

Small x evolution meets DGLAP

Yacine Mehtar-Tani (BNL)

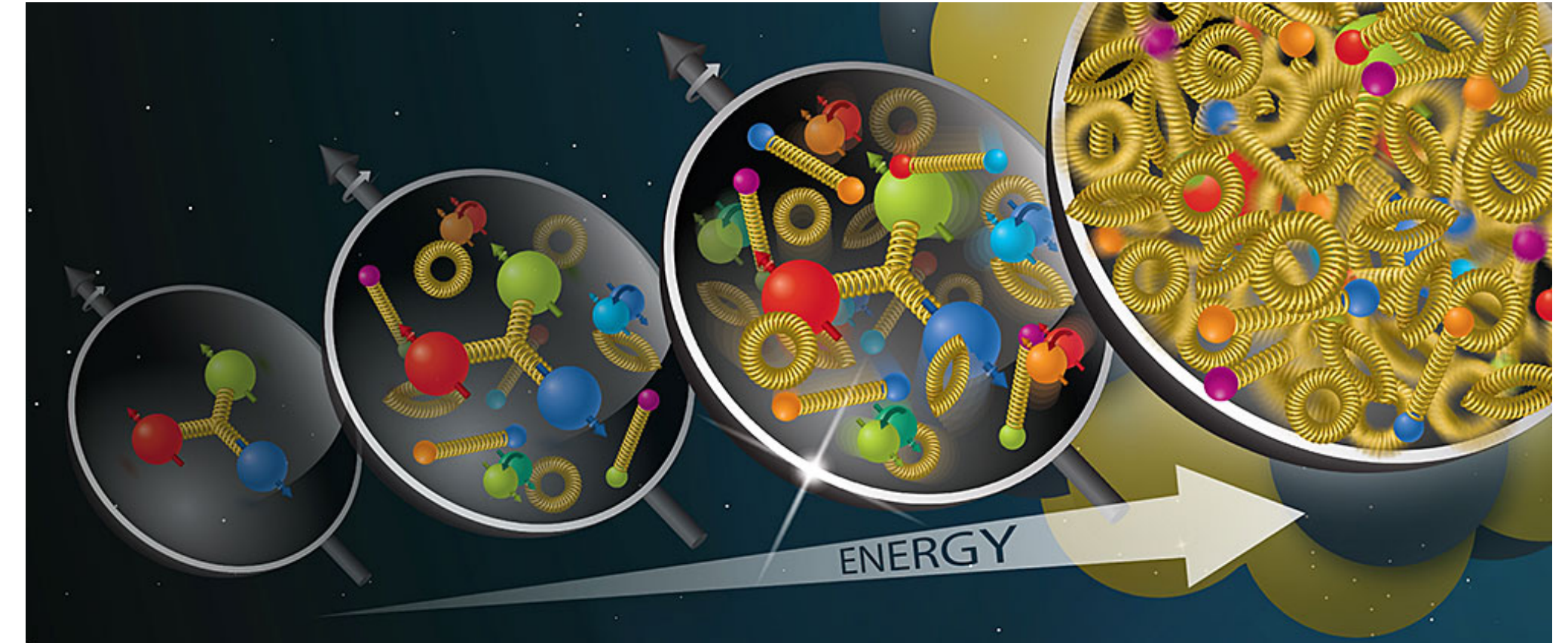
2511.07509 [hep-ph] (PRL 137 (2026) 12, 121903)

Collaboration: R. Boussarie, P. Caucal and P. Korcyl, J. Penttala

MENU 2026, Stony Brook University, October 19-23, 2026

High Energy Limit of QCD

- At high energies (small x), the gluon density rises rapidly and is expected to **saturate** leading to a new state of QCD matter
- This regime is currently being explored at **RHIC** and the **LHC**, and will be a central focus of the **EIC** experimental program
- To reach predictive power, we need **precision theoretical tools**— yet progress has been hindered by several conceptual and technical roadblocks

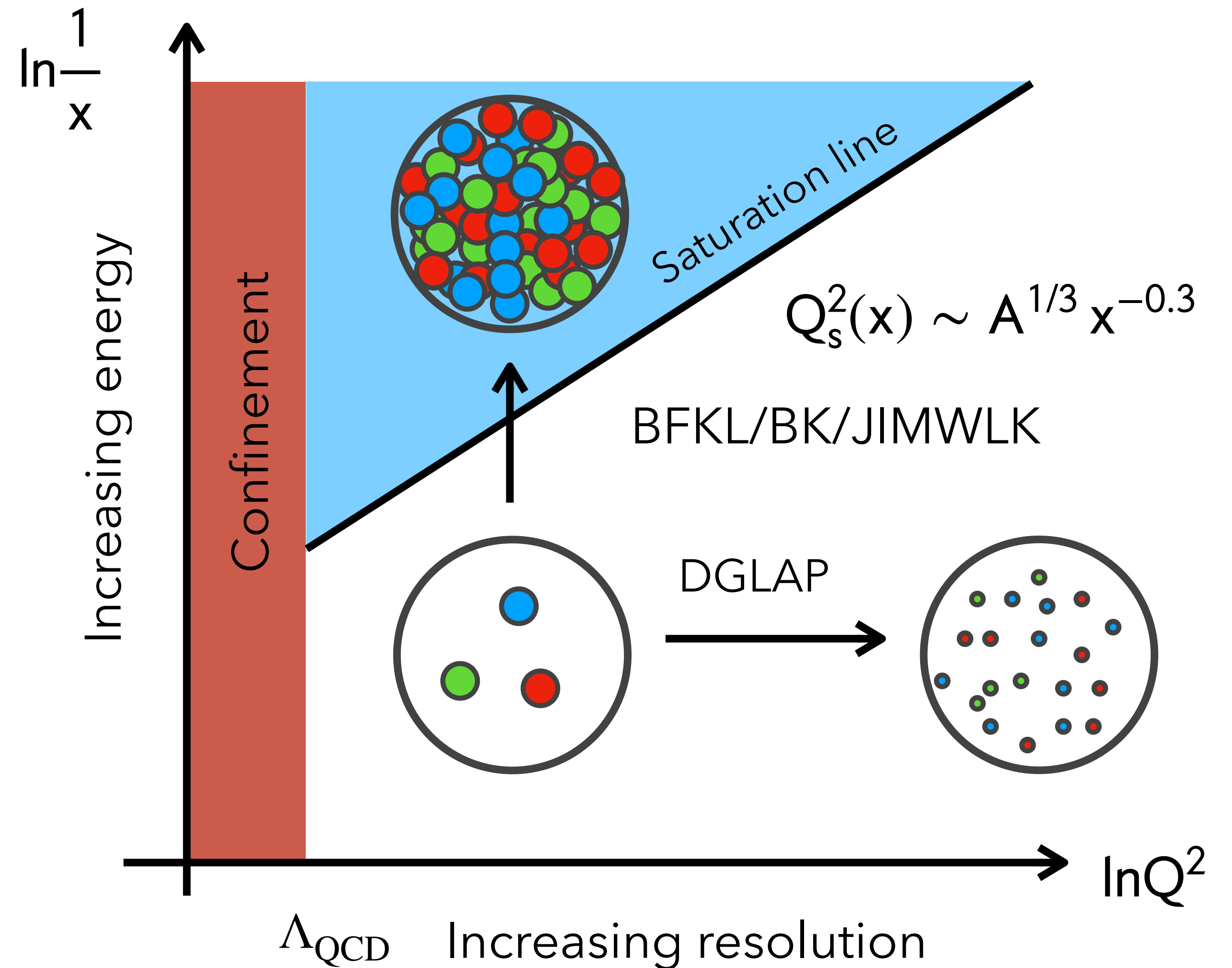


Partons $\rightarrow$ Classical gluon fields

$$x_{Bj} = \frac{Q^2}{s} \rightarrow 0$$

High Energy Limit of QCD

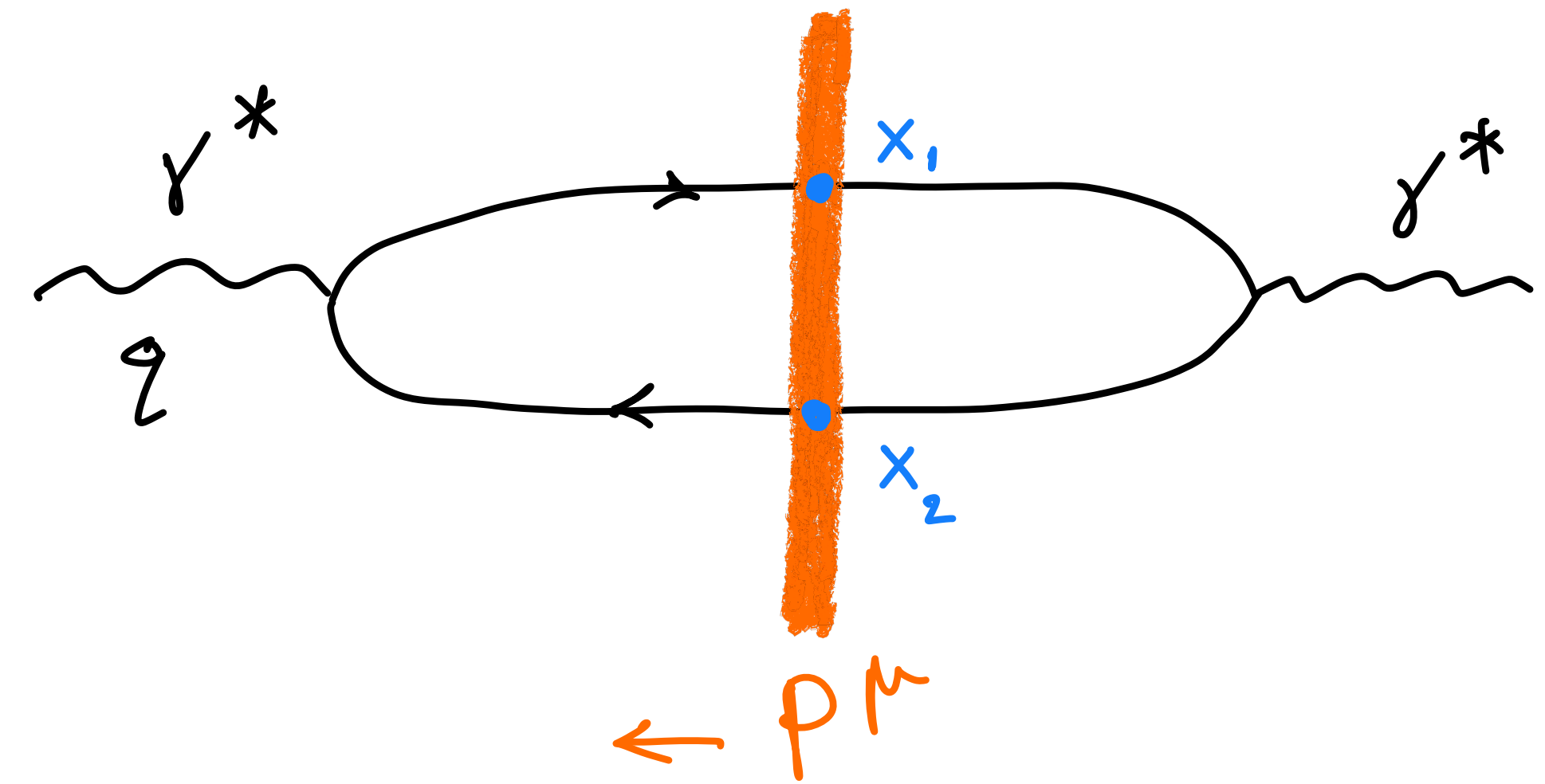
- Small x evolution resums large $\ln(1/x_{Bj})$
- At large $\ln Q^2$ - DGLAP collinear logarithms may become important
- Systematic resummation of those collinear logs may prove crucial for precision phenomenology at EIC



High Energy Factorization in DIS

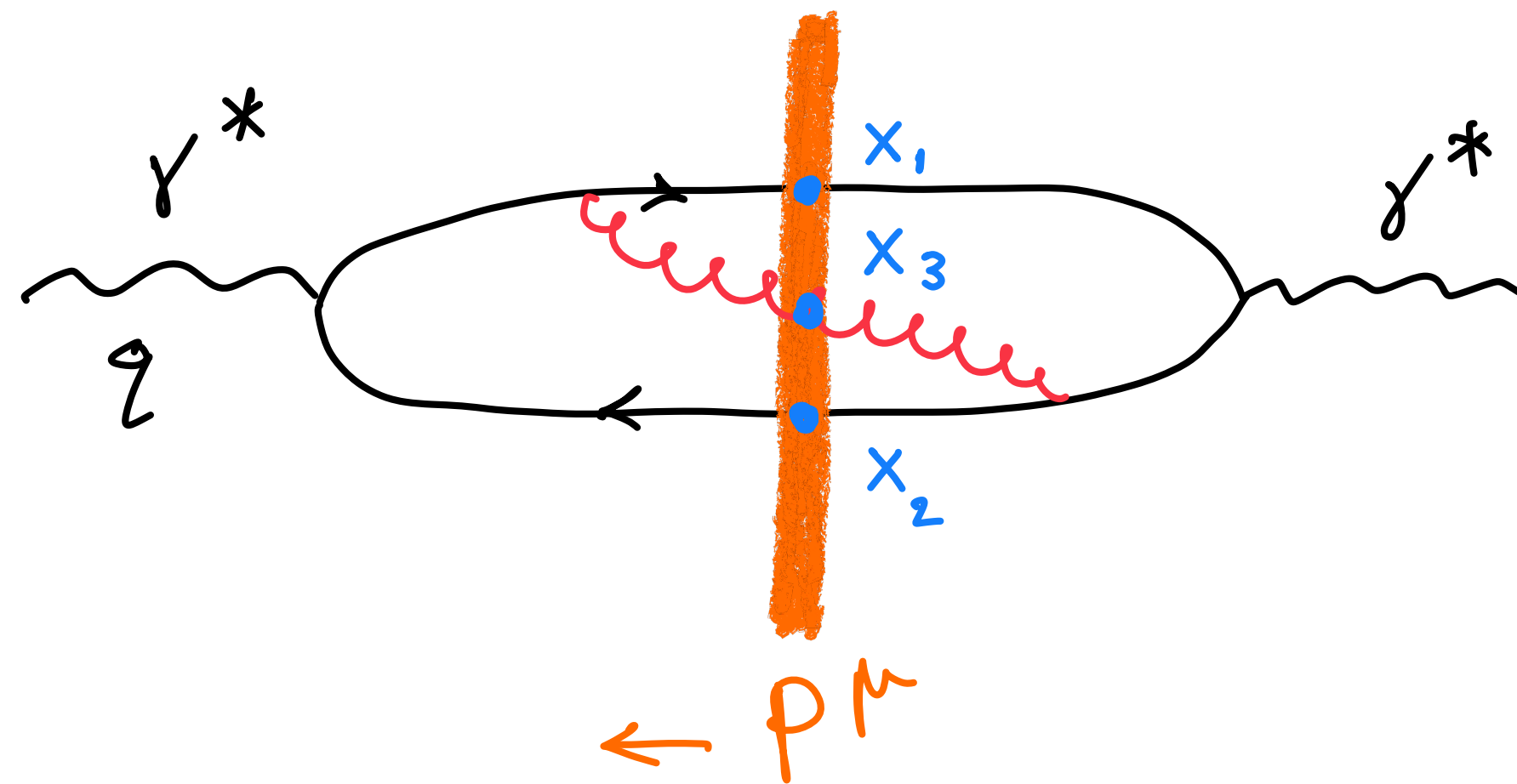
- In the dipole frame: DIS cross-section factorizes into a **hard impact factor** H and a **hadronic matrix element** S_{12}
- $S_{12} \sim \langle \text{tr } U_2^\dagger U_1 \rangle$ encodes the instantaneous interaction of a $q\bar{q}$ dipole with the target (proton or nucleus)

“Dipole scattering in the target field”

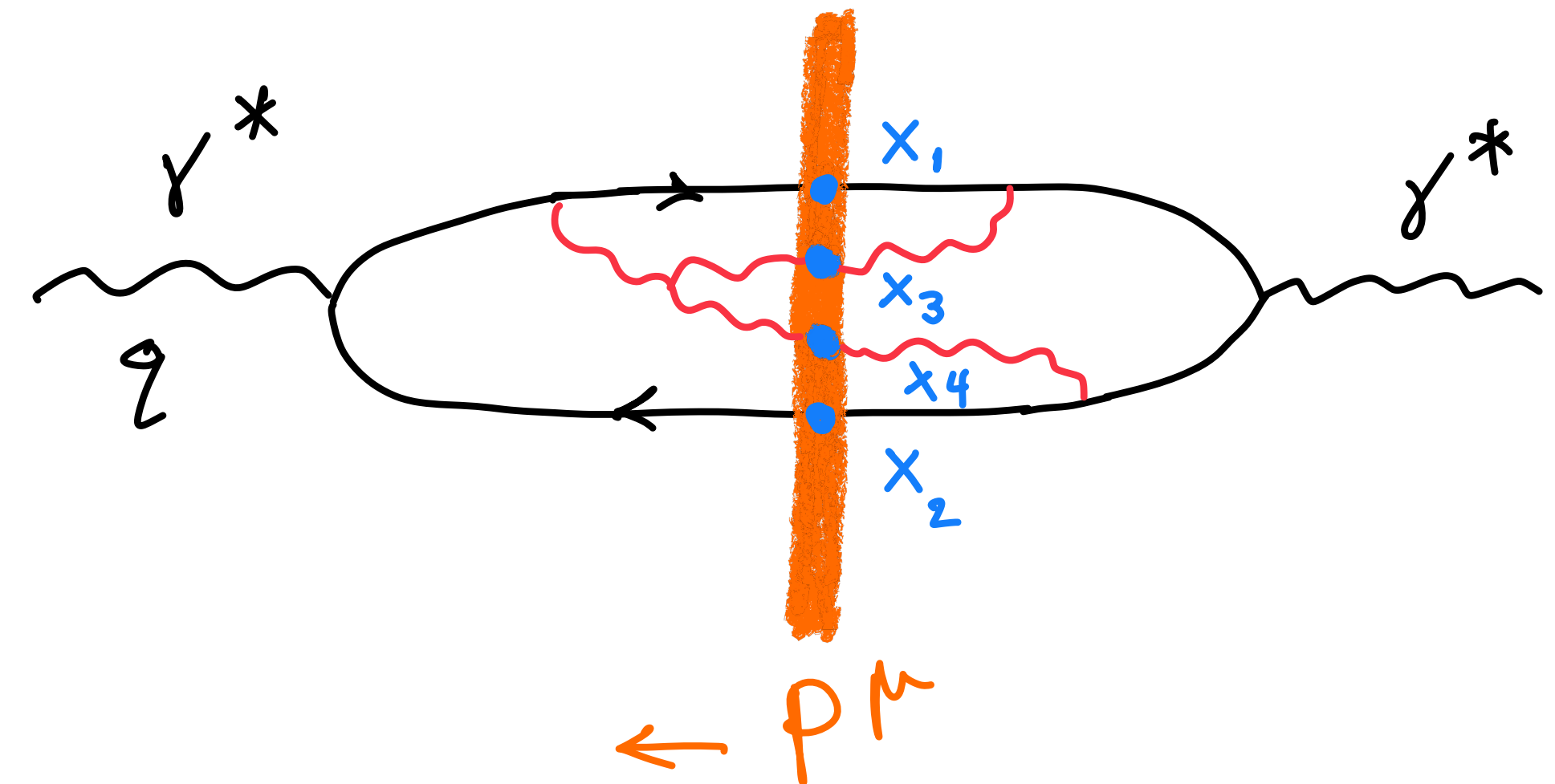


$$\sigma_{\text{LO}}(x, Q^2) = \int_{x_1, x_2} H(Q^2, \mathbf{x}_1, \mathbf{x}_2) S_{12}$$

NLO Balitsky-Kovchegov evolution



LO: Balitsky (1997)



NLO: Balitsky, Chirilli (2007)

$$\frac{\partial}{\partial \ln \rho^+} S_{12} = \bar{\alpha} \int_3 K_{123}^{\text{NLO}} [S_{13} S_{32} - S_{12}] + \bar{\alpha}^2 \int_3 K_{1234} [S_{13} S_{34} S_{42} - S_{13} S_{32}]$$

Similar color structure to LO

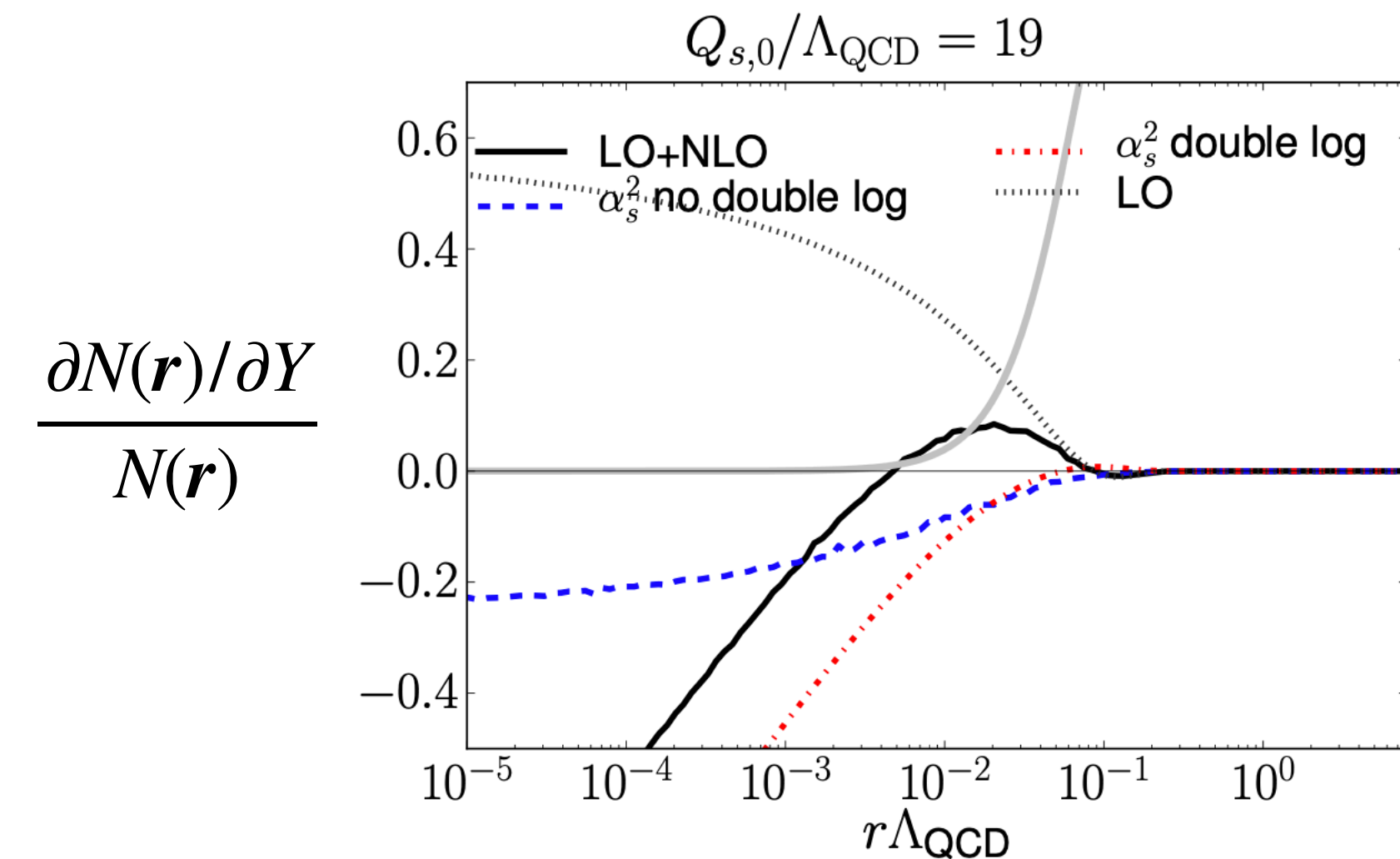
3 dipole term

$$K_{123} = \frac{x_{12}^2}{x_{13}^2 x_{32}^2}$$

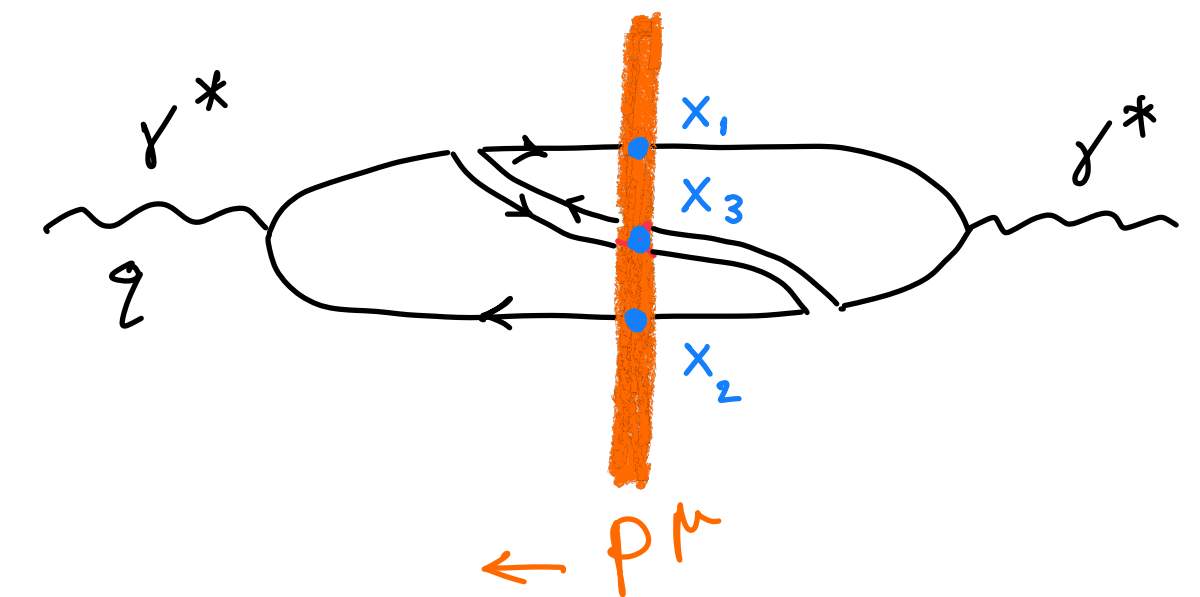
Unstable evolution due to collinear $\ln Q^2$

- NLO BK equations shown by **Lappi and Mäntysaari (2015)** to be unstable
- Instability driven by **collinear double logarithms**
- NB: $N = 1 - S$ and $\mathbf{r} \equiv \mathbf{x}_{12}$

“Evolution speed”



$$\frac{\partial}{\partial Y} S_{12} = \bar{\alpha} \int_3 K_{123} \left(1 - 2\bar{\alpha} \ln \frac{x_{13}^2}{x_{12}^2} \ln \frac{x_{32}^2}{x_{12}^2} \right) [S_{13} S_{32} - S_{12}] + O(\bar{\alpha}^2)$$



Large negative term when $\mathbf{x}_{12} \rightarrow 0$

Unstable evolution due to collinear $\ln Q^2$

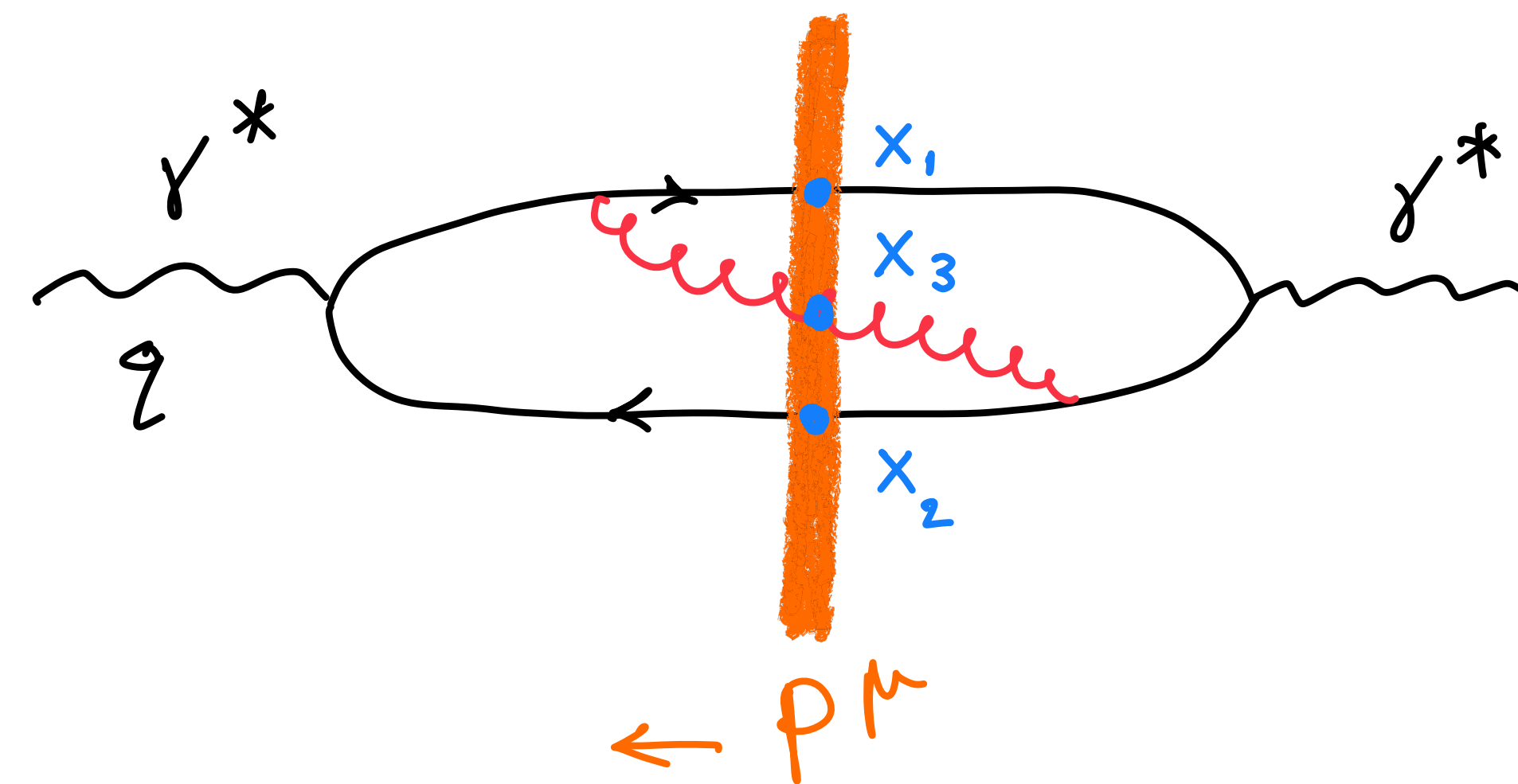
- Similar instabilities were already observed in **NLO BFKL** [Salam, Ciafaloni, Colferai (1998)]
- Related issues:
 - **Negative cross sections** in forward hadron production in pA collisions [Shi, Wang, Wei, Xiao (2021); Liu, Xie, Kang, Liu, (2022)]
 - Wrong sign of Sudakov logarithms [Caucal, Salazar, Schenke, Venugopalan, (2022); Tael, Altinoluk, Beuf, Marquet (2022)]
- Proposed solutions:
 - i) *Modification of the LO BK kernel to resum double logarithms* [Beuf (2014)]
 - ii) *Change of rapidity variable* [Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos (2015-2019)]
- So far, proposed fixes remain formally unsatisfactory and offer no insight into the all-order structure of the high-energy OPE — *its factorization, operator content, and universality*

Emergence of collinear double logs

- Standard high energy OPE ignores the constraint associated with the size of the target:

$$t_g > t_{\text{target}} \implies k^- < P^- \sim \sqrt{s}$$

- The double logarithmic phase space in high energy OPE ($k^- \sim k_{\perp}^2/k^+$)



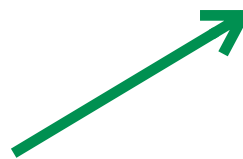
$$\bar{\alpha} \int_{Q_0^2}^{Q^2} \frac{d\mathbf{k}^2}{k^2} \int_{\mathbf{k}^2/P^-}^{\rho^+} \frac{dk^+}{k^+} = \bar{\alpha} \ln \frac{\rho^+ P^-}{Q_0^2} \ln \frac{Q^2}{Q_0^2} - \frac{\bar{\alpha}}{2} \ln^2 \frac{Q^2}{Q_0^2}$$

Emergence of collinear logs

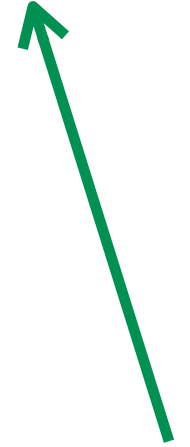
- The basic idea behind our approach is to construct a scheme transformation that eliminates the undesired collinear logarithms
- Introduce a **collinear factorization scale μ** to partition the phase-space:

$$\int_{\mathbf{k}_{\perp}^2/P^-}^{\rho^+} \frac{dk^+}{k^+} = \int_{\mu^2/P^-}^{\rho^+} \frac{dk^+}{k^+} + \int_{\mathbf{k}_{\perp}^2/P^-}^{\mu^2/P^-} \frac{dk^+}{k^+} = \ln \frac{\rho^+ P^-}{\mu^2} - \ln \frac{\mathbf{k}_{\perp}^2}{\mu^2}$$

Rapidity logarithm



Collinear logarithm

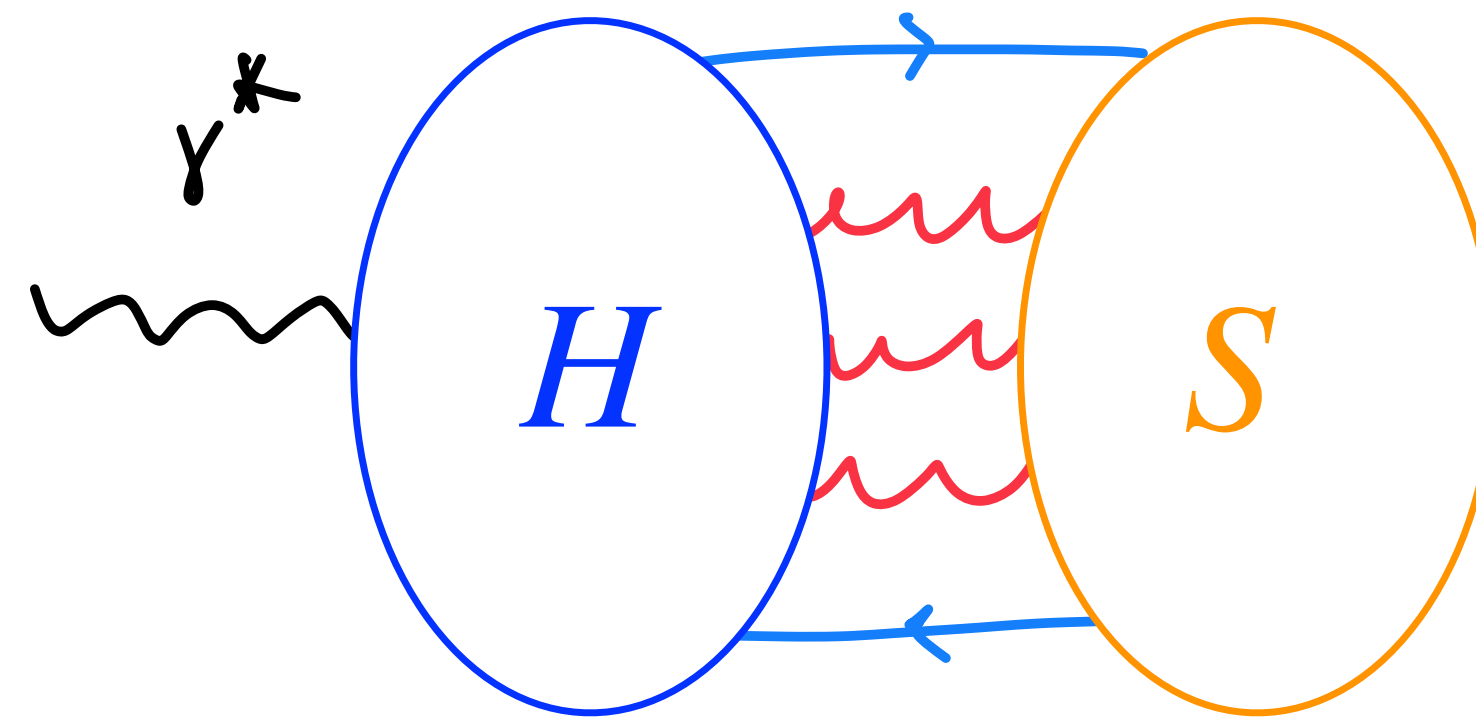


- Subtraction is not straightforward due to non-linear dynamics

Factorization and Scheme Transformation

Linear algebra for non-linear dynamics

- The all-order structure of the factorization reads:



$$\sigma(x, Q^2) = \sum_{n=2}^{\infty} \int_{\mathbf{x}_i} H^{(n)}(Q^2, \{\mathbf{x}_i\}_{i=1}^n, \zeta/s) \times (S^{(n)}(\{\mathbf{x}_i\}_{i=1}^n, \zeta) - 1)$$

- $S^{(n)} \sim \langle U_1 U_2 \dots U_n \rangle$ is the correlator of n Wilson lines and $\zeta = \rho^+ P^-$ the factorization scale

The rotated dipole

- A scheme transformation is achieved via a **nonunitary operator**

$$|\bar{S}(\zeta, \mu^2)\rangle = e^{-\bar{\alpha} L(\bar{\alpha}, \mu^2)} |S(\zeta)\rangle$$

- The L operator is constructed to all order by imposing that the **composite dipole** obeys the property (to match collinear logs with rapidity logs)

$$\frac{\partial \bar{S}}{\partial \ln \zeta} = - \frac{\partial \bar{S}}{\partial \ln \mu^2} \quad \Rightarrow \quad \bar{S}(\zeta, \mu^2) \equiv \bar{S} \left(\frac{\zeta}{\mu^2} \right)$$

- Note that choosing $\zeta = s$ and $\mu^2 = Q^2 \Rightarrow Y = \ln(\zeta/\mu^2) = \ln(1/x_{Bj})$

The rotated NLO evolution

- Transforming the NLO BK kernel using the Campbell identity:

$$\begin{aligned}\bar{K} &= e^{-\bar{\alpha}L} K e^{+\bar{\alpha}L} \\ &= K + \bar{\alpha}[K, L] + \frac{\bar{\alpha}^2}{2!} [[K, L], L] + \mathcal{O}(\bar{\alpha}^3)\end{aligned}$$

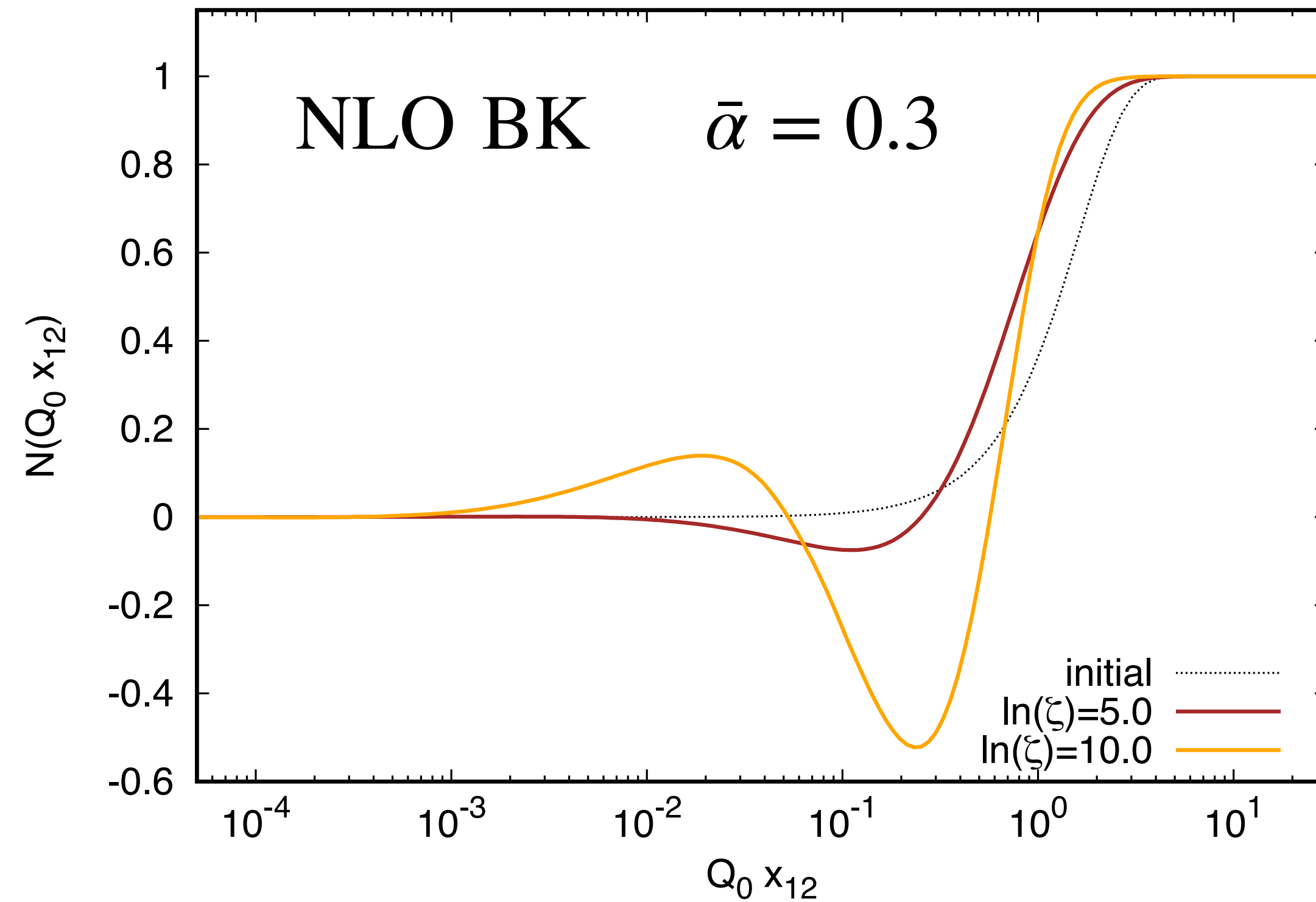
- We obtain the NLO evolution equation for the **rotated dipole**

$$\begin{aligned}\frac{\partial \bar{S}_{12}}{\partial Y} &= \bar{\alpha} K^{\text{NLO}} \otimes \bar{S} + \frac{\bar{\alpha}^2}{2} \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \ln \frac{x_{13}^2}{x_{12}^2} \ln \frac{x_{32}^2}{x_{12}^2} (\bar{S}_{13} \bar{S}_{32} - \bar{S}_{12}) \\ &\quad + \frac{\bar{\alpha}^2}{2} \int_{3,4} \frac{x_{12}^2}{x_{13}^2 x_{34}^2 x_{42}^2} \ln \left(\frac{x_{34}^2}{x_{14}^2} \right) \bar{S}_{13} (\bar{S}_{34} \bar{S}_{42} - \bar{S}_{32}) + \frac{\bar{\alpha}^2}{2} \int_{3,4} \frac{x_{12}^2}{x_{14}^2 x_{43}^2 x_{32}^2} \ln \left(\frac{x_{43}^2}{x_{42}^2} \right) (\bar{S}_{14} \bar{S}_{43} - \bar{S}_{13}) \bar{S}_{32}\end{aligned}$$

- The double logs** cancel precisely the pathological term in K^{NLO}

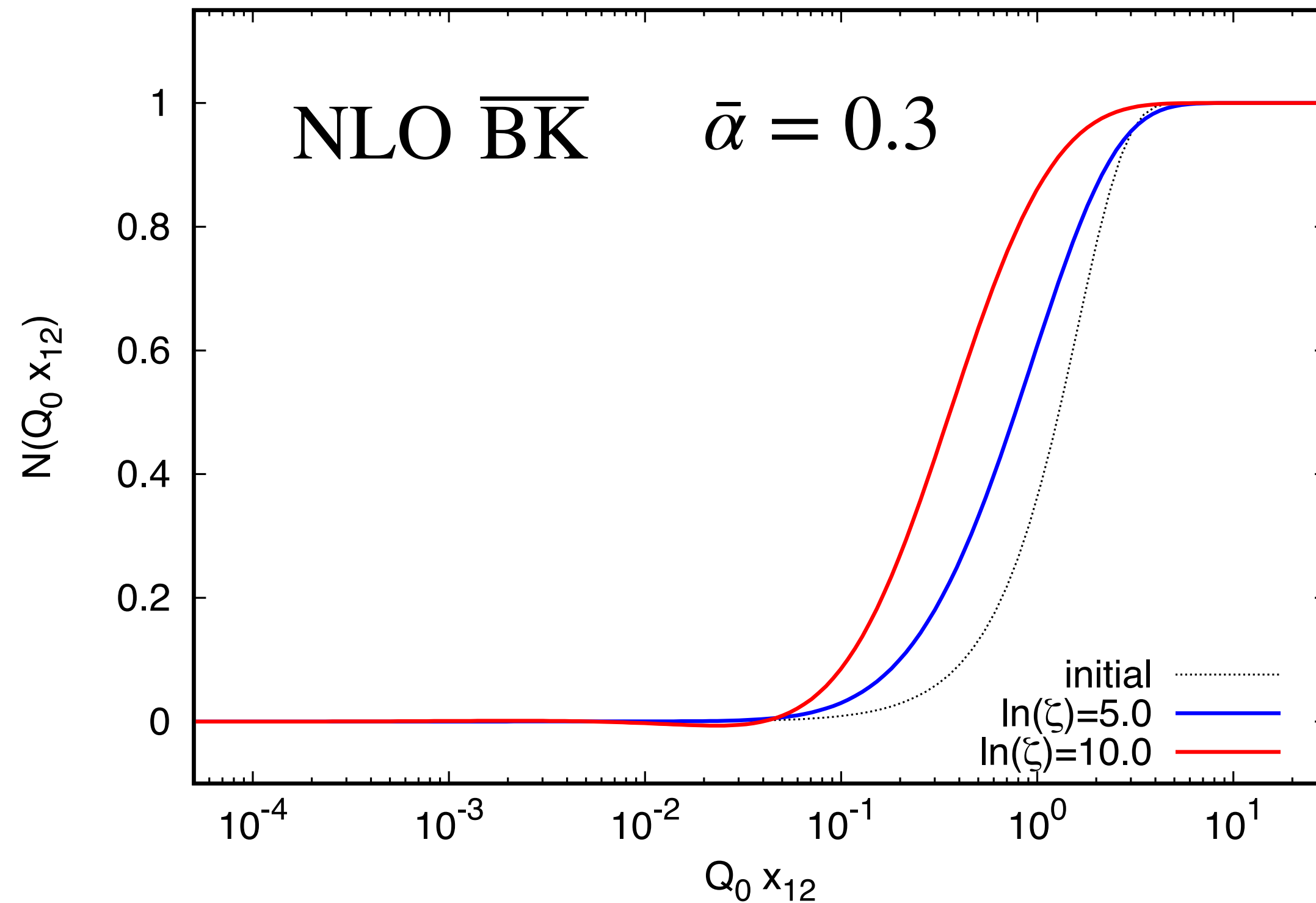
Numerical Simulations

NLO BK vs. NLO $\overline{\text{BK}}$



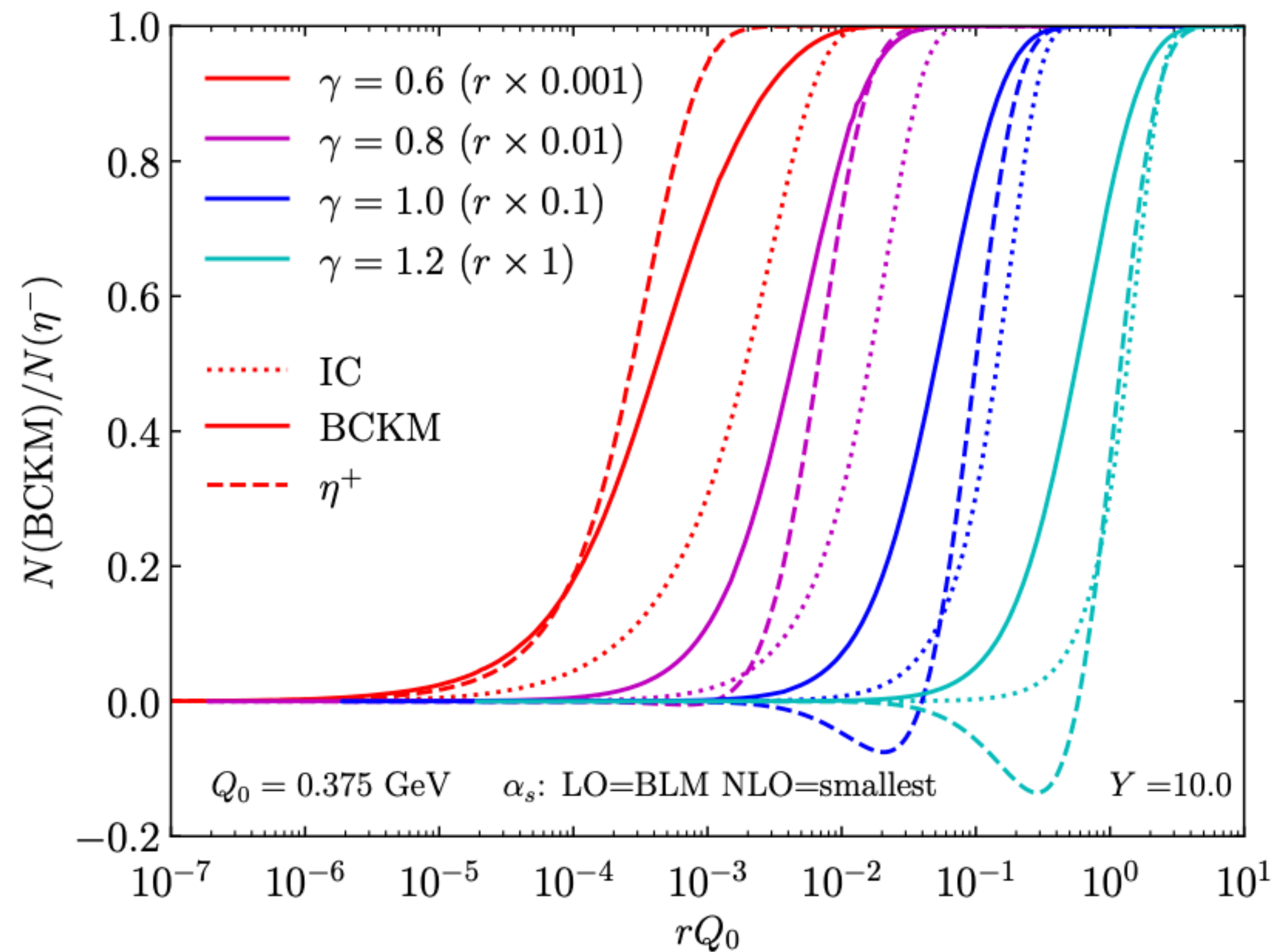
- The dipole amplitude $N = 1 - S$ becomes negative after a few units of rapidity

NLO BK vs. NLO $\overline{\text{BK}}$



- **Stable evolution** in the physical range up to $Y = \ln(\zeta) = 10$
- **Single logs** instabilities emerging at higher rapidities remain. Resolved by running coupling (still need to be addressed)

Running coupling



- **Stable evolution** in the physical range up to $Y = \ln(\zeta) = 10$

DGLAP single logs

- Single collinear logs are also present - hidden in the NLO BK kernel
- Small x expansion for the DGLAP anomalous dimension (in Mellin space)

$$\gamma_{gg}(\omega) = \bar{\alpha}_s \left[\frac{1}{\omega} + A_1 + \mathcal{O}(\omega) \right], \quad A_1 = -\frac{11}{12}, \quad \bar{\alpha}_s = \frac{\alpha_s C_A}{\pi}.$$

- Subtraction through the rotation operator acting on the dipole S_{12}

$$L_0 = \bar{\alpha}_s A_1 \ln(x_{12}^2 \mu^2)$$

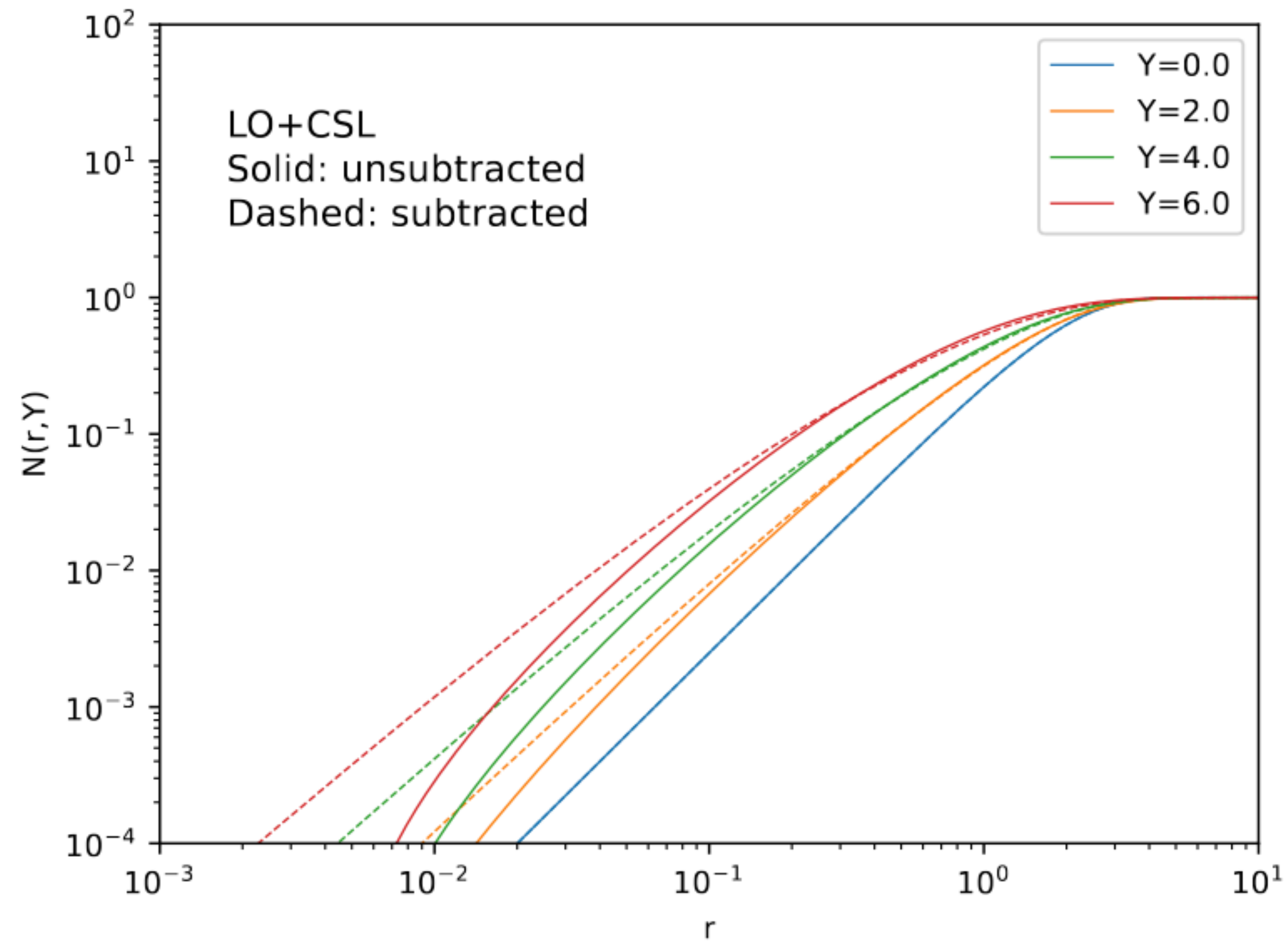
DGLAP single logs

- This transformation yields two coupled evolution equations ($T_{12} = 1 - S_{12}$)

$$\frac{\partial}{\partial Y} T_{12}(Y, \mu) = K_{\text{NLO}} \otimes T_{12} + \bar{\alpha}_s^2 A_1 \int_3 K_{123} \left[\ln \frac{x_{31}^2}{x_{12}^2} T_{31} + \ln \frac{x_{32}^2}{x_{12}^2} T_{32} - \ln \frac{x_{31}^2 x_{32}^2 \mu^2}{x_{12}^2} T_{13} T_{32} \right]$$

$$\frac{\partial}{\partial \ln \mu^2} T_{12}(Y, \mu) = -\bar{\alpha} A_1 T_{12}(Y, \mu)$$

DGLAP single logs



Summary and Outlook

- We have developed a **systematic framework, consistent with the high-energy Operator Product Expansion (OPE)**, to subtract the pathological double and single logarithms associated with DGLAP that have long plagued small- x phenomenology
- In addition to the rapidity factorization scale, we introduce a **collinear scale** that enables a consistent mapping between different rapidity variables.
- **Outlook:**
 - Compute the **full cross section**, including the **rotated impact factor**
 - **Investigate NNLO** using recent result by S. Caron-Huot (2025)
 - Extend the framework to other processes: **inclusive dijet, forward hadron production, SIDIS**, etc.

Thank you!

Linear algebra for non-linear dynamics

- In these notations the solution to the BK equation exponentiates (as in the JIMWLK approach)

$$|S(\zeta)\rangle = e^{-\bar{\alpha} K(\bar{\alpha}) \ln \frac{\zeta}{\zeta_0}} |S(\zeta_0)\rangle$$

- In the dipole chain formalism the LO BK equation reads

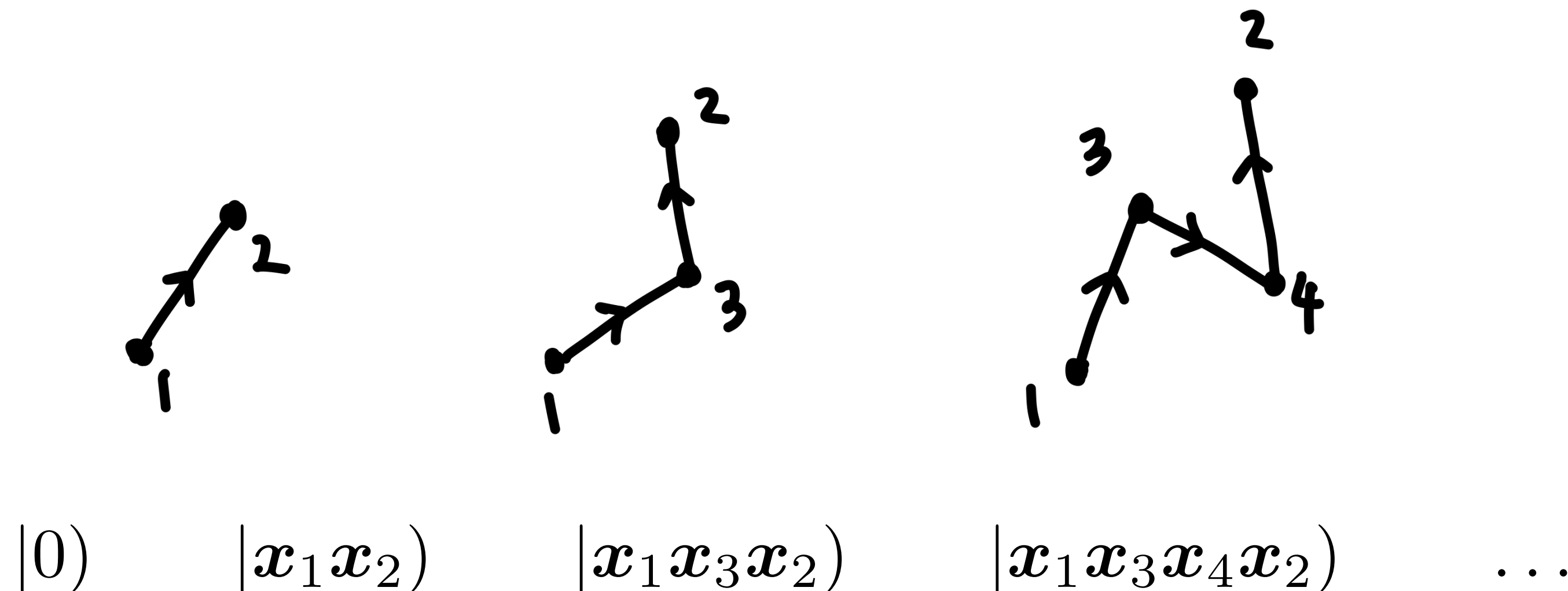
$$\frac{\partial}{\partial \ln \zeta} (2|S) = \bar{\alpha} [(2|K|3)(3|S) + (2|K|2)(2|S)]$$

- With the transition rates given by

$$(2|K|3) \sim \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \quad (2|K|2) \sim - \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \quad \text{where} \quad \begin{aligned} |2) &\equiv |x_1 x_2) \\ |3) &\equiv |x_1 x_3 x_2) \end{aligned}$$

Linear algebra for non-linear dynamics

- We introduce the “dipole chain” basis



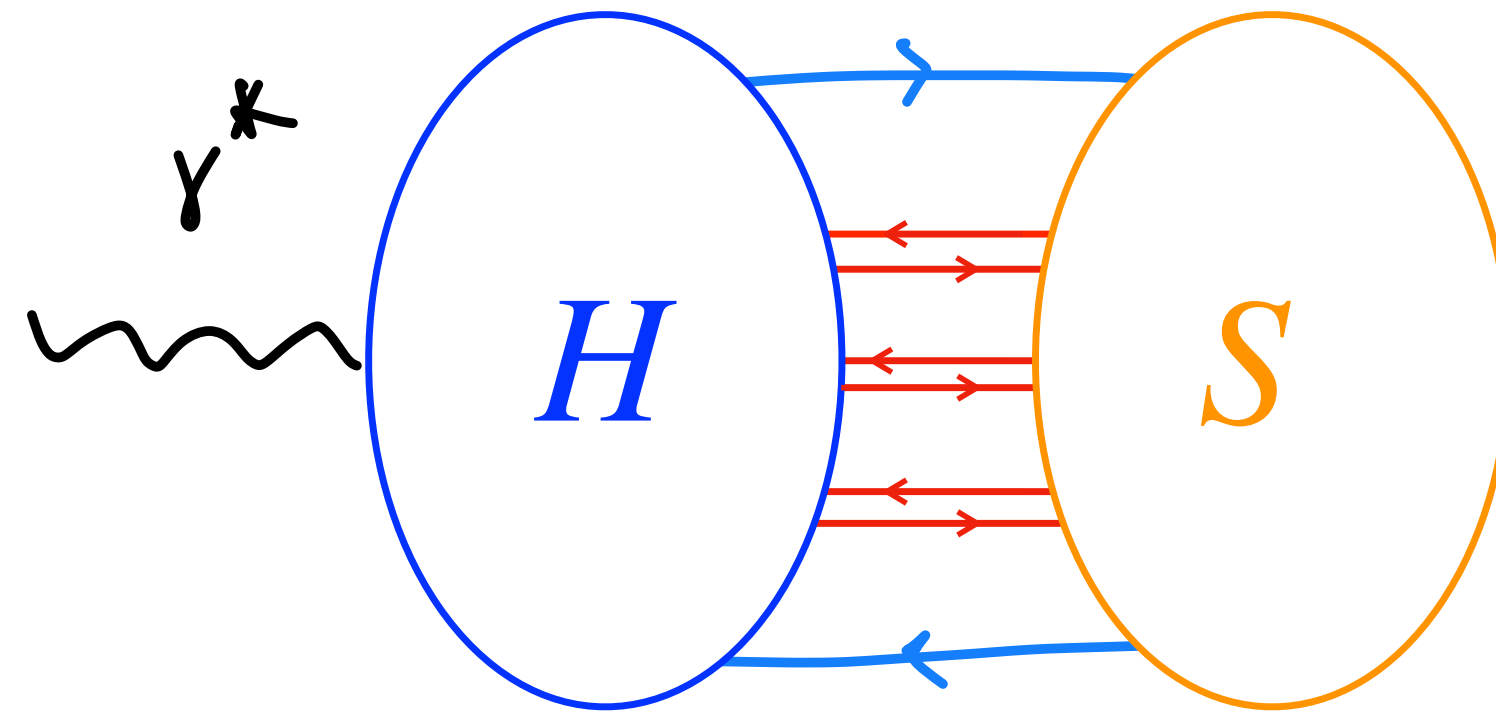
- We adopt the Dirac notation (Doi-Peliti formalism (1976-1986) for probability transitions in stochastic processes)

$$S^{(n)} \equiv (x_1 x_3 \dots x_n x_2 | S) = S_{13} S_{34} \dots S_{n2}$$

The rotated dipole and the NLO BK equation

Linear algebra for non-linear dynamics

- In the large N_c limit the d.o.f.'s are dipoles



$$\sigma(x, Q^2) = \sum_{n=2}^{\infty} \int_{\mathbf{x}_i} H^{(n)}(Q^2, \{\mathbf{x}_i\}_{i=1}^n, \zeta/s) \times (S^{(n)}(\{\mathbf{x}_i\}_{i=1}^n, \zeta) - 1)$$

- $S^{(n)} \sim \langle U_1 U_2 \dots U_n \rangle$ is the correlator of n Wilson lines and $\zeta = \rho^+ P^-$ the factorization scale

The rotated dipole

- Expanding in $\bar{\alpha}$: $K = K_1 + \bar{\alpha} K_2 + \dots$ and $L = L_1 + \bar{\alpha} L_2 + \dots$
- The scaling constraint yields at NLO

$$\frac{\partial}{\partial \ln \mu^2} L_1(\mu^2) = K_1$$

- Residual freedom in fixing the constant in L

$$(2|L_1(\mu^2)|3) \sim \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \ln(\mu^2 |x_{13}| |x_{32}|)$$

- ▶ So far, proposed fixes remain formally unsatisfactory and offer no insight into the all-order structure of the high-energy OPE — *its factorization, operator content, and universality*
- ▶ In this work, we construct a systematic, OPE-consistent framework that aims at stabilizing small- x evolution

Caucal, Boussarie, Korcyl, MT

511.07509 [hep-ph] (PRL 137 (2026) 12, 121903)

The rotated dipole at NLO

- Expanding in $\bar{\alpha}$: $K = K_1 + \bar{\alpha} K_2 + \dots$ and $L = L_1 + \bar{\alpha} L_2 + \dots$
- Expanding the exponential to NLO the composite dipole reads

$$\bar{S}_{12}(\zeta/\mu^2) = S_{12}(\zeta) + \bar{\alpha} \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \ln \left(\frac{1}{|x_{13}| |x_{32}| \mu^2} \right) [S_{13} S_{32} - S_{12}] + \mathcal{O}(\bar{\alpha}^2)$$

- This approach encompasses Balitsky's conformal dipole (2008)
- Similar transformation for the Impact factor (new NLO term)