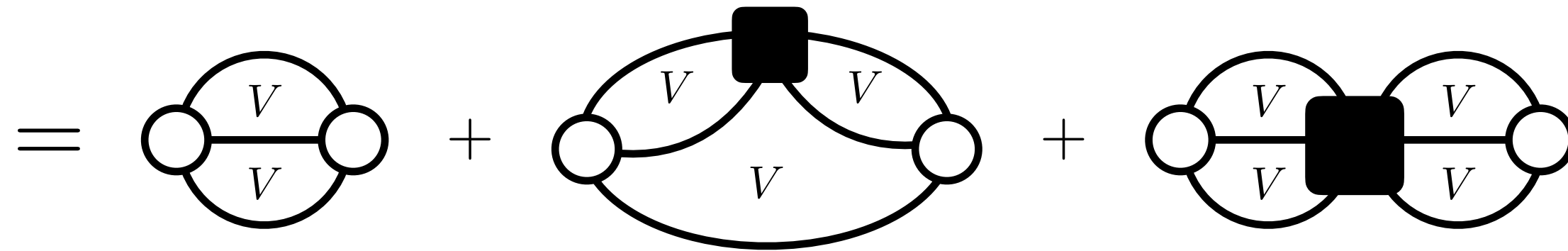
- short-distance dynamics
- cutoff-dependent (unphysical)
- constrained from Lattice QCD

Three-Hadron Systems

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- Have integral equations and more complicated analytic structures

$$\mathcal{C}_L^{(M)}(E) = i \sum_{\mathbf{n}} \frac{Z_{\mathbf{n}} Z_{\mathbf{n}}^\dagger}{E - E_{\mathbf{n}}}$$



$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_3 \cdot F_{3,L}}$$

Spectrum satisfies

Quantization Condition

$$\det [1 + \mathcal{K}_3 \cdot F_{3,L}] \Big|_{E=E_{\mathbf{n}}} = 0$$

Three-Pion Scattering

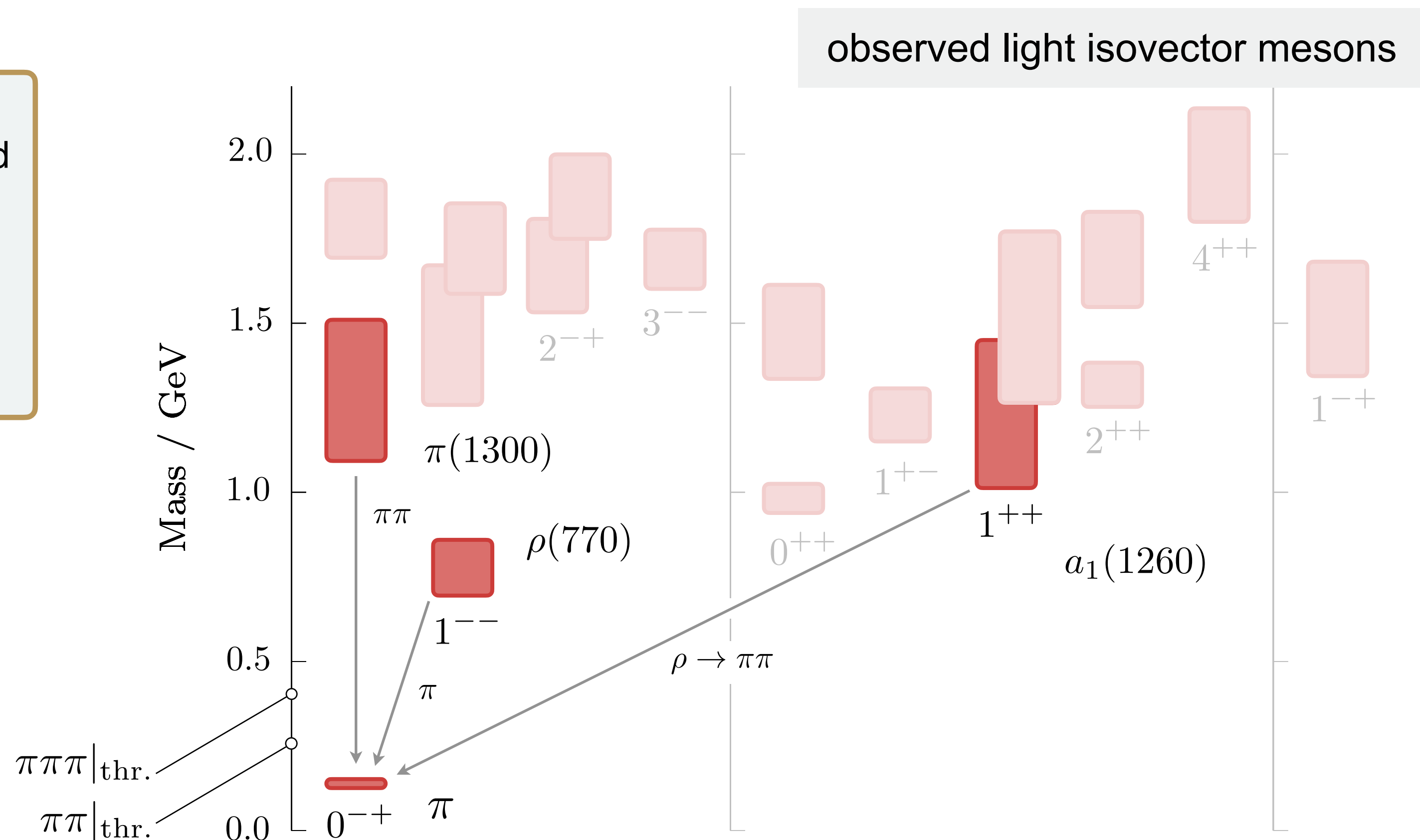
Focus on three-pion systems

- Many states couple to three pions — decades of phenomenological knowledge
- Three-pion amplitudes essential to other hadronic phenomena
- ‘Simple’ playground to test new formalisms, construct observable frameworks

Three-Hadron Systems

Studying three-hadron states is considerably more involved

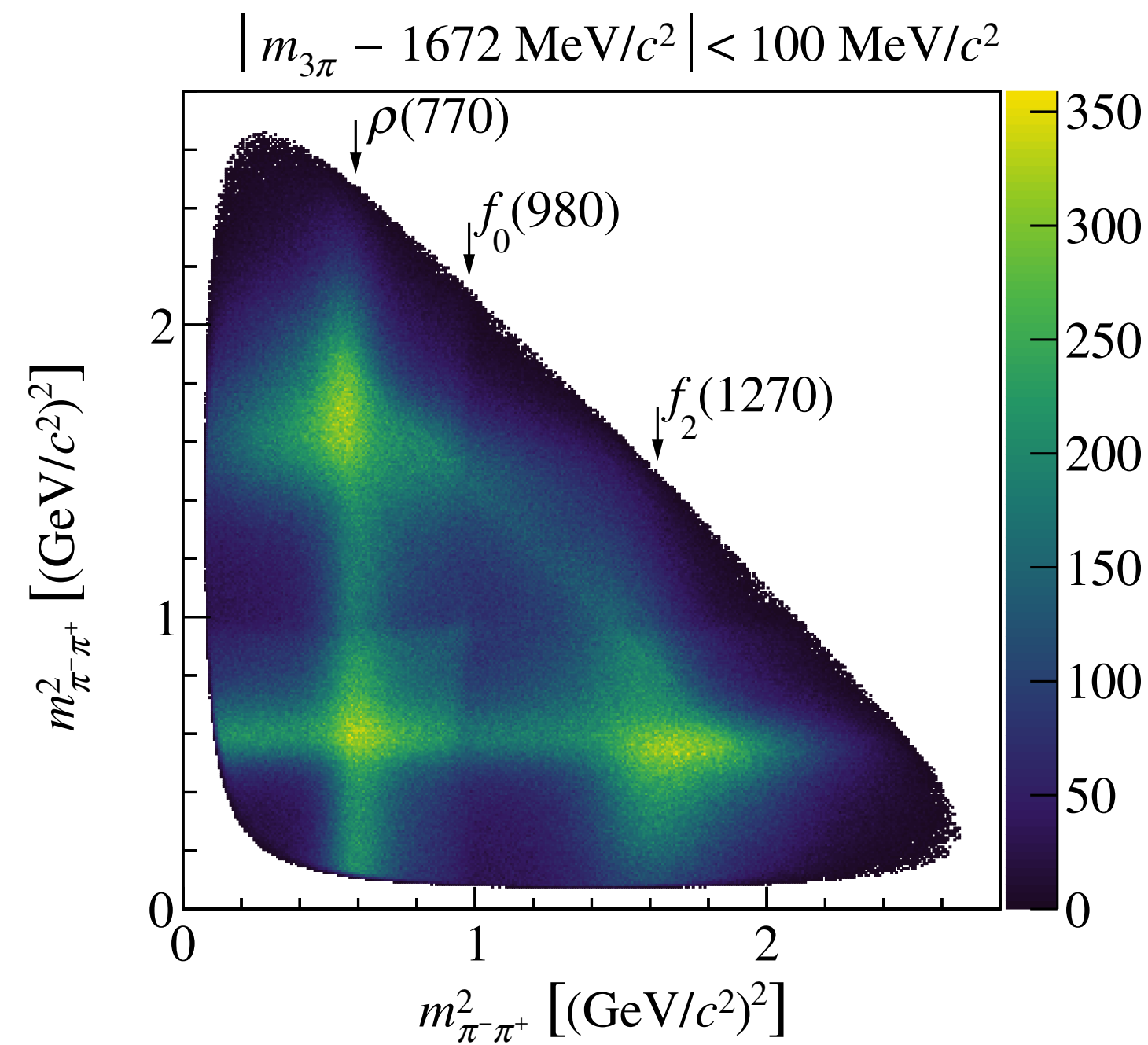
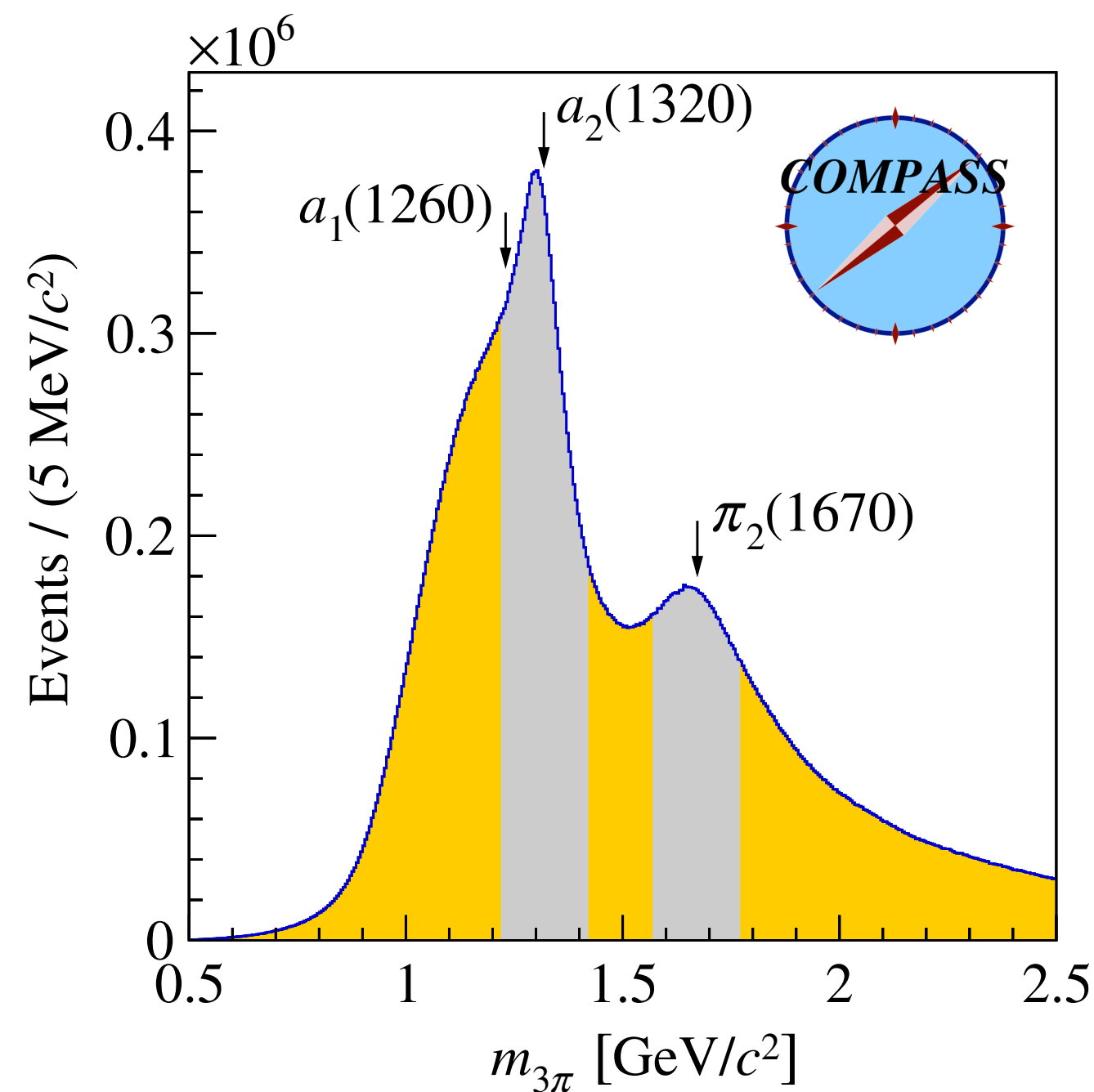
- No simple notion of *phase shift*
- Kinematic rescattering effects ubiquitous
- Relation between K matrices to amplitudes involves integral equations — more analysis required



Three-Pion Scattering

Focus on three-pion systems

- Many states couple to three pions — decades of phenomenological knowledge
- Three-pion amplitudes essential to other hadronic phenomena
- ‘Simple’ playground to test new formalisms, construct observable frameworks



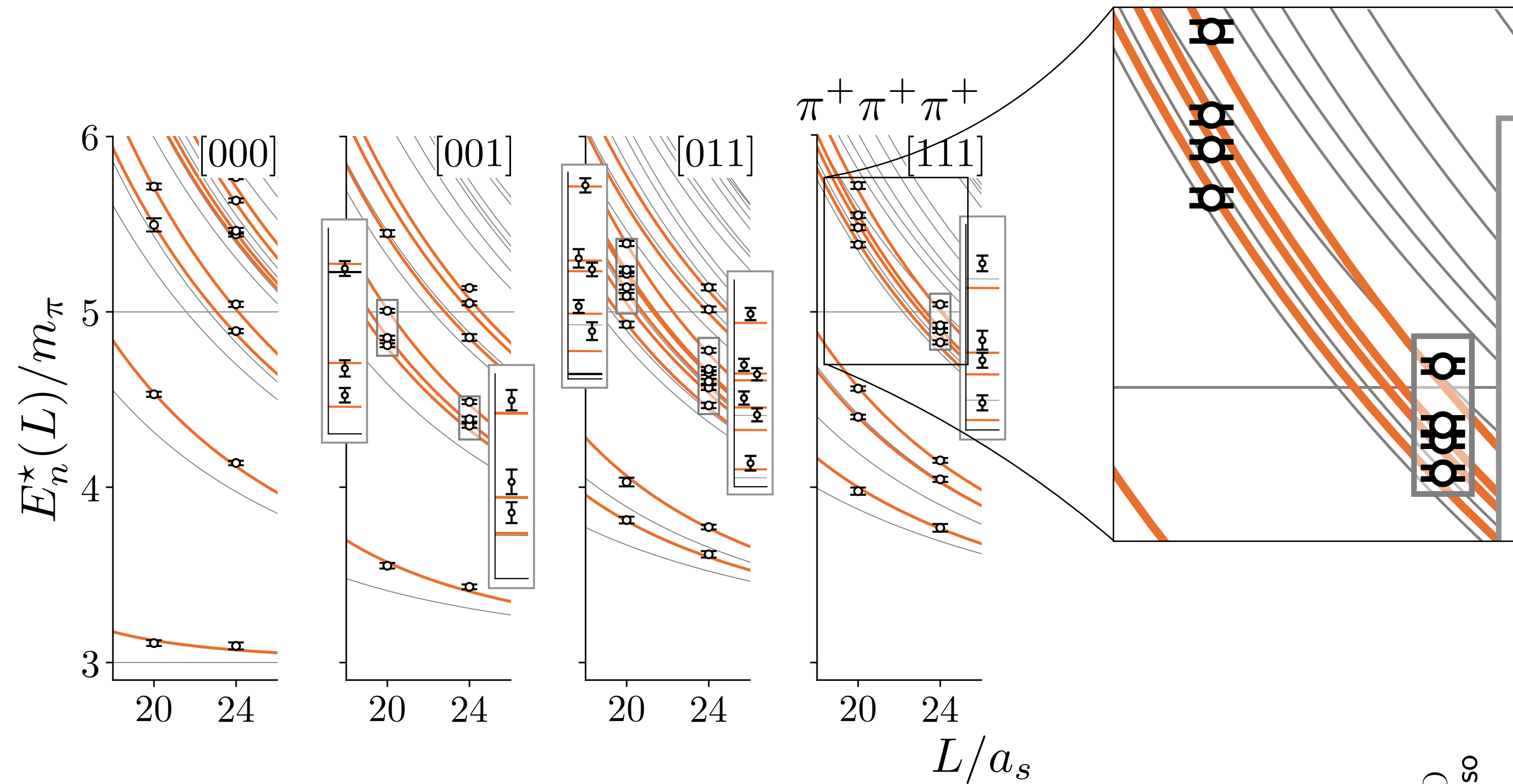
Motivation

Explore three-pion systems with lattice QCD — establish analysis framework in manner similar to experiment

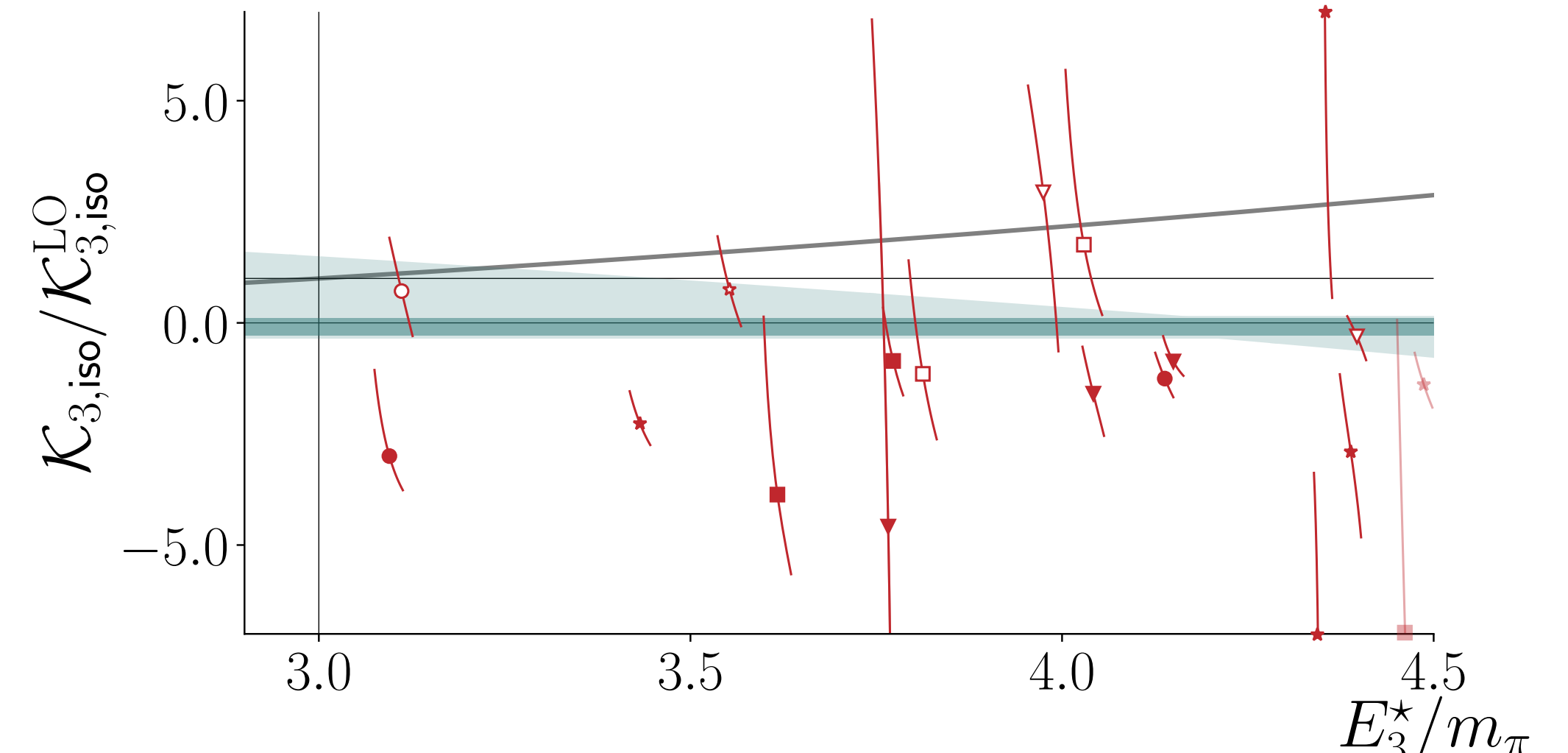
Three-Pion Scattering from Lattice QCD

First applications were in $3\pi^+$ in $I(J^P) = 3(0^-)$

- Simple application without contamination from higher partial waves



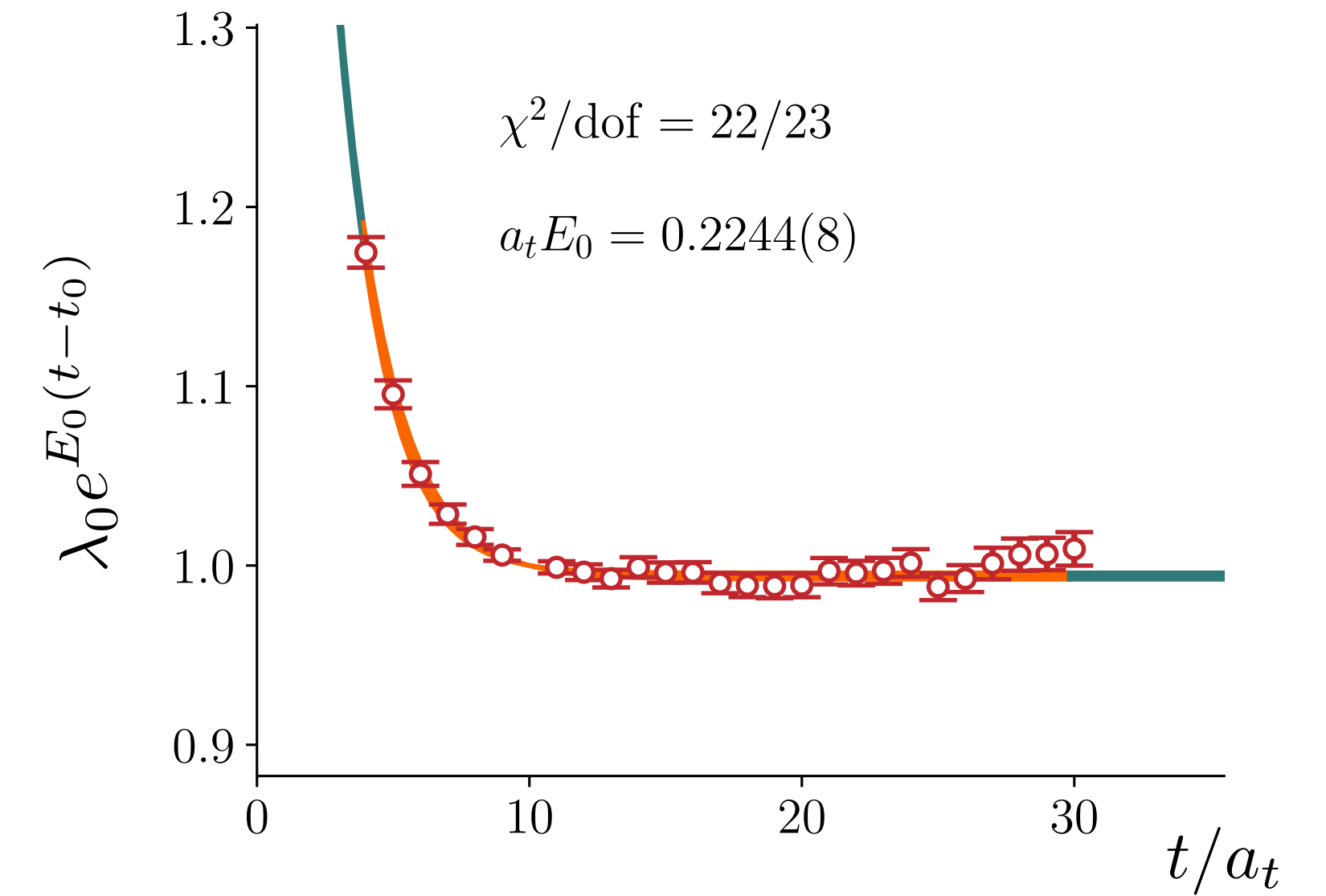
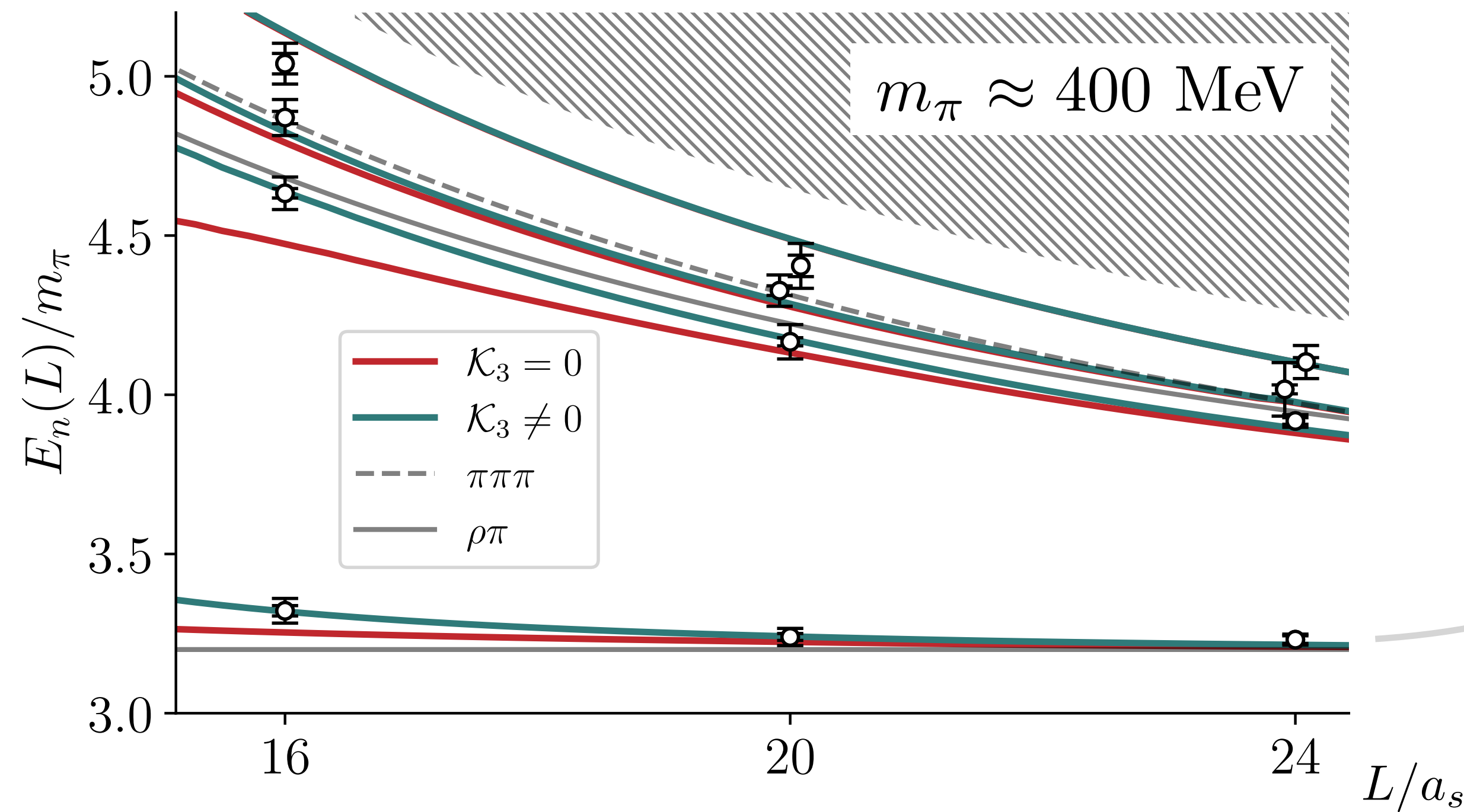
103 energy levels described by three numbers m_π , $a_{\pi\pi}^{I=2}$, $\mathcal{K}_{3,\text{iso}}^{I=3}$



Three-Pion Scattering from Lattice QCD

New results for three pions in $I = 2, J^P = 1^+$

- Two-pion sub-channel resonances, no three-pion resonances
- Strongly coupled partial waves $([\pi\pi]_S^2\pi)_P, ([\pi\pi]_P^1\pi)_S, ([\pi\pi]_P^1\pi)_D$



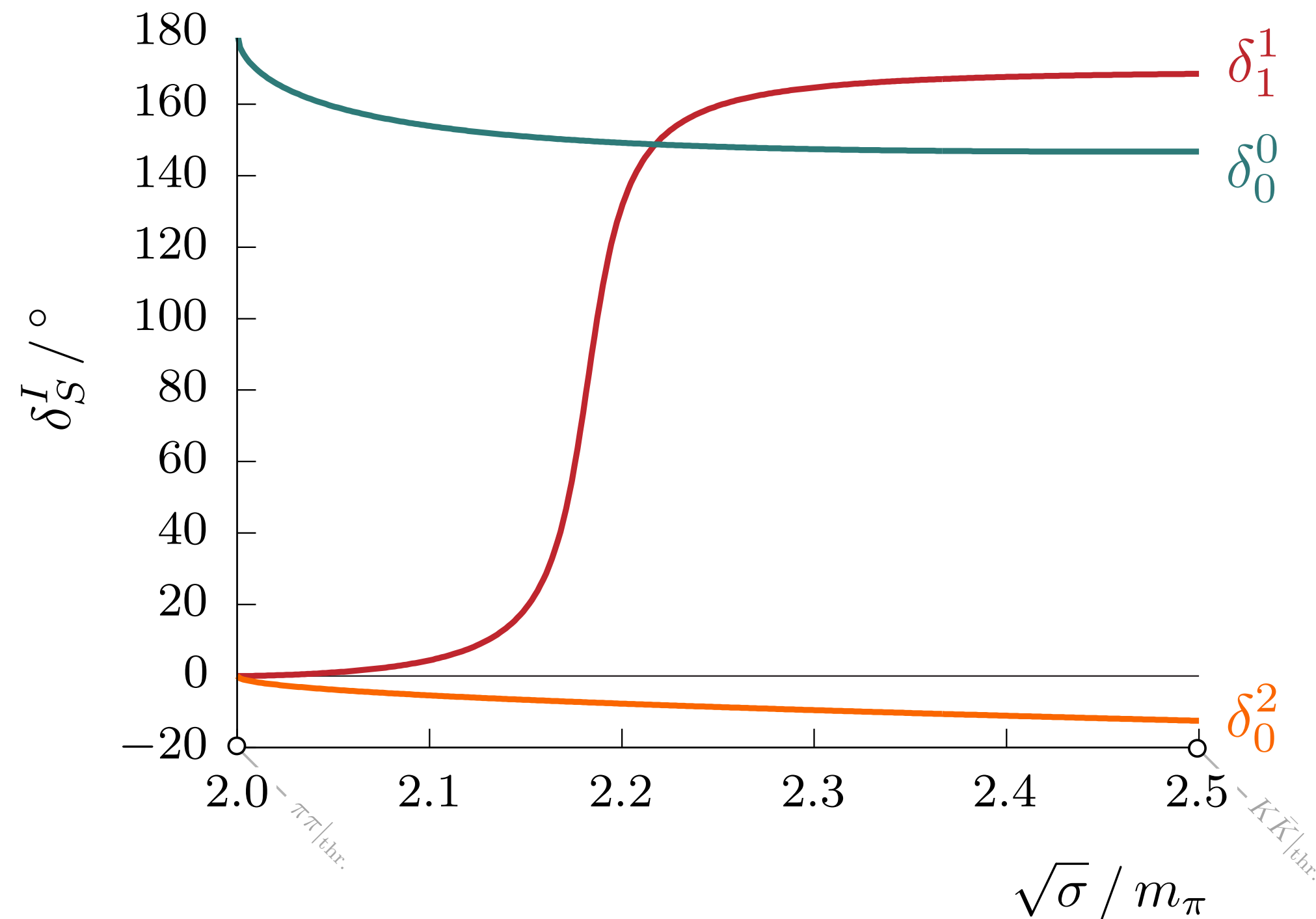
$(L/a_s)^3 \times (T/a_t)$	N_{cfgs}	N_{tsrc}	N_v
$16^3 \times 128$	479	4	64
$20^3 \times 256$	286	4	128
$24^3 \times 128$	553	1	160

Three-Pion Scattering from Lattice QCD

Map finite-volume spectrum to K matrices

- Use previously lattice QCD determined $\pi\pi$ phase shifts (input)

Two-pion phase shifts @ $m_\pi \approx 400$ MeV



$$\det_{I,\mathbf{k},S,m} [\mathbf{F}_3^{-1}(E_n, L|\boldsymbol{\eta}_2) + \mathbf{K}_3(E_n, |\boldsymbol{\eta}_3)] = 0$$

$$\mathbf{F}_3(E, L|\boldsymbol{\eta}_2) = \frac{1}{L^3} \left[\left(\mathbf{F}_2(E, L) + \mathbf{G}(E, L) \right)^{-1} + \mathbf{K}_2(E|\boldsymbol{\eta}_2) \right]^{-1}$$

J.Dudek, R.Edwards, & C.Thomas
Phys.Rev.D **86** (2012) 034031

J.Dudek, R.Edwards, & C.Thomas
Phys.Rev.D, **87** (2013) 034505

R. Briceño, J.Dudek, R.Edwards, & D.Wilson
Phys.Rev.Lett. **118** (2017) 2, 022002

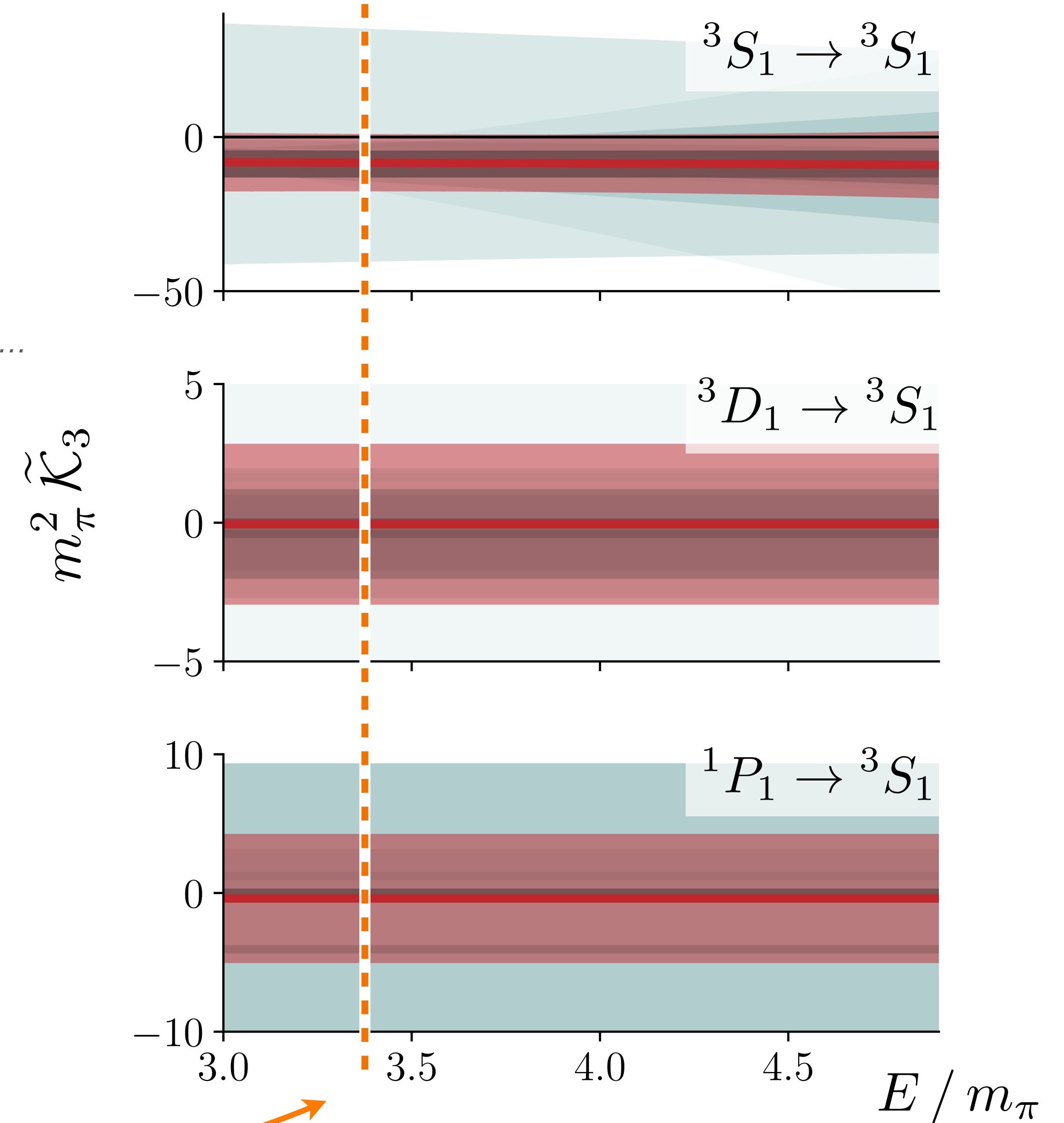
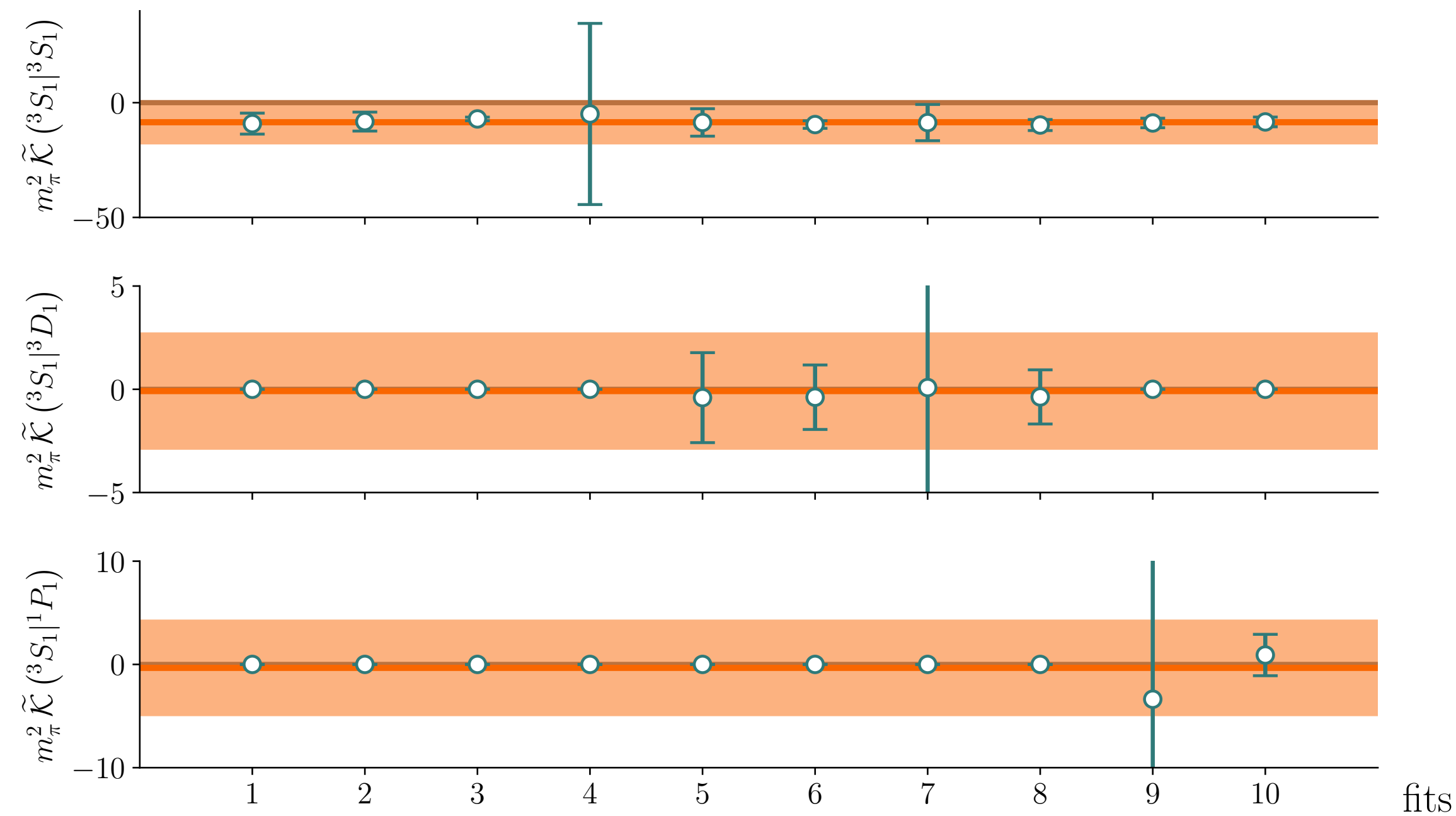
Three-Pion Scattering from Lattice QCD

Map finite-volume spectrum to K matrices

- Parameterize $\mathbf{K}_3$ with many flexible parameterizations

$$\mathbf{K}_3 = \mathbf{h} \tilde{\mathcal{K}}_3 \mathbf{h}$$

Known barrier factors, subduction coefficients,...



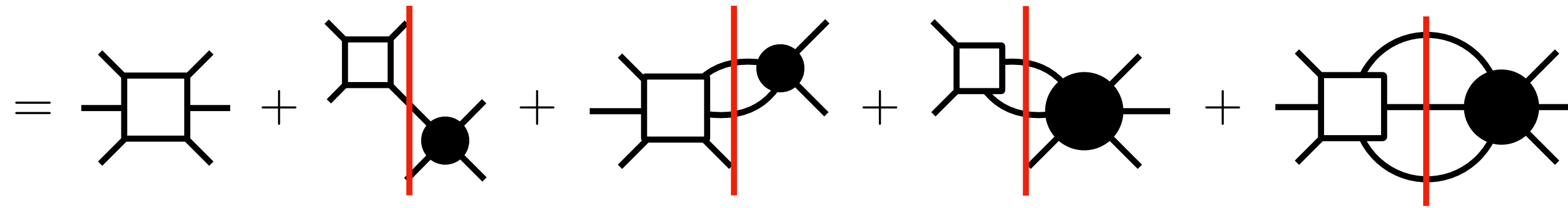
Three-Pion Scattering from Lattice QCD

Reconstruct amplitudes via set of integral equations

- Subchannel resonances, strongly coupled partial waves

$$\mathcal{M}_3 = \sum_{J^P} \mathcal{M}_3^{J^P} \mathcal{R}_{J^P}$$

K matrix representation is set of integral equations



Integral equations are solved numerically via Nyström method
— Approximate integral by quadrature, solve linear system
— care must be taken since kernels contain singularities

AJ
Phys.Rev.D **108** (2023) 3, 034505

AJ and R. Briceño
Phys.Rev.D **109** (2024) 9, 096030

R. Briceño, C. Costa, AJ
Phys.Rev.D **111** (2025) 3, 036029

S. Dawid, M. Islam, R. Briceño
Phys. Rev. A **108**, (2023) 034016

S. Dawid, M. Islam, R. Briceño, AJ
Phys. Rev. A **109**, (2024) 043325

AJ, N. Chambers, R. Briceño
Phys.Rev.D **113** (2026) 1, 014019

Three-Pion Scattering from Lattice QCD

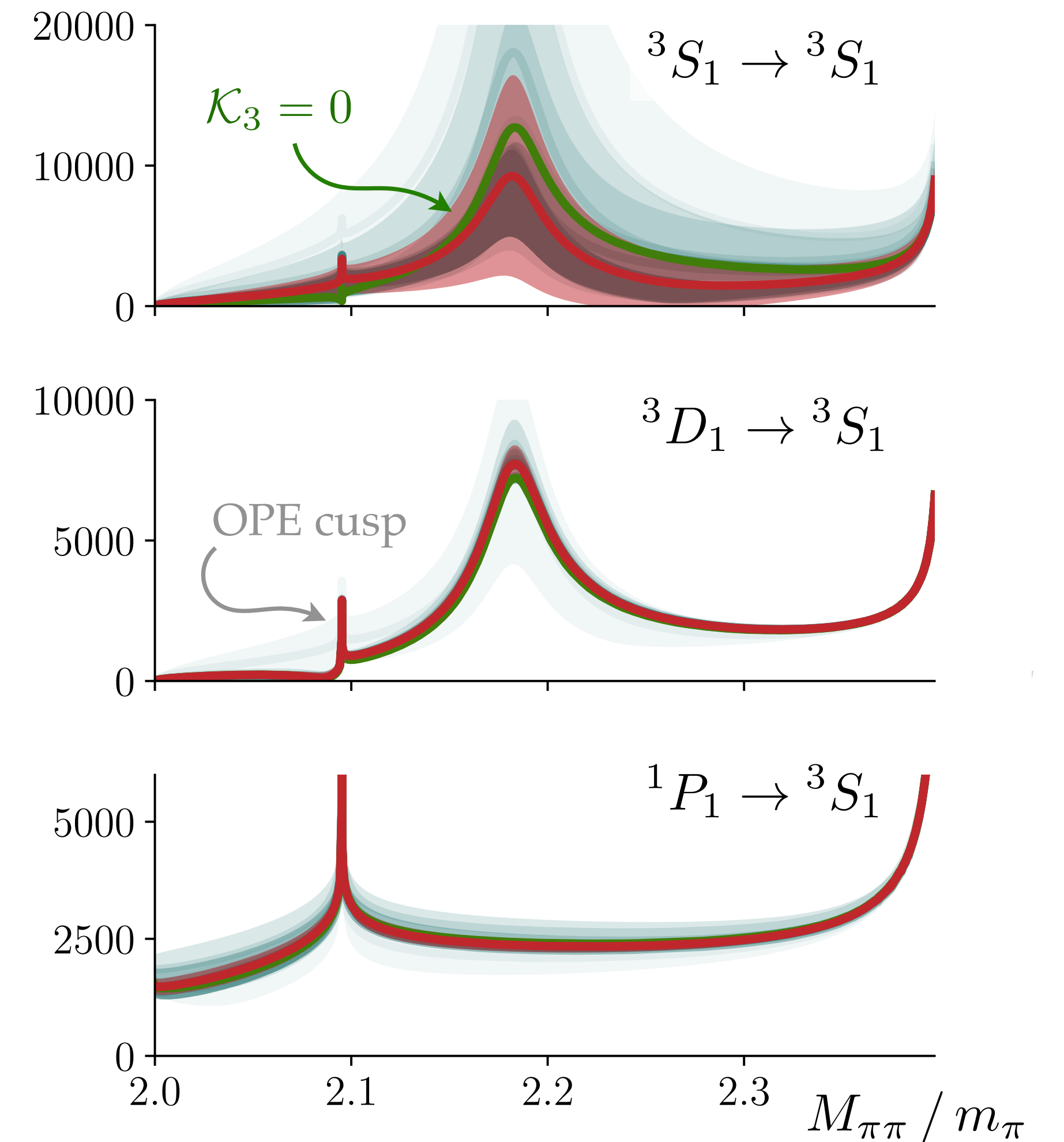
Reconstruct amplitudes via set of integral equations

- Subchannel resonances, strongly coupled partial waves

$$\mathcal{M}_3 = \sum_{J^P} \mathcal{M}_3^{J^P} \mathcal{R}_{J^P}$$

K matrix representation

$$= \text{[Feynman diagrams]} \cdot m_\pi^2 \left| \mathcal{M}_3^{(1,1)} \right|$$



AJ
Phys.Rev.D **108** (2023) 3, 034505

AJ and R. Briceño
Phys.Rev.D **109** (2024) 9, 096030

R. Briceño, C. Costa, AJ
Phys.Rev.D **111** (2025) 3, 036029

S. Dawid, M. Islam, R. Briceño
Phys. Rev. A **108**, (2023) 034016

S. Dawid, M. Islam, R. Briceño, AJ
Phys. Rev. A **109**, (2024) 043325

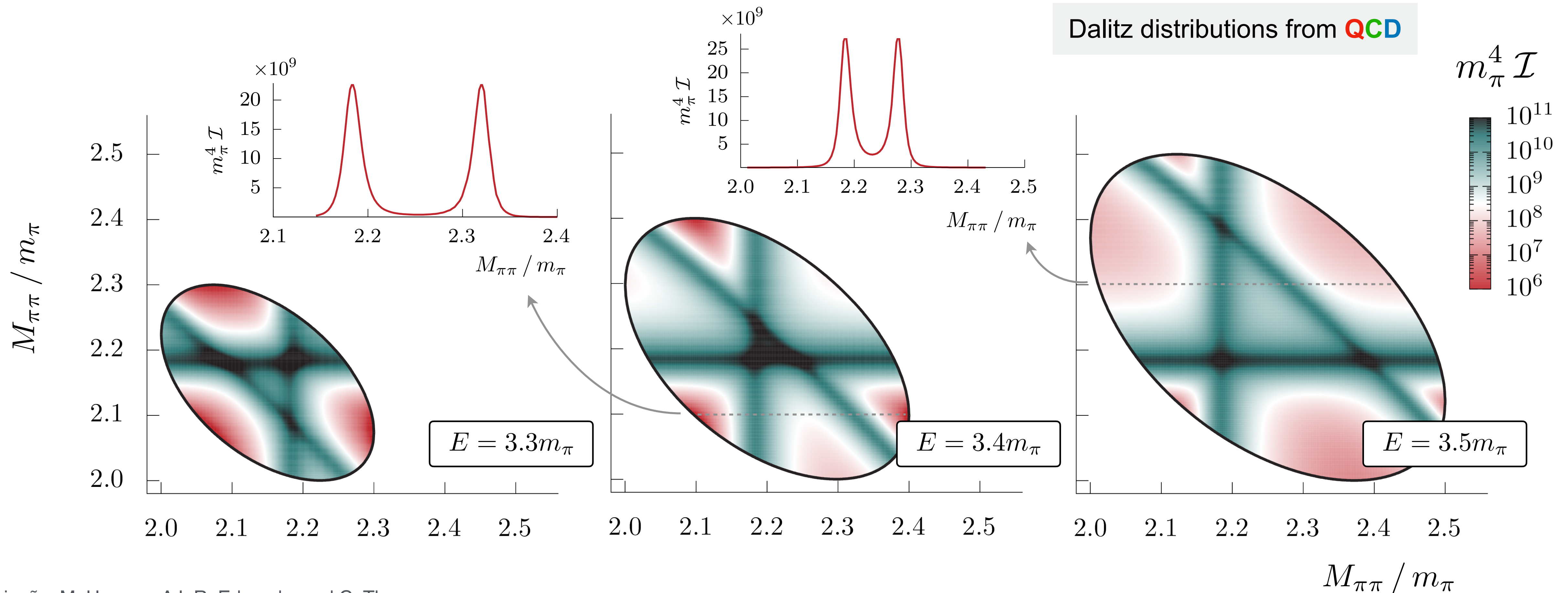
AJ, N. Chambers, R. Briceño
Phys.Rev.D **113** (2026) 1, 014019

R. Briceño, M. Hansen, AJ, R. Edwards, and C. Thomas
Phys.Rev.Lett. **137** (2026) 11, 111901

Three-Pion Scattering from Lattice QCD

Can study amplitude features through **intensity distributions**

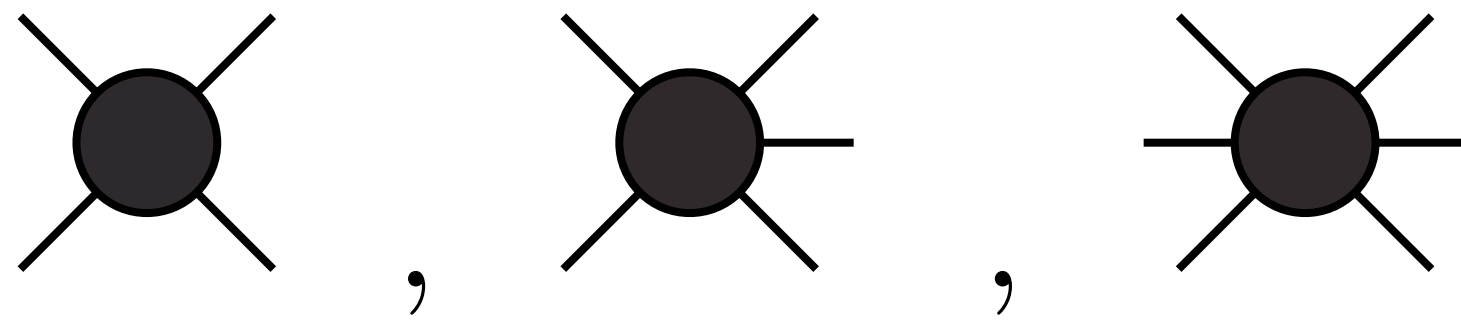
- Full amplitude requires recoupling between spectator choices (*symmetrizing*)
- Access to observables familiar with experimental analyses, e.g., Dalitz plots



Towards Structural Observables

External probes of hadronic resonances — learn about their structure

Hadronic scattering

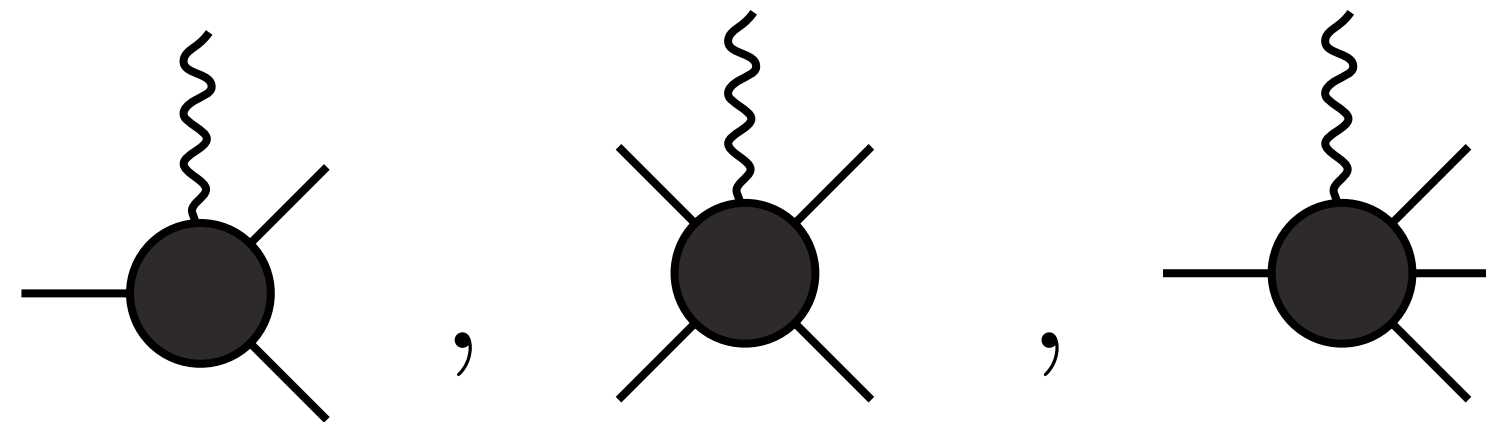


R. Briceño, M. Hansen,
Phys. Rev. D **94** 13008 (2016)

A. Baroni, R. Briceño, M. Hansen, F. Ortega-Gama,
Phys. Rev. D **100** 034511 (2019)

AJ, R. Briceño, F. Ortega-Gama, K. Sherman,
Phys. Rev. D **103** 114512 (2021)

Electroweak induced transitions

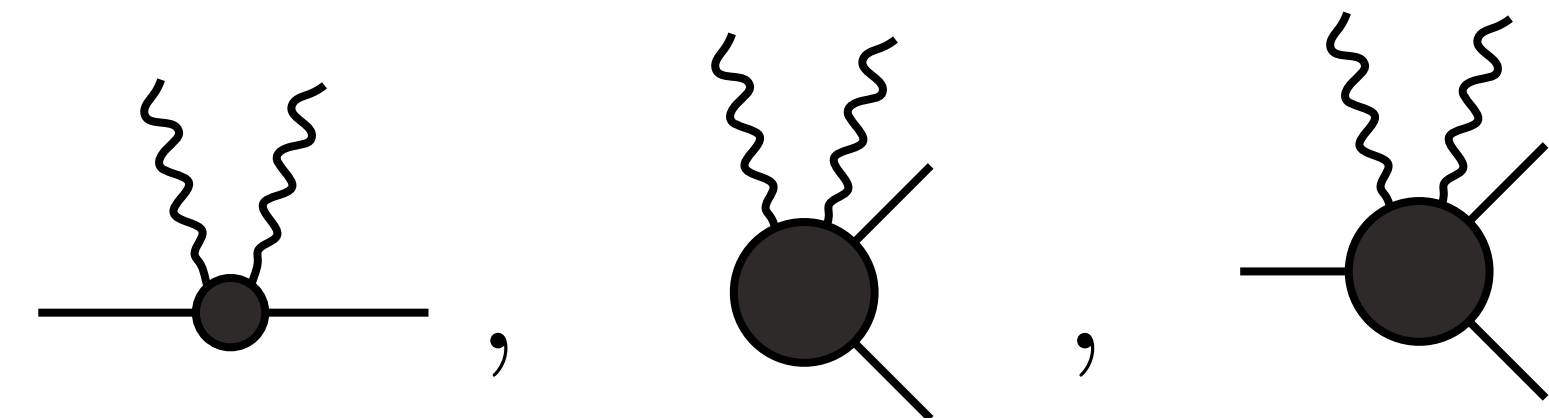


R. Briceño, Z. Davoudi, M. Hansen, M. Schindler, A. Baroni
Phys. Rev. D **101** 014509 (2020)

F. Ortega-Gama, K. Sherman, AJ, R. Briceño,
Phys. Rev. D **105** (2022)

AJ, R. Briceño, A. Rodas, J. Guerrero
Phys.Rev.D **107** 034504 (2023)

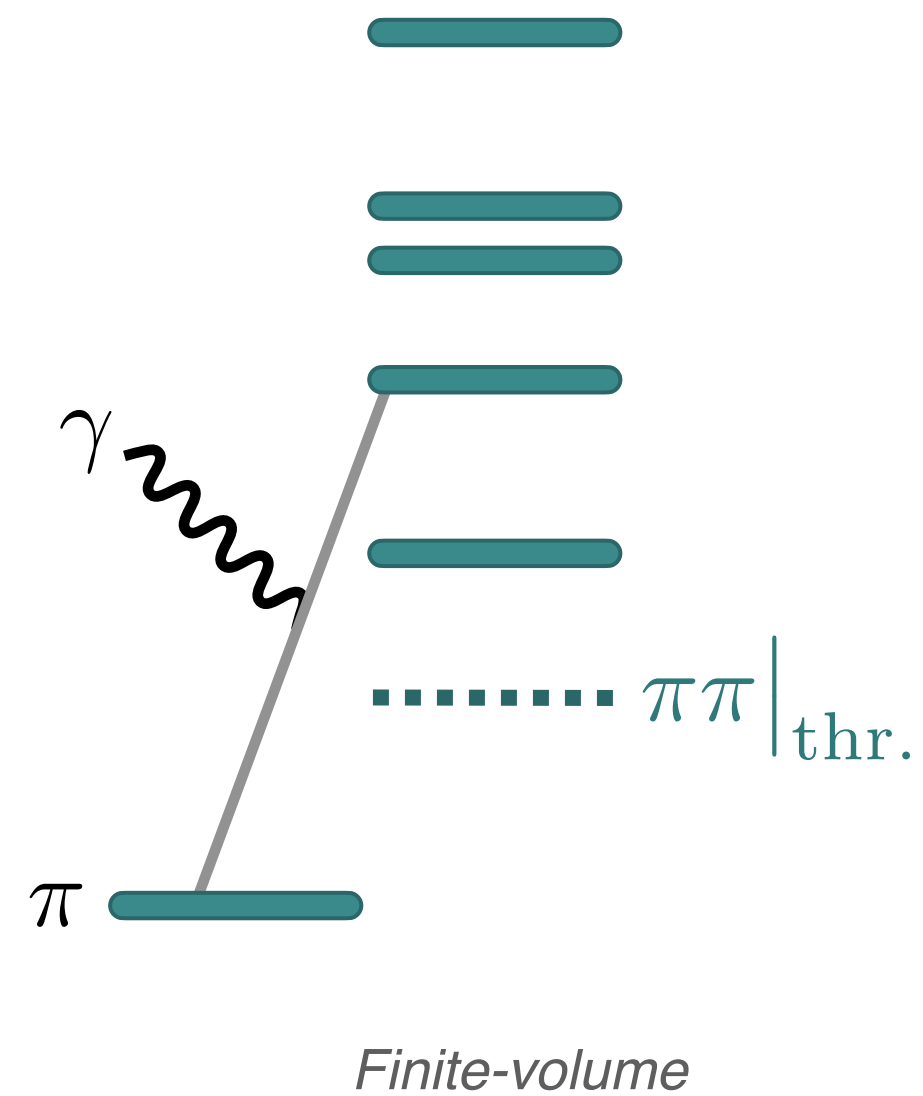
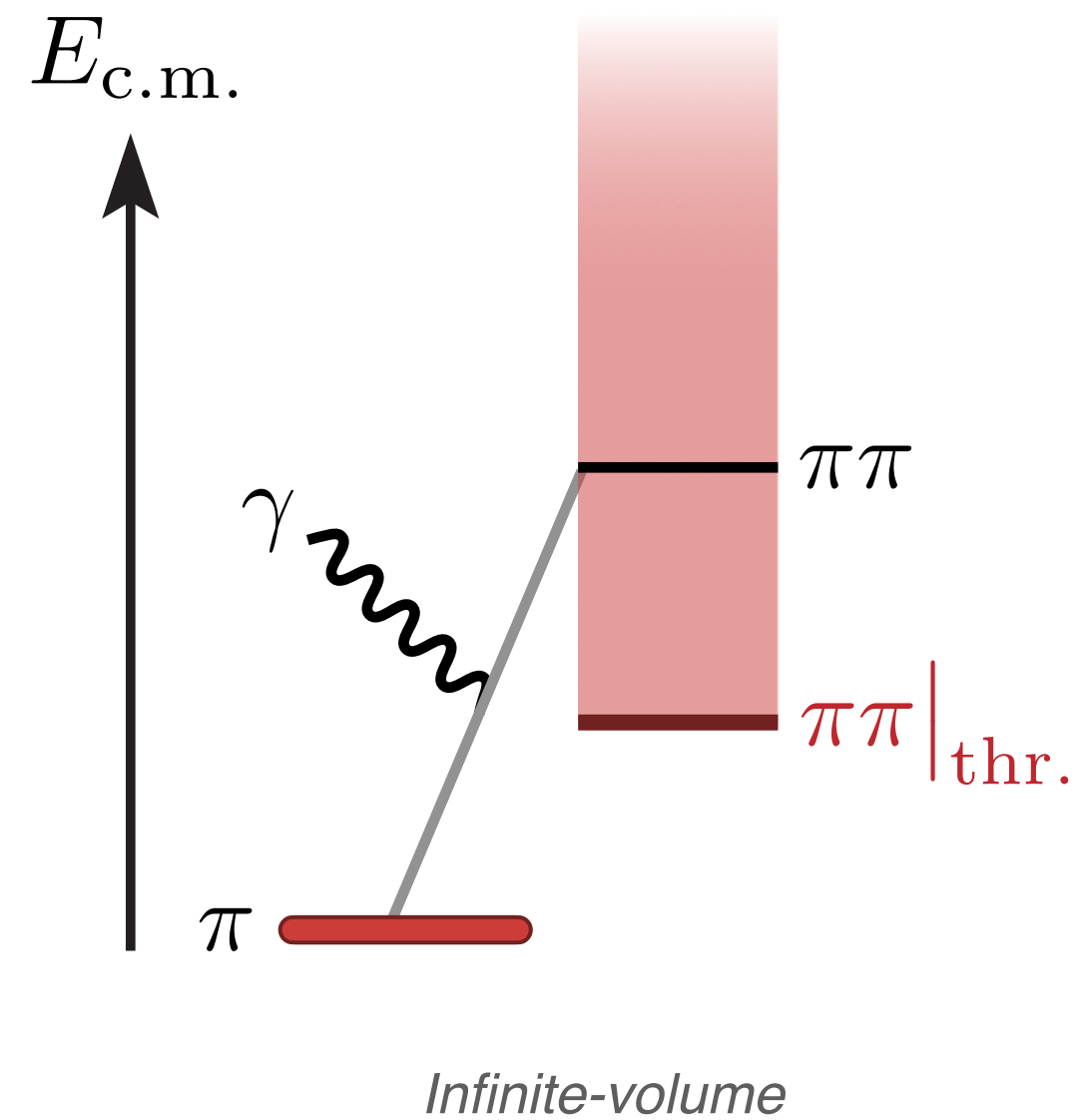
Long-range transitions



Towards Structural Observables

External probes of hadronic resonances — learn about their structure

- Photo-couplings and form-factors



$$|\pi\pi, E_n\rangle_L \sim \sqrt{\mathcal{R}_n(L)} |\pi\pi, E_{\pi\pi} = E_n\rangle_\infty$$

$$|\pi\rangle_L \sim |\pi\rangle_\infty + \mathcal{O}(e^{-m_\pi L})$$

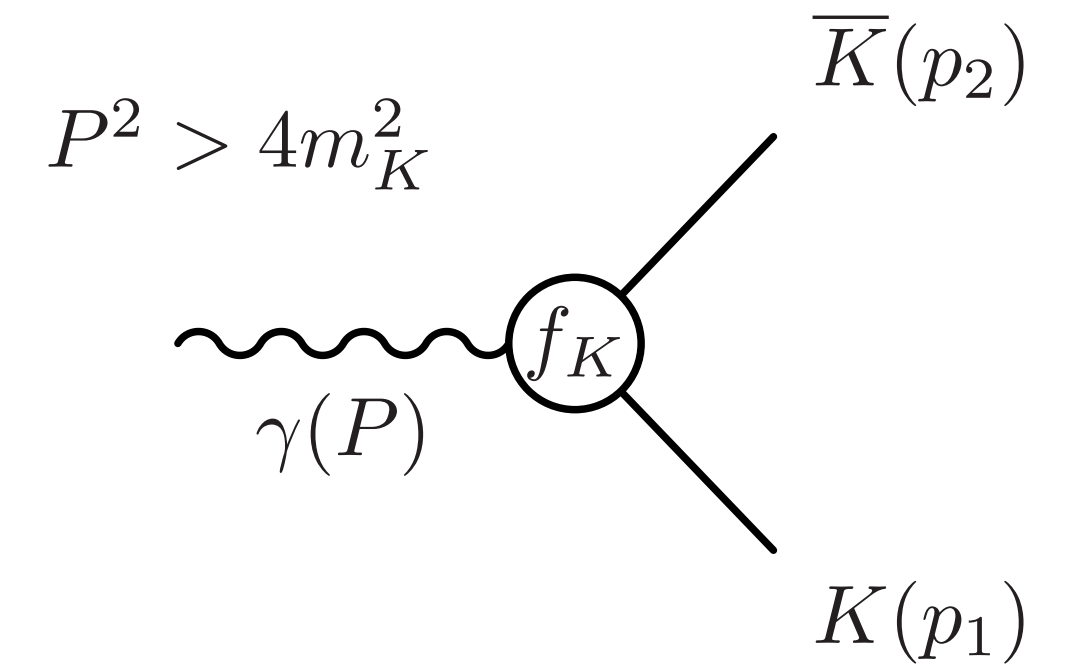
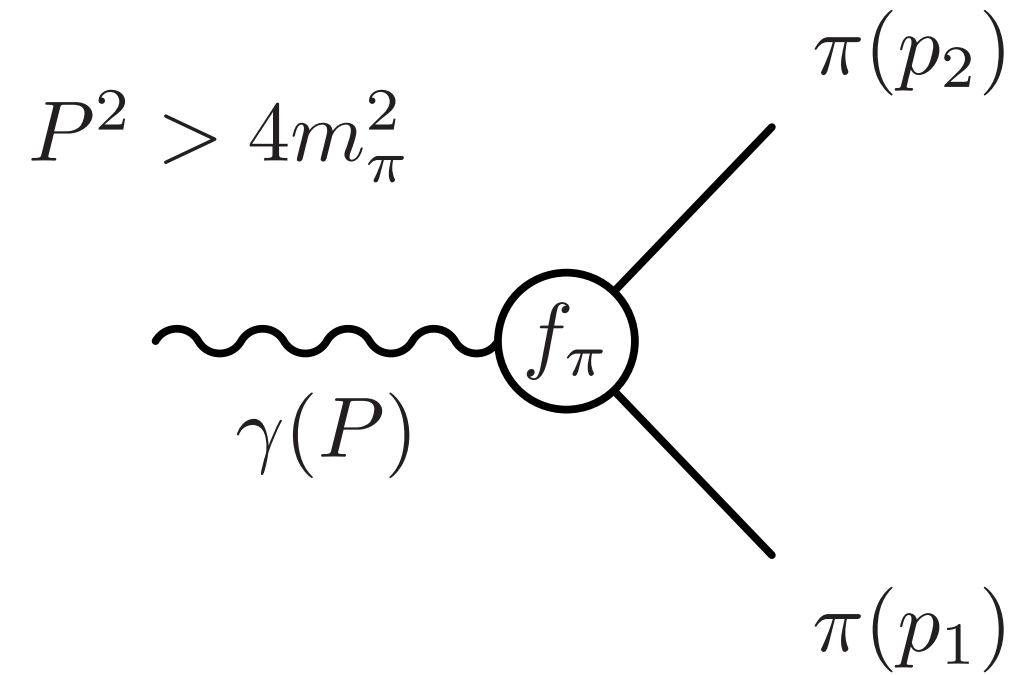
$$\mathcal{R}_n(L) \sim 2E_n \lim_{E \rightarrow E_n} \frac{E - E_n}{F^{-1}(E, L) + \mathcal{K}(E)}$$

Towards Structural Observables

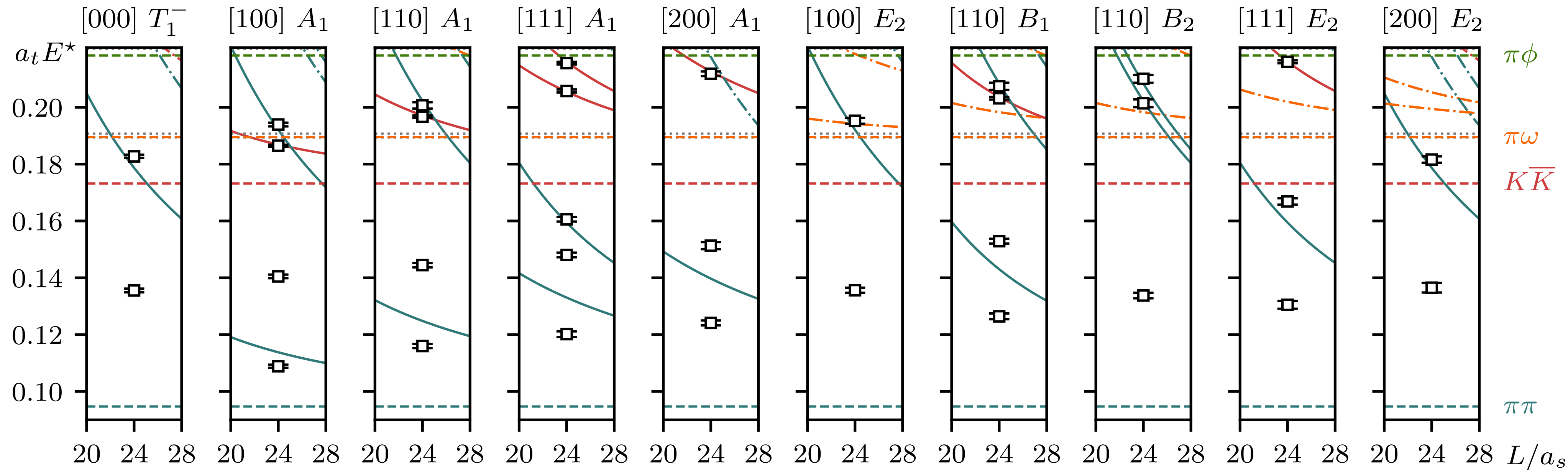
External probes of hadronic resonances — learn about their structure

- Photo-couplings and form-factors

Pion/Kaon Vector Form-Factor



$$m_\pi \approx 280 \text{ MeV}$$



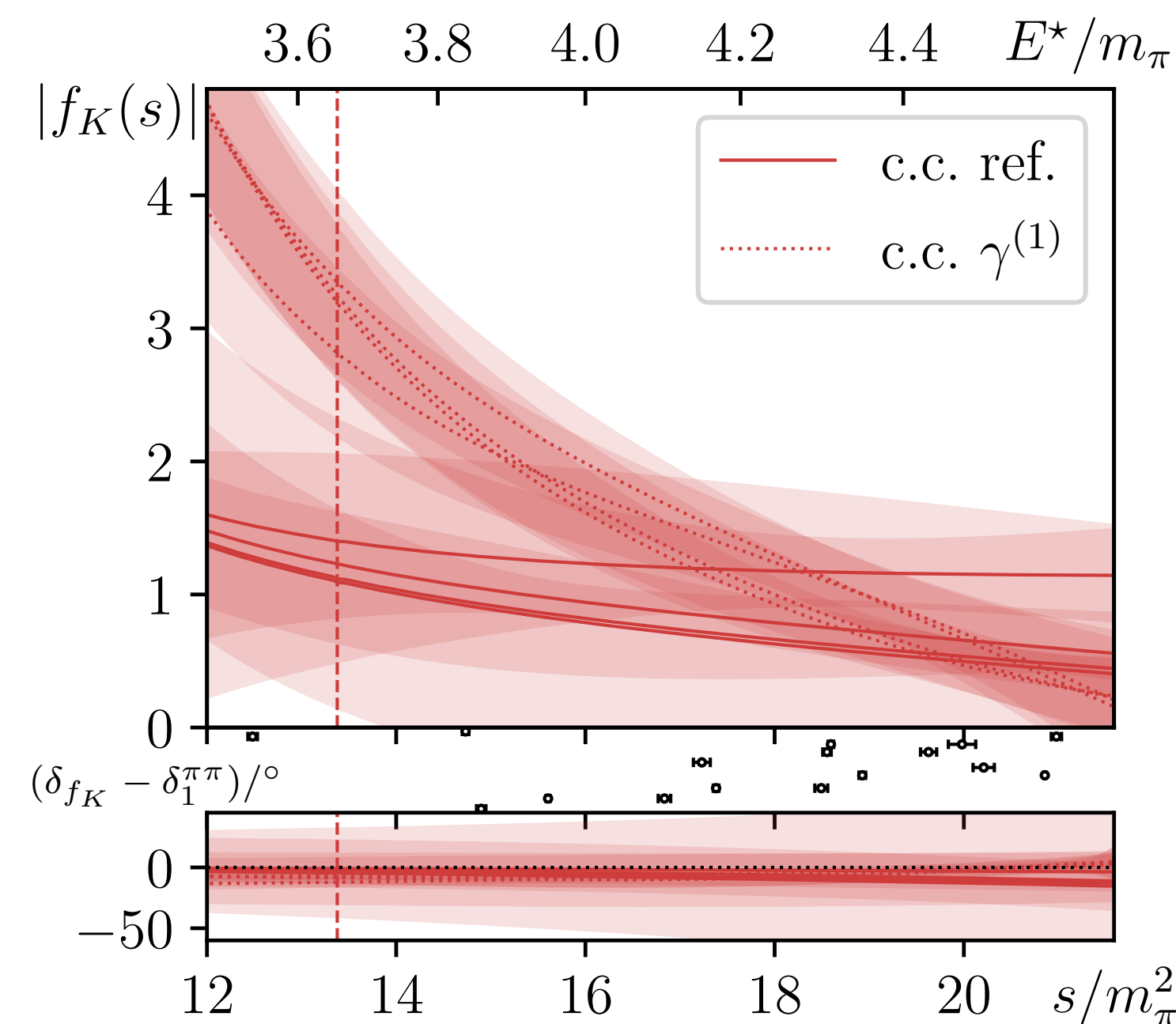
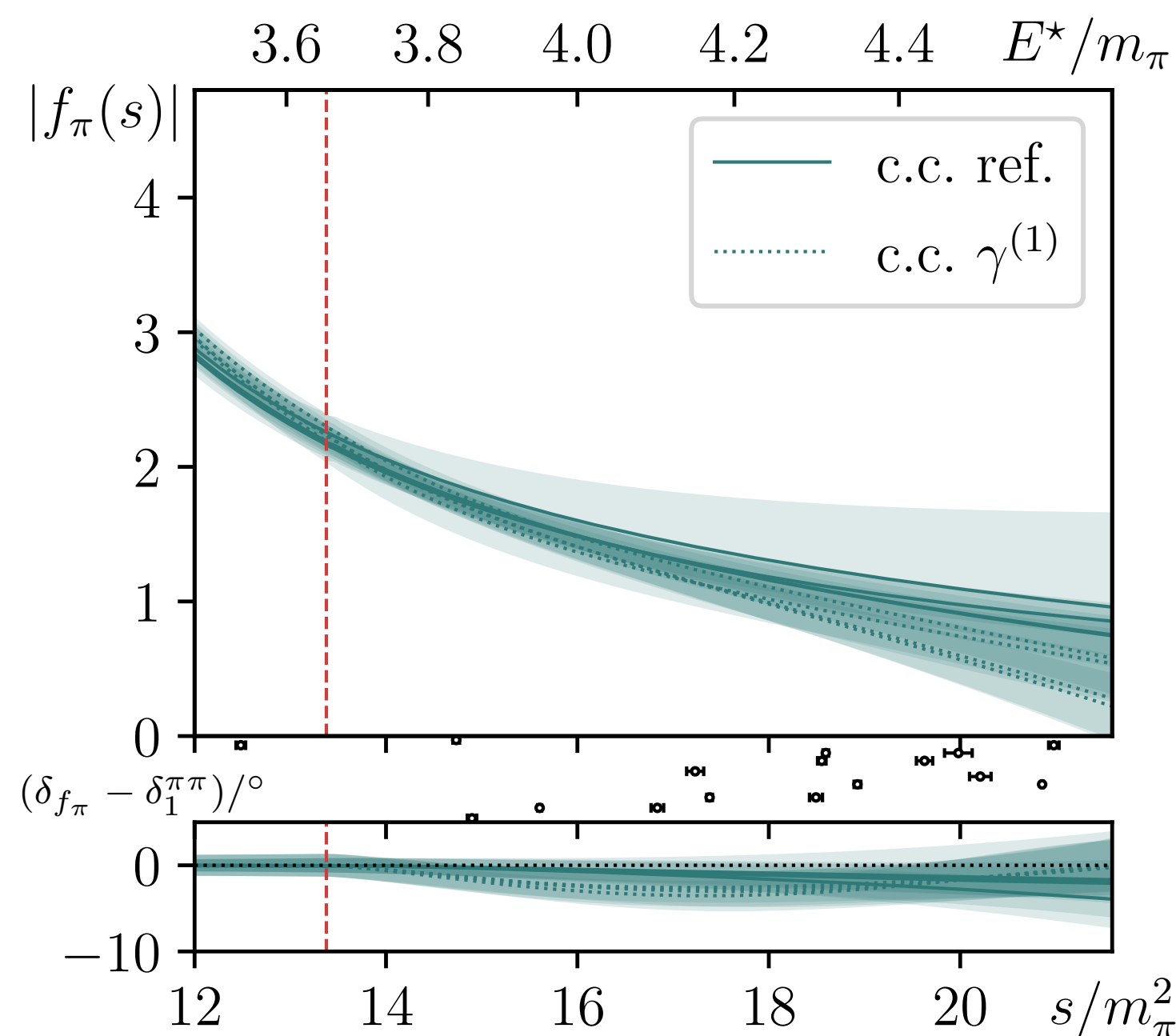
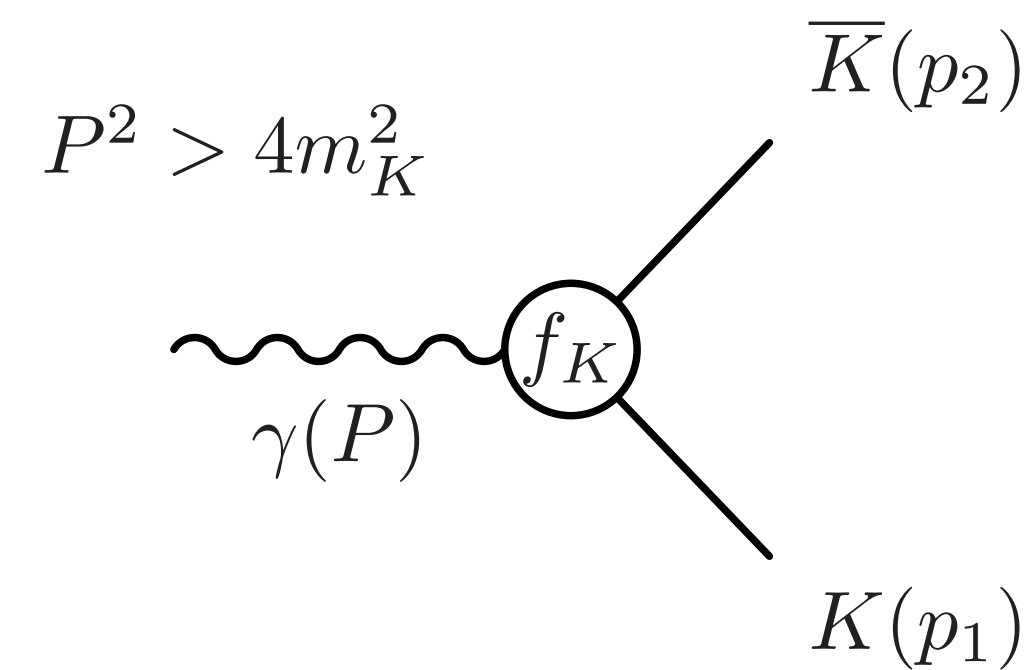
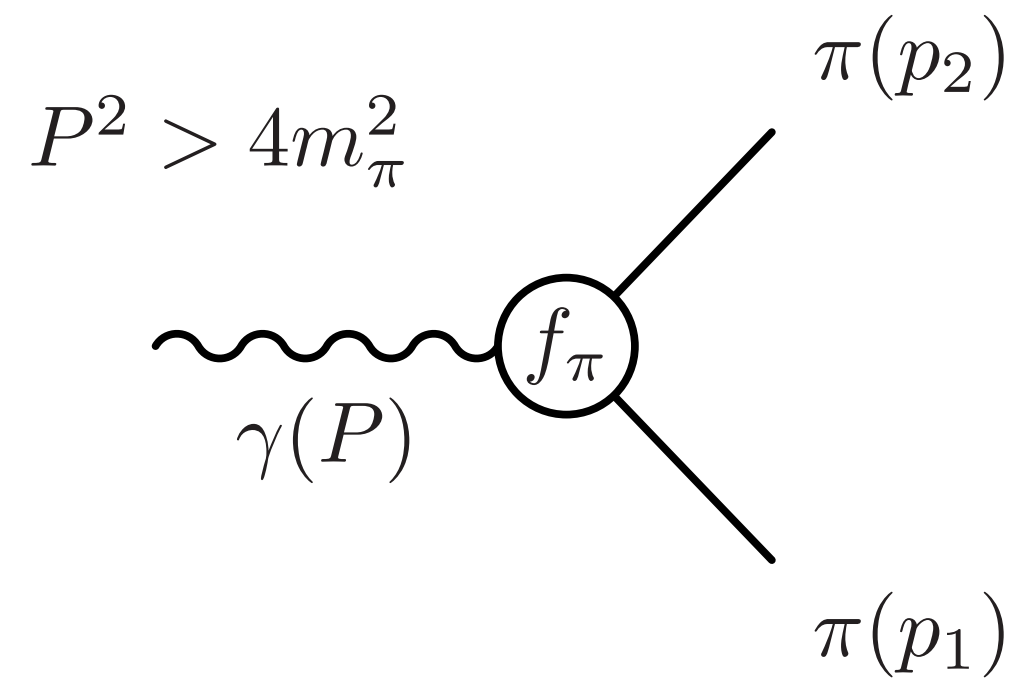
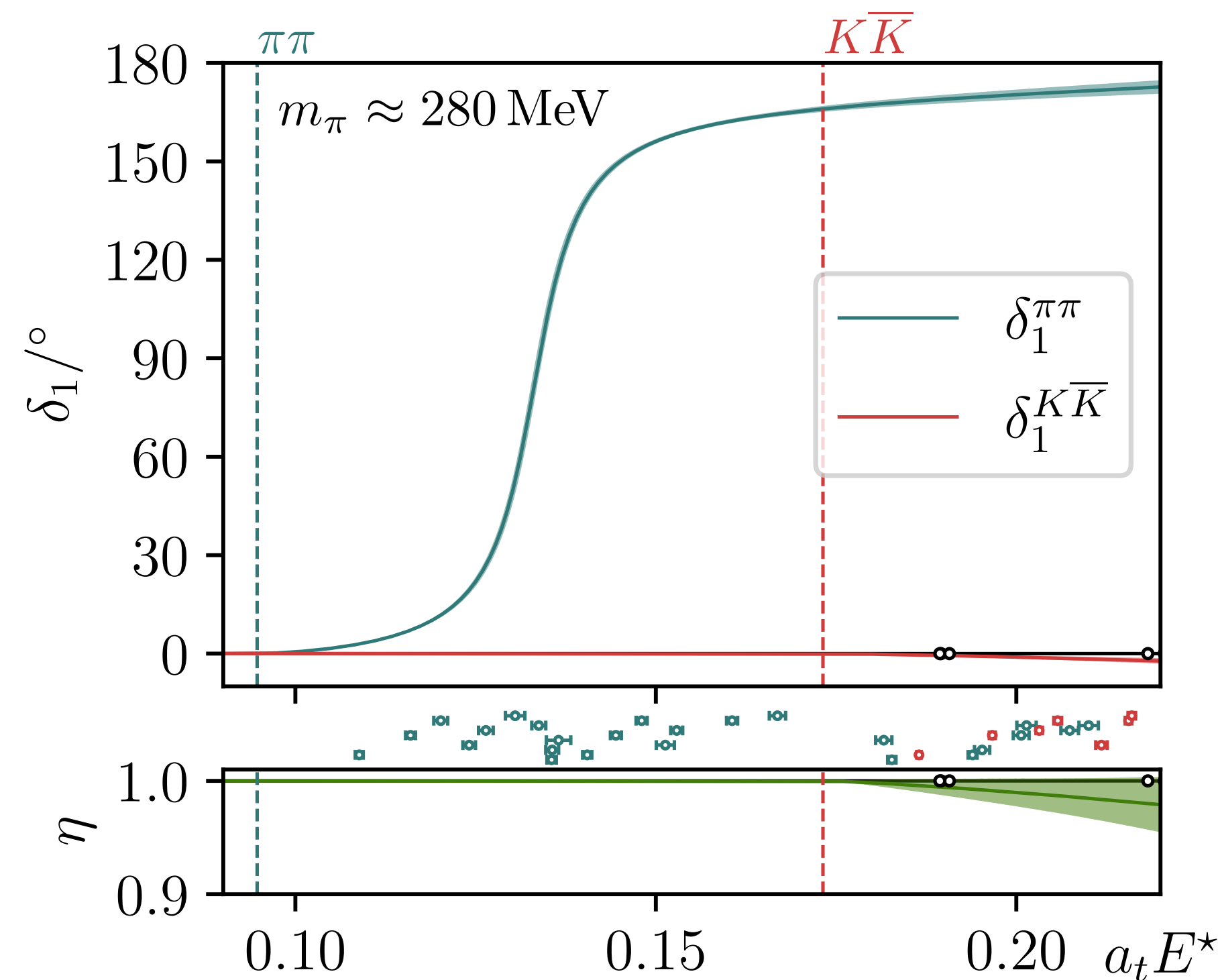
had spec

Towards Structural Observables

External probes of hadronic resonances — learn about their structure

- Photo-couplings and form-factors

Pion/Kaon Vector Form-Factor



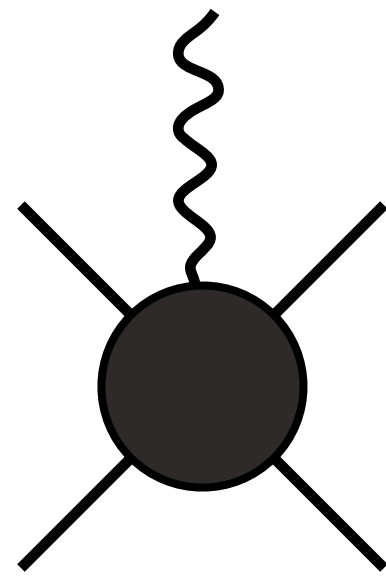
Towards Structural Observables

External probes of hadronic resonances — learn about their structure

- Access new class of observables — resonance form-factors



Felipe Ortega-Gama
UC, Berkeley



$$q \rightarrow 0 \quad \sim \quad 2P^\mu \mathcal{W}_S$$

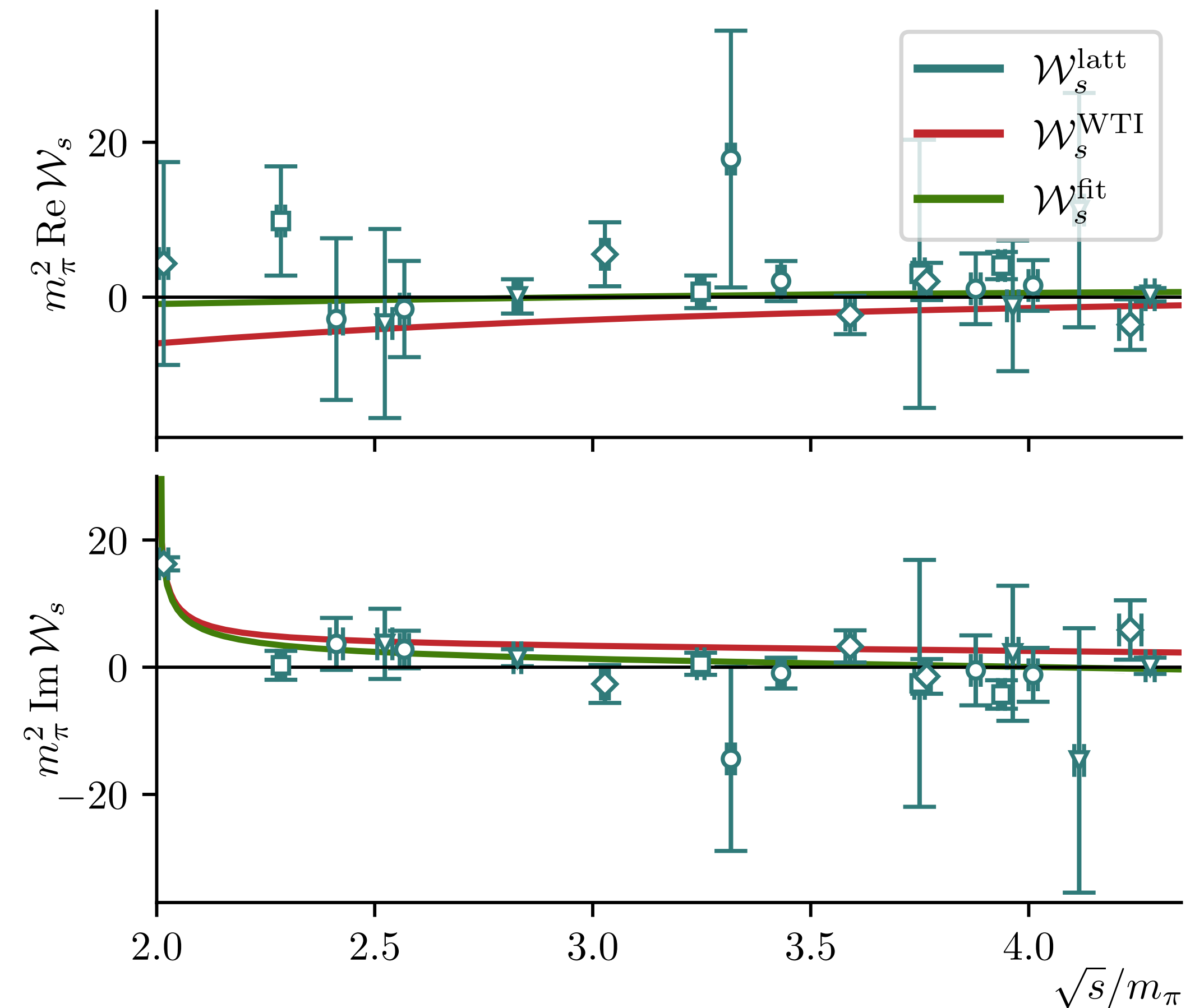
Related Talk

Felipe Ortega-Gama

Radiative two-pion amplitude from QCD

Parallel 1A: Tuesday, October 20, 9:50 — 10:15 a.m.

$\pi\pi$ isotensor form-factors



Summary

Hadron spectroscopy is a *vital* tool in understanding **QCD**

- Lattice QCD provides systematic approach to low-energy observables
- Complete end-to-end analyses from lattice observables to reaction amplitudes achievable
- Access to observables familiar with experimental analyses
- Theoretical tools allow one to probe structures previously unattainable

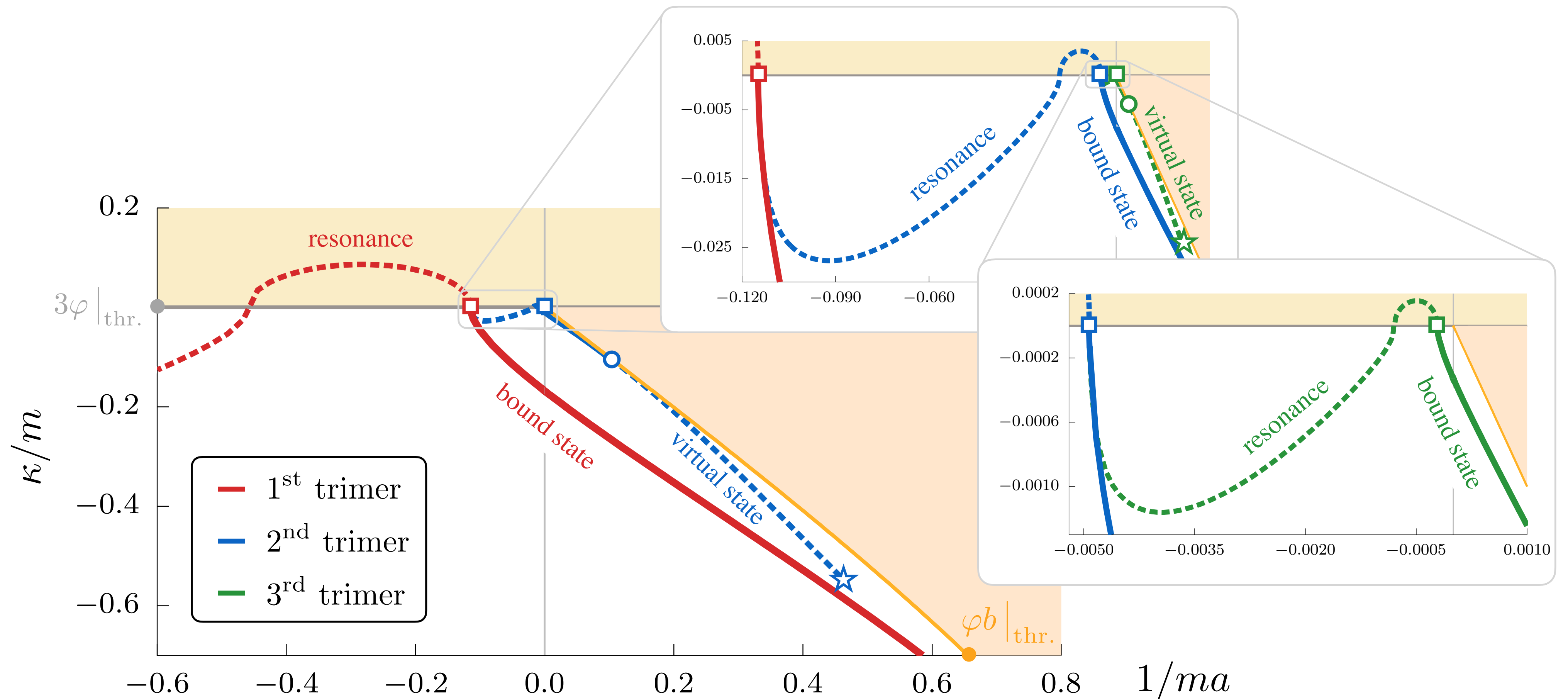


Efimov Scaling

Spectral analyses require continuing to pole positions

- Community is building confidence in robust solution strategies
- e.g., exploring Efimov physics with relativistic three-body framework

$$\mathcal{M}_3 \sim -\frac{g^2}{E^2 - E_0^2}$$

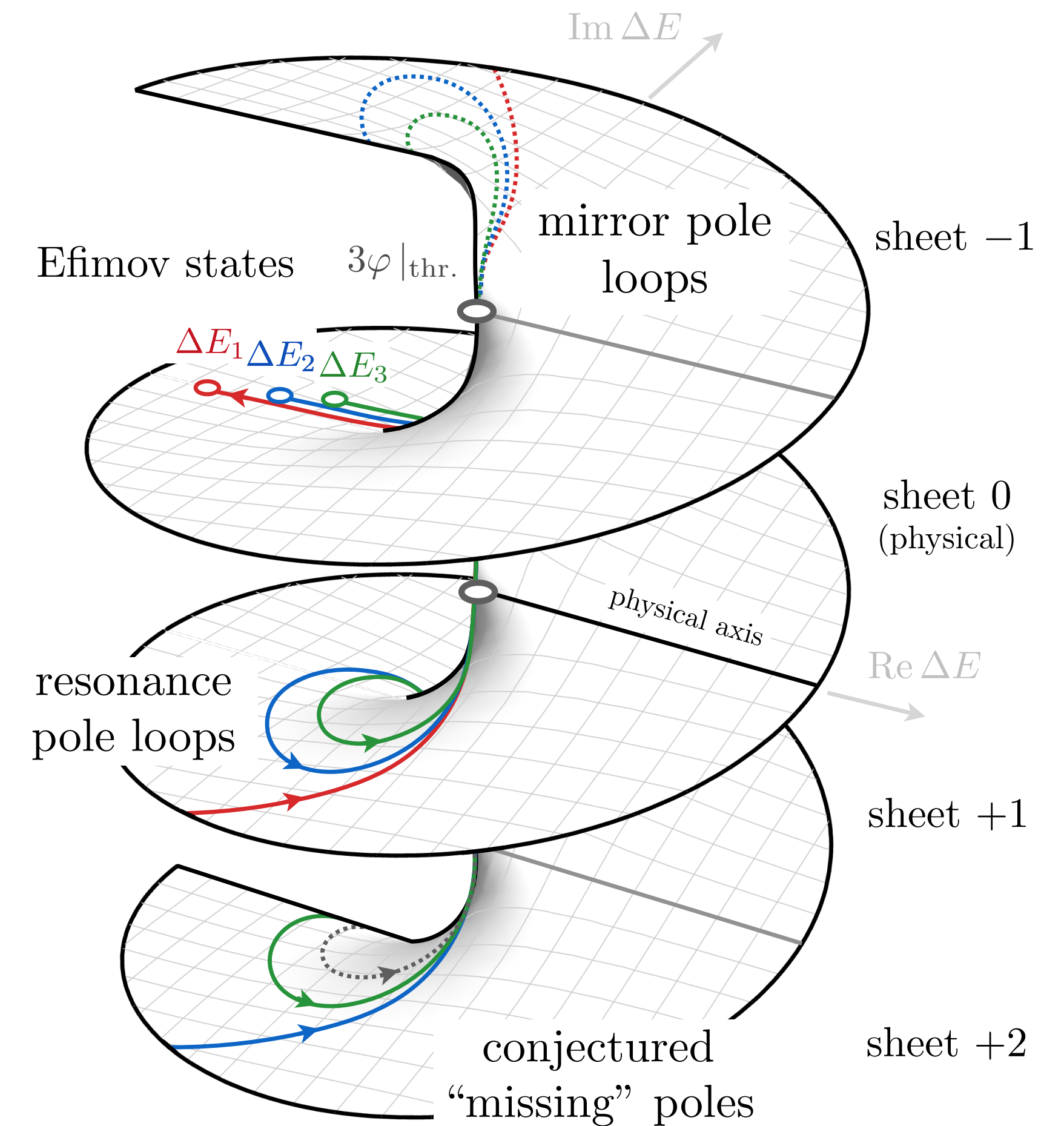


Efimov Scaling

Spectral analyses require continuing to pole positions

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- e.g., exploring Efimov physics with relativistic three-body framework

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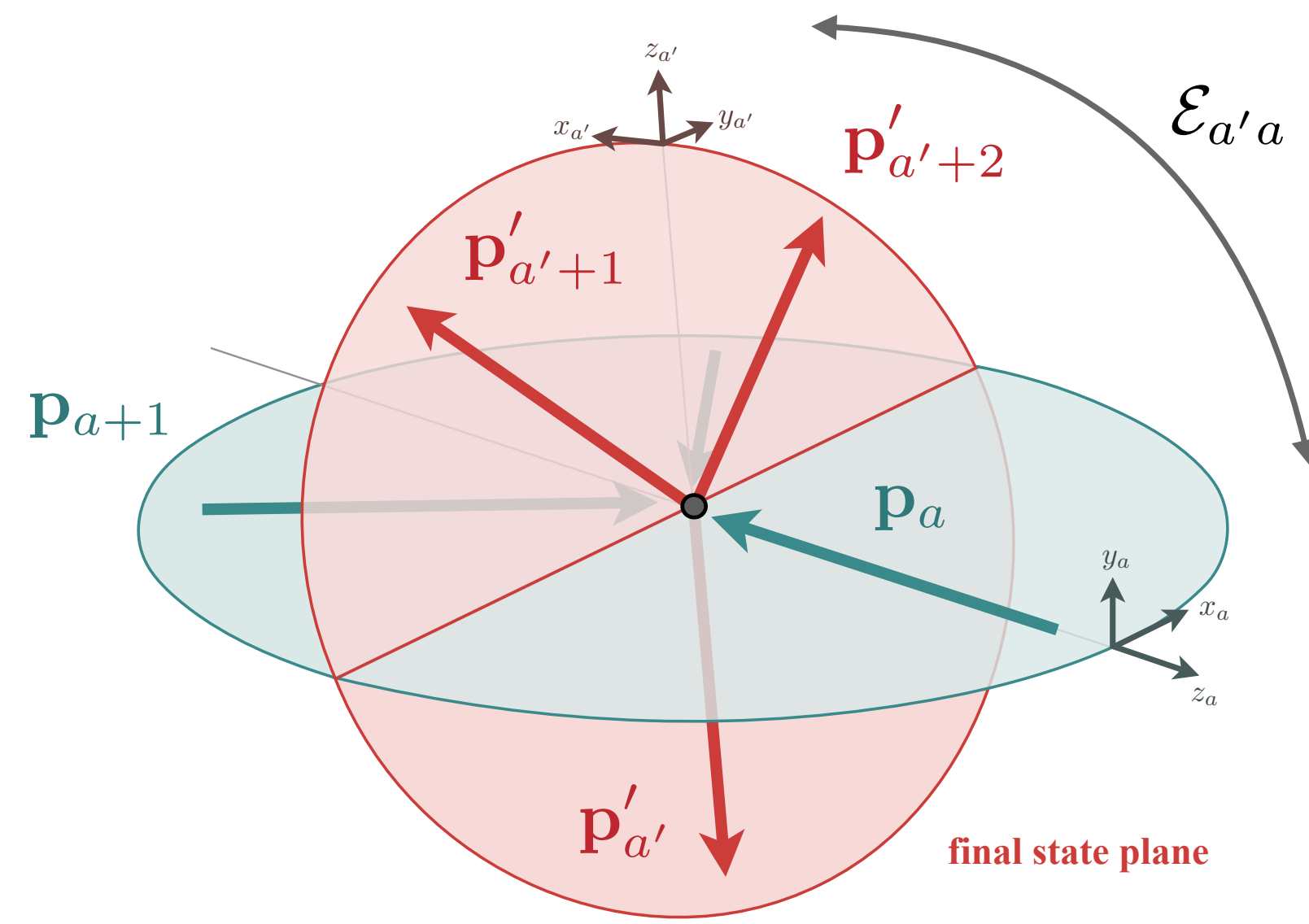
Three-Pion Scattering from Lattice QCD

Can study amplitude features through **intensity distributions**

- Full amplitude requires recoupling between spectator choices (*symmetrizing*)

$$\mathcal{I} = \sum_{I'_2, I_2} \sum_{\lambda, \lambda'} \left| \mathcal{M}_{3, I'_2 \lambda', I_2 \lambda} \right|^2$$

$$\mathcal{M}_{3, I'_2 \lambda', I_2 \lambda} = \sum_{a, b} \mathcal{R}_{I'_2 \lambda'}^{(a)} \cdot \mathcal{M}_3^{(a, b)} \cdot \mathcal{R}_{I_2 \lambda}^{(b) \top}$$



Finite-Volume Spectroscopy

Determine finite-volume spectrum with lattice QCD

- Create large basis of operators with target quantum numbers
- Form matrix of correlation functions, diagonalize (GEVP)

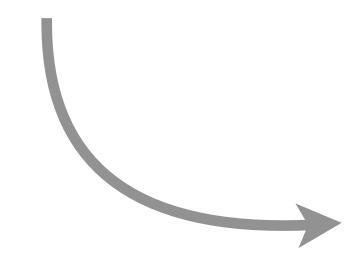
$$C_{ij}(t) = \langle \mathcal{O}_i(t) \mathcal{O}_j(0) \rangle$$

 *Hadron / multi-hadron operators*

Variationally optimize coefficients to maximize overlaps to desired states

- *Allows access to excited states*
- *Reduces contamination in low-lying states*

$$C_{ij}(t)v_{\mathbf{n}}^j = \lambda_{\mathbf{n}} C_{ij}(t_0)v_{\mathbf{n}}^j$$

 $\lambda_{\mathbf{n}} \sim e^{E_{\mathbf{n}}(t-t_0)}$

Finite-Volume Spectroscopy

Determine finite-volume spectrum with lattice QCD

- Create large basis of operators with target quantum numbers
- Form matrix of correlation functions, diagonalize (GEVP)

e.g., the pion mass

$$\mathcal{O}_\pi \sim v^1 \left(\text{op1} \right) + v^2 \left(\text{op2} \right) + v^3 \left(\text{op3} \right) + \dots$$

op1 op2 op3

$$\langle \mathcal{O}_\pi(\tau) \mathcal{O}_\pi^\dagger \rangle = e^{-m_\pi \tau} + \sum_{\mathbf{n} \neq \pi} \varepsilon_{\mathbf{n}} e^{-m_{\mathbf{n}} \tau}$$

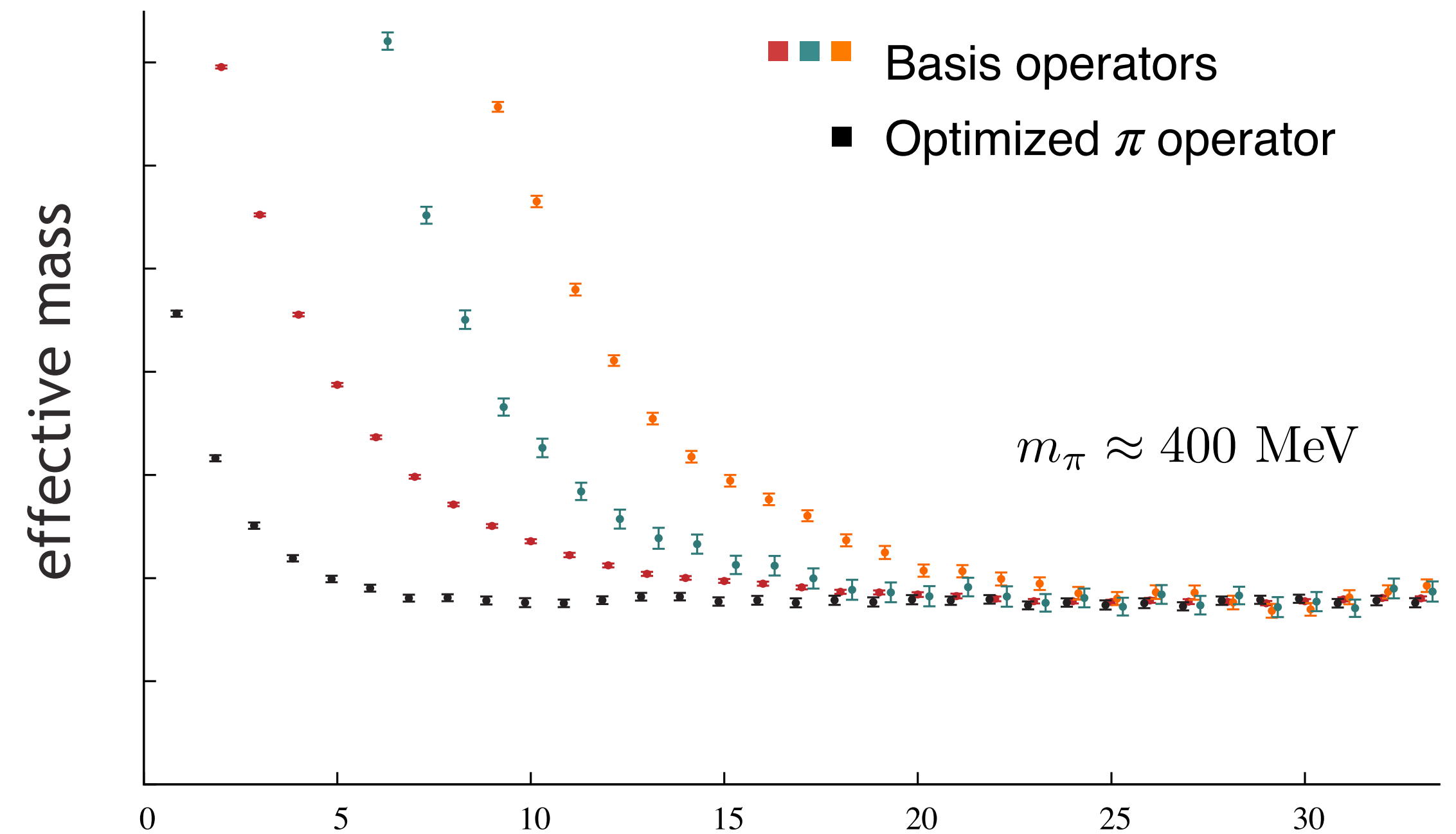
Finite-Volume Spectroscopy

Determine finite-volume spectrum with lattice QCD

- Create large basis of operators with target quantum numbers
- Form matrix of correlation functions, diagonalize (GEVP)

$$\mathcal{C}_{ij}(\tau) = \begin{bmatrix} \text{plot} & \text{plot} & \text{plot} & \text{plot} & \text{plot} \\ \text{plot} & \text{plot} & \text{plot} & \text{plot} & \text{plot} \\ \text{plot} & \text{plot} & \text{plot} & \text{plot} & \text{plot} \\ \text{plot} & \text{plot} & \text{plot} & \text{plot} & \text{plot} \\ \text{plot} & \text{plot} & \text{plot} & \text{plot} & \text{plot} \end{bmatrix}$$

op1 op2 op3 . . .



Finite-Volume Spectroscopy

Determine finite-volume spectrum with lattice QCD

- Create large basis of operators with target quantum numbers
- Form matrix of correlation functions, diagonalize (GEVP)
- For extracting multi-body physics, follow similar path

Optimal pion operator

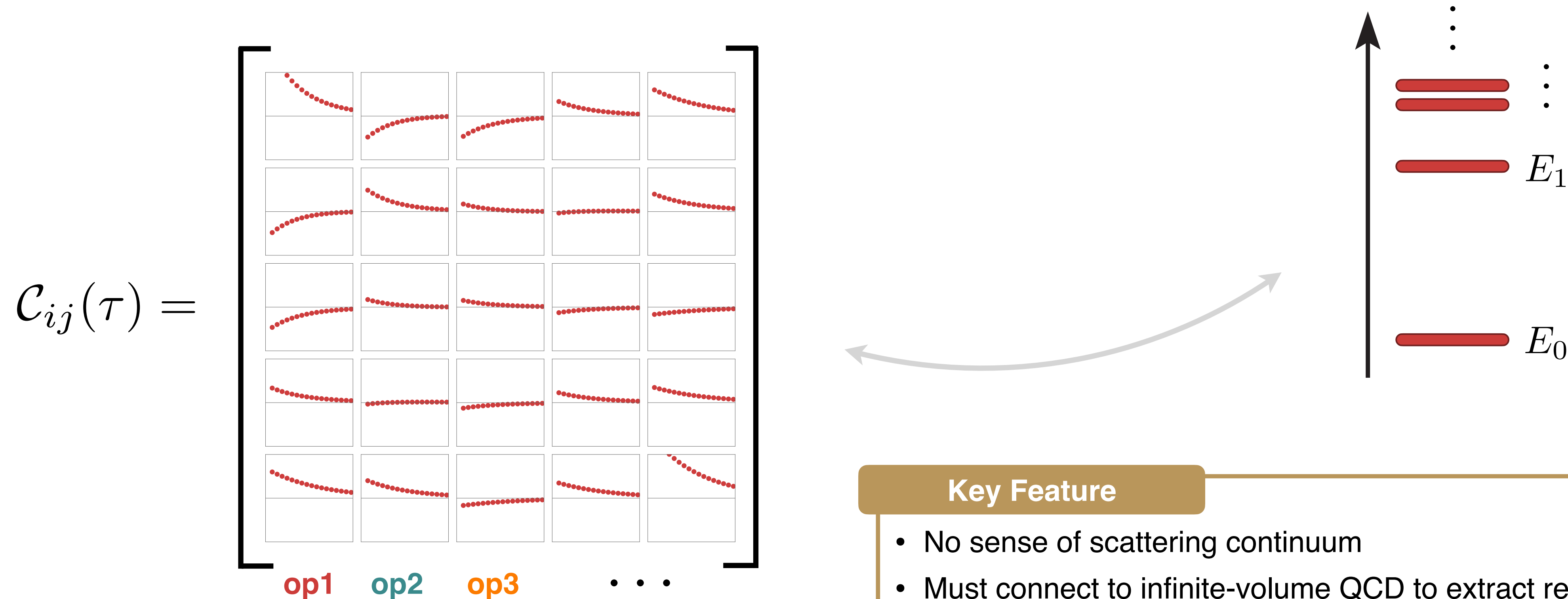
$$\mathcal{O}_{\pi\pi} = \left(\begin{array}{c} \pi \quad \pi \\ \pi \quad \pi \end{array} \right), \left(\begin{array}{c} \pi \quad \pi \\ \swarrow \quad \nearrow \end{array} \right), \left(\begin{array}{c} \pi \quad \pi \\ \swarrow \quad \nearrow \end{array} \right) \dots$$

op1 op2 op3

Finite-Volume Spectroscopy

Determine finite-volume spectrum with lattice QCD

- Create large basis of operators with target quantum numbers
- Form matrix of correlation functions, diagonalize (GEVP)
- For extracting multi-body physics, follow similar path



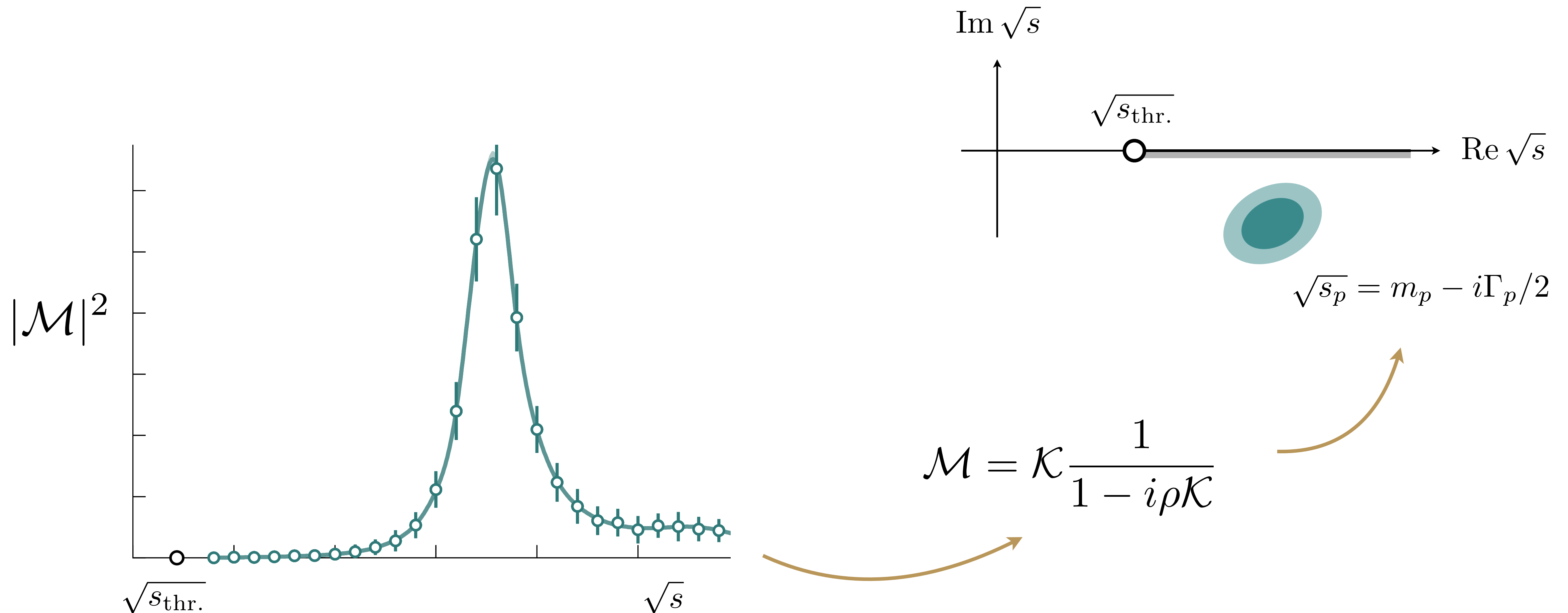
Key Feature

- No sense of scattering continuum
- Must connect to infinite-volume QCD to extract reaction observables

Two-Hadron Systems

Key concept — use fundamental S matrix principles to constrain amplitudes

- Allows for unbiased amplitude construction
- Can determine spectrum via analytic continuation

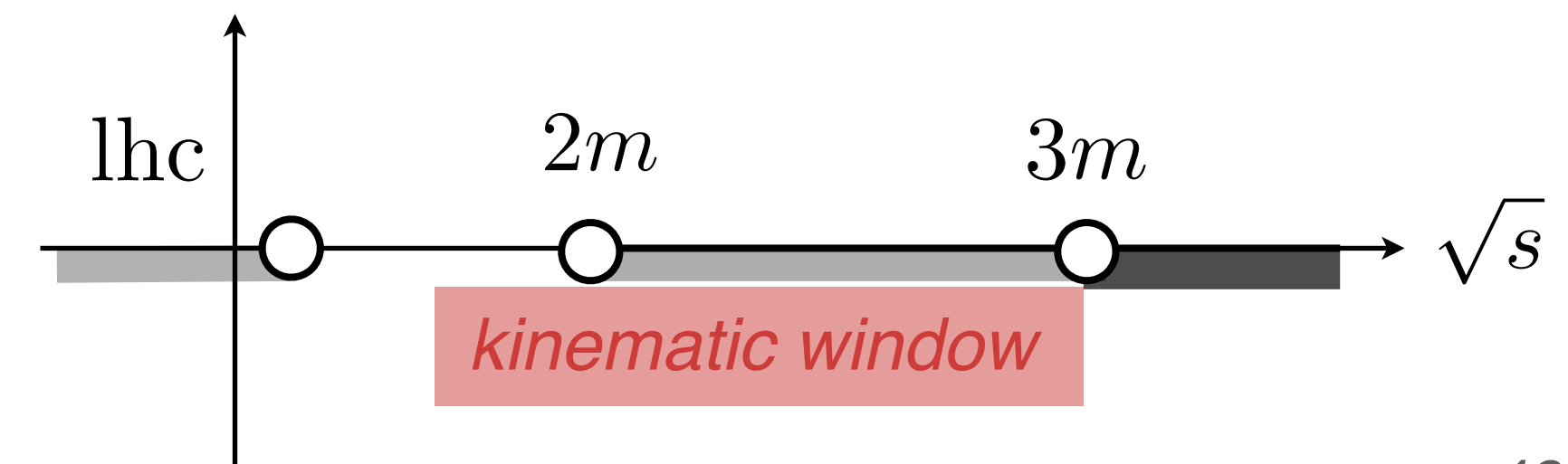


Two-Hadron Systems

Key concept — use fundamental S matrix principles to constrain amplitudes

- Allows for unbiased amplitude construction
- Can determine spectrum via analytic continuation
- e.g., elastic scattering of spinless particles, with restricted energy region

$$i\mathcal{M}_2 = \text{blob} = \text{tree} + \text{one-loop} + \text{two-loop} + \dots$$



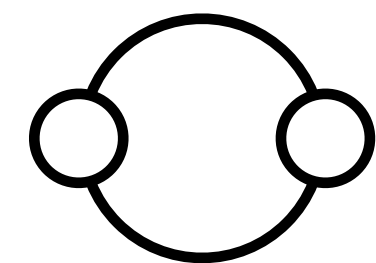
Two-Hadron Systems

Key concept — use fundamental S matrix principles to constrain amplitudes

- Allows for unbiased amplitude construction
- Can determine spectrum via analytic continuation
- e.g., elastic scattering of spinless particles, with restricted energy region

$$i\mathcal{M}_2 = \text{[solid black circle]} = \text{[white circle]} + \text{[white circle with bubble]} + \text{[white circle with two bubbles]} + \dots$$

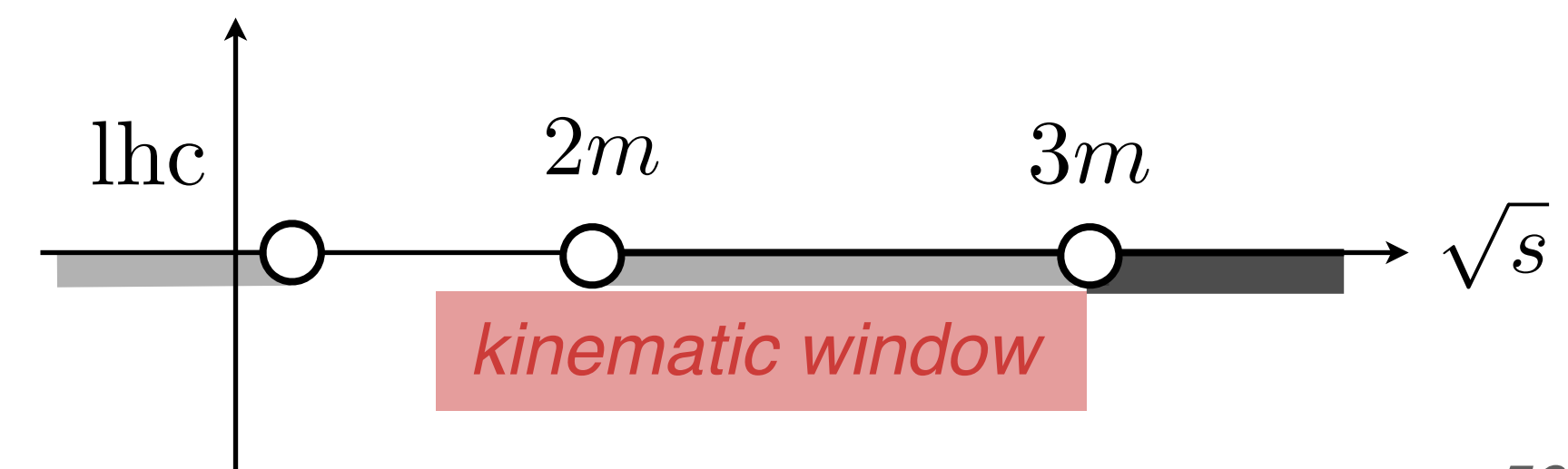
= isolate singular behavior, re-sum short-distance ‘stuff’ into new real function



$\sim$ real function $- i\rho$

two-body phase space

$$\rho \sim \sqrt{1 - \frac{4m^2}{s}}$$



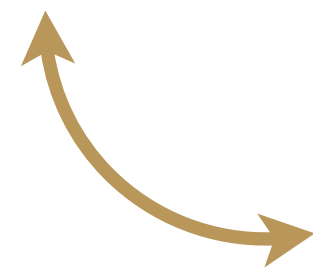
Two-Hadron Systems

Key concept — use fundamental S matrix principles to constrain amplitudes

- Allows for unbiased amplitude construction
- Can determine spectrum via analytic continuation
- e.g., elastic scattering of spinless particles, with restricted energy region

$$i\mathcal{M}_2 = \text{blob} = \text{circle} + \text{circle with bubble} + \text{circle with two bubbles} + \dots$$

$$= \underbrace{\text{square}}_{i\mathcal{K}_2} + \text{square with bubble} + \text{square with two bubbles} + \dots = i\mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$



K-Matrix

- short-distance dynamics
- constrained from data (exp. or th.)

S matrix unitarity

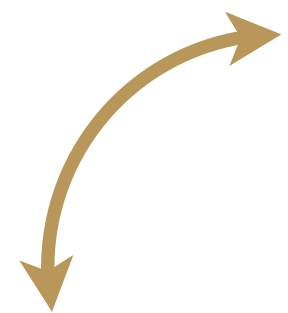
$$\text{Im } \mathcal{M}_2 = \rho |\mathcal{M}_2|^2$$

Two-Hadron Systems

Key concept — use fundamental S matrix principles to constrain amplitudes

- Allows for unbiased amplitude construction
- Can determine spectrum via analytic continuation
- e.g., elastic scattering of spinless particles, with restricted energy region
- Connect K matrix to finite-volume through correlation function

$$C_L^{(M)}(E) = i \sum_{\mathbf{n}} \frac{Z_{\mathbf{n}} Z_{\mathbf{n}}^\dagger}{E - E_{\mathbf{n}}}$$



$$\mathbf{p} = \frac{2\pi}{L} \mathbf{n}$$

$$= \text{[Diagram: a circle with two external lines and a vertical line through the center labeled V]} + \text{[Diagram: two circles connected by a vertical line, each with two external lines and a vertical line through the center labeled V]} + \text{[Diagram: three circles connected by vertical lines, each with two external lines and a vertical line through the center labeled V]} + \dots$$

= Follow similar procedure as before ...

... additional correction for finite-volume effects

$$\text{[Diagram: a circle with two external lines and a vertical line through the center labeled V]} \sim \text{[Diagram: a circle with two external lines]} + F_{2,L}$$

as before

Finite-volume correction (known)

Two-Hadron Systems

Key concept — use fundamental S matrix principles to constrain amplitudes

- Allows for unbiased amplitude construction
- Can determine spectrum via analytic continuation
- e.g., elastic scattering of spinless particles, with restricted energy region
- Connect K matrix to finite-volume through correlation function

$$C_L^{(M)}(E) = i \sum_{\mathbf{n}} \frac{Z_{\mathbf{n}} Z_{\mathbf{n}}^\dagger}{E - E_{\mathbf{n}}}$$

what we compute via lattice QCD

$$= \text{[diagram: bubble with } V \text{]} + \text{[diagram: two bubbles with } V \text{]} + \text{[diagram: three bubbles with } V \text{]} + \dots$$

$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_2 \cdot F_{2,L}}$$

what we want

Spectrum satisfies

$$\det[1 + \mathcal{K}_2 \cdot F_{2,L}] \Big|_{E=E_{\mathbf{n}}} = 0$$

Three-Hadron Systems

Repeat same procedure for three-body case

- S-matrix unitarity enforces K matrix structure
- Have integral equations and more complicated analytic structures

$$i\mathcal{M}_3 = \text{diagram} = \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram}$$

The diagram shows the decomposition of a three-body vertex $i\mathcal{M}_3$ into a sum of five diagrams. The first row shows the decomposition into a sum of five diagrams. The second row shows the same decomposition using a K-matrix structure, where the first diagram is a square loop and the others are more complex diagrams with a vertical red line.

K-Matrix

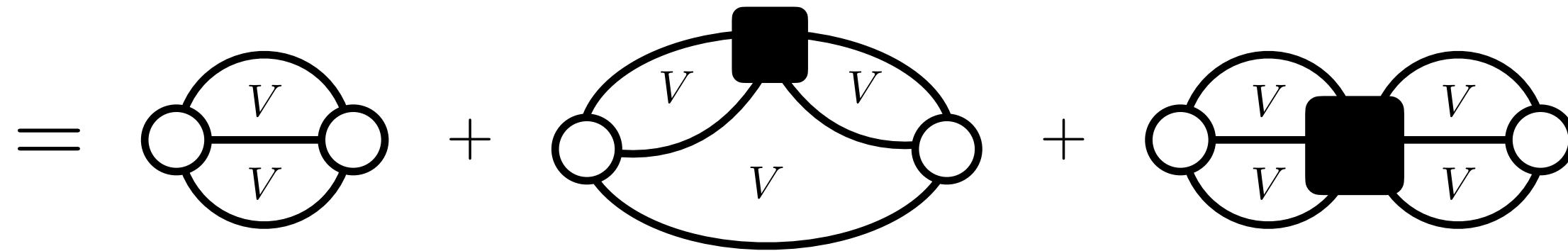
- short-distance dynamics
- cutoff-dependent (unphysical)
- constrained from Lattice QCD

Three-Hadron Systems

Repeat same procedure for three-body case

- S-matrix unitarity enforces K matrix structure
- Have integral equations and more complicated analytic structures

$$\mathcal{C}_L^{(M)}(E) = i \sum_{\mathbf{n}} \frac{Z_{\mathbf{n}} Z_{\mathbf{n}}^\dagger}{E - E_{\mathbf{n}}}$$



$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_3 \cdot F_{3,L}}$$

Spectrum satisfies

$$\det [1 + \mathcal{K}_3 \cdot F_{3,L}] \Big|_{E=E_{\mathbf{n}}} = 0$$

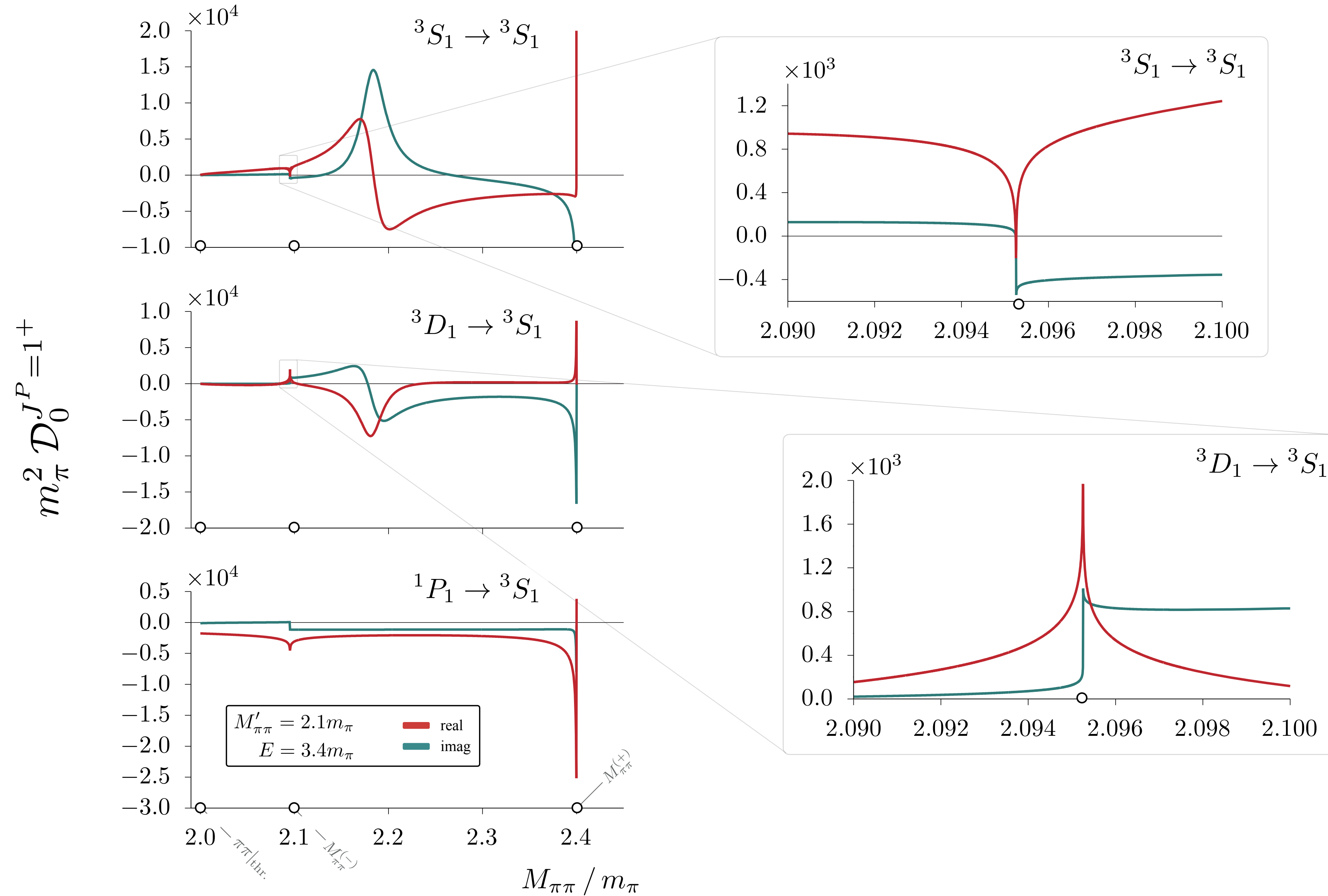
Operators

Ops.	$a_t E^{\text{non-int}}(L/a_s = 16)$	$a_t E^{\text{non-int}}(L/a_s = 20)$	$a_t E^{\text{non-int}}(L/a_s = 24)$
$[(\pi_{[001]})_{A_2} (\pi_{[00-1]})_{A_2}]_{T_1^-} (\pi_{[000]})_{A_1^-}$	—	—	0.275
$[(\pi_{[011]})_{A_2} (\pi_{[0-1-1]})_{A_2}]_{T_1^-} (\pi_{[000]})_{A_1^-}$	—	—	0.325
$[(\pi_{[-101]})_{A_2} (\pi_{[100]})_{A_2}]_{A_1} (\pi_{[00-1]})_{A_2}$	—	—	0.333
$[(\pi_{[-101]})_{A_2} (\pi_{[100]})_{A_2}]_{E_2} (\pi_{[00-1]})_{A_2}$	—	—	0.333
$[(\pi_{[111]})_{A_2} (\pi_{[000]})_{A_1^-}]_{A_1} (\pi_{[-1-1-1]})_{A_2}$	—	—	0.366
$[(\pi_{[002]})_{A_2} (\pi_{[00-1]})_{A_2}]_{A_1} (\pi_{[00-1]})_{A_2}$	—	—	0.372
$(\rho_{[000]}^{\text{n}=0})_{T_1^-} (\pi_{[000]})_{A_1^-}$	0.225	0.223	0.220
$(\rho_{[001]}^{\text{n}=0})_{A_1} (\pi_{[00-1]})_{A_2}$	0.315	0.286	0.267
$(\rho_{[001]}^{\text{n}=0})_{E_2} (\pi_{[00-1]})_{A_2}$	0.328	0.294	0.274
$(\rho_{[001]}^{\text{n}=1})_{A_1} (\pi_{[00-1]})_{A_2}$	0.346	0.307	0.282
$(\rho_{[011]}^{\text{n}=0})_{A_1} (\pi_{[0-1-1]})_{A_2}$	0.393	0.343	0.311
$(\rho_{[011]}^{\text{n}=0})_{B_1} (\pi_{[0-1-1]})_{A_2}$	0.398	0.346	0.313
$(\rho_{[011]}^{\text{n}=0})_{B_2} (\pi_{[0-1-1]})_{A_2}$	0.400	0.348	0.314

K Matrix Parameters

fit	$\alpha(^3S_1 ^3S_1)$	$\beta(^3S_1 ^3S_1)$	$\gamma(^3S_1)$	$\alpha(^3S_1 ^3D_1)$	$\alpha(^3D_1 ^3D_1)$	$\alpha(^1P_1 ^1P_1)$	dof	Corr	χ^2	$e^{-(2n_p+\chi^2)/2}$
1	-9(5)						11	1	5.1	0.029
2	-8(5)	-0.2(0.4)					10	$\begin{pmatrix} 1.00 & -0.56 \\ & 1.00 \end{pmatrix}$	5.0	0.011
3	-7.1(0.8)		-4(2)				10	$\begin{pmatrix} 1.00 & -0.29 \\ & 1.00 \end{pmatrix}$	4.7	0.013
4	-6(40)	-0.1(0.9)	-4(40)				9	$\begin{pmatrix} 1.00 & -0.62 & -0.97 \\ & 1.00 & 0.42 \\ & & 1.00 \end{pmatrix}$	4.6	0.005
5	-8(4)	0(1)		0(2)			9	$\begin{pmatrix} 1.00 & 0.52 & 0.70 \\ & 1.00 & 0.94 \\ & & 1.00 \end{pmatrix}$	4.6	0.005
6	-10(2)			0(2)			10	$\begin{pmatrix} 1.00 & 0.63 \\ & 1.00 \end{pmatrix}$	4.7	0.013
7	-8(2)	0(3)		0(20)	0(5)		8	$\begin{pmatrix} 1.00 & 0.82 & -0.83 & -0.83 \\ & 1.00 & -1.00 & -0.99 \\ & & 1.00 & 1.00 \\ & & & 1.00 \end{pmatrix}$	4.4	0.002
8	-10(2)			0(1)	0(1)		9	$\begin{pmatrix} 1.00 & 0.02 & -0.53 \\ & 1.00 & -0.29 \\ & & 1.00 \end{pmatrix}$	4.7	0.0047
9	-9(2)					-3(10)	10	$\begin{pmatrix} 1.00 & -0.21 \\ & 1.00 \end{pmatrix}$	5.0	0.011
10	-8(2)	-0.2(0.3)				1(2)	9	$\begin{pmatrix} 1.00 & -0.01 & 0.04 \\ & 1.00 & 0.03 \\ & & 1.00 \end{pmatrix}$	5.0	0.0042

OPE Singularities



OPE Dalitz Plots

