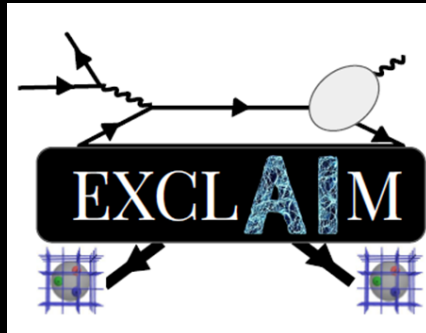


Beyond χ^2 : Likelihood, Correlations and Uncertainty in Hadron Physics

Simonetta Liuti

Precision QCD ad hoc meeting
CFNS, Stony Brook, September 2026



UVA sub-group



Douglas Q. Adams



Saraswati Pandey

Physics Framework/Motivation

Deeply Virtual Exclusive Processes

- Deeply Virtual Compton Scattering (DVCS)
- Deeply Virtual Meson Production (DVMP)

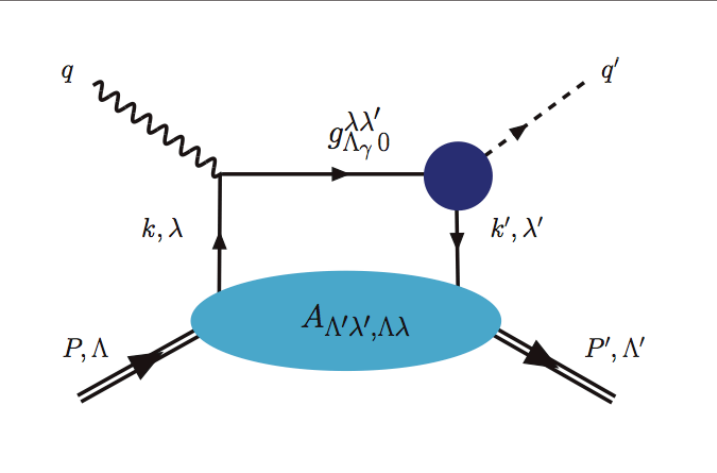
Inclusive processes

- Polarized structure functions

Focus on Deeply virtual pseudoscalar meson production

first suggested in S. Ahmad, G. Goldstein and SL, Phys.Rev. D79 (2009) 054014

$$e p \rightarrow e' p' \pi^0$$



$$\langle P' | \bar{u}(\xi) \sigma_{\mu\nu} u(0) | P \rangle$$

$$W_{\Lambda', \Lambda}^{[i\sigma^{i+}\gamma_5]}(x, \xi, t) = \frac{1}{2P^+} \bar{U}(p', \Lambda') \left(i\sigma^{+i} H_T(x, \xi, t) + \frac{\gamma^+ \Delta^i - \Delta^+ \gamma^i}{2M} E_T(x, \xi, t) \right. \\ \left. + \frac{p^+ \Delta^i - \Delta^+ p^i}{M^2} \tilde{H}_T(x, \xi, t) + \frac{\gamma^+ p^i - p^+ \gamma^i}{2M} \tilde{E}_T(x, \xi, t) \right) U(p, \Lambda)$$

$$g_1 = g_{\pi}^{V, odd}(Q), \quad g_2 = \frac{g_{\pi}^{V, odd}(Q) + g_{\pi}^{A, odd}(Q)}{2}, \quad g_3 = \frac{g_{\pi}^{V, odd}(Q) - g_{\pi}^{A, odd}(Q)}{2}, \quad g_4 = g_{\pi}^{V, odd}(Q).$$

Cross Section

$$\begin{aligned}
 \frac{d^4\sigma}{dx_{Bj}dyd\phi dt} = & \Gamma \left\{ F_{UU,T} + \epsilon F_{UU,L} + \epsilon \cos 2\phi F_{UU}^{\cos 2\phi} + \sqrt{2\epsilon(\epsilon+1)} \cos \phi F_{UU}^{\cos \phi} + h \sqrt{2\epsilon(1-\epsilon)} \sin \phi F_{LU}^{\sin \phi} \right\} \\
 & + S_{\parallel} \left[\sqrt{2\epsilon(\epsilon+1)} \sin \phi F_{UL}^{\sin \phi} + \epsilon \sin 2\phi F_{UL}^{\sin 2\phi} + h \left(\sqrt{1-\epsilon^2} F_{LL} + \sqrt{2\epsilon(1-\epsilon)} \cos \phi F_{LL}^{\cos \phi} \right) \right] \\
 & + S_{\perp} \left[\sin(\phi - \phi_S) \left(F_{UT,T}^{\sin(\phi-\phi_S)} + \epsilon F_{UT,L}^{\sin(\phi-\phi_S)} \right) + \epsilon \left(\sin(\phi + \phi_S) F_{UT}^{\sin(\phi+\phi_S)} + \sin(3\phi - \phi_S) F_{UT}^{\sin(3\phi-\phi_S)} \right) \right. \\
 & + \left. \sqrt{2\epsilon(1+\epsilon)} \left(\sin \phi_S F_{UT}^{\sin \phi_S} + \sin(2\phi - \phi_S) F_{UT}^{\sin(2\phi-\phi_S)} \right) \right] \\
 & + S_{\perp} h \left[\sqrt{1-\epsilon^2} \cos(\phi - \phi_S) F_{LT}^{\cos(\phi-\phi_S)} + \sqrt{2\epsilon(1-\epsilon)} \left(\cos \phi_S F_{LT}^{\cos \phi_S} + \cos(2\phi - \phi_S) F_{LT}^{\cos(2\phi-\phi_S)} \right) \right] \Big\}
 \end{aligned}$$

GPDs will be included in
helicity amplitudes

$$F_{UU,T} = \mathcal{N} [|f_{10}^{++}|^2 + |f_{10}^{+-}|^2 + |f_{10}^{-+}|^2 + |f_{10}^{--}|^2]$$

$$F_{UU,L} = \mathcal{N} [|f_{00}^{++}|^2 + |f_{00}^{+-}|^2]$$

$$F_{UU}^{\cos 2\phi} = -\mathcal{N} 2\Re [(f_{10}^{++})^* (f_{10}^{--}) - (f_{10}^{+-})^* (f_{10}^{-+})]$$

$$F_{UU}^{\cos \phi} = -\mathcal{N} \Re [(f_{00}^{+-})^* (f_{10}^{+-} + f_{10}^{-+}) + (f_{00}^{++})^* (f_{10}^{++} - f_{10}^{--})]$$

$$F_{LU}^{\sin \phi} = \mathcal{N} \Im [(f_{00}^{+-})^* (f_{10}^{+-} + f_{10}^{-+}) + (f_{00}^{++})^* (f_{10}^{++} - f_{10}^{--})]$$

Putting all together

$$f_{++}^{10} \equiv f_1 = g_1(Q^2) e^{i\phi} \frac{\sqrt{t_0 - t}}{4M} \left[2\tilde{\mathcal{H}}_T + (1 + \xi) (\mathcal{E}_T - \tilde{\mathcal{E}}_T) \right]$$

$$f_{+-}^{10} \equiv f_2 = g_2(Q^2) \sqrt{1 - \xi^2} \left[\mathcal{H}_T + \frac{t_0 - t}{4M^2} \tilde{\mathcal{H}}_T - \frac{\xi^2}{1 - \xi^2} \mathcal{E}_T + \frac{\xi}{1 - \xi^2} \tilde{\mathcal{E}}_T \right]$$

$$f_{-+}^{10} \equiv f_3 = g_3(Q^2) e^{-i2\phi} \sqrt{1 - \xi^2} \frac{t_0 - t}{4M^2} \tilde{\mathcal{H}}_T$$

$$f_{--}^{10} \equiv f_4 = g_4(Q^2) e^{i\phi} \frac{\sqrt{t_0 - t}}{4M} \left[2\tilde{\mathcal{H}}_T + (1 - \xi) (\mathcal{E}_T + \tilde{\mathcal{E}}_T) \right]$$

CFF

$$\tilde{\mathcal{F}}(\xi, t, Q^2) = \int_{-1}^1 dx C^+(x, \xi) F_T(x, \xi, t, Q^2)$$

GPD

Compton Form Factor

Generalized Parton Distributions

Forward limit

$$H_T(x, 0, 0) = q_{\uparrow\uparrow}^{\uparrow}(x) - q_{\uparrow\uparrow}^{\downarrow}(x) = h_1(x)$$

transversity

Form Factors/properties

$$\int H_T(x, \xi, t) dx = \delta_T(t)$$

Tensor charge

$$\int \bar{E}_T(x, \xi, t) dx = \int (2\tilde{H}_T + E_T) dx = \kappa_T$$

Tensor magnetic moment

$$\int \tilde{E}_T(x, \xi, t) dx = 0$$

The fundamental tensor charge can be measured in differential decay distribution for polarized neutron decay

T.D. Lee, Chen-Ning Yang, Phys. Rev. 104 (1956)

$$n \rightarrow p + e^- + \bar{\nu}$$

$$\frac{d\Gamma}{dE_e d\Omega_e d\Omega_\nu} = \frac{(G_F^{(0)})^2 |V_{ud}|^2}{(2\pi)^5} (1 + 2\epsilon_L + 2\epsilon_R) \times (1 + 3\tilde{\lambda}^2) \cdot w(E_e) \cdot D(E_e, \mathbf{p}_e, \mathbf{p}_\nu, \boldsymbol{\sigma}_n),$$

$$\begin{aligned} D(E_e, \mathbf{p}_e, \mathbf{p}_\nu, \boldsymbol{\sigma}_n) = & 1 + c_0 + c_1 \frac{E_e}{M_N} + \boxed{\frac{m_e}{E_e} \bar{b}} \quad \text{Fierz term} \\ & + \bar{a}(E_e) \frac{\mathbf{p}_e \cdot \mathbf{p}_\nu}{E_e E_\nu} + \bar{A}(E_e) \frac{\boldsymbol{\sigma}_n \cdot \mathbf{p}_e}{E_e} \\ & + \boxed{\bar{B}(E_e) \frac{\boldsymbol{\sigma}_n \cdot \mathbf{p}_\nu}{E_\nu}} + \bar{C}_{(aa)}(E_e) \left(\frac{\mathbf{p}_e \cdot \mathbf{p}_\nu}{E_e E_\nu} \right)^2 \\ & + \bar{C}_{(aA)}(E_e) \frac{\mathbf{p}_e \cdot \mathbf{p}_\nu}{E_e E_\nu} \frac{\boldsymbol{\sigma}_n \cdot \mathbf{p}_e}{E_e} \\ & + \bar{C}_{(aB)}(E_e) \frac{\mathbf{p}_e \cdot \mathbf{p}_\nu}{E_e E_\nu} \frac{\boldsymbol{\sigma}_n \cdot \mathbf{p}_\nu}{E_\nu}, \end{aligned} \quad (9)$$

These terms can contain tensor corrections

- The precision with which ε_T (ε_S) can be measured depends on the uncertainty on g_T (g_S)
- The observable is always the product of the fundamental coupling times a hadronic matrix element

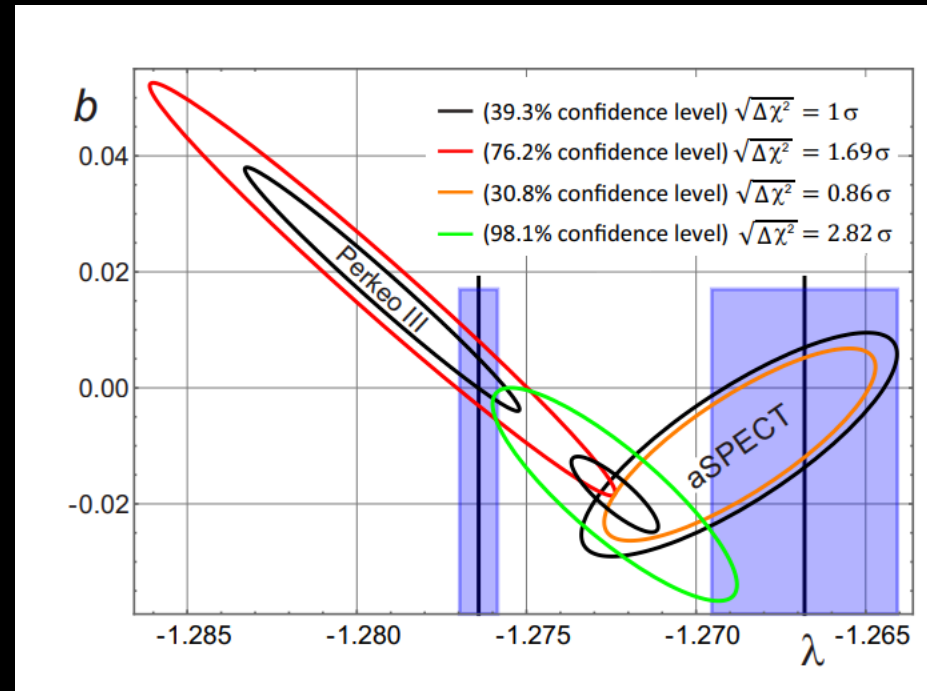
$$C_T = \frac{G_F}{\sqrt{2}} V_{ud} g_T \varepsilon_T$$

- Polarized hard scattering processes at Jlab @12 GeV and at EIC can provide the hadronic matrix elements to extract the BSM **tensor, scalar and pseudo-scalar** effective couplings entering the neutron beta decay cross section

Reanalysis of aSpect experiment

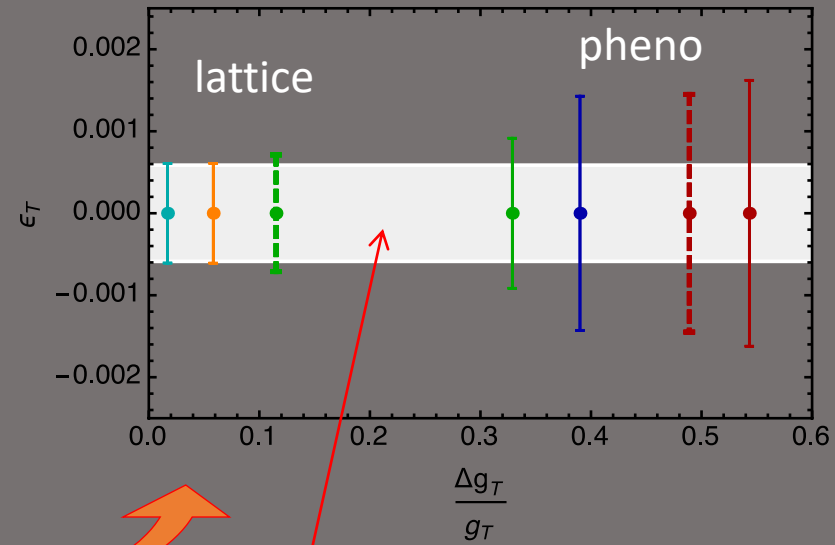
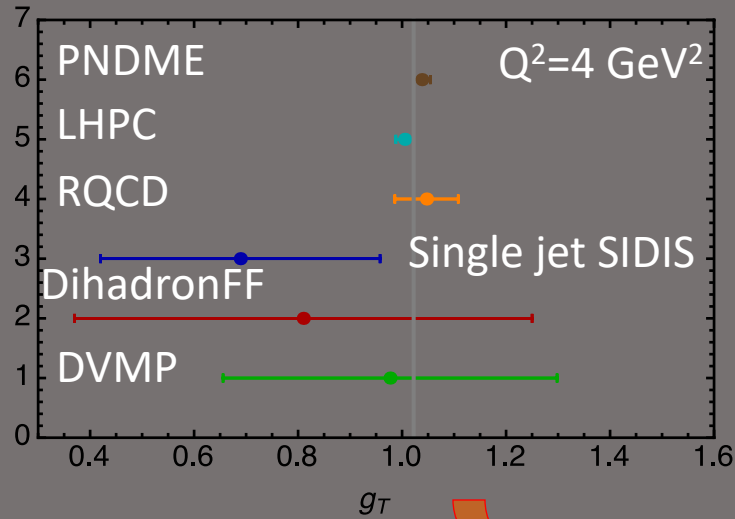
M. Beck, W. Heil, Ch. Schmidt, S. Baeßler, F. Gluck, G. Konrad,
and U. Schmidt: arXiv [2308.16170](https://arxiv.org/abs/2308.16170) (Phys.Rev.Lett.)

Correlated analysis of Fierz term and $\lambda = g_A/g_V$



b is found to be non-zero

Impact on BSM searches...



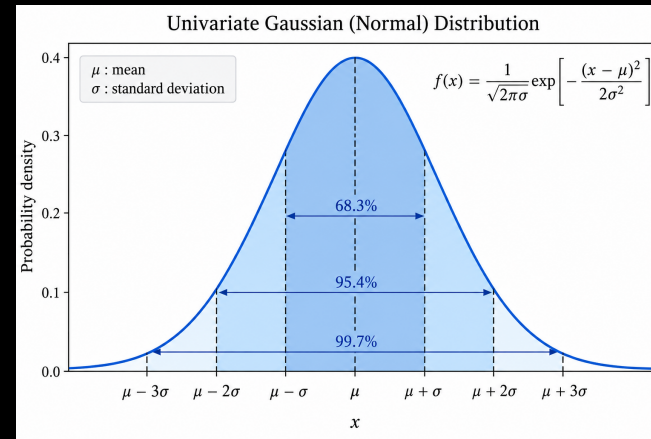
$$|\epsilon_T g_T| < 6.4 \times 10^{-4}$$

Pattie et al, PRC88 (2013)

A. Courtoy, S. Baessler, M. Gonzalez-Alonso and S. Liuti,
arXiv:1503.06814 [hep-ph], Phys Rev. Lett (2015).

Analysis Framework

Estimating parameters and their errors in a model through maximum likelihood



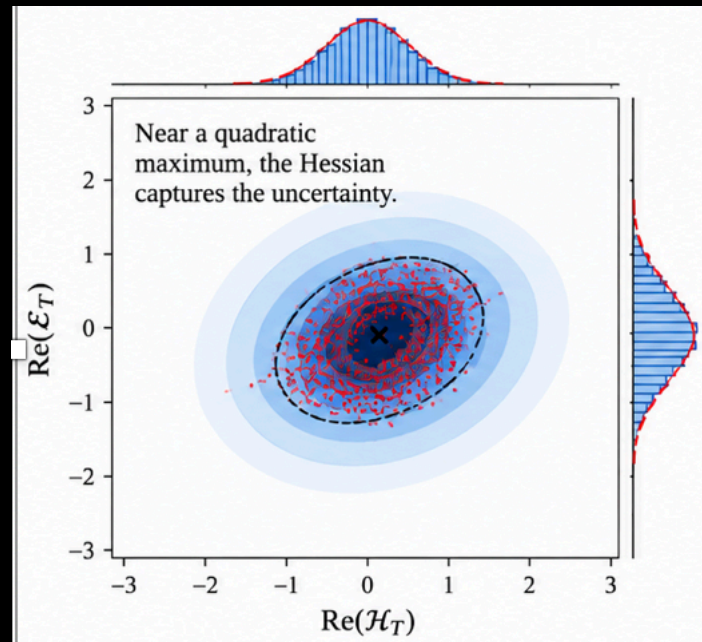
$$p(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

probability of measuring x – true value is μ given σ – the error bar

Likelihood

$$\mathcal{L}(\mu) = \prod_{i=1}^n p(x_i|\mu, \sigma) \rightarrow \left. \frac{d \ln \mathcal{L}(\mu)}{d\mu} \right|_{\mu_o} = 0 \rightarrow \mu_o = \frac{1}{n} \sum_{i=1}^n x_i$$

Can generalize to multivariate gaussians

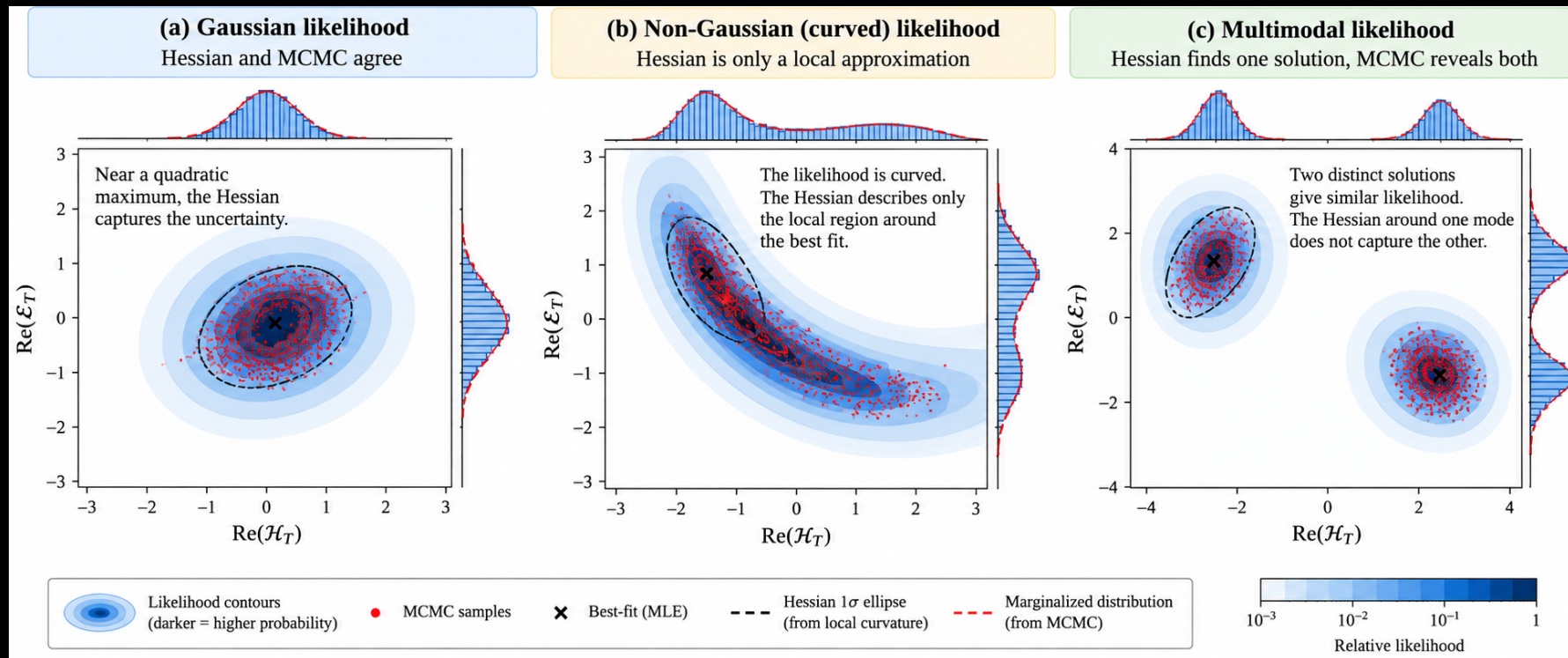


But this can go wrong

The likelihood may exhibit curved valleys, flat directions, asymmetric regions, strong correlations, and disconnected/multiple separated solutions

Likelihood sampling

Rather than approximating the likelihood by its curvature at the maximum, we explore the likelihood/posterior throughout the multidimensional parameter space.



Hessian approach

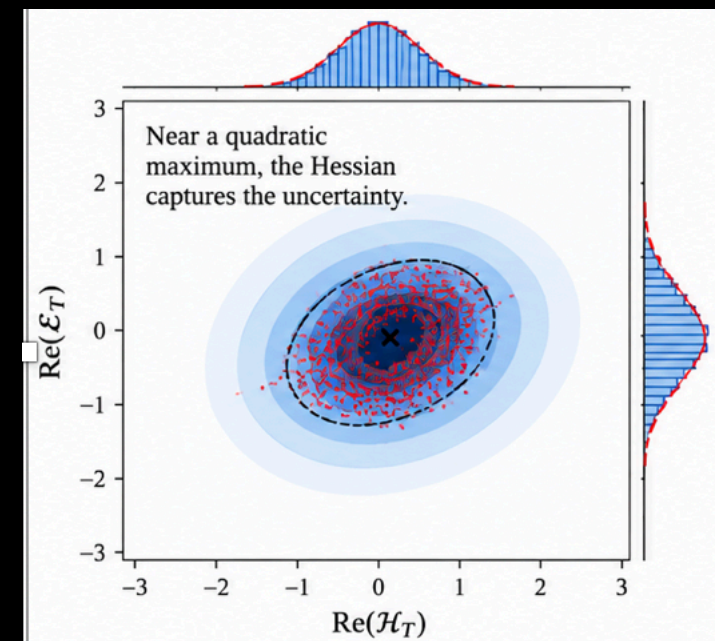
$$\mathcal{L}(\theta) = p(D|\theta)$$

D=data
 θ =parameters

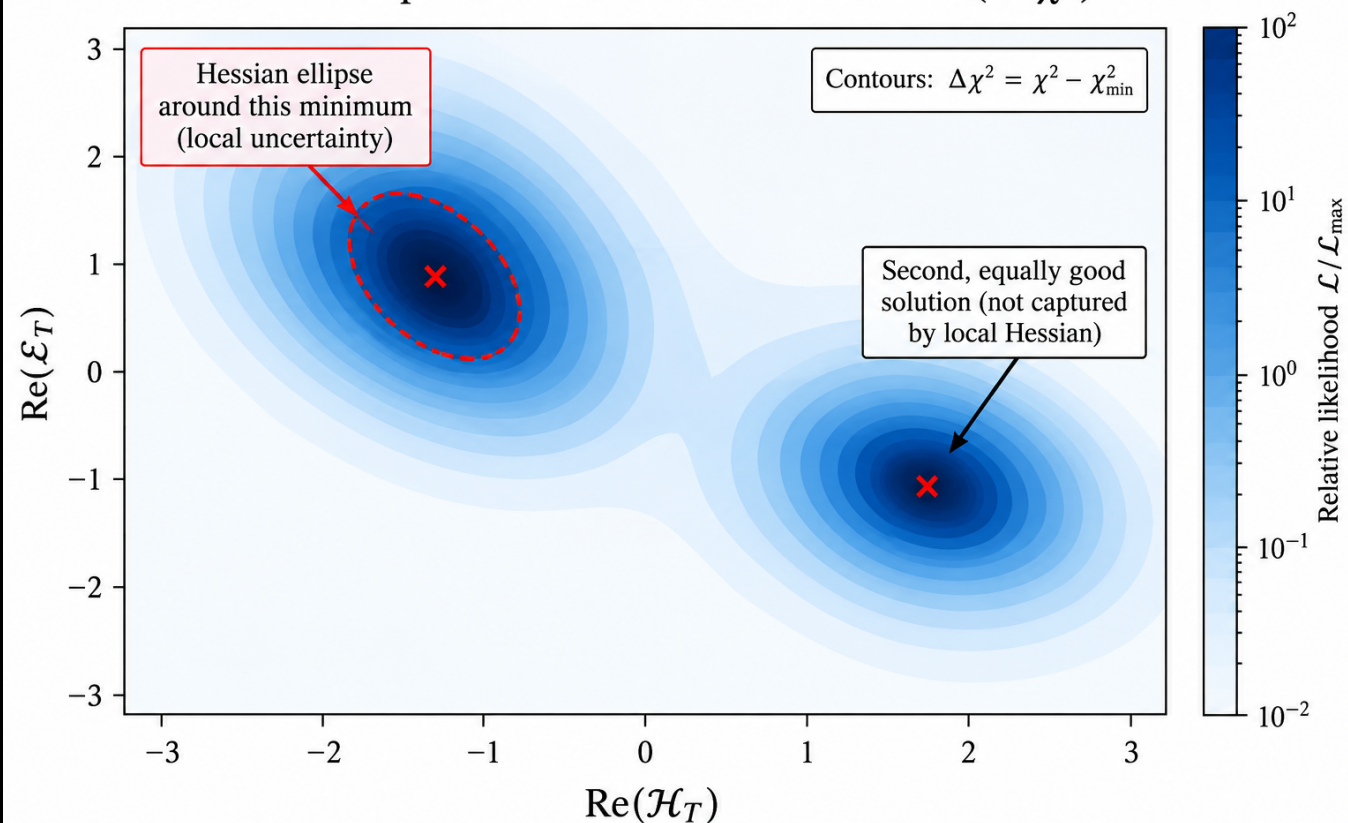
Expand the likelihood locally around the best-fit point:

$$-2 \ln \mathcal{L}(\theta) = -2 \ln \mathcal{L}(\theta_o) + (\theta - \theta_o)^T H (\theta - \theta_o)$$

- This assumes that the likelihood is approximately quadratic in the region relevant for the uncertainty determination
- Even at the global minimum, the Hessian can give a misleading uncertainty so you can have the correct global best fit and nevertheless obtain misleading uncertainties
- If the minimizer lands at a local rather than global minimum, the Hessian will calculate uncertainties around the wrong minimum



Two separated minima in the likelihood (or χ^2)



Working in a Bayesian framework

Probability Distribution Functions are in principle unknown but related to one another in a measurement process through Bayes theorem

- Probabilities describe the subjective belief → not just referring to data
- Parameters are also affected by probabilities → defining the uncertainty of knowledge on any parameter

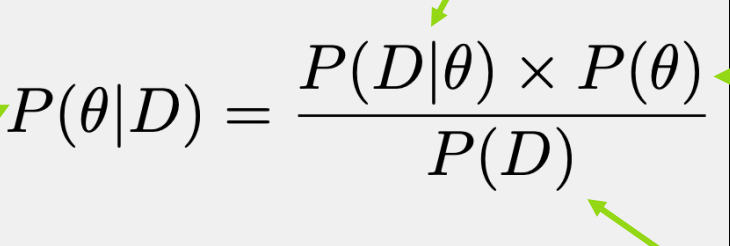
Likelihood : PDF of Data given Model (parameters θ)

Posterior : PDF of Model (parameters θ) given Data (D)

$$P(\theta|D) = \frac{P(D|\theta) \times P(\theta)}{P(D)}$$

Prior: PDF of Model

Evidence/Normalization: PDF of Data

A diagram showing the Bayes' theorem equation with four labels and arrows. A green arrow points from 'Likelihood' to 'P(D|θ)'. A green arrow points from 'Prior' to 'P(θ)'. A green arrow points from 'Posterior' to 'P(θ|D)'. A green arrow points from 'Evidence/Normalization' to 'P(D)'. The equation is enclosed in a light blue box.

Bayesian fwrk. is broader and more comprehensive but the computational part is much more complicated

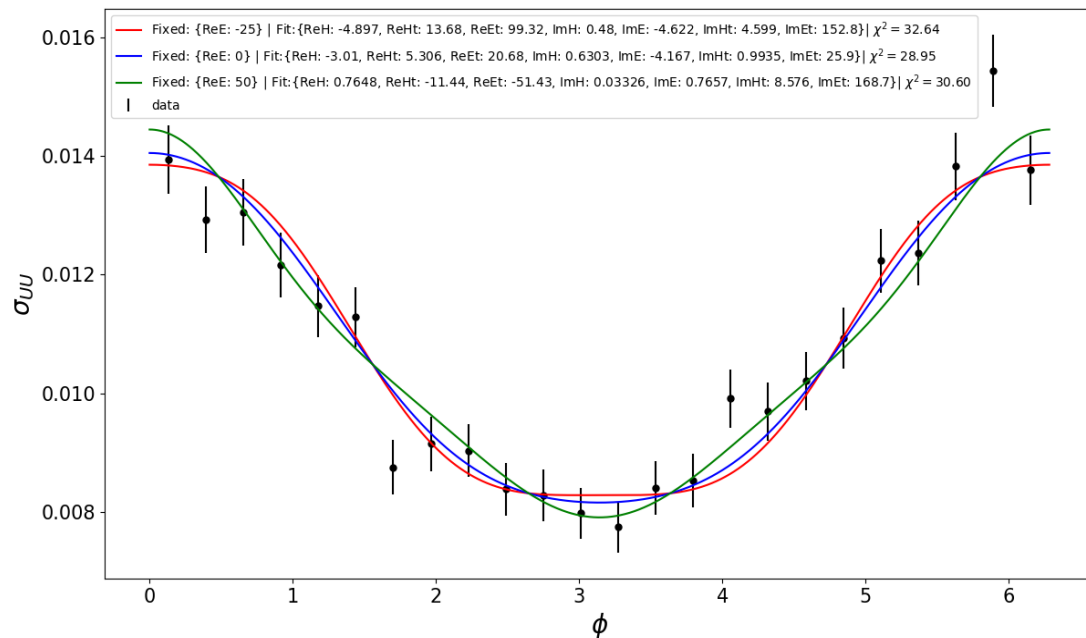
When the number of parameters is large, direct exploration of the posterior probability distribution becomes impractical

We introduce other methods than an exhaustive grid search: Markov Chain Monte Carlo(MCMC)

Deeply Virtual Compton Scattering (DVCS) data

Performing CURVE FIT with “standard methods” results in degeneracy

{Eb: 10.591, x: 0.369, Q: 2.1284, t: -0.2094}



The extracted CFFs are significantly sensitive to the way the fit is done!

— Fixed: {ReE: -25}
— Fixed: {ReE: 0} |
— Fixed: {ReE: 50}

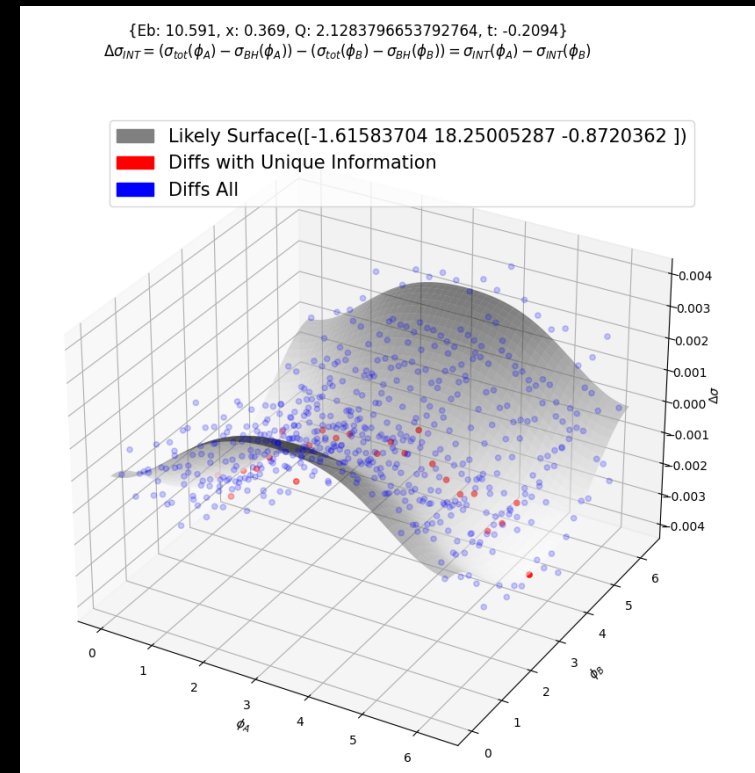
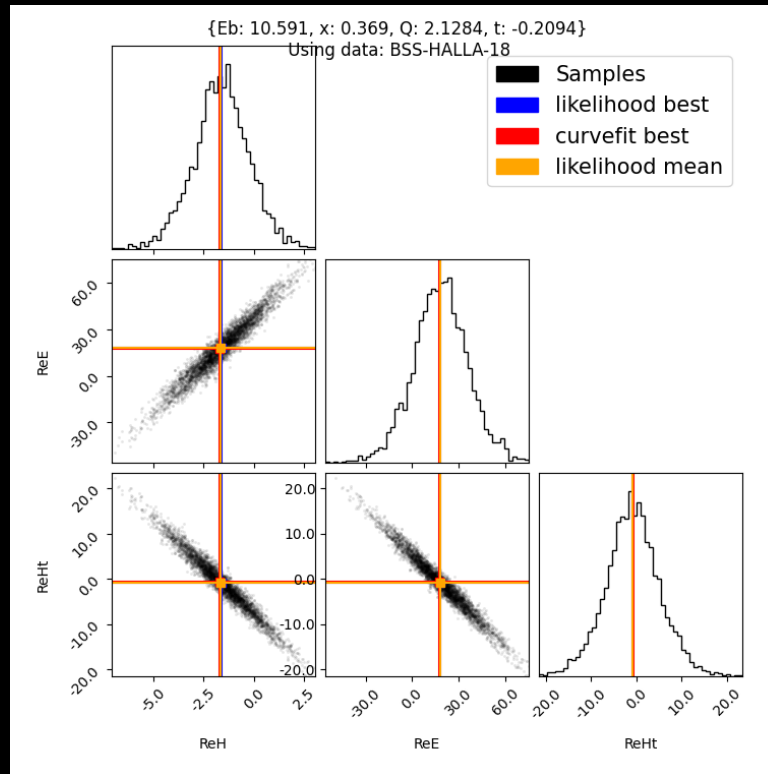
$$Posterior\left(CFFs\right) \approx \frac{Likelihood \times Prior}{Evidence} \approx \frac{Likelihood \times 1}{1}$$

Uninformative prior

Covariance of CFF Results

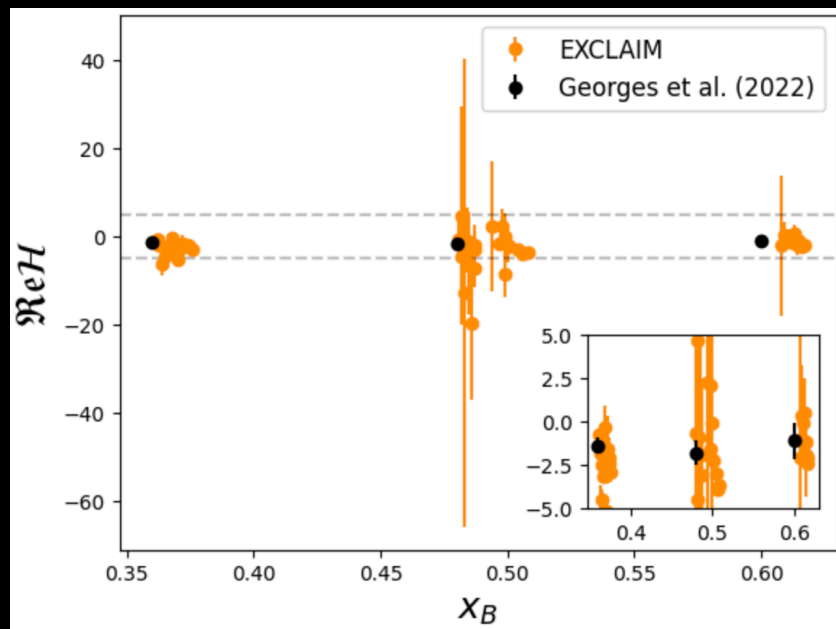
- Here the maximum likelihood is achieved allowing 3 CFFs to vary.
- Only 23 combinations of 2 angles are used.

Douglas Adams et al., arXiv:2410.23469

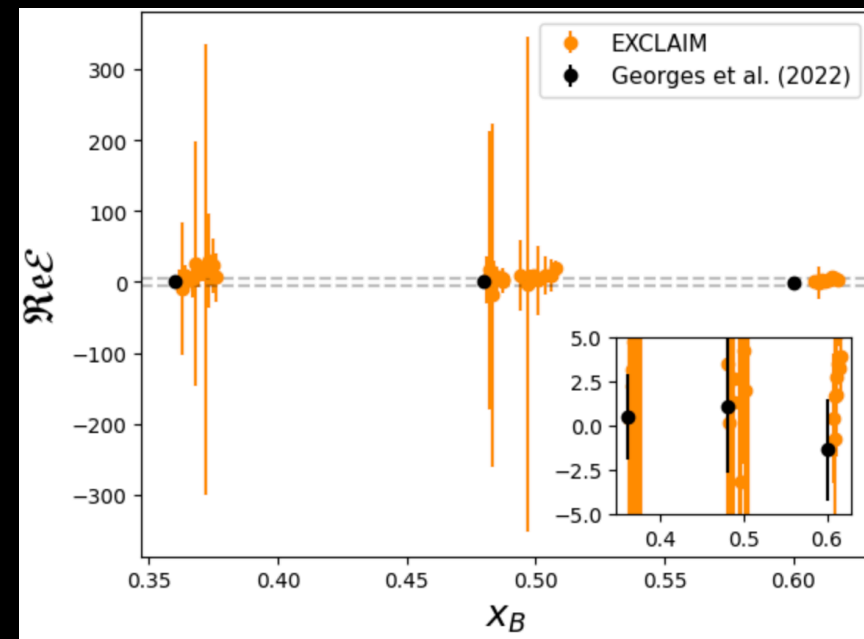


Marginal Posterior CFF Results

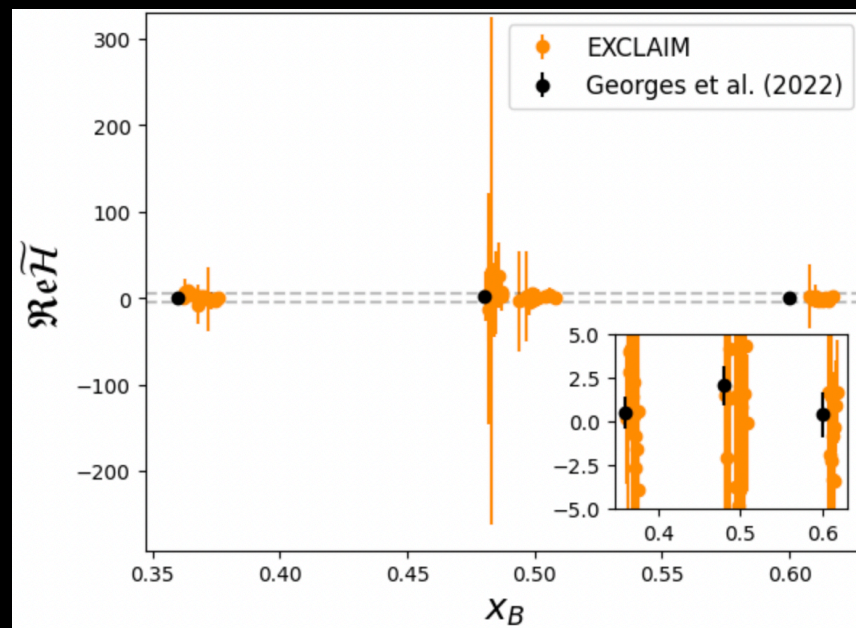
$\mathcal{H}$



$\mathcal{E}$

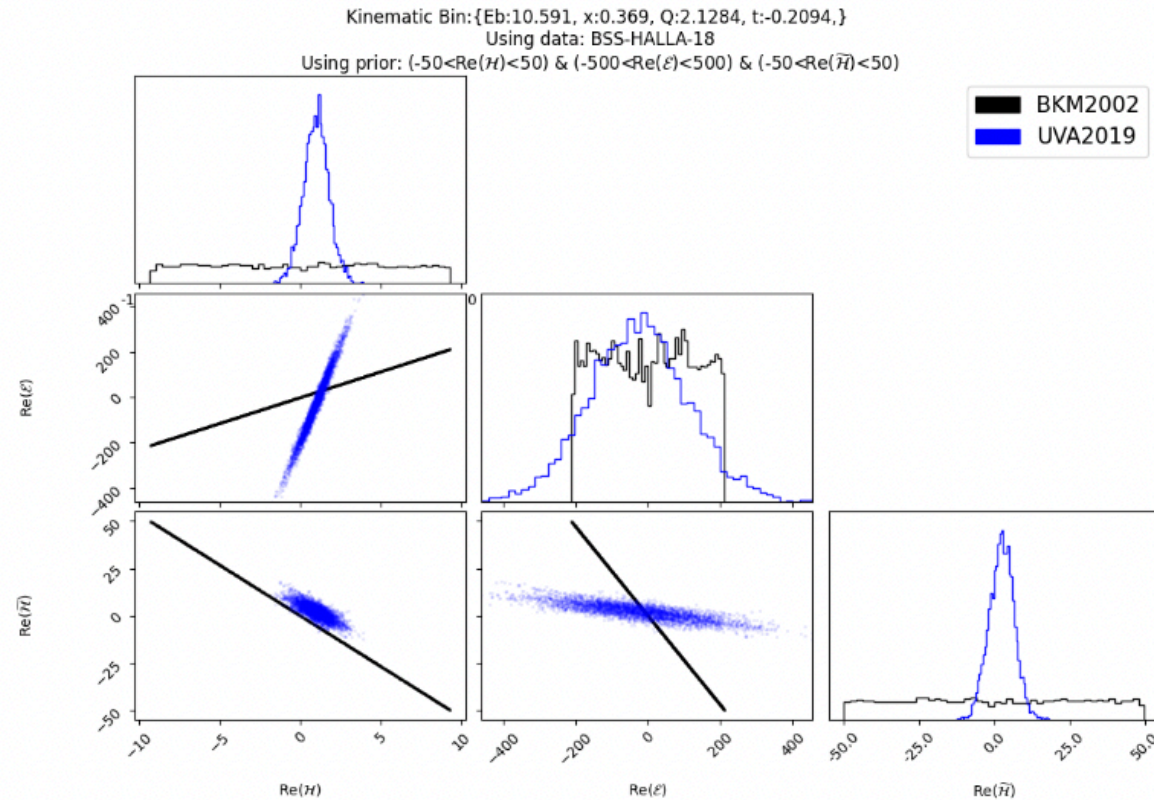


$\tilde{\mathcal{H}}$

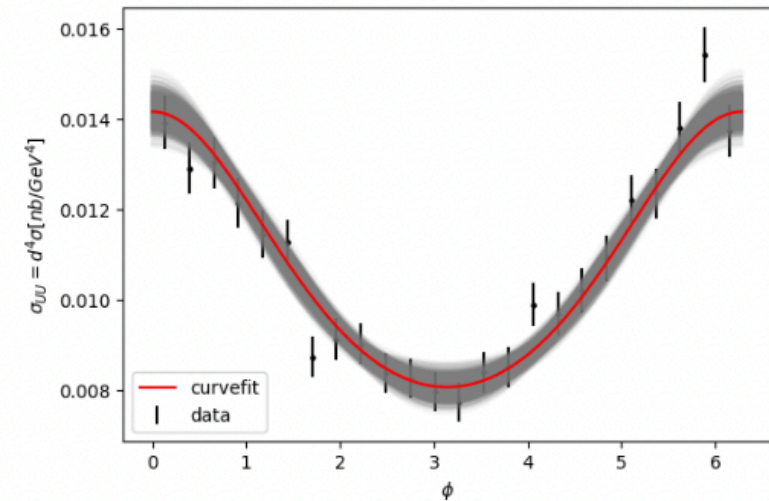


Comparison with harmonics formalism

(Belitsky, Mueller, Kirchner (NPB(2001,2010) Braun et al.)



BKM2002 Kinematic Bin: $\{E_b:10.591, x_{bj}:0.369, Q:2.1284, t:-0.2094,\}$
 Best Curve Fit: $\{\text{Re}(\mathcal{H}):123.1, \text{Re}(\mathcal{E}):2795.0, \text{Re}(\tilde{\mathcal{H}}):-660.2, \sigma_{\text{DVCS}}:0.0023,\}$
 Using data: BSS-HALLA-18



$\mathcal{H}(\xi, t)$

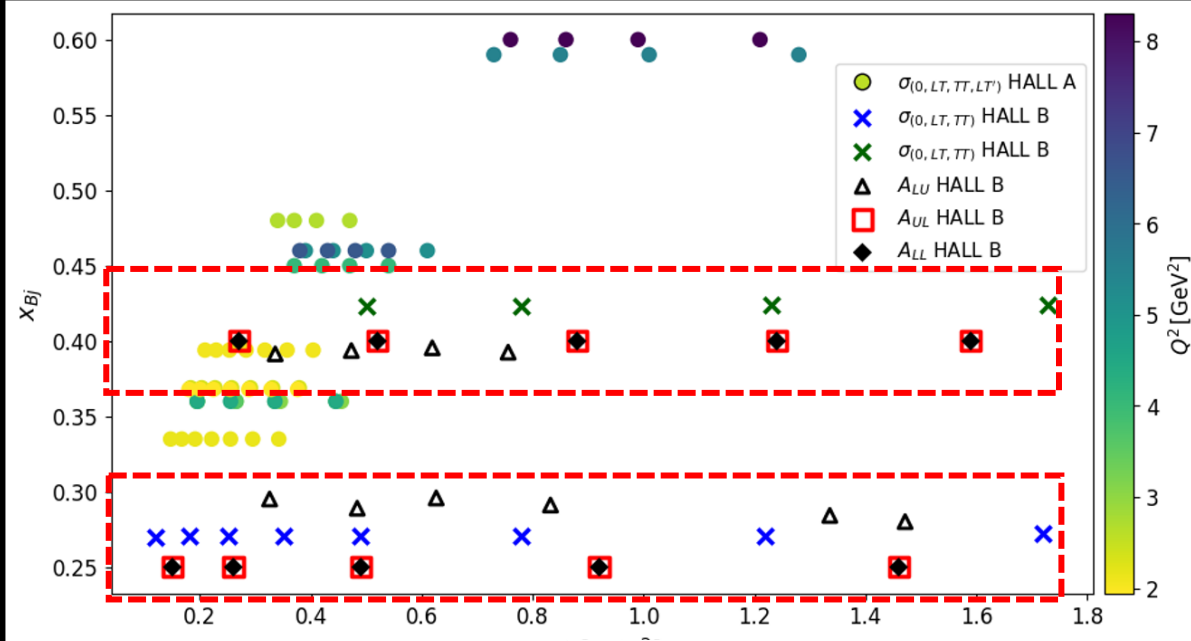
$\mathcal{E}(\xi, t)$

Next: we identified structure functions that help resolve the degeneracy

$$F_{LL}^{\cos\phi} = A_{LL}^{\mathcal{I}} \Re \left(F_1 (\tilde{\mathcal{H}} - \xi \tilde{\mathcal{E}}) + \tau F_2 \tilde{\mathcal{E}} \right) + B_{LL}^{\mathcal{I}} G_M \Re \tilde{\mathcal{H}} + C_{LL}^{\mathcal{I}} G_M \Re (\mathcal{H} + \mathcal{E})$$

LL type data: E.Seder et al. [CLAS], PRL114 (2015)

DV π^0 P data (Jefferson Lab Hall A, Hall B)



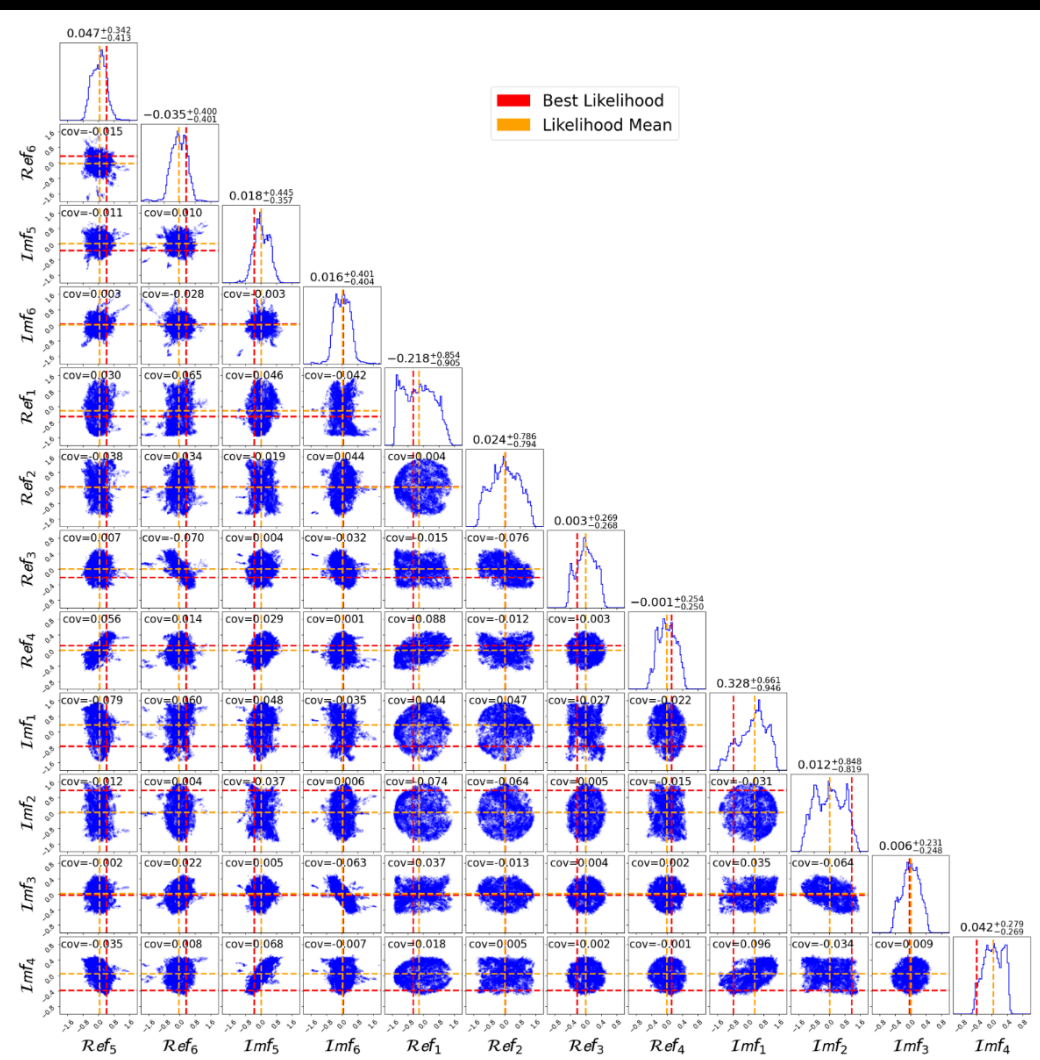
$$\frac{1}{\Gamma} \sigma_{UU} = (\mathbf{F}_{UU,T} + \epsilon \mathbf{F}_{UU,L}) + \epsilon \cos 2\phi \mathbf{F}_{UU}^{\cos 2\phi} + \sqrt{\epsilon(\epsilon+1)} \cos \phi \mathbf{F}_{UU}^{\cos \phi} + h \sqrt{\epsilon(1-\epsilon)} \sin \phi \mathbf{F}_{LU}^{\sin \phi}$$

$$\mathbf{A}_{LU} = \sqrt{\epsilon(1-\epsilon)} \left[\frac{F_{LU}^{\sin \phi}}{F_{UU,T} + \epsilon F_{UU,L}} \right]$$

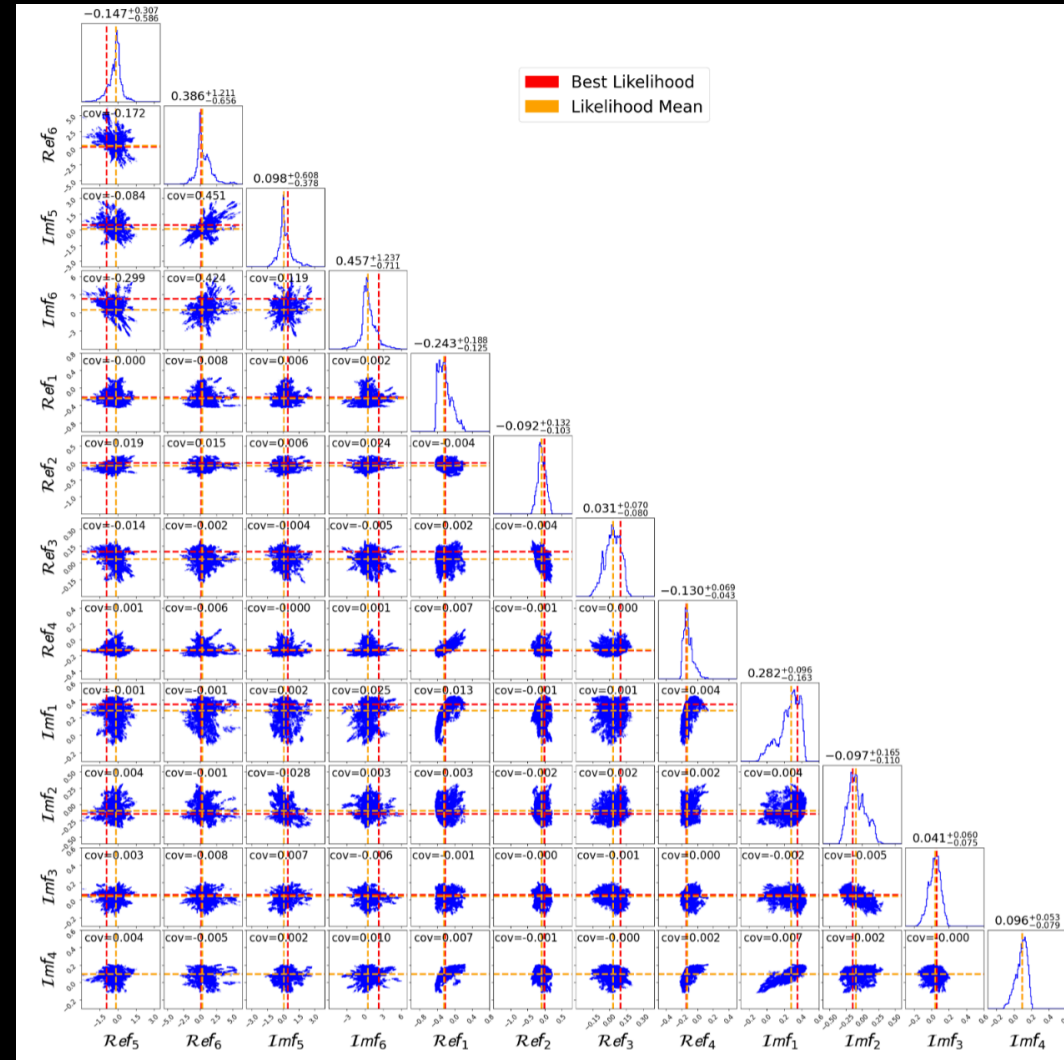
$$A_{UL} = \mathbf{A}_{UL}^{\sin \phi} \sin \phi + \mathbf{A}_{UL}^{\sin 2\phi} \sin 2\phi = \sqrt{\epsilon(\epsilon+1)} \left[\frac{F_{UL}^{\sin \phi}}{F_{UU,T} + \epsilon F_{UU,L}} \right] \sin \phi + \epsilon \left[\frac{F_{UL}^{\sin 2\phi}}{F_{UU,T} + \epsilon F_{UU,L}} \right] \sin 2\phi$$

$$A_{LL} = \mathbf{A}_{LL}^{\text{const}} + \mathbf{A}_{LL}^{\cos \phi} \cos \phi = \sqrt{1-\epsilon^2} \left[\frac{F_{LL}}{F_{UU,T} + \epsilon F_{UU,L}} \right] + \sqrt{\epsilon(1-\epsilon)} \left[\frac{F_{LL}^{\cos \phi}}{F_{UU,T} + \epsilon F_{UU,L}} \right] \cos \phi$$

Multimodality: Hall A and Hall B data are inconsistent

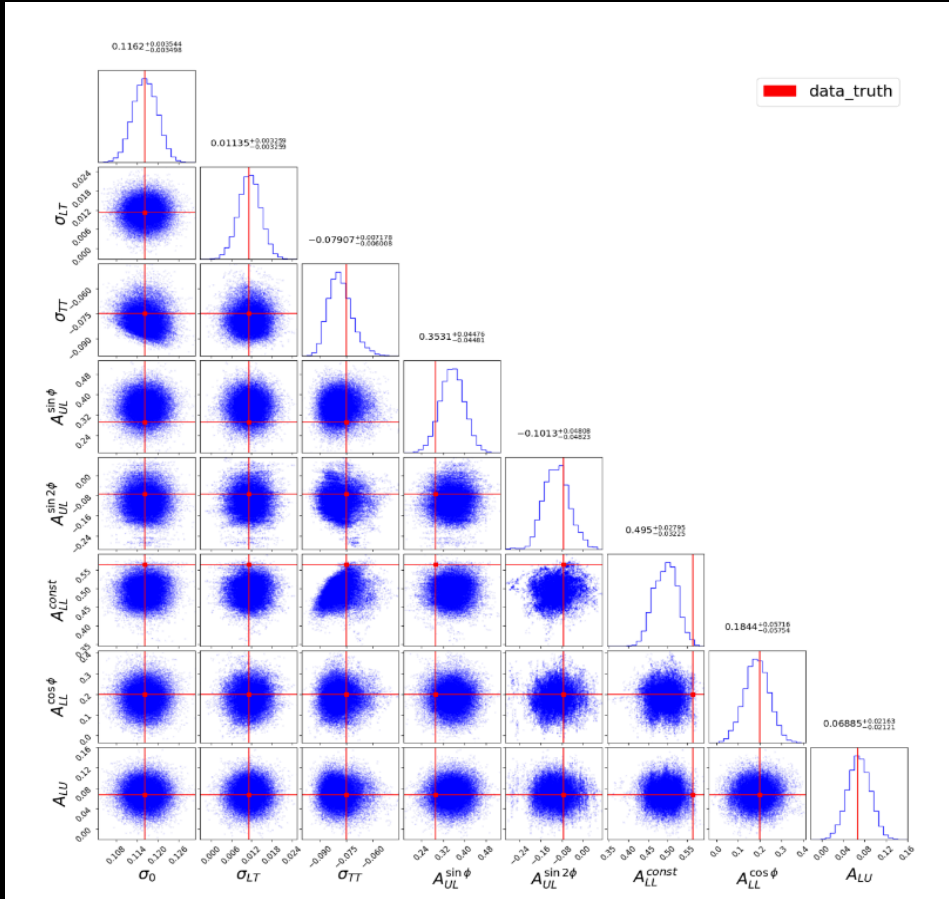


Multimodality is solved if only e.g. Hall B data are analyzed

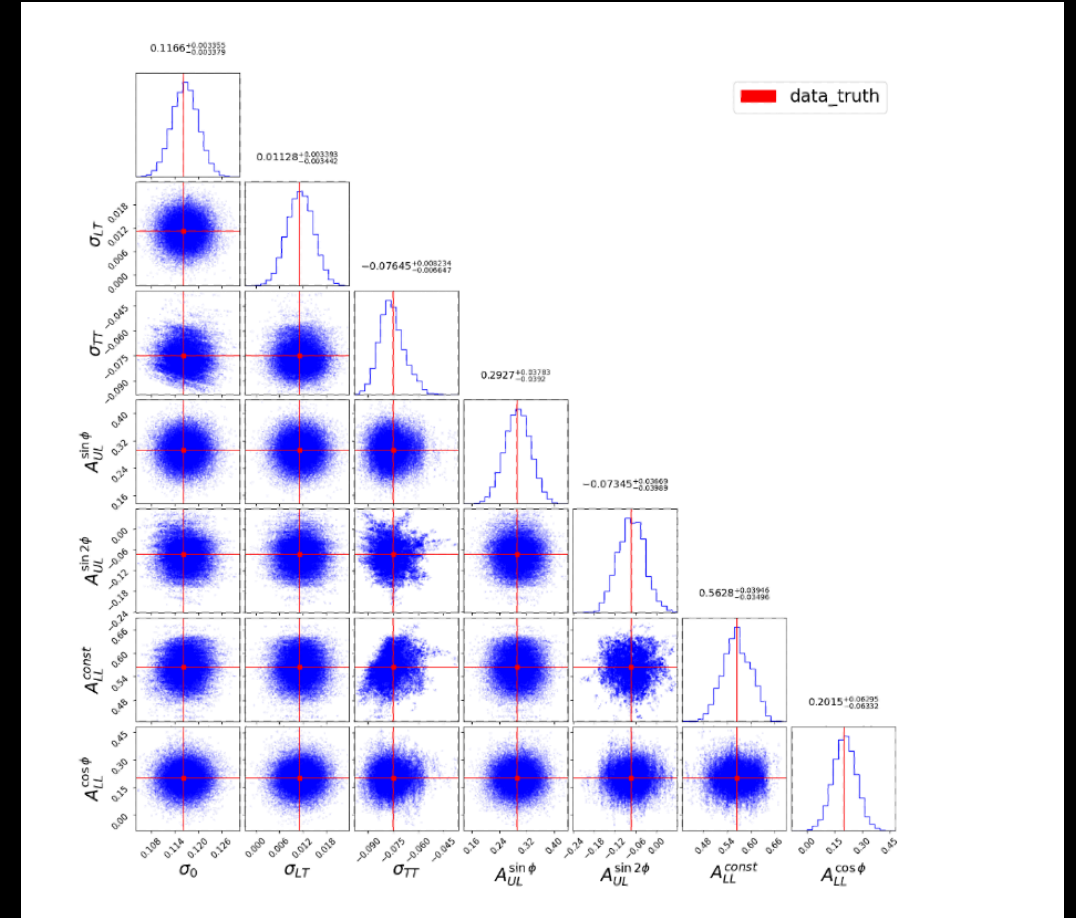


NEW!! Introduce Consistency Check

can we reconstruct the
observables given the MCMC
results on amplitudes

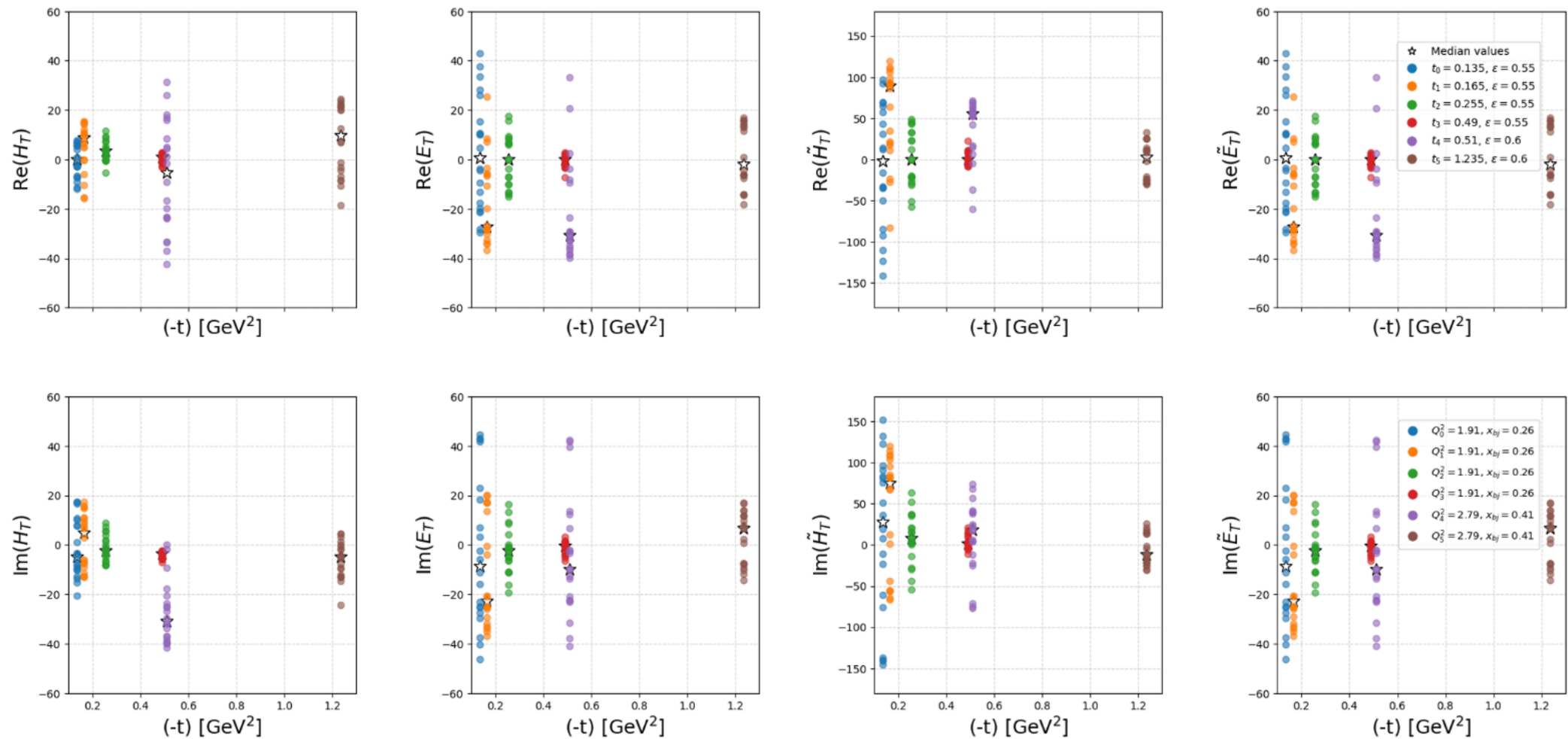


Bad result

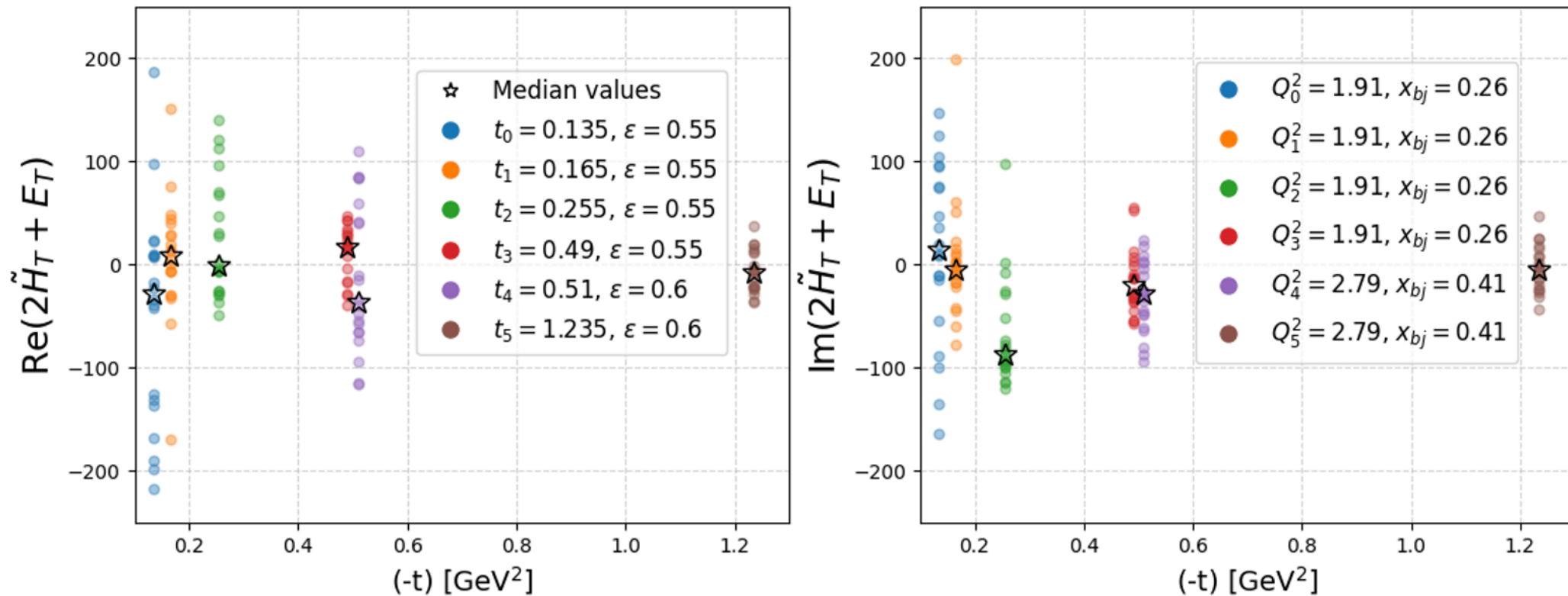


Good result

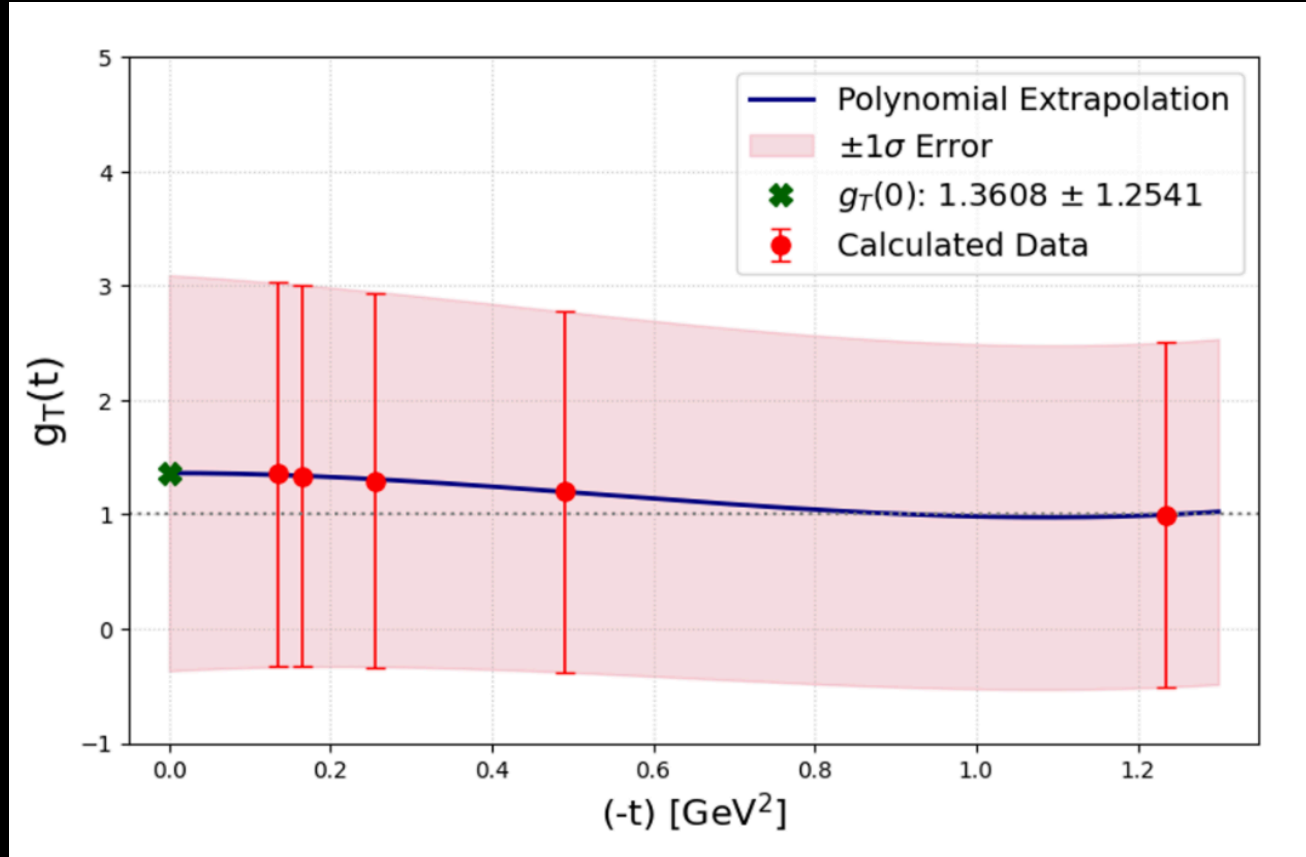
Final spread of Chiral-Odd Compton Form Factors across t -values with $f_2 = 0$



Final spread of transversity GPD $\tilde{E}_T$ across t -values from all 4 HAs



First extraction of the tensor charge



$$H(\xi, \xi, t) = P(t) \left(\mathcal{N} \left(\xi^{-(\alpha + \alpha' (1 - \xi)^p)} t \right) \right)$$

Conclusions and Outlook

- We performed a **Bayesian likelihood analysis with MCMC sampling** of available Jefferson Lab observables on DVES experimental data providing quantitative constraints on the corresponding chiral-even and chiral-odd CFFs.
- This is a nonlinear inverse problem: the measured cross sections/asymmetries constrain bilinear combinations of helicity amplitudes and therefore do not, in general, map uniquely onto individual CFFs.
- **Exploring the full probability distribution is fundamental.** The analysis reveals correlations, asymmetric uncertainties, weakly constrained directions, and possible multiple solutions that cannot necessarily be represented by a local Hessian approximation.
- **The existing data show the way towards meaningful constraints on specific combinations of CFFs.**
- **We provide a new avenue to obtaining the tensor charge from experiment without having to model the x-dependence**
- Additional **polarization observables** and **increased kinematic coverage** will be crucial for resolving degeneracies and improving the determination of CFFs